



HAL
open science

Independent sets in graphs with an excluded clique minor

David R. Wood

► **To cite this version:**

David R. Wood. Independent sets in graphs with an excluded clique minor. *Discrete Mathematics and Theoretical Computer Science*, 2007, Vol. 9 no. 1 (1), pp.171–175. 10.46298/dmtcs.385 . hal-00966507

HAL Id: hal-00966507

<https://inria.hal.science/hal-00966507v1>

Submitted on 26 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Independent Sets in Graphs with an Excluded Clique Minor

David R. Wood[†]

Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya
Barcelona, Spain
david.wood@upc.es

received December 7, 2006, accepted August 22, 2007.

Let G be a graph with n vertices, with independence number α , and with no K_{t+1} -minor for some $t \geq 5$. It is proved that $(2\alpha - 1)(2t - 5) \geq 2n - 5$. This improves upon the previous best bound whenever $n \geq \frac{2}{5}t^2$.

Keywords: graph, minor, independent set, Hadwiger's Conjecture.

Mathematics Subject Classification: 05C15 (Coloring of graphs and hypergraphs)

1 Introduction

In 1943, Hadwiger [7] made the following conjecture, which is widely considered to be one of the most important open problems in graph theory⁽ⁱ⁾; see [19] for a survey.

Hadwiger's Conjecture. For every integer $t \geq 1$, every graph with no K_{t+1} -minor is t -colourable. That is, $\chi(G) \leq \eta(G)$ for every graph G .

Hadwiger's Conjecture is trivial for $t \leq 2$, and is straightforward for $t = 3$; see [4, 7, 22]. In the cases $t = 4$ and $t = 5$, Wagner [20] and Robertson et al. [16] respectively proved that Hadwiger's Conjecture is equivalent to the Four-Colour Theorem [2, 3, 6, 15]. Hadwiger's Conjecture is open for all $t \geq 6$.

[†]Research supported by a Marie Curie Fellowship from the European Commission under contract MEIF-CT-2006-023865, and by the projects MEC MTM2006-01267 and DURSI 2005SGR00692.

⁽ⁱ⁾ All graphs considered in this note are undirected, simple and finite. Let G be a graph with vertex set $V(G)$. Let $X \subseteq V(G)$. X is *connected* if the subgraph of G induced by X is connected. X is *dominating* if every vertex of $G \setminus X$ has a neighbour in X . X is *independent* if no two vertices in X are adjacent. The *independence number* $\alpha(G)$ is the maximum cardinality of an independent set of G . X is a *clique* if every pair of vertices in X are adjacent. The *clique number* $\omega(G)$ is the maximum cardinality of a clique in G . A k -*colouring* of G is a function that assigns one of k colours to each vertex of G such that adjacent vertices receive distinct colours. The *chromatic number* $\chi(G)$ is the minimum integer k such that G is k -colourable. A *minor* of G is a graph that can be obtained from a subgraph of G by contracting edges. The *Hadwiger number* $\eta(G)$ is the maximum integer n such that the complete graph K_n is a minor of G .

Progress on the $t = 6$ case has been recently been obtained by Kawarabayashi and Toft [10] (without using the Four-Colour Theorem). The best known upper bound is $\chi(G) \leq c \cdot \eta(G) \sqrt{\log \eta(G)}$ for some constant c , independently due to Kostochka [11] and Thomason [17, 18].

Woodall [21] observed that since $\alpha(G) \cdot \chi(G) \geq |V(G)|$ for every graph G , Hadwiger's Conjecture implies that

$$\alpha(G) \cdot \eta(G) \geq |V(G)|. \quad (1)$$

Equation (1) holds for $\eta(G) \leq 5$ since Hadwiger's Conjecture holds for $t \leq 5$. For example, $\alpha(G) \geq \frac{1}{4}|V(G)|$ for every planar graph G . It is interesting that the only known proof of this result depends on the Four-Colour Theorem. The best bound not using the Four-Colour Theorem is $\alpha(G) \geq \frac{2}{9}|V(G)|$ due to Albertson [1].

Equation (1) is open for $\eta(G) \geq 6$. In general, (1) is weaker than Hadwiger's Conjecture, but for graphs with $\alpha(G) = 2$ (that is, graphs whose complements are triangle-free), Plummer et al. [13] proved that (1) is in fact equivalent to Hadwiger's Conjecture. The first significant progress towards (1) was made by Duchet and Meyniel [5] (also see [12]), who proved that

$$(2\alpha(G) - 1) \cdot \eta(G) \geq |V(G)|. \quad (2)$$

This result was improved by Kawarabayashi et al. [8] to

$$(2\alpha(G) - 1) \cdot \eta(G) \geq |V(G)| + \omega(G). \quad (3)$$

Assuming $\alpha(G) \geq 3$, Kawarabayashi et al. [8] proved that

$$(4\alpha(G) - 3) \cdot \eta(G) \geq 2|V(G)|, \quad (4)$$

which was further improved by Kawarabayashi and Song [9] to

$$(2\alpha(G) - 2) \cdot \eta(G) \geq |V(G)|. \quad (5)$$

The following theorem is the main contribution of this note.

Theorem 1 *Every graph G with $\eta(G) \geq 5$ satisfies*

$$(2\alpha(G) - 1)(2\eta(G) - 5) \geq 2|V(G)| - 5.$$

Observe that Theorem 1 represents an improvement over (2), (4) and (5) whenever $\eta(G) \geq 5$ and $|V(G)| \geq \frac{2}{5}\eta(G)^2$. For example, Theorem 1 implies that $\alpha(G) > \frac{1}{7}|V(G)|$ for every graph G with $\eta(G) \leq 6$, whereas each of (2), (4) and (5) imply that $\alpha(G) > \frac{1}{12}|V(G)|$.

2 Proof of Theorem 1

Theorem 1 employs the following lemma by Duchet and Meyniel [5]. The proof is included for completeness.

Lemma 1 ([5]) *Every connected graph G has a connected dominating set D and an independent set $S \subseteq D$ such that $|D| = 2|S| - 1$.*

Proof: Let D be a maximal connected set of vertices of G such that D contains an independent set S of G and $|D| = 2|S| - 1$. There is such a set since $D := S := \{v\}$ satisfies these conditions for each vertex v . We claim that D is dominating. Otherwise, since G is connected, there is a vertex v at distance 2 from D , and there is a neighbour w of v at distance 1 from D . Let $D' := D \cup \{v, w\}$ and $S' := S \cup \{v\}$. Thus D' is connected and contains an independent set S' such that $|D'| = 2|S'| - 1$. Hence D is not maximal. This contradiction proves that D is dominating. \square

The next lemma is the key to the proof of Theorem 1.

Lemma 2 *Suppose that for some integer $t \geq 1$ and for some real number $p \geq t$, every graph G with $\eta(G) \leq t$ satisfies $p \cdot \alpha(G) \geq |V(G)|$. Then every graph G with $\eta(G) \geq t$ satisfies*

$$\alpha(G) \geq \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2}.$$

Proof: We proceed by induction on $\eta(G) - t$. If $\eta(G) = t$ the result holds by assumption. Let G be a graph with $\eta(G) > t$. We can assume that G is connected. By Lemma 1, G has a connected dominating set D and an independent set $S \subseteq D$ such that $|D| = 2|S| - 1$. Now $\alpha(G) \geq |S| = \frac{|D|+1}{2}$. Thus we are done if

$$\frac{|D| + 1}{2} \geq \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2}. \quad (6)$$

Now assume that (6) does not hold. That is,

$$|D| \leq \frac{2|V(G)| - p}{2\eta(G) + p - 2t}.$$

Thus

$$|V(G \setminus D)| = |V(G)| - |D| \geq \frac{(2\eta(G) + p - 2t)|V(G)| + p}{2\eta(G) + p - 2t}.$$

Since D is dominating and connected, $\eta(G \setminus D) \leq \eta(G) - 1$. Thus by induction,

$$\begin{aligned} \alpha(G) \geq \alpha(G \setminus D) &\geq \frac{2|V(G \setminus D)| - p}{4\eta(G \setminus D) + 2p - 4t} + \frac{1}{2} \\ &\geq \frac{2(2\eta(G) + p - 2t)|V(G)| + 2p}{(2\eta(G) + p - 2t)(4\eta(G) - 4 + 2p - 4t)} - \frac{p}{4\eta(G) - 4 + 2p - 4t} + \frac{1}{2} \\ &= \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2}. \end{aligned}$$

This completes the proof. \square

Lemma 3 *Suppose that Hadwiger's Conjecture is true for some integer t . Then every graph G with $\eta(G) \geq t$ satisfies*

$$(2\eta(G) - t)(2\alpha(G) - 1) \geq 2|V(G)| - t.$$

Proof: If Hadwiger's Conjecture is true for t then $t \cdot \alpha(G) \geq |V(G)|$ for every graph G with $\eta(G) \leq t$. Thus Lemma 2 with $p = t$ implies that every graph G with $\eta(G) \geq t$ satisfies

$$\alpha(G) \geq \frac{2|V(G)| - t}{4\eta(G) - 2t} + \frac{1}{2},$$

which implies the result. □

Theorem 1 follows from Lemma 3 with $t = 5$ since Hadwiger's Conjecture holds for $t = 5$ [16].

3 Concluding Remarks

The proof of Theorem 1 is substantially simpler than the proofs of (3)–(5), ignoring its dependence on the proof of Hadwiger's Conjecture with $t = 5$, which in turn is based on the Four-Colour Theorem. A bound that still improves upon (2), (4) and (5) but with a completely straightforward proof is obtained from Lemma 3 with $t = 3$: Every graph G with $\eta(G) \geq 3$ satisfies $(2\eta(G) - 3)(2\alpha(G) - 1) \geq 2|V(G)| - 3$.

We finish with an open problem. The method of Duchet and Meyniel [5] was generalised by Reed and Seymour [14] to prove that the fractional chromatic number $\chi_f(G) \leq 2\eta(G)$. For sufficiently large $\eta(G)$, is $\chi_f(G) \leq 2\eta(G) - c$ for some constant $c \geq 1$?

References

- [1] MICHAEL O. ALBERTSON. A lower bound for the independence number of a planar graph. *J. Combinatorial Theory Ser. B*, 20(1):84–93, 1976.
- [2] KENNETH APPEL AND WOLFGANG HAKEN. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.
- [3] KENNETH APPEL, WOLFGANG HAKEN, AND JOHN KOCH. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.
- [4] GABRIEL A. DIRAC. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1952.
- [5] PIERRE DUCHET AND HENRI MEYNIEL. On Hadwiger's number and the stability number. *Annals of Discrete Mathematics*, 13:71–73, 1982.
- [6] GEORGES GONTHIER. A computer-checked proof of the four colour theorem. 2005. Microsoft Research, Cambridge, U.S.A.
- [7] HUGO HADWIGER. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943.
- [8] KEN-ICHI KAWARABAYASHI, MICHAEL D. PLUMMER, AND BJARNE TOFT. Improvements of the theorem of Duchet and Meyniel on Hadwiger's conjecture. *J. Combin. Theory Ser. B*, 95(1):152–167, 2005.
- [9] KEN-ICHI KAWARABAYASHI AND ZI-XIA SONG. Independence number and clique minors. *J. Graph Theory*, to appear.

- [10] KEN-ICHI KAWARABAYASHI AND BJARNE TOFT. Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. *Combinatorica*, 25(3):327–353, 2005.
- [11] ALEXANDR V. KOSTOCHKA. The minimum Hadwiger number for graphs with a given mean degree of vertices. *Metody Diskret. Analiz.*, 38:37–58, 1982.
- [12] FRÉDÉRIC MAFFRAY AND HENRI MEYNIEL. On a relationship between Hadwiger and stability numbers. *Discrete Math.*, 64(1):39–42, 1987.
- [13] MICHAEL D. PLUMMER, MICHAEL STIEBITZ, AND BJARNE TOFT. On a special case of Hadwiger’s conjecture. *Discuss. Math. Graph Theory*, 23(2):333–363, 2003.
- [14] BRUCE REED AND PAUL SEYMOUR. Fractional colouring and Hadwiger’s conjecture. *J. Combin. Theory Ser. B*, 74(2):147–152, 1998.
- [15] NEIL ROBERTSON, DANIEL P. SANDERS, PAUL D. SEYMOUR, AND ROBIN THOMAS. The four-colour theorem. *J. Combin. Theory Ser. B*, 70(1):2–44, 1997.
- [16] NEIL ROBERTSON, PAUL D. SEYMOUR, AND ROBIN THOMAS. Hadwiger’s conjecture for K_6 -free graphs. *Combinatorica*, 13(3):279–361, 1993.
- [17] ANDREW THOMASON. An extremal function for contractions of graphs. *Math. Proc. Cambridge Philos. Soc.*, 95(2):261–265, 1984.
- [18] ANDREW THOMASON. The extremal function for complete minors. *J. Combin. Theory Ser. B*, 81(2):318–338, 2001.
- [19] BJARNE TOFT. A survey of Hadwiger’s conjecture. *Congr. Numer.*, 115:249–283, 1996.
- [20] KLAUS WAGNER. Über eine Eigenschaft der ebene Komplexe. *Math. Ann.*, 114:570–590, 1937.
- [21] DOUGLAS R. WOODALL. Subcontraction-equivalence and Hadwiger’s conjecture. *J. Graph Theory*, 11(2):197–204, 1987.
- [22] DOUGLAS R. WOODALL. A short proof of a theorem of Dirac’s about Hadwiger’s conjecture. *J. Graph Theory*, 16(1):79–80, 1992.

