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# On randomly colouring locally sparse graphs

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We consider the problem of generating a random  $q$ -colouring of a graph  $G = (V, E)$ . We consider the simple Glauber Dynamics chain. We show that if for all  $v \in V$  the average degree of the subgraph  $H_v$  induced by the neighbours of  $v \in V$  is  $\ll \Delta$  where  $\Delta$  is the maximum degree and  $\Delta > c_1 \ln n$  then for sufficiently large  $c_1$ , this chain mixes rapidly provided  $q/\Delta > \alpha$ , where  $\alpha \approx 1.763$  is the root of  $\alpha = e^{1/\alpha}$ . For this class of graphs, which includes planar graphs, triangle free graphs and random graphs  $G_{n,p}$  with  $p \ll 1$ , this beats the  $11\Delta/6$  bound of Vigoda [20] for general graphs.

**Keywords:** Counting Colourings, Sampling, Markov Chains

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## 1 Introduction

Markov Chain Monte Carlo (MCMC) is an important tool in sampling from complex distributions. It has been successfully applied in several areas of Computer Science, most notably volume computation [3], [15], [16] and estimating the permanent of a non-negative matrix [12]. It was used by Jerrum [10] to generate a random  $q$ -colouring of a graph  $G$ , provided  $q > 2\Delta$ . This has led to the challenging problem of determining the smallest value of  $q$  for which it is possible to generate a (near)-uniform sample from the set  $\mathcal{Q}$  of proper  $q$ -colourings of  $G$  in polynomial time. We cannot expect the chain to mix for  $q \leq \Delta + 1$  and at present it is unknown as to whether or not it mixes rapidly for say  $q = \Delta + 2$ . Vigoda [20] improved Jerrum's result by reducing the lower bound on  $q$  to  $11\Delta/6$ . This is still the best lower bound on  $q$  for general graphs.

The lack of complete success on the general problem has led to the analysis of restricted classes of graphs. Suppose that we consider *Glauber dynamics* on the set  $\mathcal{Q}$ . Specifically we will consider the *heat bath* dynamics, which may be described as follows. We start from an arbitrary proper  $q$ -colouring  $X_0 \in \mathcal{Q}$ . At step  $t > 0$  of the process, in state  $X_{t-1} \in \mathcal{Q}$ , we choose a vertex  $v_t \in V$  uniformly

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at random. Then we choose  $j_t$  uniformly at random from the colours with which  $v_t$  may be properly coloured, given  $X_{t-1}(V \setminus v_t)$ . We recolour  $v_t$  with  $j_t$  to give  $X_t \in \mathcal{Q}$ .

Dyer and Frieze [2] considered this process restricted to the class of graphs  $\mathcal{G}(c_1, c_2)$ : the set of graphs with  $n$  vertices, maximum degree  $\Delta \geq c_1 \log n$  and girth  $g \geq c_2 \log \Delta$ . They showed using the idea of “burn-in” that for  $c_1, c_2$  sufficiently large, Glauber Dynamics mixed in  $O(n \log n)$  time, provided  $q > \alpha \Delta$  where  $\alpha \approx 1.763$  is the root of  $\alpha = e^{1/\alpha}$ . Molloy [17] improved this result by reducing the lower bound on  $q$  to being more than  $\beta \Delta$  where  $\beta \approx 1.489$  is the root of  $(1 - e^{-1/\beta})^2 + \beta e^{-1/\beta} = 1$ . The girth assumptions were then relaxed by Hayes [7] to  $g \geq 5$  for  $k/\Delta > \alpha$  and  $g \geq 6$  for  $k/\Delta > \beta$ . Subsequently, Hayes and Vigoda [8] made considerable progress, using a non-Markovian coupling, and reduced the lower bound on  $k/\Delta$  to  $(1 + \epsilon)$  for all  $\epsilon > 0$ , which is nearly optimal. Their result requires girth  $g \geq 9$ . However, the large maximum degree restriction remained. This was replaced by  $\Delta \geq \Delta_0$  in Dyer, Frieze, Hayes and Vigoda [5], with the same restrictions on girth as in [7]. Dyer, Flaxman, Frieze and Vigoda [4] show that for sparse random graphs, the number of colours required for rapid mixing is of order the average rather than maximum degree **whp**. Goldberg, Martin and Paterson [6] prove results on the related notion of strong spatial mixing.

In this paper we avoid girth restrictions and consider locally sparse graphs instead. We say that a graph  $G = (V, E)$  is  $\gamma$ -locally sparse if for all  $v \in V$ , the average degree of the graph induced by the neighbourhood  $N(v)$  is at most  $\gamma$ . Thus planar graphs are always 6-locally-sparse and triangle free graphs are 0-locally-sparse.

**Theorem 1.1** *Suppose that  $q \geq (\alpha + \epsilon)\Delta$  where  $\epsilon$  is a small positive constant. Let  $G$  be an  $n$ -vertex  $\gamma$ -locally sparse graph with  $\gamma \leq \epsilon^2 \Delta / 10$  and  $\Delta \geq c_1 \log n$ . If  $c_1 = c_1(\epsilon)$  is sufficiently large then the Glauber dynamics converges to within variation distance  $e^{-1}$  from uniform over  $\mathcal{Q}$  in at most  $O(n \ln n)$ .*

Notice that if  $G = G_{n,p}$  and  $\frac{c_1 \log n}{n} \leq p \leq \epsilon^2 / 11$  then **whp**  $G$  satisfies the conditions of the theorem. Note also that the chromatic number of a triangle-free graph is  $O(\Delta / \log \Delta)$  – see Johansson [14] or Molloy and Reed [18] or Alon, Krivelevich and Sudakov [1] or Vu [21].

Our proof uses coupling and relies on a recent idea from Hayes and Vigoda [9] that utilises the fact that we can couple against the steady state distribution of the chain. Note that the theorem generalises Theorem 4 of [9].

In what follows we will assume that  $n$  is sufficiently large and  $\epsilon$  is sufficiently small to satisfy our inequalities.

## 2 Preliminaries

We will consider two copies of Glauber Dynamics,  $(X_t, t \geq 0)$  and  $(Y_t, t \geq 0)$ . Here  $X_0$  is an arbitrary colouring and  $Y_0$  is chosen from the uniform (*stationary*) distribution over  $\mathcal{Q}$ . At time  $t$ , the Hamming distance between  $X_t, Y_t$  is defined by

$$H(X_t, Y_t) = \sum_{v \in V} 1_{X_t(v) \neq Y_t(v)}.$$

We will couple the two processes as in Jerrum [10]. Here  $v_t$  is the same in both processes and then the choice of colours is maximally coupled. For vertex  $w$  let

$$A(X_t, w) = \{c \in [q] : c \notin X_t(N(w))\}$$

be the set of colours available to colour  $w$  in  $X_t$  if  $v_t = w$ .

Let  $a(X_t, w) = |A(X_t, w)|$  and define the terms  $A(Y_t, w)$ ,  $a(Y_t, w)$  analogously.

It is shown in [9] that

$$\mathbf{E}(H(X_{t+1}, Y_{t+1}) - H(X_t, Y_t)) \leq -\frac{1}{n}H(X_t, Y_t) + \frac{1}{n} \sum_{w \in V} \frac{|\{u \in N(w) : X_t(u) \neq Y_t(u)\}|}{\max\{a(X_t, w), a(Y_t, w)\}}. \quad (1)$$

We will show that for  $w \in V$  and  $\delta = \epsilon/10$ ,

$$\Pr(a(Y_t, w) \leq \Delta/(1 - \delta)) \leq n^{-4}. \quad (2)$$

Assuming that  $a(Y_t, w) \geq \Delta/(1 - \delta)$  in (1) we get

$$\begin{aligned} \mathbf{E}(H(X_{t+1}, Y_{t+1}) - H(X_t, Y_t)) &\leq -\frac{1}{n}H(X_t, Y_t) + \frac{1}{n} \frac{H(X_t, Y_t)\Delta}{\Delta/(1 - \delta)} \\ &\leq -\frac{\delta}{n}H(X_t, Y_t). \end{aligned}$$

So conditional on an event of probability  $1 - O(n^{-3})$ , we have

$$\mathbf{E}(H(X_{t+1}, Y_{t+1}) \mid X_t, Y_t) \leq \left(1 - \frac{\delta}{n}\right) H(X_t, Y_t).$$

Thus if  $T = n(1 + \ln n)\delta^{-1}$  then conditional on an event of probability  $1 - O(n^{-2} \log n)$ , we have

$$\mathbf{E}(H(X_T, Y_T)) \leq e^{-1}$$

and so unconditionally

$$\mathbf{E}(H(X_T, Y_T)) \leq e^{-1} + o(1).$$

Hence the mixing time of the Glauber Dynamics is  $O(n \ln n)$  as claimed.

### 3 Bounding the number of available colours

Fix  $v \in W$  and let  $H_v$  be the subgraph of  $G$  induced by  $N(v)$ . Let  $B(v)$  be the vertices of  $N(v)$  that have degree at least  $\gamma\delta^{-1}$  in  $H_v$ . Note that  $\gamma\delta^{-1} \leq \epsilon\Delta$  and

$$|B(v)| \leq \delta|N(v)|, \quad (3)$$

since  $G$  is  $\gamma$ -locally-sparse.

Let

$$N^*(v) = N(v) \setminus B(v) = \{w_1, w_2, \dots, w_d\}.$$

Now let us fix the colours  $\kappa(v)$  used at

$$v \in W_v = V \setminus N^*(v).$$

Let us use the term *allowable* for colorings of  $N^*(v)$  which respect this conditioning. Let  $\Omega$  be the set of allowable colourings of  $N^*(v)$ .

Let  $a^*(\sigma, v)$  be the number of colours not used on  $N^*(v)$ . Note that (3) implies

$$a(\sigma, v) \geq a^*(\sigma, v) - \delta|N(v)|. \quad (4)$$

Now consider the following process  $\mathcal{P}_\sigma$  for producing an allowable colouring of  $H_v$ . Here  $\sigma \in \Omega$ . We let  $\sigma_0 = \sigma$  and for  $j = 1, 2, \dots, d$  let  $\sigma_j$  be obtained from  $\sigma_{j-1}$  as follows: Keep  $\sigma_j(w_k) = \sigma_{j-1}(w_k)$  for  $k \neq j$  and choose  $\sigma_j(w_j)$  randomly from what is available to it.

Let  $Z_\sigma$  be the number of colours not appearing on a vertex in  $N^*(v)$  if we start with  $\sigma_0 = \sigma$ .

**Lemma 3.1** *If  $\sigma$  is chosen uniformly from  $\Omega$  then for any  $c > 0$ ,*

$$\Pr(a^*(\sigma, v) \geq c) = \Pr(Z_\sigma \geq c).$$

**Proof** We first prove that

$$\text{If } \sigma_0 \text{ is chosen uniformly from } \Omega \text{ then } \sigma_d \text{ is also uniform over } \Omega. \quad (5)$$

We do this by induction on  $j$ , with base case  $j = 0$ .

$$\begin{aligned} \Pr(\sigma_j = \sigma) &= \sum_{\sigma' \in \Omega} \Pr(\sigma_j = \sigma \mid \sigma_{j-1} = \sigma') \Pr(\sigma_{j-1} = \sigma') \\ &= \frac{1}{|\Omega|} \sum_{\sigma' \sim \sigma} \Pr(\sigma_j = \sigma \mid \sigma_{j-1} = \sigma') \end{aligned}$$

Here  $\sigma' \sim \sigma$  if  $\sigma, \sigma'$  differ only at  $w_j$ .

$$\begin{aligned} &= \frac{1}{|\Omega|} \sum_{\sigma' \sim \sigma} \frac{1}{|\{\sigma' : \sigma' \sim \sigma\}|} \\ &= \frac{1}{|\Omega|}. \end{aligned}$$

Now  $a^*(\sigma_d, v) = Z_{\sigma_0}$  and so

$$\Pr(a^*(\sigma_d, v) \geq c) = \Pr(Z_{\sigma_0} \geq c)$$

and the lemma follows from (5) □

For  $w \in N^*(v)$  let

$$L(w) = [q] \setminus \{\kappa(u) : u \in N(w) \setminus N^*(v)\}$$

be the colours not specifically barred from  $w$  by the current conditioning. Then let

$$L^*(w_j) = [q] \setminus \{\sigma_{j-1}(u) : u \neq w_j\} \quad \text{for } j = 1, 2, \dots, d$$

be the colours available to  $w_j$  when it is re-coloured by  $\sigma_j$ .

We will first estimate the (conditional) expectation of  $Z_\sigma$  for arbitrary  $\sigma$ . Suppose that  $x \in [q]$ . Let  $\theta_{x,j} = 1_{x \in L(w_j)}$  and let  $\theta_{x,j}^* = 1_{x \in L^*(w_j)}$ . Then we have

$$\begin{aligned} \Pr(x \notin \sigma_d(N^*(v))) &= \prod_{j=1}^d \Pr(\sigma_d(w_j) \neq x \mid \sigma_d(w_i) \neq x, 1 \leq i < j) \\ &= \prod_{j=1}^d \mathbf{E} \left( \left( 1 - \frac{1}{|L^*(w_j)|} \right)^{\theta_{x,j}^*} \right) \\ &\geq \prod_{j=1}^d \left( 1 - \frac{1}{|L(w_j)| - \gamma\delta^{-1}} \right)^{\theta_{x,j}} \end{aligned}$$

since  $|L^*(w_j)| \geq |L(w_j)| - \gamma\delta^{-1}$  and  $L^*(w_j) \subseteq L(w_j)$  implying  $\theta_{x,j}^* \leq \theta_{x,j}$ .

Then, following [2],

$$\begin{aligned} \mathbf{E}(Z_\sigma) &\geq \sum_{x \in [q]} \prod_{j=1}^d \left( 1 - \frac{1}{|L(w_j)| - \gamma\delta^{-1}} \right)^{\theta_{x,j}} \\ &\geq q \left( \prod_{x \in [q]} \prod_{j=1}^d \left( 1 - \frac{1}{|L(w_j)| - \gamma\delta^{-1}} \right)^{\theta_{x,j}} \right)^{1/q} \\ &= q \left( \prod_{j=1}^d \left( 1 - \frac{1}{|L(w_j)| - \gamma\delta^{-1}} \right)^{|L(w_j)|} \right)^{1/q} \\ &\geq q \exp \left\{ -\frac{1}{q} \sum_{j=1}^d \frac{|L(w_j)|}{|L(w_j)| - 1 - \gamma\delta^{-1}} \right\}, \quad \text{using } 1 - x \geq e^{-x/(1-x)} \text{ for } 0 < x < 1, \\ &\geq q \exp \left\{ -\frac{\Delta}{q} \cdot \frac{q - \Delta}{q - \Delta - 1 - \gamma\delta^{-1}} \right\} \\ &\geq \left( 1 + \frac{\epsilon}{2} \right) \Delta. \end{aligned} \tag{6}$$

(If  $f(x) = xe^{-1/x}$  then  $f(\alpha) = 1$  and  $f'(\alpha) \sim .891$ .)

We will now prove that for all  $\sigma \in \Omega$ ,  $Z_\sigma$  is concentrated around its mean via the use of the Azuma-Hoeffding martingale inequality. To this end, let  $x_1, x_2, \dots, x_d$  be the colours assigned to  $w_1, w_2, \dots, w_d$ . Thus we can write  $Z_\sigma = Z_\sigma(x_1, x_2, \dots, x_d)$ . Now let

$$Z_{\sigma,i} = Z_{\sigma,i}(x_1, x_2, \dots, x_i) = \mathbf{E}(Z \mid x_1, x_2, \dots, x_i).$$

We will show next that for all feasible colours  $x_1, x_2, \dots, x_i, x'_i$  that

$$|Z_{\sigma,i}(x_1, x_2, \dots, x_{i-1}, x_i) - Z_{\sigma,i}(x_1, x_2, \dots, x_{i-1}, x'_i)| \leq 2. \tag{7}$$

The aforementioned inequality will then imply that for any  $t \geq 0$ ,

$$\Pr(Z_\sigma - \mathbf{E}(Z_\sigma) \leq -t) \leq e^{-t^2/(2d)}$$

and then taking  $t = \epsilon\Delta/4$  and using (6) we get

$$\Pr\left(Z_\sigma \leq \left(1 + \frac{\epsilon}{4}\right)\Delta\right) \leq e^{-\epsilon^2\Delta/32}.$$

This together with Lemma 3.1 and (4) implies (2).

To prove (7), fix  $i, x_1, x_2, \dots, x_i, x_i^*$ . In one instance of  $\mathcal{P}_\sigma$  we start by colouring  $w_1, w_2, \dots, w_i$  with  $x_1, x_2, \dots, x_i$  to produce colouring  $\tau$ . In another instance we start by colouring  $w_1, w_2, \dots, w_i$  with  $x_1, x_2, \dots, x_i^*$  to produce colouring  $\tau^*$ .

We couple these two constructions in order to minimise the expected difference in the number of vertices  $U$  with a different colour. A *paths of disagreement* argument gives that

$$\mathbf{E}(U) \leq 1 + \sum_{j=i+1}^d \left( \frac{\gamma\delta^{-1}}{|L(w_j)| - \gamma\delta^{-1}} \right)^{j-i} \leq 2 \quad (8)$$

and (7) follows.  $\square$

**Explanation of (8):** We claim that if  $c_j, c_j^*$  is the colour of  $v_j$  in  $\sigma_d, \sigma_d^*$  respectively, then

$$\Pr(c_j \neq c_j^*) \leq \left( \frac{\gamma\delta^{-1}}{|L(w_j)| - \gamma\delta^{-1}} \right)^{j-i}.$$

This is because if  $c_j \neq c_j^*$  then there is a path of disagreements  $v_{i_1}, v_{i_2}, \dots, v_{i_s}$  where  $i = i_1 < i_2 < \dots < i_s = j$  such that  $c_{i_r} \neq c_{i_r}^*$  for  $1 \leq r \leq s$ . There are at most  $(\lambda\delta^{-1})^{j-i}$  such paths and each has probability at most  $(|L(w_j)| - \gamma\delta^{-1})^{i-j}$  of all vertices being coloured differently.  $\square$

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