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# On the $k^{\text{th}}$ Eigenvalues of Trees with Perfect Matchings

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Let  $\mathcal{T}_{2p}^+$  be the set of all trees on  $2p$  ( $p \geq 1$ ) vertices with perfect matchings. In this paper, we prove that for any tree  $T$  in  $\mathcal{T}_{2p}^+$ , the  $k$ th largest eigenvalue  $\lambda_k(T)$  satisfies  $\lambda_k(T) \leq \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor - 1} + \sqrt{\lfloor \frac{p}{k} \rfloor + 3} \right)$  ( $k = 1, 2, \dots, p$ ). This upper bound is known to be best possible when  $k = 1$ . The set of trees obtained from a tree on  $p$  vertices by joining a pendent vertex to each vertex of the tree is denoted by  $\mathcal{T}_{2p}^*$ . We also prove that for any tree  $T$  in  $\mathcal{T}_{2p}^*$ , its  $k$ th largest eigenvalue  $\lambda_k(T)$  satisfies  $\lambda_k(T) \leq \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor - 1} + \sqrt{\lfloor \frac{p}{k} \rfloor + 3} \right)$  ( $k = 1, 2, \dots, p$ ) and show that this upper bound is best possible when  $k = 1$  or  $p \not\equiv 0 \pmod{k}$ . We further give the following inequality

$$\lambda_k^*(2p) > \frac{1}{2} \left( \sqrt{t-1 - \sqrt{\frac{k-1}{t-k}}} + \sqrt{t+3 - \sqrt{\frac{k-1}{t-k}}} \right) \quad t = \lfloor \frac{p}{k} \rfloor,$$

where  $\lambda_k^*(2p)$  is the maximum value of the  $k$ th largest eigenvalue of the trees in  $\mathcal{T}_{2p}^*$ . By this inequality, it is easy to see that the above upper bound on  $\lambda_k(T)$  for  $T \in \mathcal{T}_{2p}^*$  turns out to be asymptotically tight when  $p \equiv 0 \pmod{k}$ .

**Keywords:** tree, eigenvalue, perfect matching.

## 1 Introduction

Let  $G$  be a simple graph, i.e., a graph without loops or multiple edges. Suppose the vertex set of  $G$  is  $V(G) = \{v_1, v_2, \dots, v_n\}$ . The adjacency matrix of  $G$  is an  $n \times n$  matrix  $A(G) = (a_{ij})$ , where  $a_{ij} = 1$  if  $v_i$  is adjacent to  $v_j$ , and  $a_{ij} = 0$  otherwise. The characteristic polynomial of  $G$  is  $\det(\lambda I - A(G))$ , which is denoted by  $P(G; \lambda)$ . Since  $A(G)$  is symmetric, its eigenvalues are real; moreover, they are independent of the ordering of the vertices of  $G$ . As usual, we write them in non-increasing order as  $\lambda_1(G) \geq \lambda_2(G) \geq \lambda_3(G) \geq \dots \geq \lambda_n(G)$  and call them the eigenvalues of  $G$ . If  $G$  is a bipartite graph, then  $\lambda_i(G) = -\lambda_{n-i+1}(G)$  for  $i = 1, 2, \dots, \lfloor n/2 \rfloor$  (see [6]), where  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ , i.e., the floor function of  $x$  when  $x$  is a real number. Similarly,  $\lceil x \rceil$  denotes the least integer greater than or equal to  $x$ , i.e., the ceiling function of  $x$ .

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Two distinct edges in a graph  $G$  incident with the same vertex will be called *adjacent edges*. A *matching* of  $G$  is a set of edges in  $G$  such that no two of them are adjacent. A largest matching is called a *maximum matching*. The cardinality of a maximum matching of  $G$  is commonly known as its *matching number*, denoted by  $\mu(G)$ . Let  $M$  be a matching of  $G$ .  $M$  is called an *s-matching* of  $G$  if  $M$  contains exactly  $s$  edges of  $G$ . A vertex  $v \in V(G)$  is said to be *M-saturated* if it is incident with an edge of  $M$ , otherwise  $v$  is called an *M-unsaturated vertex*. The matching  $M$  of  $G$  is called a *perfect matching* if all vertices of  $G$  are  $M$ -saturated. Trees are connected acyclic graphs, and it is obvious that they are also bipartite graphs. So we only need to investigate those eigenvalues  $\lambda_k(T)$  of a tree  $T$  with  $n$  vertices for  $k = 1, 2, \dots, \lfloor n/2 \rfloor$ .

Throughout this paper, we denote by  $\mathcal{T}_n$  and  $\mathcal{T}_{2p}^+$  the set of trees on  $n$  vertices and the set of trees on  $2p$  vertices with perfect matchings. For simplicity, a tree with  $n$  vertices is often called a tree of order  $n$ . For symbols and concepts not defined in this paper we refer to the book [2].

The investigation on the eigenvalues of trees in  $\mathcal{T}_n$  is one of the oldest problems in the spectral theory of graphs and has been intensively studied by many authors (see [1, 6, 11, 12, 13, 15]). A classic result is that for any  $T \in \mathcal{T}_n$ ,  $\lambda_1(T) \leq \sqrt{n-1}$  and equality holds if and only if  $T$  is the star  $K_{1,n-1}$ . In particular, H. Yuan [12] studied the  $k$ th eigenvalue of a tree  $T \in \mathcal{T}_n$  and obtained the following upper bound.

**Theorem 1.1 ([12])** *Let  $T$  be a tree in  $\mathcal{T}_n$ . Then*

$$\lambda_k(T) \leq \sqrt{\left\lfloor \frac{n-2}{k} \right\rfloor} \quad \left(2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\right)$$

and the upper bound is best possible if  $n \equiv 1 \pmod{k}$ .

J.Y. Shao [15] improved the above result.

**Theorem 1.2 ([15])** *Let  $T$  be a tree in  $\mathcal{T}_n$ . Then*

$$\lambda_k(T) \leq \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} - 1 \quad \left(1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\right).$$

Moreover, the bound is best possible when  $n \not\equiv 0 \pmod{k}$  and it is an asymptotically tight bound when  $n \equiv 0 \pmod{k}$  ( $2 \leq k \leq \lfloor n/2 \rfloor$ ).

Concerning the trees in  $\mathcal{T}_{2p}^+$  there are lots of results on the first two largest eigenvalues (see [3, 4, 5, 8, 9, 10, 16, 17, 18]).

Frucht and Harary [7] gave the following construction of graphs. Given two graphs  $G$  and  $H$ , the *corona of  $G$  with  $H$* , denoted by  $G \odot H$ , is the graph with

$$\begin{aligned} V(G \odot H) &= V(G) \cup \bigcup_{i \in V(G)} V(H_i), \\ E(G \odot H) &= E(G) \cup \bigcup_{i \in V(G)} \left( E(H_i) \cup \{iu_i \mid u_i \in V(H_i)\} \right), \end{aligned}$$

where  $H_i \cong H$  for all  $i \in V(G)$ .

Let  $T_{2p}^1 = K_{1,p-1} \odot N_1$  (see Fig. 1.1), where  $N_s$  is the null graph (i.e., edgeless graph) of order  $s$ . G.H. Xu [17] got the following initial result.

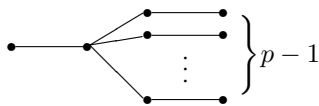


Fig. 1.1: The tree  $T_{2p}^1$

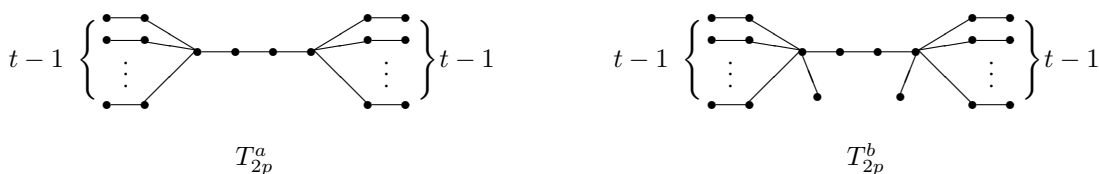


Fig. 1.2: Two graphs  $T_{2p}^a$  and  $T_{2p}^b (= T_{2p}^{2'})$

**Theorem 1.3 ([17])** Let  $T$  be a tree in  $\mathcal{T}_{2p}^+$ . Then

$$\lambda_1(T) \leq \frac{1}{2}(\sqrt{p-1} + \sqrt{p+3}) = \lambda_1(T_{2p}^1) \quad p = 1, 2, 3, \dots$$

and equality holds if and only if  $T \cong T_{2p}^1$ .

A. Chang [3] studied bounds for the second largest eigenvalue of trees in  $\mathcal{T}_{2p}^+$  and proposed the following conjecture:

Let  $p$  be a positive integer, and  $T$  be a tree in  $\mathcal{T}_{2p}^+$ . Then

$$\lambda_2(T) \leq \begin{cases} r' & \text{if } p = 2t \\ r'' & \text{if } p = 2t + 1 \end{cases} \quad \text{for } t = 1, 2, 3, \dots,$$

where  $r'$  and  $r''$  are the maximum positive roots of the equations  $x^3 - (t+1)x + 1 = 0$  and  $x^4 - (t+2)x^2 + x + 1 = 0$ , respectively. Equality holds in the first inequality if and only if  $T \cong T_{2p}^a$ , and equality holds in the second inequality if and only if  $T \cong T_{2p}^b$ , where  $T_{2p}^a$  and  $T_{2p}^b$  are the trees shown in Fig. 1.2

More recently, J-M. Guo and S-W. Tan [9] proved that the second inequality holds but the first one does not hold. A correct version of the first inequality was given by J-M. Guo and S-W. Tan in [10]. Their results can be stated as follows.

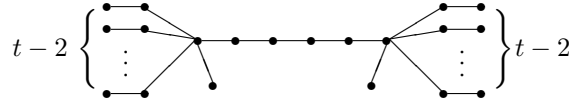


Fig. 1.3: The tree  $T_{2p}^2$

**Theorem 1.4 ([9, 10])** Let  $p$  be a positive integer, and  $T$  be a tree in  $\mathcal{T}_{2p}^+$ . Then

$$\lambda_2(T) \leq \begin{cases} r_1 & \text{if } p = 2t \\ r_2 & \text{if } p = 2t + 1 \end{cases} \text{ for } t = 2, 3, \dots,$$

where  $r_1$  and  $r_2$  are the maximum positive roots of the equations  $(x^4 - (t + 1)x^2 + 1)(x^2 + x - 1) + x = 0$  and  $x^4 - (t + 2)x^2 + x + 1 = 0$ , respectively. Equality holds in the first inequality if and only if  $T \cong T_{2p}^2$ , and equality holds in the second inequality if and only if  $T \cong T_{2p}^{2'}$ , where  $T_{2p}^2$  and  $T_{2p}^{2'} \cong T_{2p}^b$  are the trees shown in Fig. 1.3 and 1.2, respectively.

It is natural to consider the problem of determining upper and lower bounds of the  $k$ th eigenvalues of the trees in  $\mathcal{T}_{2p}^+$ . This is the purpose of our paper.

## 2 Main results

We need some groundwork before giving the main result. Before we recall the well-known *Cauchy Interlacing Theorem* [6, Theorem 0.10], we introduce some notation and terminology first. A vertex subset with  $k$  vertices is called a  $k$ -vertex subset. Suppose  $V'$  is a subset of vertices.  $G - V'$  is the subgraph of  $G$  obtained by deleting all vertices in  $V'$  together with their incident edges. Cauchy Interlacing Theorem usually plays an important role in the estimation of the  $k$ th eigenvalue of trees.

**Theorem 2.1 (Cauchy Interlacing Theorem)** For every graph  $G$  and every  $k$ -vertex subset  $V'$  we have

$$\lambda_i(G) \geq \lambda_i(G - V') \geq \lambda_{i+k}(G), \quad i = 1, 2, \dots, n - k.$$

**Lemma 2.2 ([1])** Let  $G$  be a graph and  $H$  a subgraph of  $G$ . Then  $\lambda_1(H) \leq \lambda_1(G)$ .

**Lemma 2.3 ([15])** Let  $T$  be a tree in  $\mathcal{T}_n$ . Then for any positive integer  $s$ , there exists a vertex  $v \in V(T)$  such that the largest component of  $T - v$  has order at most  $\max\{n - 1 - s, s\}$  and all other components of  $T - v$  have orders at most  $s$ .

It is worth mentioning that when the tree  $T$  considered in Lemma 2.3 is in  $\mathcal{T}_{2p}^+$ , i.e.,  $T$  is a tree with a perfect matching, then obviously all components of  $T - v$  but one have perfect matchings. The only component without perfect matching, say  $T_0$ , has matching number  $\mu(T_0) = \frac{1}{2}(V(T_0) - 1)$ , and the only unsaturated vertex of  $T_0$  is the vertex  $w$  which is adjacent with  $v$  in  $T$  and  $wv$  is an edge of the perfect matching of  $T$ . This fact leads us to get the following useful lemma.

**Lemma 2.4** *Let  $T \in \mathcal{T}_n^+$ , and let  $s$  be a positive even integer not greater than  $n$ . Then there exist a vertex  $v \in V(T)$  and a subtree  $U$  of  $T$  such that*

1.  $U$  has a perfect matching;
2. either  $U$  is a component of  $T - v$  when  $v \notin V(U)$ , or  $U - v$  is a component of  $T - v$  when  $v \in V(U)$ ;
3.  $|V(U)| \leq \max\{n - s, s\}$ ;
4. all components of  $(T - V(U)) - v$  have order at most  $s$ , and all but at most one of them have a perfect matching.

**Proof:** Let  $M$  be a perfect matching of  $T$ . By Lemma 2.3, there exists a vertex  $v \in V(T)$  such that one component  $T'$  of  $T - v$  has order  $|V(T')| \leq \max\{n - 1 - s, s\}$ , and all other components of  $T - v$  have orders not exceeding  $s$ . We know that only one component, say  $T_0$ , has no perfect matching and all the others have perfect matchings.

Suppose  $T' \neq T_0$ . Then  $M \cap E(T')$  is a perfect matching of  $T'$ . Since  $s$  and  $n$  are even,  $|V(T')| \leq \max\{n - 2 - s, s\}$ . Let  $U = T'$ . Then  $T_0$  is a component of  $(T - V(U)) - v$  and its matching number is  $\mu(T_0) = \frac{1}{2}(|V(T_0)| - 1)$ . Let  $w$  be the only unsaturated vertex of  $T_0$  which is adjacent with  $v$  in  $T$  and  $vw$  is an edge of  $M$ . Now let  $T'_0$  be the tree obtained from  $T_0$  by joining a pendant vertex  $u$  to  $w$ . Actually, we can view this vertex  $u$  as the removed vertex  $v$ . Obviously,  $T'_0$  has a perfect matching  $(E(T_0) \cap M) \cup \{vw\}$  and order  $|V(T'_0)| = |V(T_0)| + 1 \leq s$ , and  $T'_0$  is a subtree of  $T$ .

Suppose  $T' = T_0$ . Then  $T'$  has a maximum matching  $M_1 = E(T') \cap M$  and its matching number is  $\mu(T') = \frac{1}{2}(|V(T')| - 1)$ . Since  $s$  is even,  $|V(T')| \leq \max\{n - 1 - s, s - 1\}$ . Let  $w \in V(T')$  be the only  $M_1$ -unsaturated vertex. Then  $vw$  is an edge of  $M$ . Let  $U$  be the tree obtained from  $T'$  by joining a pendant vertex  $u$  to  $w$ . Actually, we can view this vertex  $u$  as the removed vertex  $v$ . Then  $U$  is a subtree of  $T$  and is of order not greater than  $\max\{n - s, s\}$ . Clearly,  $M_1 \cup \{vw\}$  is a perfect matching of  $U$  and  $U - v = T'$ . □

**Lemma 2.5** *Let  $T$  be a tree in  $\mathcal{T}_{2p}^+$ . Then for any positive integer  $k$  with  $1 \leq k \leq p$ , there exists a  $(k - 1)$ -vertex subset  $V' \subset V(T)$  such that all components of  $T - V'$  have the largest eigenvalues not greater than  $\lambda_1(T_{2t}^1)$ , where  $T_{2t}^1$  is the tree shown in Fig. 1.1 and  $t = \lceil p/k \rceil$ .*

**Proof:** When  $k = 1$ , the result is actually Theorem 1.3. So we may assume that  $k \geq 2$ . Let  $s = 2t = 2\lceil p/k \rceil$ , and  $T_0 = T$ ,  $n_0 = 2p$ . Since  $k \geq 2$ , we have  $n_0 > s$ . We perform the following procedure:

By Lemma 2.4, there are a vertex  $v_1 \in V(T)$  and a subtree  $T_1$  of order not greater than  $\max\{n_0 - s, s\}$  such that  $T_1$  has a perfect matching,  $T_1 - v_1$  is a component of  $T - v_1$  and the other components of  $T - v_1$  have orders not greater than  $s$ . Note that  $v_1$  may not be a vertex of  $T_1$ .

Let  $n_1 = |V(T_1)|$ . If all components of  $T - v_1$  and  $T_1$  are of orders not greater than  $s$ , then we stop the procedure. If not, then  $n_1 > s$ . By applying Lemma 2.4 to  $T_1$  there are a vertex  $v_2 \in V(T_1)$  and a subtree  $T_2$  of  $T_1$  such that the order of  $T_2$  is not greater than  $\max\{n_1 - s, s\}$ ,  $T_2$  has a perfect matching,  $T_2 - v_2$  is a component of  $T_1 - v_2$  and the other components of  $T_1 - v_2$  have orders not greater than  $s$ .

Let  $n_2 = |V(T_2)|$ . If all components of  $T - \{v_1, v_2\}$  and  $T_2$  are of orders not greater than  $s$ , then we stop the procedure. If not, we continue to perform the above procedure. Since  $n_0$  is finite, there are  $h$  subtrees  $T_0 \supset T_1 \supset \dots \supset T_h$  and vertices  $v_1, \dots, v_h$  (not necessary distinct) such that all components of

$T - \{v_1, v_2, \dots, v_h\}$  are of orders not greater than  $s$ ,  $n_i = |V(T_i)| \leq \max\{n_{i-1} - s, s\}$  and  $v_i \in V(T_{i-1})$  for  $1 \leq i \leq h$ . Hence we have  $n_i > s$  for  $1 \leq i \leq h - 1$ . Since  $s = 2\lceil p/k \rceil$ ,

$$ks = 2k\lceil p/k \rceil \geq 2k(p/k) = 2p.$$

Since  $n_i \leq n_{i-1} - s$ , ( $i = 1, 2, \dots, h$ ),

$$n_{h-1} - n_0 = \sum_{i=1}^{h-1} (n_i - n_{i-1}) \leq -(h-1)s.$$

Hence

$$s < n_{h-1} \leq 2p - (h-1)s \leq ks - hs + s = (k-h+1)s.$$

Thus  $h \leq k - 1$ .

Now we may choose a  $(k - 1)$ -vertex subset  $V'$  containing  $\{v_1, v_2, \dots, v_h\}$  such that the components of  $T - V'$  are of orders not exceeding  $s$ . By Lemma 2.2 and Theorem 1.3, all components of  $T - V'$  have their largest eigenvalues not great than  $\lambda_1(T_{2t}^1)$ . The proof is completed.  $\square$

Combining Lemma 2.5 with the Cauchy Interlacing Theorem, we obtain the following main result.

**Theorem 2.6** *Let  $T$  be a tree in  $\mathcal{T}_{2p}^+$ . Then for any positive integer  $k$  with  $1 \leq k \leq p$ , we have*

$$\lambda_k(T) \leq \frac{1}{2} \left( \sqrt{\lceil \frac{p}{k} \rceil - 1} + \sqrt{\lceil \frac{p}{k} \rceil + 3} \right) \tag{2.1}$$

and this upper bound is best possible when  $k = 1$ .

**Proof:** Suppose that  $T \in \mathcal{T}_{2p}^+$ . By Lemma 2.5, we have a  $(k - 1)$ -vertex subset  $V' \subset V(T)$  such that all components, say  $T_1, T_2, \dots, T_q$ , of  $T - V'$  are trees with the largest eigenvalues not exceeding  $\lambda_1(T_{2t}^1)$ ,  $t = \lceil \frac{p}{k} \rceil$ . By Theorems 2.1 and 1.3, we obtain

$$\begin{aligned} \lambda_k(T) &\leq \lambda_1(T - V') = \max_{1 \leq i \leq q} \lambda_1(T_i) \\ &\leq \max_{1 \leq i \leq q} \frac{1}{2} \left( \sqrt{\frac{|V(T_i)|}{2} - 1} + \sqrt{\frac{|V(T_i)|}{2} + 3} \right) \\ &\leq \frac{1}{2} \left( \sqrt{\lceil \frac{p}{k} \rceil - 1} + \sqrt{\lceil \frac{p}{k} \rceil + 3} \right) \end{aligned}$$

This proves the upper bound (2.1). Obviously, for  $k = 1$ , (2.1) is just the upper bound  $\lambda_1(T) \leq \frac{1}{2}(\sqrt{p-1} + \sqrt{p+3})$  in Theorem 1.3, and it is best possible upper bound.  $\square$

**Example 2.1** *For any  $T \in \mathcal{T}_{10}^+$ , from Theorem 2.6 we get that  $\lambda_1(T) \leq \frac{1}{2}(\sqrt{4} + \sqrt{8}) \approx 2.414$ ,  $\lambda_2(T) \leq \frac{1}{2}(\sqrt{2} + \sqrt{6}) \approx 1.932$ ,  $\lambda_3(T) \leq \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ ,  $\lambda_4(T) \leq \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$  and  $\lambda_5(T) \leq 1$ . We find that  $\lambda_1(T)$  and  $\lambda_5(T)$  are tight, which can be verified by the table of the spectra of all trees with  $n$  vertices ( $2 \leq n \leq 10$ ) in [6].  $\square$*

There is a relationship between the characteristic polynomial  $P(G \odot N_s; \lambda)$  of  $G \odot N_s$  and that of  $G$  as follows.

**Lemma 2.7 ([6])**  $P(G \odot N_s; \lambda) = \lambda^{ps} P(G; \lambda - \frac{s}{\lambda})$ .

Let  $\mathcal{T}_{2p}^*$  be the set of the coronas of trees of order  $p$  with  $N_1$ , i.e.,

$$\mathcal{T}_{2p}^* = \{T \odot N_1 \mid T \in \mathcal{T}_p\}.$$

Obviously, any graph in  $\mathcal{T}_{2p}^*$  is a tree and has a perfect matching. Thus we have  $\mathcal{T}_{2p}^* \subset \mathcal{T}_{2p}^+$ . Note that for any  $T^* \in \mathcal{T}_{2p}^*$ , there is a unique tree  $T$  with  $T^* = T \odot N_1$ . The tree  $T$  is called the *contracted tree* of the tree  $T^*$ . Now we prove an upper bound on the  $k$ th eigenvalue of trees in  $\mathcal{T}_{2p}^*$ .

**Lemma 2.8** *Let  $T^* \in \mathcal{T}_{2p}^*$  and let  $T$  be the contracted tree of  $T^*$ . Then*

$$\lambda_k(T^*) = \frac{1}{2} \left( \sqrt{\lambda_k(T)^2 + 4} + \lambda_k(T) \right).$$

**Proof:** By Lemma 2.7, we have  $P(T^*; \lambda) = \lambda^p P(T; \lambda - \frac{1}{\lambda})$ . Since  $\lambda_k(T)$  is the  $k$ th eigenvalue of  $T$  for  $k = 1, 2, \dots, p$ ,

$$P(T^*; \lambda) = \lambda^p \prod_{i=1}^p \left( \lambda - \frac{1}{\lambda} - \lambda_i(T) \right) = \prod_{i=1}^p (\lambda^2 - \lambda_i(T)\lambda - 1).$$

So the positive eigenvalues of  $T^*$  are  $\frac{1}{2}(\sqrt{\lambda_i(T)^2 + 4} + \lambda_i(T))$ ,  $i = 1, 2, \dots, p$ . Since  $f(x) = \frac{1}{2}(\sqrt{x^2 + 4} + x)$  is an increasing function of the variable  $x$ , the result follows immediately.  $\square$

**Theorem 2.9** *Let  $T^*$  be a tree in  $\mathcal{T}_{2p}^*$ . Then*

$$\lambda_k(T^*) \leq \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor - 1 + \sqrt{\left\lfloor \frac{p}{k} \right\rfloor + 3}} \right), \tag{2.2}$$

for  $k = 1, 2, \dots, p$ . Moreover, this upper bound is best possible when  $k = 1$  or  $p \not\equiv 0 \pmod{k}$ .

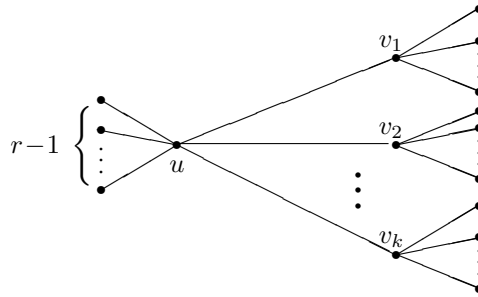
**Proof:** Suppose that  $T$  is the contracted tree of the tree  $T^*$ . Then  $T$  is a tree of order  $p$ . By Theorem 1.2, Lemma 2.8 and its proof, we have

$$\begin{aligned} \lambda_k(T^*) &= \frac{1}{2} \left( \sqrt{\lambda_k(T)^2 + 4} + \lambda_k(T) \right) \\ &\leq \frac{1}{2} \left( \sqrt{\left\lfloor \frac{p}{k} \right\rfloor - 1 + \sqrt{\left\lfloor \frac{p}{k} \right\rfloor + 3}} \right) \end{aligned}$$

for  $k \leq \lfloor \frac{p}{2} \rfloor$ . For  $k > \lfloor \frac{p}{2} \rfloor$ , since  $\lambda_k(T) \leq 0$ , we have  $\lambda_k(T^*) \leq 1$ . The Equation (2.2) holds, since the right hand side of (2.2) is equal to 1.

When  $k = 1$ , it is known that this bound is best possible. To show tightness for  $k \geq 2$  and  $p \not\equiv 0 \pmod{k}$ , we shall construct a corona of a tree with  $N_1$ . First, we write  $p = \lfloor p/k \rfloor k + r$ , where  $1 \leq r \leq k - 1$ . Set  $t = 2\lfloor p/k \rfloor$  and thus  $2p = tk + 2r$ . Let  $T$  be the tree obtained by joining edges from the





**Fig. 2.1:** A tree  $T$  in the proof of Theorem 2.9

center  $u$  of a star  $K_{1,r-1}$  to the centers  $v_1, v_2, \dots, v_k$  of  $k$  disjoint stars  $K_{1, \frac{t}{2}-1}$  (see Fig. 2.1). Then let  $T^* = T \odot N_1 \in \mathcal{T}_{2p}^*$ . It is easy to see that

$$\lambda_1(T - u) = \lambda_2(T - u) = \dots = \lambda_k(T - u) = \lambda_1(K_{1, \frac{t}{2}-1}) = \sqrt{\frac{t}{2} - 1}.$$

By Lemma 2.2 and the Cauchy Interlacing Theorem we have

$$\lambda_k(T - u) \leq \lambda_k(T) \leq \lambda_{k-1}(T - u).$$

Therefore,

$$\lambda_k(T) = \sqrt{\frac{t}{2} - 1}.$$

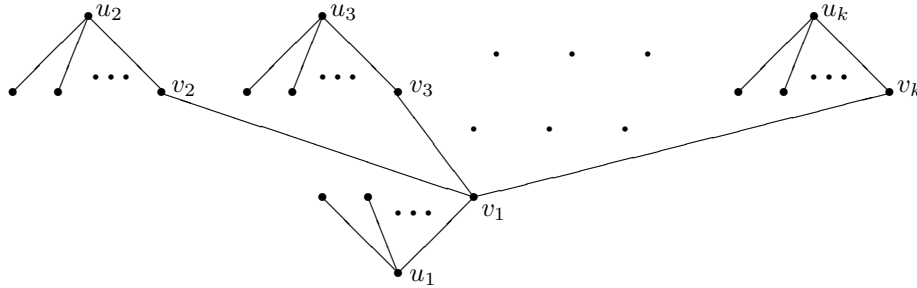
By Lemma 2.8, we have

$$\lambda_k(T^*) = \frac{1}{2} \left( \sqrt{\frac{t}{2} - 1} + \sqrt{\frac{t}{2} + 3} \right) = \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor - 1} + \sqrt{\lfloor \frac{p}{k} \rfloor + 3} \right).$$

This shows that the upper bound (2.2) is best possible when  $p \not\equiv 0 \pmod{k}$ . □

**Example 2.2** For any tree  $T^* \in \mathcal{T}_{10}^*$ , by Theorem 2.6 we have  $\lambda_1(T^*) \leq \frac{1}{2}(\sqrt{4} + \sqrt{8}) \approx 2.414$ ,  $\lambda_2(T^*) \leq \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ ,  $\lambda_3(T^*) \leq 1$ ,  $\lambda_4(T^*) \leq 1$  and  $\lambda_5(T^*) \leq 1$ . It can be verified from the table of the spectra of all trees with  $n$  vertices ( $2 \leq n \leq 10$ ) in [6] that these bounds are tight. □

**Example 2.3** For any tree  $T^* \in \mathcal{T}_8^*$ , by Theorem 2.6 we have  $\lambda_1(T^*) \leq \frac{1}{2}(\sqrt{3} + \sqrt{7}) \approx 2.189$ ,  $\lambda_2(T^*) \leq \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$ ,  $\lambda_3(T^*) \leq 1$  and  $\lambda_4(T^*) \leq 1$ . We know that the upper bounds on  $\lambda_1$  and  $\lambda_3$  are tight but on  $\lambda_2$  and  $\lambda_4$  they are not. Actually, the maximum values of  $\lambda_2$  and  $\lambda_4$  are approximately 1.356 and 0.477, respectively. □



**Fig. 2.2:** The tree  $T$  in Lemma 2.10

Example 2.3 shows that for those  $k$  satisfying  $p \equiv 0 \pmod{k}$ , we usually only have

$$\lambda_k(T^*) < \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor} - 1 + \sqrt{\lfloor \frac{p}{k} \rfloor + 3} \right),$$

and especially, the upper bound is not as good as that in Theorem 1.2 when  $k = 2$ . However, the upper bound in Theorem 2.6 will be shown to be asymptotically tight when  $p \equiv 0 \pmod{k}$ .

**Lemma 2.10 ([14])** *Let  $v$  be a vertex of  $G$ , and  $\mathcal{C}(v)$  be the set of all cycles containing  $v$ . Then the characteristic polynomial of  $G$  satisfies*

$$P(G; \lambda) = \lambda P(G - v; \lambda) - \sum_{uv \in E(G)} P(G - v - u; \lambda) - 2 \sum_{Z \in \mathcal{C}(v)} P(G - V(Z); \lambda).$$

We first take  $k$  copies of the star  $K_{1, t-1}$  (say  $S_1, S_2, \dots, S_k$ ) with centers  $u_1, u_2, \dots, u_k$ , respectively, and choose  $v_i \in V(S_i) \setminus \{u_i\}$  ( $i = 1, 2, \dots, k$ ). Then add the edges  $v_1 v_i$  ( $i = 2, 3, \dots, k$ ) to obtain tree  $T$  with  $tk$  vertices as shown in Fig. 2.2

The next lemma follows by direct calculation from Lemma 2.10 with  $v = v_1$ , observing that the last term in the lemma becomes zero because there are no cycles containing  $v_1$ .

**Lemma 2.11 ([15])** *Denoting  $f(x) = x^3 + (t-k)x^2 - 2(k-1)x - (k-1)$ , the characteristic polynomial of the tree  $T$  shown in Fig. 2.2 is*

$$P(T; \lambda) = \lambda^{tk-2(k+1)} (\lambda^2 - t + 1)^{k-2} f(\lambda^2 - t + 1)$$

and the  $k$ th eigenvalue of  $T$  satisfies

$$\lambda_k(T) = \sqrt{t-1 + \lambda_2(f)} > \sqrt{t-1 - \sqrt{\frac{k-1}{t-k}}},$$

where  $\lambda_2(f)$  is the second largest root of the equation  $f(x) = 0$ .

Denote the maximum value of the  $k$ th largest eigenvalue of the trees in  $\mathcal{T}_{2p}^*$  by  $\lambda_k^*(2p)$ . Then Theorem 2.9 tells us that  $\lambda_k^*(2p) \leq \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor - 1 + \sqrt{\lfloor \frac{p}{k} \rfloor + 3}} \right)$ . We shall give a lower bound for  $\lambda_k^*(2p)$ , which shows that as  $k$  gets large, the upper bound in Theorem 2.6 is asymptotically tight for the value of  $\lambda_k^*(2p)$  when  $p \equiv 0 \pmod k$ .

**Theorem 2.12** *Let  $p$  and  $k$  be integers with  $1 \leq k \leq p$ . If  $t = \lfloor \frac{p}{k} \rfloor > k$ , then*

$$\lambda_k^*(2p) > \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor - 1 - \sqrt{\frac{k-1}{t-k}}} + \sqrt{\lfloor \frac{p}{k} \rfloor + 3 - \sqrt{\frac{k-1}{t-k}}} \right).$$

**Proof:** Let  $T^* = T \odot N_1 \in \mathcal{T}_{2tk}^*$  by taking the tree  $T$  with  $tk$  vertices described in Fig. 2.2 From Lemma 2.11, it is easy to see that the second largest root  $\lambda_2(f)$  of  $f(x) = 0$  is negative. Note that  $f(0) = -(k-1) < 0$ ,  $f\left(-\sqrt{\frac{k-1}{t-k}}\right) > 0$  and  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ . So we know that  $\lambda_2(f) > -\sqrt{\frac{k-1}{t-k}}$ . Moreover, the expression  $\lambda = \frac{1}{2} (\sqrt{t-1+\alpha} + \sqrt{t+3+\alpha})$  can be regarded as a strictly increasing function of the variable  $\alpha$ . Thus, by Lemmas 2.8 and 2.11, we have

$$\begin{aligned} \lambda_k(T^*) &= \frac{1}{2} \left( \sqrt{t-1+\lambda_2(f)} + \sqrt{t+3+\lambda_2(f)} \right) \\ &> \frac{1}{2} \left( \sqrt{t-1-\sqrt{\frac{k-1}{t-k}}} + \sqrt{t+3-\sqrt{\frac{k-1}{t-k}}} \right). \end{aligned}$$

There is a tree  $U$  of order  $p$  containing  $T$  described above. Hence  $U^* = U \odot N_1 \in \mathcal{T}_{2p}^*$  and

$$\begin{aligned} \lambda_k(U^*) &\geq \lambda_k(T^*) > \frac{1}{2} \left( \sqrt{t-1-\sqrt{\frac{k-1}{t-k}}} + \sqrt{t+3-\sqrt{\frac{k-1}{t-k}}} \right) \\ &= \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor - 1 - \sqrt{\frac{k-1}{t-k}}} + \sqrt{\lfloor \frac{p}{k} \rfloor + 3 - \sqrt{\frac{k-1}{t-k}}} \right). \end{aligned}$$

Thus we get the theorem. □

**Remark:** If we let  $t \rightarrow \infty$  (that is,  $2p \rightarrow \infty$ ) for a fixed  $k$ , then  $\sqrt{\frac{k-1}{t-k}} \rightarrow 0$ , i.e.,

$$\lambda_k^*(2p) \rightarrow \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor - 1 + \sqrt{\lfloor \frac{p}{k} \rfloor + 3}} \right) \text{ as } t \rightarrow \infty.$$

So we can say that our upper bound (2.2) is asymptotically tight. Of course, if we denote the maximum value of the  $k$ th eigenvalues of trees in  $\mathcal{T}_{2p}^+$  by  $\lambda_k^+(2p)$ , then by Theorem 2.9, we have  $\lambda_k^+(2p) \geq \frac{1}{2} \left( \sqrt{\lfloor \frac{p}{k} \rfloor - 1 + \sqrt{\lfloor \frac{p}{k} \rfloor + 3}} \right)$ . So the upper bound (2.1) is also asymptotically tight in a sense.

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