



Design and Analysis of Distributed Averaging with Quantized Communication

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Design and Analysis of Distributed Averaging with Quantized Communication

Mahmoud El Chamie*,
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Abstract: Consider a network whose nodes (say there are n of them) have some initial values, and it is desired to design an algorithm that builds on neighbor to neighbor interactions with the ultimate goal of convergence to the average of all initial node values or to some value close to that average. Such an algorithm is called generically “distributed averaging”, and our goal in this paper is to study the performance of a subclass of deterministic distributed averaging algorithms where the information exchange between neighboring nodes (agents) is subject to uniform quantization. With such quantization, the precise average cannot be achieved (except in exceptional cases), but some value close to it, called quantized consensus. It is shown in this paper that in finite time, the algorithm will either cause all n agents to reach a quantized consensus where the consensus value is the largest integer not greater than the average of their initial values, or will lead all n variables to cycle in a small neighborhood around the average, depending on initial conditions. In the latter case, it is further shown that the neighborhood can be made arbitrarily small by adjusting the algorithm’s parameters in a distributed manner.

Key-words: distributed averaging, quantization, finite state automata, cycle, quantized consensus

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Conception et Analyse d'Algorithmes Distribués de Moyennage avec Valeurs Échangées Discrétisées

Résumé : Nous allons nous intéresser à un réseau dont les nœuds, ou agents, ont des valeurs initiales. Le nombre de nœuds est n . Nous souhaitons concevoir un algorithme ayant pour objectif la convergence vers une valeur qui est la plus proche possible de la moyenne de toutes les valeurs initiales des nœuds. Cette algorithme est basée sur les interaction entre les nœuds, où un nœud interagit avec un autre nœud si ils sont voisins dans le graphe. Un tel algorithme est communément appelé “moyenne distribuée”. L’objectif de cet article est d’étudier les performances d’une sous-classe d’algorithmes déterministes de calcul de la moyenne distribuée, où l’échange d’informations entre les nœuds voisins est soumis à la quantification uniforme. Avec une telle quantification, la moyenne précise ne peut être atteinte (sauf dans des cas exceptionnels), mais une valeur proche d’elle peut être atteinte. Cette valeur est appelée consensus quantifié. Nous montrons dans ce papier que, dans un temps fini, soit tous les n agents parviennent à un consensus quantifié où la valeur de consensus est le plus grand entier qui n’est pas supérieur à la moyenne de leurs valeurs initiales; ou soit tous les n agents cyclent dans un petit voisinage autour de la moyenne, en fonction des conditions initiales. Dans ce dernier cas, il est démontré que le voisinage peut être rendue arbitrairement faible en ajustant les paramètres de l’algorithme de manière distribuée.

Mots-clés : distributed averaging, quantization, finite state automata, cycle, quantized consensus

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1 Introduction

There has been considerable interest recently in developing algorithms for distributing information among members of interactive agents via local interactions (e.g., a group of sensors [2] or mobile autonomous agents [29]), especially for the scenarios where agents or sensors are constrained by limited sensing, computation, and communication capabilities. Notable among these are those algorithms intended to cause such a group to reach a consensus in a distributed manner [7, 23, 37]. Consensus processes play an important role in many other problems such as Google's PageRank [22], clock synchronization [33], rendezvous [16] and formation control [19].

One particular type of consensus process, distributed averaging, has received much attention lately [17, 38]. In its simplest form, distributed averaging deals with a network of $n > 1$ agents and the constraint that each agent i is able to communicate only with certain other agents called agent i 's neighbors. Neighbor relations are conveniently characterized by a simple, connected graph in which vertices correspond to agents and edges indicate neighbor relations. Each agent i initially has or acquires a real number z_i which might be a measurement value. The *distributed averaging problem* is to devise an algorithm which will enable each agent to compute the average $z_{ave} = \frac{1}{n} \sum_{i=1}^n z_i$ using only information acquired from its neighbors.

Most existing algorithms for precise distributed averaging require that agents are able to send and receive real values with infinite precision. However, a realistic network can only allow messages with limited length to be transmitted between agents due to constraints on the capacity of communication links. With such a constraint, when a real value is sent from an agent to its neighbors, this value will be truncated and only a quantized version will be received by the neighbors. With such quantization, the precise average cannot be achieved (except in particular cases), but some value close to it can be achieved, called quantized consensus. A number of papers have studied this quantized consensus problem and various *probabilistic* strategies have been proposed to cause all the agents in a network to reach a quantized consensus with probability one (or at least with high probability) [3–6, 18, 24–26, 34]. Notwithstanding this, the problem of how to design and analyze *deterministic* algorithms for quantized consensus remains open [12, 20].

In this paper, we thoroughly analyze the performance of a deterministic distributed averaging algorithm where the information exchange between neighboring agents is subject to uniform quantization. It is shown that in finite time, the algorithm will either cause all n agents to reach a quantized consensus where the consensus value is the largest integer not greater than the average of their initial values, or will lead all n agents' variables to cycle in a small neighborhood around the average, depending on initial conditions. In the latter case, it is further shown that the neighborhood can be arbitrarily small by adjusting the algorithm's parameters in a distributed manner.

The rest of the paper is organized as follows: in Section 2 we review the existing literature related to our work. In section 3 we introduce some preliminaries of distributed averaging. A network model for quantized communications is given in Section 4. In Section 5, we formulate the problem considered in this paper and present the equation model of the quantized system. The design and analysis of the system, including the main results of the paper, are given in Section 6. A further discussion is given in Section 7. Section 8 provides some simulations supporting our analytic results and Section 9 concludes the paper.

2 Literature Review

Most of the related works for distributed averaging with quantized communication propose either a deterministic algorithm (as our approach in this paper) or a probabilistic one.

There are only a few publications which study deterministic algorithms for quantized consensus. In [27] the distributed averaging problem with quantized communication is formulated as a feedback control design problem for coding/decoding schemes; the paper characterizes the amount of information needed to be sent for the agents to reach a consensus and shows that with an appropriate scaling function and some carefully chosen control gain, the proposed protocol can solve the distributed averaging problem, but some spectral properties of the Laplacian matrix of the underlying fixed undirected graph have to be known in advance. More sophisticated coding/decoding schemes were proposed in [28] for time-varying undirected graphs and in [41] for time-varying directed graphs, all requiring carefully chosen parameters. Recently a novel dynamic quantizer has been proposed in [36] based on dynamic quantization intervals for coding of the exchanged messages in wireless sensor networks leading to asymptotic convergence to consensus. In [15] a biologically inspired algorithm was proposed which will cause all n agents to reach some consensus with arbitrary precision, but at the cost of not preserving the desired average. Control performance of logarithmic quantizers was studied in [13] and quantization effects were considered in [31]. A deterministic algorithm of the same form as in this paper has been only partially analyzed in [20] where the authors have approximated the system by a probabilistic model and left the design of the weights as an open problem.

Over the past decade quite a few probabilistic quantized consensus algorithms have been proposed. The probabilistic quantizer in [4] ensures almost surely consensus at a common but random quantization level for fixed (strongly connected) directed graphs; although the expectation of the consensus value equals the desired average, the deviation of the consensus value from the desired average is not tightly bounded. An alternative algorithm which gets around this limitation was proposed in [24]; the algorithm adds dither to the agents' variables before quantization and the mean square error can be made arbitrarily small by tuning the parameters. The probabilistic algorithm in [5, 6], called "interval consensus gossip", causes all n agents to reach a consensus in finite time almost surely on the interval in which the average lies, for time-varying (jointly connected) undirected graphs. A stochastic quantized gossip algorithm was shown to work properly in [26]. The effects of quantized communication on the standard randomized gossip algorithm [9] were analyzed in [14]. An alternative approach to analyze the quantization effect was introduced in [3, 34] which model the effect as noise following certain probability.

Another thread of research has studied quantized consensus with the additional constraint that the value at each node is an integer. The probabilistic algorithm in [25] causes all n agents to reach quantized consensus almost surely for a fixed (connected) undirected graph; convergence time of the algorithm was studied in [18], with strong bounds on its expected value. In [11] a probabilistic algorithm was proposed to solve the quantized consensus problem for fixed (strongly connected) directed graphs using the idea of "surplus".

We should note that, in addition, our work in this paper is also related to the literature on the problem of load balancing [1, 21, 35].

3 Distributed Averaging

Consider a group of $n > 1$ agents labeled 1 to n . Each agent i has control over a real-valued scalar quantity x_i called an *agreement variable* which the agent is able to update its value from time to time. Agents may only communicate with their "neighbors". Agent j is a *neighbor* of agent i if $(i, j) \in \mathcal{E}$ is an edge in a given simple, undirected n -vertex graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, 2, \dots, n\}$ is the vertex set and \mathcal{E} is the edge set. We assume that the graph \mathbb{G} is connected and does not change over time. Initially each agent i has a real number $x_i(0)$. Let

$$x_{ave}(k) = \frac{1}{n} \sum_{i \in \mathcal{V}} x_i(k),$$

be the average of values of all agreement variables in the network, we will refer to $x_{ave}(0)$ simply as x_{ave} . The purpose of the distributed averaging problem is to devise an algorithm which enables all n agents to asymptotically determine in a decentralized manner, the average of the initial values of their scalar variables, i.e.,

$$\lim_{k \rightarrow \infty} x_i(k) = x_{ave}.$$

A well studied approach to the problem is for each agent to use a linear iterative update rule of the form

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j \in \mathcal{N}_i} w_{ij}x_j(k), \quad \forall i \in \mathcal{V}, \quad (1)$$

where k is a discrete time index, \mathcal{N}_i is the set of neighbors of agent i and the w_{ij} are real-valued weights to be designed. In [8] several methods are proposed for choosing the weights w_{ij} with the goal of obtaining algorithms with improved convergence rates. One particular choice, which

defines what has come to be known as the Metropolis algorithm, requires only local information to define the w_{ij} [39, 40]. The corresponding Metropolis weights are chosen as follows:

$$\begin{aligned} w_{ij} &= \frac{1}{\max\{d_i, d_j\} + 1}, \quad \forall (i, j) \in \mathcal{E}, \\ w_{ii} &= 1 - \sum_{j \in N_i} w_{ij}, \quad \forall i \in \mathcal{V}, \end{aligned}$$

where d_i is the degree of agent i .

Eq. (1) can be written in a matrix form as

$$\mathbf{x}(k+1) = W\mathbf{x}(k),$$

where $\mathbf{x}(k)$ is the state vector of agreement values whose i th element equals $x_i(k)$, and W is the weight matrix whose ij th entry equals w_{ij} . It should be clear that $w_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $w_{ij} = 0$ otherwise. A necessary and sufficient condition for the convergence of Eq. (1) to the desired average for any initial values is that W is a doubly stochastic matrix and all eigenvalues of W , with the exception of a single eigenvalue of value 1, have magnitude strictly less than unity [38]. It is easy to verify that the Metropolis weights satisfy this condition. Thus the Metropolis weights guarantee the desired convergence, i.e.,

$$\lim_{k \rightarrow \infty} \mathbf{x}(k) = x_{ave} \mathbf{1},$$

where $\mathbf{1}$ is the vector in \mathbb{R}^n whose entries all equal one. It is worth noting that since W is doubly stochastic, the summation of all n values of agreement variables is kept constant, so is the average of the variables, namely

$$\mathbf{1}^T \mathbf{x}(k) = \mathbf{1}^T \mathbf{x}(0) = nx_{ave}, \quad \forall k.$$

4 Quantized Communication

In a network where links have constraints on the capacity and have limited bandwidth (e.g., digital communication networks), messages cannot have infinite length. But the distributed averaging algorithm requires sending real (infinite precision) values through these communication links. Therefore, the messages transmitted between neighboring agents will have to be truncated. If the communication bandwidth was limited, the more truncation of agents' values, the higher would be the deviation of agent's value from the desired average consensus x_{ave} .

To model the effect of quantized communication, we assume that the links perform a quantization effect on the values transmitted between agents. The network model is given by Fig. 1. As we can see from the model, each agent i can have infinite bandwidth to store its latest value $x_i(k)$ and perform computations. However, when agent i sends its value at time k through the communication network, its neighbors will receive a value $\hat{x}_i(k)$ which is the quantized value of $x_i(k)$. A *quantizer* is a function $\mathcal{Q} : \mathbb{R} \rightarrow \mathbb{Z}$ that maps a real value to an integer. Quantizers can be of different forms. We present here three most widely used quantizers in the literature [14, 30, 31]:

1. Rounding quantizer \mathcal{Q}_r which rounds a real number to its nearest integer:

$$\mathcal{Q}_r(x) = \begin{cases} \lfloor x \rfloor & \text{if } x - \lfloor x \rfloor < 1/2 \\ \lceil x \rceil & \text{if } x - \lfloor x \rfloor \geq 1/2 \end{cases} \quad (2)$$

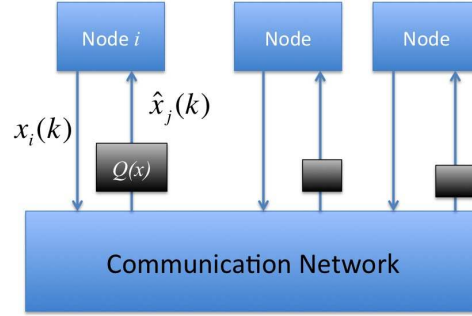


Figure 1: The network model for the quantized system.

2. Truncation quantizer \mathcal{Q}_t which truncates the decimal part of a real number and keeps the integer part:

$$\mathcal{Q}_t(x) = \lfloor x \rfloor \quad (3)$$

3. Probabilistic quantizer \mathcal{Q}_p which is defined as follows: For any real number x ,

$$\mathcal{Q}_p(x) = \begin{cases} \lfloor x \rfloor & \text{with probability } \lceil x \rceil - x \\ \lceil x \rceil & \text{with probability } x - \lfloor x \rfloor \end{cases} \quad (4)$$

The above quantizers map \mathbb{R} into \mathbb{Z} and have quantization jumps of size 1. Quantizers having a generic real positive quantization step ϵ can be simply recovered by a suitable scaling: $\mathcal{Q}^{(\epsilon)}(x) = \epsilon \mathcal{Q}(x/\epsilon)$ [14].

In this paper we study the effect of the deterministic quantizers ($\mathcal{Q}_t(x)$ and $\mathcal{Q}_r(x)$) on the performance of the distributed averaging algorithms and we show the distance that the agents' stored values deviates from the initial average x_{ave} .

5 Problem Formulation

Suppose that all n agents adhere to the same update rule of Eq. (1). Then with a quantizer $\mathcal{Q}(x)$, the network equation would be

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j \in \mathcal{N}_i} w_{ij}\mathcal{Q}(x_j(k)), \quad \forall i \in \mathcal{V}. \quad (5)$$

Simple examples show that this algorithm can cause the system to shift away from the initial average x_{ave} .

Since agents know exactly the effect of the quantizer, for the agents not to lose any information caused by quantization, at each iteration k each agent i can send out the quantized value $\mathcal{Q}(x_i(k))$ (instead of sending $x_i(k)$) and store in a *local* scalar $c_i(k)$ the difference between the real value $x_i(k)$ and its quantized version, i.e.,

$$c_i(k) = x_i(k) - \mathcal{Q}(x_i(k)).$$

Then, the next iteration update of agent i can be modified to be

$$x_i(k+1) = w_{ii}\mathcal{Q}(x_i(k)) + \sum_{j \in \mathcal{N}_i} w_{ij}\mathcal{Q}(x_j(k)) + c_i(k), \quad \forall i \in \mathcal{V}. \quad (6)$$

A major difference in this equation from (5) is that no information is being lost; i.e., the total average is being conserved in the network as we will show in a moment. The state equation of the system becomes,

$$\mathbf{x}(k+1) = W\mathcal{Q}(\mathbf{x}(k)) + \mathbf{x}(k) - \mathcal{Q}(\mathbf{x}(k)). \quad (7)$$

For any W where each column sums to 1 ($\mathbf{1}^T W = \mathbf{1}^T$ where $\mathbf{1}$ is the vector of all ones), the total sum of all n agreement variables does not change over time if agents followed the protocol of Eq. (7):

$$\begin{aligned} \mathbf{1}^T \mathbf{x}(k+1) &= \mathbf{1}^T (W\mathcal{Q}(\mathbf{x}(k)) + \mathbf{x}(k) - \mathcal{Q}(\mathbf{x}(k))) \\ &= \mathbf{1}^T \mathcal{Q}(\mathbf{x}(k)) + \mathbf{1}^T \mathbf{x}(k) - \mathbf{1}^T \mathcal{Q}(\mathbf{x}(k)) \\ &= \mathbf{1}^T \mathbf{x}(k) \\ &= \mathbf{1}^T \mathbf{x}(0) = nx_{ave}, \end{aligned} \quad (8)$$

Thus the average is also conserved ($x_{ave}(k) = x_{ave}, \forall k$). Equation (7) would be our model of distributed averaging with deterministic quantized communication where the quantizer can take the form of the truncation \mathcal{Q}_t or the rounding one \mathcal{Q}_r . In fact, it is sufficient to study only the truncation quantizer because the two quantizers can be related by the following equation:

$$\mathcal{Q}_r(x) = \mathcal{Q}_t(x + 1/2).$$

Given a model with the rounding quantizer \mathcal{Q}_r in equation (7), by taking $\mathbf{y}(k) = \mathbf{x}(k) + \frac{1}{2}\mathbf{1}$, then the system evolves as follows:

$$\begin{aligned} \mathbf{y}(k+1) &= \mathbf{y}(k) + W\mathcal{Q}_t(\mathbf{y}(k)) - \mathcal{Q}_t(\mathbf{y}(k)) \\ \mathbf{y}(0) &= \mathbf{x}(0) + \frac{1}{2}\mathbf{1}. \end{aligned}$$

Therefore, by the analysis of the system $\mathbf{y}(k)$ that satisfies equation (7) with a truncation quantizer \mathcal{Q}_t , we can deduce the performance of $\mathbf{x}(k)$ because they are related by a simple translation equation ($\mathbf{y}(k) = \mathbf{x}(k) + \frac{1}{2}\mathbf{1}$). Therefore the effects of the two quantizers are essentially the same. With this nontrivial observation in mind, we focus on the analysis of the truncation quantizer only in the rest of this paper. The results can then be easily extended to the case of the rounding quantizer.

In the sequel we will fully characterize the behavior of system (7) and its convergence properties. But first, let us give the following definition:

Definition 1. We say that a network of n agents reaches quantized consensus if there is an iteration k_0 such that

$$\mathcal{Q}(x_i(k)) = \mathcal{Q}(x_j(k)), \quad \forall i, j \in \mathcal{V}, \quad \forall k \geq k_0.$$

6 Design and Analysis of the System

In this section, we carry out the analysis of the proposed quantized system equation. By considering the truncation quantizer \mathcal{Q}_t in (7), the system equation becomes:

$$\mathbf{x}(k+1) = W\lfloor \mathbf{x}(k) \rfloor + \mathbf{x}(k) - \lfloor \mathbf{x}(k) \rfloor. \quad (9)$$

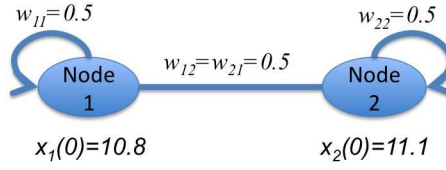


Figure 2: Network of 2 nodes where quantized communication does not converge.

This can be written in a distributed way for every $i \in \mathcal{V}$ as follows:

$$x_i(k+1) = x_i(k) + \sum_{j \in N_i} w_{ji} (\lfloor x_j(k) \rfloor - \lfloor x_i(k) \rfloor), \quad (10)$$

$$= x_i(k) + \sum_{j \in N_i} w_{ji} L_{ji}(k), \quad (11)$$

where

$$L_{ji}(k) \triangleq \lfloor x_j(k) \rfloor - \lfloor x_i(k) \rfloor = -L_{ij}(k).$$

The non-linearity of the system due to quantization complicates the analysis, and traditional stability analysis of linear systems (ergodicity, product of stochastic matrices, ...) cannot be applied here. This is mainly due to the fact that the system is not asymptotically stable (the product of the matrices might not converge). As demonstrated by the 2 nodes example of Fig. 2, suppose that $x_1(0) = 10.8$, $x_2(0) = 11.1$, and $w_{12} = w_{21} = 0.5$. Then the corresponding system equations are as follows:

$$x_1(k+1) = x_1(k) + 0.5 \times (\lfloor x_2(k) \rfloor - \lfloor x_1(k) \rfloor) \quad (12)$$

$$x_2(k+1) = x_2(k) + 0.5 \times (\lfloor x_1(k) \rfloor - \lfloor x_2(k) \rfloor). \quad (13)$$

Note that $x_1(1) = 11.3$ and $x_2(1) = 10.6$, while after one more iteration the values of nodes are back to their initial values (i.e., $x_1(2) = x_1(0) = 10.8$ and $x_2(2) = x_2(0) = 11.1$). Therefore, even though the weight matrix of the system given as:

$$W = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix},$$

has second largest eigenvalue $\lambda_2(W) = 0$ which on the linear system (1) can cause convergence in one iteration starting from any initial condition, we can see that with quantization the system is not even converging to a fixed value but entering into a periodic cycle. However, if $w_{12} = w_{21} = 0.15$, the system converges in one iteration, i.e., $x_1(k) = x_2(k) = 10.95$ for all $k \geq 1$.

The system behavior depends of course on the design of the weight matrix. Since distributed averaging algorithms are distributed by nature, it is important to consider in our design weights that can be chosen locally and guarantees some convergence properties.

Assumption 1: The weight matrix in our design have the following properties:

- W is a symmetric doubly stochastic matrix:

$$w_{ij} = w_{ji} \geq 0 \quad \forall i, j \in \mathcal{V}$$

$$\sum_i w_{ij} = \sum_j w_{ij} = 1,$$

- Diagonal entries of W : $w_{ii} > 1/2$ for all $i \in \mathcal{V}$,
- For any link $(i, j) \in \mathcal{E}$ we have $w_{ij} \in \mathbb{Q}^+$, where \mathbb{Q}^+ is the set of *positive* rational numbers in the interval $[0, 1]$,
- If $(i, j) \notin \mathcal{E}$, then $w_{ij} = 0$.

These are also sufficient conditions for the linear system (1) to converge. The restriction of the weights to the class of rational numbers is just because of a technical reason to prove convergence results. Let $\delta > 0$ be a positive lower bound on all the link weights w_{ij} where $(i, j) \in \mathcal{E}$, namely:

$$\delta = \min_{(i,j) \in \mathcal{E}} w_{ij}.$$

We now state the main result of this paper which will be proved in the following subsections.

Main Convergence Result 1. *Consider the quantized system (9). Suppose that Assumption 1 holds. Then for any initial value $\mathbf{x}(0)$, there is a finite time iteration where either*

1. *the system reaches quantized consensus, or*
2. *the nodes' values cycle in a small neighborhood around the average, where the neighborhood can be made arbitrarily small by a decentralized design of the weights (having trade-off with the speed of convergence).*

6.1 Cyclic States

We study in this subsection the convergence properties of the system equation (9) under Assumption 1. Let us first show that due to quantized communication, the states of the agents lie in a discrete set. Since $w_{ij} \in \mathbb{Q}^+$ for any link (i, j) , we can write

$$w_{ij} = \frac{a_{ij}}{b_{ij}},$$

where a_{ij} and b_{ij} are co-prime positive integers. Suppose that B_i is the Least Common Multiple (LCM) of the integers $\{b_{ij}; (i, j) \in \mathcal{E}, j \in N_i\}$. Let $c_i(k) = x_i(k) - \lfloor x_i(k) \rfloor$; then we have $c_i(k) \in [0, 1]$. Let us see how $c_i(k)$ evolves:

$$\begin{aligned} c_i(k) &= x_i(k) - \lfloor x_i(k) \rfloor \\ &= x_i(k-1) + \sum_{j \in N_i} w_{ij} \times (\lfloor x_j(k-1) \rfloor - \lfloor x_i(k-1) \rfloor) \\ &\quad - \lfloor x_i(k) \rfloor \\ &= \lfloor x_i(k-1) \rfloor + c_i(k-1) \\ &\quad + \sum_{j \in N_i} \frac{a_{ij}}{b_{ij}} \times (\lfloor x_j(k-1) \rfloor - \lfloor x_i(k-1) \rfloor) - \lfloor x_i(k) \rfloor \\ &= c_i(k-1) + \frac{Z(k)}{B_i}, \end{aligned} \tag{14}$$

where $Z(k) \in \mathbb{Z}$ is an integer. Then with a simple recursion, we can see that for any iteration k we have:

$$c_i(k) = c_i(0) + \frac{\tilde{Z}(k)}{B_i}, \tag{15}$$

where $\tilde{Z}(k) \in \mathbb{Z}$. Since $c_i(k) \in [0, 1)$, this equation shows that the states of the nodes are quantized, and the decimal part can have maximum B_i quantization levels. Equation (15) also implies that there exist three fixed¹ strictly positive constants $\gamma_1, \gamma_2, \gamma_3 > 0$ as follows:

- For any i and any iteration k such that $c_i(k) > \left(\sum_{j \in N_i} w_{ij}\right)$, we have:

$$c_i(k) - \sum_{j \in N_i} w_{ij} \geq \gamma_1 > 0,$$

- For any i and any iteration k such that $\bar{c}_i(k) > \left(\sum_{j \in N_i} w_{ij}\right)$, we have:

$$\bar{c}_i(k) - \sum_{j \in N_i} w_{ij} \geq \gamma_2 > 0,$$

- For any i and any iteration k , we have:

$$\bar{c}_i(k) \geq \gamma_3 > 0.$$

where $\bar{c}_i(k) = 1 - c_i(k)$.

Remark: The above three properties do not hold for the simple linear model of (1). Consider a linear model that does not reach consensus in finite time, and suppose that $x_{ave} \in \mathbb{Z}$. Then, since $\lim_{k \rightarrow \infty} x_i(k) = x_{ave}$, we have that $c_i(k)$ can be as close to 1 as desired, and hence we cannot bound $\bar{c}_i(k)$ by a fixed positive value.

We now give the following definition,

Definition 2. The quantized system (9) is cyclic if there exists a positive integer P and a finite time k_0 such that

$$\mathbf{x}(k + P) = \mathbf{x}(k) \quad \forall k \geq k_0,$$

where P is the cycle period.

Proposition 1. Suppose Assumption 1 holds. Then, the quantized system (9), starting from any initial value $\mathbf{x}(0)$, is cyclic.

Proof. Let $m(k)$ and $M(k)$ be defined as follows:

$$m(k) \triangleq \min_{i \in \mathcal{V}} \lfloor x_i(k) \rfloor, \quad M(k) \triangleq \max_{i \in \mathcal{V}} \lfloor x_i(k) \rfloor. \quad (16)$$

Notice that for any k , we have

$$\begin{aligned} x_i(k+1) &= x_i(k) + \sum_{j \in N_i} w_{ji} L_{ji} \\ &\leq c_i(k) + \lfloor x_i(k) \rfloor + \left(\sum_{j \in N_i} w_{ji} \right) (M(k) - \lfloor x_i(k) \rfloor) \\ &\leq c_i(k) + M(k), \end{aligned}$$

¹By ‘fixed’ we mean that the value is independent of time and it only depends on initial values and the network structure.

from which it follows that $\lfloor x_i(k+1) \rfloor \leq M(k)$, and hence $M(k+1) \leq M(k)$. By a simple recursion we can see that the maximum cannot increase, $M(k) \leq M(0)$. Similarly, we have $m(k) \geq m(0)$.

As a result, $\lfloor x_i(k) \rfloor \in \{m(0), m(0) + 1, \dots, M(0) - 1, M(0)\}$ is a finite set. Moreover, from equation (15), $c_i(k)$ belongs to a finite set that can have at most B_i elements. Since $x_i(k) = \lfloor x_i(k) \rfloor + c_i(k)$, and each of the elements in the sum belongs to a finite set, $x_i(k)$ belongs to a finite set. But from equation (9), we have $\mathbf{x}(k+1) = f(\mathbf{x}(k))$ where the function $f(\cdot)$ is a deterministic function of the input state at iteration k , so the system is a deterministic finite state automata. States of deterministic automata enter a cycle in finite time [32], and therefore the system is cyclic. \square

6.2 Lyapunov Stability

In this subsection, we will study the stability of the above system using a Lyapunov function. Let $m(k)$ and $M(k)$ be defined as in (16). Let us define the following set:

$$S_k = \{\mathbf{y} \in \mathbb{R}^n, |y_i - m(k) - 1| \leq \alpha_i\},$$

where $\alpha_i = 1 - w_{ii} + \gamma$, $\gamma = \min\{\frac{\gamma_1}{2}, \frac{\gamma_2}{2}, \gamma_3\}$, and let $\alpha = \max_i \alpha_i$. The set S_k depends on the iteration k because the value m does. Since according to the system (9), $m(k)$ cannot decrease and $M(k)$ cannot increase as indicated earlier, then S_k can only belong to one of the $M(0) - m(0)$ possible compact sets at each iteration k . Furthermore, if S_k changes to a different compact set due to an increase in m , it cannot go back to the old one as m cannot decrease.

Let us define the following candidate Lyapunov function:

$$\begin{aligned} V(k) &= d(\mathbf{x}(k), S_k) \\ &= \min_{\mathbf{y} \in S_k} \|\mathbf{y} - \mathbf{x}(k)\|_1 \\ &= \min_{\mathbf{y} \in S_k} \sum_{i \in \mathcal{V}} |y_i - x_i(k)| \end{aligned} \tag{17}$$

By minimizing along each component of \mathbf{y} independently, we get

$$V(k) = \sum_i \max\{|x_i(k) - m(k) - 1| - \alpha_i, 0\}.$$

Let us determine the change in the proposed candidate Lyapunov function. In order to understand the evolution of $\nabla V_k = V(k+1) - V(k)$, we group the nodes depending on their values at iteration k into 6 sets, $X_1(k)$, $X_2(k)$, $X_3(k)$, $X_4(k)$, $X_5(k)$, and $X_6(k)$ (see Fig. 3):

- Node $i \in X_1(k)$ if $m(k) \leq x_i(k) < m(k) + 1 - \alpha_i$,
- Node $i \in X_2(k)$ if $m(k) + 1 - \alpha_i \leq x_i(k) < m(k) + 1$,
- Node $i \in X_3(k)$ if $m(k) + 1 \leq x_i(k) \leq m(k) + 1 + \alpha_i$,
- Node $i \in X_4(k)$ if $m(k) + 1 + \alpha_i < x_i(k) < m(k) + 2$,
- Node $i \in X_5(k)$ if $m(k) + 2 \leq x_i(k) < m(k) + 2 + \alpha_i$,
- Node $i \in X_6(k)$ if $m(k) + 2 + \alpha_i \leq x_i(k)$.

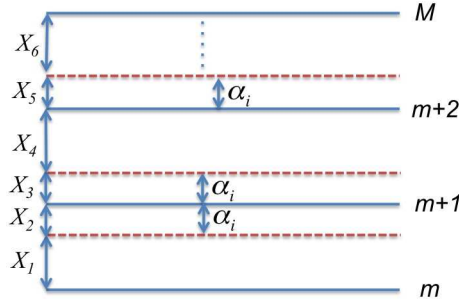


Figure 3: Dividing the nodes into sets according to their local values.

For simplicity we will drop the index k in the notation of the sets and $m(k)$ when there is no confusion. With every iteration, nodes can change their sets. Note that any node can jump in one iteration to a higher set, but the other way around is not always possible. For example, a node at iteration k in X_1 can jump at iteration $k+1$ to X_6 , but no node outside X_1 can get back to it as we will show next. Let us define L_i^k be the level of node i at iteration k , i.e. $L_i^k = \lfloor x_i(k) \rfloor - m(k)$. If $i \notin X_1(k_0)$, then $i \notin X_1(k)$ for all $k \geq k_0$ since,

$$\begin{aligned}
 x_i(k+1) &= x_i(k) + \sum_{j \in N_i} w_{ji} L_{ji} \\
 &\geq c_i(k) + \lfloor x_i(k) \rfloor + \left(\sum_{j \in N_i} w_{ji} \right) (m(k) - \lfloor x_i(k) \rfloor) \\
 &= c_i(k) + L_i^k + m(k) + \left(\sum_{j \in N_i} w_{ji} \right) (-L_i^k) \\
 &= m(k) + c_i(k) + w_{ii} L_i^k \\
 &\geq m(k) + 1 - \alpha_i,
 \end{aligned}$$

and $i \notin X_1(k+1)$. The last inequality is due to two possibilities,

- if $i \in X_2(k)$ then $L_i^k = 0$, and $m(k) + c_i(k) = x_i(k) \geq m(k) + 1 - \alpha_i$,
- otherwise $L_i^k \geq 1$, so $m(k) + c_i(k) + w_{ii} L_i^k \geq m(k) + w_{ii} \geq m(k) + 1 - \alpha_i$.

Therefore, the increase ∇V_k is due to nodes changing to a higher set. However, any node changing its set to a higher one, should have neighbors in the higher sets that cause ∇V_k to decrease by at least the same amount. To make this a formal argument we give the following lemma:

Lemma 1. *Consider the quantized system (9). Suppose that Assumption 1 holds. If $m(k+1) = m(k)$, we have*

$$\nabla V_k \leq 0.$$

Proof. See the Appendix. □

Lemma 1 implies that ∇V_k is non-increasing with time. Now we present two situations under which ∇V_k is strictly decreasing. The two situations will play an important role in the proof of the main result.

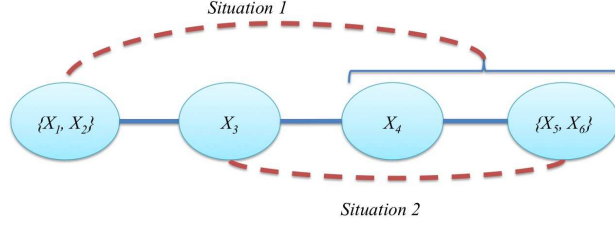


Figure 4: The solid lines (blue links) identify the network structure at any iteration $k_0 \leq k < k_0 + R(k_0)$, while if a dotted link (in red) appears, then $k = k_0 + R(k_0)$ and one of the situations occurs.

- **Situation 1 (S1)** occurs if at iteration k there exists a link in the network between a node $i \in X_4 \cup X_5 \cup X_6$ and a node $j \in X_1 \cup X_2$, in this case we have,

$$\begin{aligned} \nabla V_k &\leq -\min\{x_i(k) - m - 1 - \alpha_i, w_{ij}, \bar{c}_j\} \\ &\leq -\min\{\gamma, \delta\}, \end{aligned} \quad (18)$$

- **Situation 2 (S2)** occurs if at iteration k there exists any link in the network between a node $i \in X_5 \cup X_6$ and a node $j \in X_3$, in this case we have,

$$\begin{aligned} \nabla V_k &\leq -\min\{\alpha_j - c_j(k), w_{ij}\} \\ &\leq -\min\{\gamma, \delta\}. \end{aligned} \quad (19)$$

6.3 Proof of Main Result

To show that $V(k)$ is eventually decreasing, we have to introduce some more notation. Let

$$R(k_0) = \min\{k - k_0; k \geq k_0, \text{ either S1 or S2 occurs}\},$$

be the minimum time for at least one of the two situations to occur. We will show that if there exists at least one node in $\{X_1, X_2, X_3\}$ at k_0 and $m(k) = m(k_0)$ for $k \leq R(k_0)$, then we can have a fixed upper bound on $R(k_0)$. If we looked at the values of the nodes in the network at any iteration k_0 , we can see that if $k < k_0 + R(k_0)$, the network have special structure: only nodes in $\{X_1, X_2, X_3\}$ have links between each other, nodes in X_3 can also have links to X_4 , but not to $\{X_5, X_6\}$. Nodes in $\{X_5, X_6\}$ can only be connected to X_4 (see Fig. 4). Moreover, the values of nodes in X_3 cannot increase due to the link between X_3 and X_4 . To see this, let $i \in X_3$ and $s \in X_4$ where $s \in N_i$. Then we have:

$$x_i(k+1) = x_i(k) + w_{is}L_{si} + \sum_{j \in N_i - \{s\}} w_{ij}L_{ji},$$

but since $\lfloor x_i(k) \rfloor = \lfloor x_s(k) \rfloor$, we have $L_{is} = 0$ and thus $x_i(k+1) = x_i(k) + \sum_{j \in N_i - \{s\}} w_{ij}L_{ji}$, so nodes in X_4 do not have any effect on nodes in X_3 and the values of nodes in X_3 cannot increase for all $k < k_0 + R(k_0)$ (we will get back to this issue later).

To find the number of iterations for a red link to appear, we define the following function for nodes in $\{X_1, X_2, X_3\}$:

$$f(i, k) = \begin{cases} 1 & \text{if } i \in \{X_1(k), X_2(k)\}, \\ 0 & \text{if } i \in X_3(k), \end{cases} \quad (20)$$

and let $T_i(k_0, k)$ be the number of times a node i is in $\{X_1, X_2\}$ in the duration $k - k_0$, i.e.

$$T_i(k_0, k) = \sum_{t=k_0}^{t=k} f(i, t).$$

In fact, we can partition the nodes in $\{X_1, X_2, X_3\}$ depending on their distance to nodes in X_4 . Let r_i be the shortest path of all possible paths from a node $i \in \{X_1, X_2, X_3\}$ to some node in X_4 . We define the set D_u where $u = 1, \dots, r$ and $r = \max_i r_i$ as the set of nodes such that $i \in D_u$ if and only if $u = r_i$. For example, D_1 contains nodes that have direct neighbors in X_4 , D_2 contains the nodes that do not have direct neighbors in X_4 but there is a node in X_4 found 2 hops away, and so on. Moreover, for any node $i \in D_u$, we can find at least one neighbor $j \in D_{u-1}$. Let $P(i)$ be any one of these neighbors, referred to as the parent of i . We can now obtain the following lemma:

Lemma 2. *If $\{X_4, X_5, X_6\} \neq \emptyset$ at an iteration k_0 , and $m(k) = m(k_0)$ for $k_0 \leq k \leq k_0 + R(k_0)$, then for any integer $N \in \mathbb{N}$:
if*

$$T_i(k_0, k) \geq N \times \left(\frac{\alpha_{P(i)}}{w_{iP(i)}} + 1 \right),$$

then

$$T_{P(i)}(k_0, k) \geq N.$$

Proof. The proof is based on the observation we mentioned earlier. For any node $s \in X_3$, its neighbors in X_4 do not have any effect on $x_s(k+1)$ and it cannot have any neighbor in $\{X_5, X_6\}$ otherwise one of the situations (S1 or S2) occurs. Therefore, the decrease of the node s from X_3 to X_2 can only be due to its neighbors in $\{X_1, X_2\}$. Let $i \in \{X_1, X_2\}$ be the neighbor of node s , then

$$\begin{aligned} x_s(k+1) &= x_s(k) + \sum_{j \in N_s} w_{js} L_{js} \\ &= x_s(k) + w_{is} \times (-1) + \sum_{j \in N_s \cap \{X_1, X_2\} - \{i\}} w_{js} L_{js} \\ &\leq x_s(k) - w_{is} \\ &= 1 + c_s(k) - w_{is}, \end{aligned}$$

and the node s can either drop to X_2 or stay in X_3 depending on the resulting value $x_s(k+1)$. And since $c_s(k) \leq \alpha_s$ and $x_s(k+1)$ cannot increase if s was in X_3 at iteration k , then we are sure that if i was in $\{X_1, X_2\}$ for more than $\frac{\alpha_s}{w_{is}}$ iterations (i.e. $T_i(k_0, k) \geq \frac{\alpha_s}{w_{is}} + 1$), then s has dropped to X_2 at least once (i.e. $T_s(k_0, k) \geq 1$). Thus since $P(i) \in N_i$, we have

$$T_i(k_0, k) \geq \left(\frac{\alpha_{P(i)}}{w_{iP(i)}} + 1 \right) \implies T_{P(i)}(k_0, k) \geq 1. \quad (21)$$

If $T_i(k_0, k_N) \geq N \times \left(\frac{\alpha_{P(i)}}{w_{iP(i)}} + 1 \right)$, then we can find $N - 1$ iterations, k_1, k_2, \dots, k_{N-1} , such that

$$T_i(k_{v-1}, k_v - 1) \geq \left(\frac{\alpha_{P(i)}}{w_{iP(i)}} + 1 \right) \quad \text{for } v = 1, \dots, N.$$

By (21), we have $T_{P(i)}(k_{v-1}, k_v - 1) \geq 1$. Therefore,

$$\begin{aligned} T_{P(i)}(k_0, k) &= \sum_{v=1}^{N-1} T_{P(i)}(k_{v-1}, k_v - 1) + T_{P(i)}(k_{N-1}, k) \\ &\geq \left(\sum_{v=1}^{N-1} 1 \right) + 1 \\ &\geq N, \end{aligned}$$

and the lemma is proved. \square

Now we show that there is a fixed upper bound on the time for either of the situations to occur,

Lemma 3. *If $\{X_4, X_5, X_6\} \neq \phi$ at an iteration k_0 , and $m(k) = m(k_0)$ for $k \geq k_0$, then*

$$R(k_0) \leq n \left(1 + \frac{1}{2\delta} \right)^{n-1}.$$

Proof. Notice first that for any iteration $\bar{k} \geq k_0$, if $T_i(k_0, \bar{k}) \geq 1$ where $i \in D_1$, then situation 1 has occurred and $R(k_0) \leq \bar{k} - k_0$.

Moreover, since $m(k) = m(k_0)$ for $k \geq k_0$, then at every iteration k there is at least one node in $\{X_1, X_2\}$, leading to

$$\sum_{i \in \{X_1, X_2, X_3\}} T_i(k_0, k) \geq k - k_0.$$

Let $\bar{k} = k_0 + n \left(1 + \frac{1}{2\delta} \right)^{n-1}$; then we have

$$\sum_{i \in \{X_1, X_2, X_3\}} T_i(k_0, \bar{k}) \geq n \left(1 + \frac{1}{2\delta} \right)^{n-1},$$

and there must be a node $i \in D_u$ in this sum such that

$$T_i(k_0, \bar{k}) \geq \left(1 + \frac{1}{2\delta} \right)^{n-1}.$$

Without loss of generality, we can suppose $\frac{1}{2\delta} \in \mathbb{N}$. So applying Lemma 2, we can see that

$$\begin{aligned} T_i(k_0, \bar{k}) &\geq \left(1 + \frac{1}{2\delta} \right)^{n-1} \\ &\geq \left(1 + \frac{\alpha_{P(i)}}{w_{iP(i)}} \right) \times \left(1 + \frac{1}{2\delta} \right)^{n-2}, \\ &= \left(1 + \frac{\alpha_{P(i)}}{w_{iP(i)}} \right) \times N, \end{aligned}$$

where $N = \left(1 + \frac{1}{2\delta} \right)^{n-2}$, which implies

$$T_j(k_0, \bar{k}) \geq \left(1 + \frac{1}{2\delta} \right)^{n-2},$$

where $j = P(i)$ and $j \in D_{u-1}$. Doing this recursively ($u - 1$ times), we see that there is a node $s \in D_1$ such that,

$$T_s(k_0, \bar{k}) \geq \left(1 + \frac{1}{2\delta}\right)^{n-u},$$

but since $u \leq r \leq n$, we have $T_s(k_0, \bar{k}) \geq 1$ which means situation S1 occurred because $s \in D_1$. Therefore,

$$\begin{aligned} R(k_0) &\leq \bar{k} - k_0 \\ &\leq n \left(1 + \frac{1}{2\delta}\right)^{n-1}, \end{aligned}$$

and the lemma is proved. \square

We also need the following lemma,

Lemma 4. *Suppose Assumption 1 holds. Let $\beta = \min\{\gamma, \delta\}$, then for the quantized system (9), at any time k_0 , there is a finite time $k_1 \geq k_0$ such that for $k \geq k_1$, either $\{X_4, X_5, X_6\} = \phi$ or $m(k) > m(k_0)$. Moreover,*

$$k_1 \leq k_0 + n \left(\frac{V(k_0)}{\beta} + 1 \right) \left(\frac{1}{2\delta} + 1 \right)^{n-1}.$$

Proof. Let us prove it by contradiction. Suppose that $\{X_4, X_5, X_6\} \neq \phi$ and $m(k) = m(k_0)$ for $k \geq k_0$. Therefore we can apply Lemma 3 to show that there is a finite time $R(k_0)$ for situations S1 or S2 to occur. Whenever one of the situations occurs, we have $\nabla V_k \leq -\beta$, otherwise $\nabla V_k \leq 0$. For $k > k_0 + n \left(\frac{V(k_0)}{\beta} + 1 \right) \left(\frac{1}{2\delta} + 1 \right)^{n-1}$, we have that situations S1 or S2 have occurred at least $\left(\frac{V(k_0)}{\beta} + 1 \right)$ times; then

$$V(k) \leq V(k_0) - \beta \times \left(\frac{V(k_0)}{\beta} + 1 \right) \leq -\beta < 0,$$

which is a contradiction since $V(k) \geq 0$ is a Lyapunov function. As a result, there exists an iteration k_1 satisfying $k_1 \leq k_0 + n \left(\frac{V(k_0)}{\beta} + 1 \right) \left(\frac{1}{2\delta} + 1 \right)^{n-1}$ such that for $k \geq k_1$, either $\{X_4, X_5, X_6\} = \phi$ or $m(k) > m(k_0)$. \square

We are now ready to prove the following propositions,

Proposition 2. *Consider the quantized system (9). Suppose that Assumption 1 holds. Then for any initial value $\mathbf{x}(0)$, there is a finite time iteration where $\{X_4, X_5, X_6\} = \phi$.*

Proof. The value $m(k)$ cannot increase more than $M(0) - m(0)$ number of times because $M(k)$ is non-increasing. Therefore, applying Lemma 4 for $M(0) - m(0)$ times, we see that $\{X_4, X_5, X_6\} = \phi$ in a finite number of iterations. \square

Proposition 2 shows that in fact the nodes are restricted in a finite number of iterations to the sets $\{X_1, X_2, X_3\}$. In fact, we can even show a stronger result, that either X_1 or X_3 can be nonempty, but not both. This is given in the next proposition.

Proposition 3. *Consider the quantized system (9). Suppose that Assumption 1 holds. Then for any initial value $\mathbf{x}(0)$, there is a finite time iteration where either $\{X_3, X_4, X_5, X_6\} = \phi$ or $\{X_1, X_4, X_5, X_6\} = \phi$.*

Proof. Due to Proposition 2, we can find a finite time T such that $\{X_4, X_5, X_6\} = \phi$. Let us study the system for $k \geq T$. If we look at the system $\mathbf{y}(k) = -\mathbf{x}(k)$ (we call it system- y while the initial one we call it system- x), we see that $\mathbf{y}(k)$ satisfies equation (9). We define Y_1, \dots, Y_6 the sets corresponding to different nodes values with analogy to X_1, \dots, X_6 . Since system- x has nodes in $\{X_1, X_2, X_3\}$ only, this implies that system- y has nodes in $\{Y_2, Y_3, Y_4\}$. Since system- y has nodes in $\{Y_2, Y_3, Y_4\}$, by the definition of the sets, no node can get back to Y_1 unless $m(k)$ strictly increases which means in this case $M(k) = m(k)$. By applying Proposition 2 to system- y , there is a finite time for Y_4 to be empty, which leads us to be in one of the following possibilities:

- there is a finite time where nodes are in $\{Y_2, Y_3\}$, or
- there is a finite time where $m(k) = M(k)$, i.e. nodes are in $\{Y_1, Y_2\}$.

Projecting back these observations on the system- x , we see that there is a finite time for nodes to be either in $\{X_2, X_3\}$ or in $\{X_1, X_2\}$. \square

Proposition 4. *Consider the quantized system (9). Suppose that Assumption 1 holds. Then for any initial value $\mathbf{x}(0)$, there is a finite time iteration where either*

- *the values of nodes are cycling in a small neighborhood around the average such that :*

$$\begin{cases} |x_i(k) - x_j(k)| \leq \alpha_i + \alpha_j \text{ for all } i, j \in \mathcal{V} \\ |x_i(k) - x_{ave}| \leq 2\alpha \text{ for all } i \in \mathcal{V}, \end{cases} \quad (22)$$

- *or the quantized values have reached consensus, i.e.*

$$\begin{cases} \lfloor x_i(k) \rfloor = \lfloor x_j(k) \rfloor \text{ for all } i, j \in \mathcal{V} \\ |x_i(k) - x_{ave}| < 1 \text{ for all } i \in \mathcal{V}. \end{cases} \quad (23)$$

Proof. The two possibilities are consequence of the two possible cases of Proposition 3,

- Case $\{X_1, X_4, X_5, X_6\} = \phi$. Then all nodes are in $\{X_2, X_3\}$ and by the definition of the sets we have $|x_i(k) - x_j(k)| \leq \alpha_i + \alpha_j$ for all $i, j \in \mathcal{V}$, so nodes are cycling (due to Proposition 1) around $m + 1$. Moreover, since the average is conserved from Eq. (8), we have:

$$\begin{aligned} |x_i(k) - x_{ave}| &= |x_i(k) - x_{ave}(k)| \\ &\leq |\max_i x_i(k) - \min_i x_i(k)| \\ &\leq 2 \max_i \alpha_i \\ &= 2\alpha, \end{aligned}$$

- Case $\{X_3, X_4, X_5, X_6\} = \phi$. Then all nodes are in $\{X_1, X_2\}$ and by the definition of the sets we have reached quantized consensus. Since for any i and j we have $c_i(k), c_j(k) \in [0, 1)$, then $|x_i(k) - x_j(k)| < 1$ and as in the above due to Eq. (8), we have $|x_i(k) - x_{ave}| < 1$.

\square

6.4 Design of weights with arbitrarily small error

If the system has reached quantized consensus, the values of the agents' agreement variables become stationary and the deviation of these values from the average is no larger than 1. In the case when the system does not reach quantized consensus but becomes cyclic, Proposition 4 shows that the deviation of nodes' values from the average is upper bounded by 2α . Moreover the deviation can be made arbitrarily small by adjusting the weights in a distributed manner. Toward that end, we propose the following modified Metropolis weights:

$$\begin{aligned} w_{ij} &= \frac{1}{C(\max\{d_i, d_j\} + 1)}, \quad \forall (i, j) \in \mathcal{E} \\ w_{ii} &= 1 - \sum_{j \in N_i} w_{ij}, \quad \forall i \in \mathcal{V} \end{aligned}$$

where C is any rational constant such that $C \geq 2$. It can be easily checked that the proposed weights satisfy Assumption 1. Moreover, in addition to its distributed nature, the choice of C can be used to define the error. Notice that for any $i \in \mathcal{V}$, we have $w_{ii} > 1 - \frac{1}{C} \geq 1 - \frac{1}{C} + \gamma$, so

$$\alpha \leq \frac{1}{C},$$

which shows that given an arbitrary level of precision known to all the agents, the agents can choose the weights with large enough C in a distributed manner, so that the neighborhood of the cycle will be close to the average with the given precision.

It is worth mentioning that this arbitrarily small neighborhood weight design has a trade-off with the speed of convergence of quantized consensus protocol (small error weight design leads to slower convergence).

7 Discussion

Propositions 1 shows that the uniform quantization on communications given by the model of this paper can have a very important cyclic property. Up to our knowledge, this is the first work in deterministic quantized algorithms that shows this cyclic effect of nodes' values and it is also shown by Proposition 4 that the cyclic values can be control by a simple distributed adjustment of the weights. This can have an important impact on the design of quantized communication algorithms.² For example, due to the cyclic effect, nodes can use the history of their values to reach asymptotic convergence as the following proposition shows:

Corollary 1. *Consider the quantized system (9). Suppose that Assumption 1 holds. Then for any initial value $\mathbf{x}(0)$, if $y_i(k)$ is an estimate of the average at node i following the recursion:*

$$y_i(k) = \frac{k}{k+1}y_i(k-1) + \frac{1}{k+1}x_i(k), \quad \forall i \in \mathcal{V}, \quad (24)$$

where $y_i(0) = x_i(0)$, then $y_i(k)$ is converging,

$$\lim_{k \rightarrow \infty} y_i(k) = y_i^*, \quad \forall i \in \mathcal{V}, \quad (25)$$

having

$$|y_i^* - x_{ave}| \leq 1.$$

²Pattern generation (as for cyclic systems) plays an important role in the design of many mechanical and electrical systems [10].

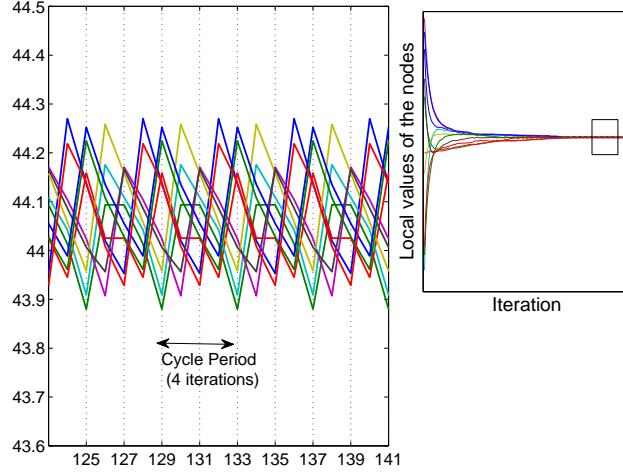


Figure 5: The nodes' values are entering into a cycle.

Proof. See the Appendix. □

Moreover, since the final behavior of the system depends on the initial values as shown by Proposition 4, we give here a condition on the initial values for the nodes to reach quantized consensus in networks:

Corollary 2. *Consider the quantized system (9). Suppose that Assumption 1 holds. If the initial values $\mathbf{x}(0)$ satisfy,*

$$\alpha < x_{ave} - \lfloor x_{ave} \rfloor < 1 - \alpha, \quad (26)$$

then the network reaches quantized consensus.

Proof. If the system was cyclic, then for any node $i \in \mathcal{V}$, we have $i \in \{X_1, X_2\}$, so $x_i(k) \in [m + 1 - \alpha_i, m + 1 + \alpha_i]$. This implies that $x_{ave}(k) \in [m + 1 - \alpha_i, m + 1 + \alpha_i]$, but since the average is conserved (from equation (8)), it also implies that $x_{ave} \in [m + 1 - \alpha_i, m + 1 + \alpha_i]$. From the latter condition, we see that if $\alpha < x_{ave} - \lfloor x_{ave} \rfloor < 1 - \alpha$, the system cannot be cyclic, and by Proposition 4, it must reach quantized consensus. □

8 Simulations

In this section, we present some simulations to demonstrate the theoretical results in the previous section. The weights for the simulations satisfy Assumption 1 and are the modified Metropolis weights with $C = 2$, i.e.

$$w_{ij} = \frac{1}{2(\max\{d_i, d_j\} + 1)} \quad \forall (i, j) \in \mathcal{E}.$$

8.1 Simple Network with n=10 Nodes

Proposition 4 shows that depending on the initial state $\mathbf{x}(0)$, the system reaches in finite time one of the two possibilities: 1) cyclic, 2) quantized consensus. We show on a network of 10 nodes

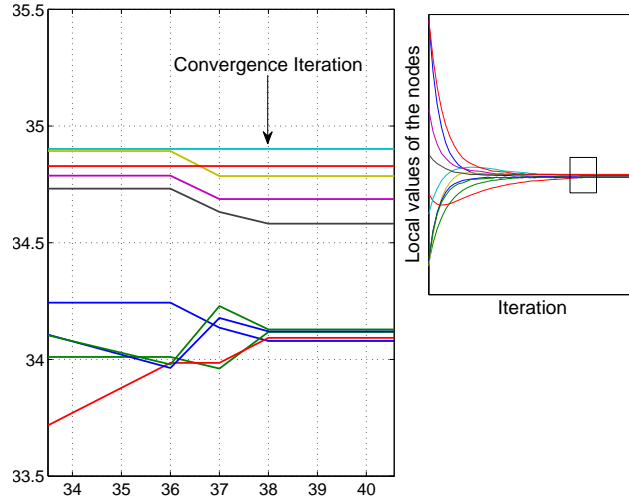


Figure 6: The nodes' values are converging.

with initial values selected uniformly at random from the interval $[0, 100]$ that both of these are possible. Fig. 5, shows that after a certain iteration, the nodes' values enter into a cycle of period 4 iterations, while Fig. 6 shows that starting from different initial values, all the 10 nodes reach quantized consensus in finite time. Mainly, at iteration 38, all nodes' values are between 34 and 35; therefore, we have

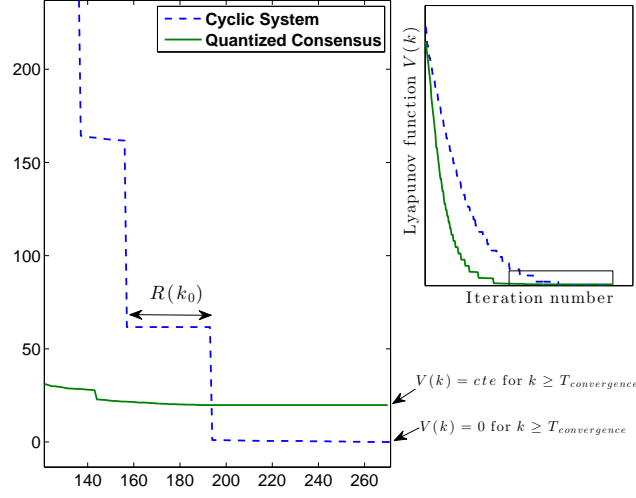
$$\lfloor x_i(k) \rfloor = 34 \quad \forall i = 1, \dots, 10, \quad \forall k \geq 38.$$

8.2 Random Graphs Simulations

To further simulate our theoretical results, we need to select some network model. The simulations are done on random graphs: Erdős-Renyi (ER) graphs and Random Geometric Graphs (RGG), given that they are connected. The random graphs are generated as follows:

- For the ER random graphs, we start from n nodes fully connected graph, and then every link is removed from the graph by a probability $1 - P$ and is left there with a probability P . We have tested the performance for different probabilities P given that the graph is connected.
- For the RGG random graphs, n nodes are thrown uniformly at random on a unit square area, and any two nodes within a connectivity radius R are connected by a link (the connectivity radius R is selected as $R = \sqrt{c \times \frac{\log(n)}{n}}$ where c is a constant that is studied by wide literature on RGG for connectivity). We have tested the performance for different connectivity radii given that the graph is connected. It is known that for a small connectivity radius, the nodes tend to form clusters.

Since Proposition 4 shows that the system would reach one of the cases in finite time, let us define T_{conv} be this time. Notice that if nodes enter the cyclic states (case 1), the Lyapunov function is null because for all $i \in \mathcal{V}$ and $k \geq T_{conv}$, we have $x_i(k) \in [m + 1 - \alpha_i, m + 1 + \alpha_i]$,

Figure 7: The system Lyapunov function $V(k)$.

so we can write,

$$V(k) = 0 \quad \forall k \geq T_{conv}.$$

However, if nodes reached quantized convergence (case 2), then the Lyapunov function is a constant because for all $i \in \mathcal{V}$ and $k \geq T_{conv}$, we have $x_i(k) \in [m, m+1]$, so we can write,

$$V(k) = cte \quad \forall k \geq T_{conv}.$$

8.2.1 Lyapunov Function

Fig. 7 shows the Lyapunov functions for the two different cases on an RGG with 100 nodes and $R = 0.2146$, where each case corresponds to initial values of nodes selected uniformly at random from the interval $[0, 100]$. The figure also shows $R(k_0)$ which is the number of iterations after k_0 up till $V(k)$ decreases (S1 or S2 occurs).

8.2.2 Quantized Consensus

Given that we are considering Metropolis weights with $C = 2$, then the system satisfies (26) if initial states are such that $x_{ave} - \lfloor x_{ave} \rfloor = 0.5$. We considered RGG and ER graphs of 100 nodes, where the initial condition is chosen as follows: the first 99 nodes are given uniformly random initial values from the interval $[0, 100]$, while the last node is given an initial value such that $x_{ave} - \lfloor x_{ave} \rfloor = 0.5$ is satisfied. Therefore, with these initial values, by applying Corollary 2, the system reaches quantized consensus in finite time T_{conv} . Table I shows the mean value over 100 runs of the T_{conv} for the RGG with different connectivity radii, $R_1 < R_2 < R_3 < R_4 < R_5$, where $R \in \{0.1357, 0.1517, 0.1858, 0.2146, 0.3717\}$. The results show that the more the graph is connected, the faster the convergence. These results are also shown to be true on ER graphs. Table II shows the mean value over 100 runs of the T_{conv} for the ER with different probability P , $P_1 < P_2 < P_3 < P_4$, where $P \in \{0.04, 0.06, 0.08, 0.10\}$.

	RGG $n = 100$				
	R_1	R_2	R_3	R_4	R_5
T_{conv}	1965.3	1068.9	364.3	233.3	55.9

Table 1: Convergence time for Random Geometric Graphs (RGG) with different connectivity radii (averaged over 100 runs).

	ER $n = 100$			
	$P_1 = 0.04$	$P_2 = 0.06$	$P_3 = 0.08$	$P_4 = 0.10$
T_{conv}	161.49	99.38	66.58	43.43

Table 2: Convergence time for Erdos Renyi (ER) with different probabilities of link existence (averaged over 100 runs).

9 Conclusion

In this paper, we studied the performance of deterministic distributed averaging protocols subject to communication quantization. We have shown that quantization due to links can force quantization on the state. Depending on initial conditions, the system converges in finite time to either a quantized consensus, or the nodes' values are entering into a cyclic behavior oscillating around the average.

Since the quantized consensus can be considered as a cyclic state with cycle period equal to zero, we will be investigating in future work the cycle period of the system. Moreover, we have just considered in this paper fixed networks with synchronous iterations, but since the weights for the quantized distributed averaging are selected in a totally distributed way, we are planning on extending this study to include asynchronous updates on time varying networks.

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A Proof of Lemma 1

We define $\nabla_i V_k$ as follows:

$$\begin{aligned} \nabla_i V_k \triangleq & \max\{|x_i(k+1) - m - 1| - \alpha_i, 0\} \\ & - \max\{|x_i(k) - m - 1| - \alpha_i, 0\}, \end{aligned} \quad (27)$$

from which it is evident that $\nabla V_k = \sum_{i \in \mathcal{V}} \nabla_i V_k$. Since only nodes moving from a set X_s to a higher set X_t where $t \geq \max\{s, 4\}$ can increase ∇V_k , then we can enumerate all the possible transitions of nodes that can cause ∇V_k to increase:

1. $X_1(k) \rightarrow X_m(k+1)$, $m \geq 4$,

$$\begin{aligned} \nabla_i V_k &= \max\{|x_i(k+1) - m - 1| - \alpha_i, 0\} \\ &\quad - \max\{|x_i(k) - m - 1| - \alpha_i, 0\} \\ &= (x_i(k+1) - m - 1 - \alpha_i) \\ &\quad - (1 + m - x_i(k) - \alpha_i) \\ &= x_i(k) + \sum_{j \in N_i} w_{ij} (\lfloor x_j(k) \rfloor - \lfloor x_i(k) \rfloor) - m - 1 \\ &\quad - m - 1 + x_i(k) \\ &= \sum_{j \in N_i} w_{ij} L_{ji} - 2(m + 1 - x_i(k)) \\ &= \sum_{j \in N_i} w_{ij} L_{ji} - 2\bar{c}_i(k) \\ &= \left(\sum_{j \in N_{i1}} w_{ij} \right) + \left(\sum_{j \in N_{i2}} w_{ij} \times 2 \right) + \left(\sum_{j \in N_{i3}} w_{ij} L_{ji} \right) \\ &\quad - 2 \left(\sum_{j \in N_i} w_{ij} + \gamma + (\bar{c}_i(k) - \alpha_i) \right) \\ &\leq \underbrace{\left(\sum_{j \in N_i \cap X_6} w_{ij} L_{ji} \right)}_{\geq 0} - 4\gamma, \end{aligned}$$

where $N_{i1} = N_i \cap \{X_3, X_4\}$, $N_{i2} = N_i \cap X_5$, and $N_{i3} = N_i \cap X_6$.

2. $X_2(k) \rightarrow X_m(k+1)$, $m \geq 4$, and the change in the Lyapunov function due to these nodes

is as follows:

$$\begin{aligned}
\nabla_i V_k &= \max\{|x_i(k+1) - m - 1| - \alpha_i, 0\} \\
&\quad - \max\{|x_i(k) - m - 1| - \alpha_i, 0\} \\
&= (x_i(k+1) - m - 1 - \alpha_i) - 0 \\
&= \sum_{j \in N_i} w_{ij} L_{ji} - \alpha_i - \bar{c}_i(k) \\
&= \left(\sum_{j \in N_i \cap \{X_3, X_4\}} w_{ij} \right) + \left(\sum_{j \in N_i \cap X_5} w_{ij} \times 2 \right) \\
&\quad + \left(\sum_{j \in N_i \cap X_6} w_{ij} L_{ji} \right) - \sum_{j \in N_i} w_{ij} - \gamma - \bar{c}_i(k) \\
&\leq \underbrace{\left(\sum_{j \in N_i \cap X_5} w_{ij} \right)}_{\geq 0} + \underbrace{\left(\sum_{j \in N_i \cap X_6} w_{ij} L_{ji} \right)}_{\geq 0} - 2\gamma.
\end{aligned}$$

3. $X_3(k) \rightarrow X_m(k+1)$, $m \geq 4$, then

$$\begin{aligned}
\nabla_i V_k &= \sum_{j \in N_i} w_{ij} L_{ji} - (\alpha_i - \bar{c}_i(k)) \\
&= \left(\sum_{j \in N_i \cap \{X_1, X_2\}} w_{ij} \times (-1) \right) + \left(\sum_{j \in N_i \cap X_5} w_{ij} \right) \\
&\quad + \left(\sum_{j \in N_i \cap X_6} w_{ij} L_{ji} \right) - (\alpha_i - \bar{c}_i(k)) \\
&\leq \underbrace{\left(\sum_{j \in N_i \cap X_5} w_{ij} \right)}_{\geq 0} + \underbrace{\left(\sum_{j \in N_i \cap X_6} w_{ij} L_{ji} \right)}_{\geq 0} - \gamma.
\end{aligned}$$

4. $X_4(k) \rightarrow X_m(k+1)$, $m \geq 4$, then

$$\begin{aligned}
\nabla_i V_k &= \sum_{j \in N_i} w_{ij} L_{ji} \\
&\leq \underbrace{\left(\sum_{j \in N_i \cap X_5} w_{ij} \right)}_{\geq 0} + \underbrace{\left(\sum_{j \in N_i \cap X_6} w_{ij} L_{ji} \right)}_{\geq 0}.
\end{aligned}$$

5. $X_5(k) \rightarrow X_m(k+1)$, $m \geq 5$, then

$$\begin{aligned}
\nabla_i V_k &= \sum_{j \in N_i} w_{ij} L_{ji} \\
&= \underbrace{\left(\sum_{j \in N_i \cap X_6} w_{ij} L_{ji} \right)}_{\geq 0} + \underbrace{\left(\sum_{j \in N_i, j \notin X_6} w_{ij} L_{ji} \right)}_{\leq 0}.
\end{aligned}$$

6. $X_6(k) \rightarrow X_6(k+1)$, then

$$\begin{aligned}\nabla_i V_k &= \sum_{j \in N_i} w_{ij} L_{ji} \\ &= \underbrace{\left(\sum_{j \in N_i \cap \bar{X}_6^i} w_{ij} L_{ji} \right)}_{\geq 0} + \underbrace{\left(\sum_{j \in N_i, j \notin \bar{X}_6^i} w_{ij} L_{ji} \right)}_{\leq 0}.\end{aligned}$$

where the expression $X_s \rightarrow X_t$ means the transition of a node that belongs to the set X_s at iteration k to the set X_t at iteration $k+1$, and the set \bar{X}_6^i is the set of nodes such that $j \in \bar{X}_6^i$ if $x_j(k) \geq x_i(k)$.

Notice that the increase in ∇V_k because of a node s belonging to one of the presented 6 possibilities is only due to a neighbor p in $\{X_5(k), X_6(k)\}$ such that $x_p(k) \geq x_s(k)$. Then p can belong to two possible sets: X_5 or X_6 .

Suppose first that $p \in X_6(k)$, let A be the increase in $\nabla_s V_k$, then this increase is as follows:

$$A = w_{ps} L_{ps} > 0,$$

but this increase is decreased again in $\nabla_p V_k$ since a node in $X_6(k)$ cannot drop below $X_4(k+1)$, we can write:

$$\begin{aligned}\nabla_p V_k &= \max\{|x_p(k+1) - m - 1| - \alpha_p, 0\} \\ &\quad - \max\{|x_p(k) - m - 1| - \alpha_p, 0\} \\ &= (x_p(k+1) - m - 1 - \alpha_p) - (x_p(k) - 1 - m - \alpha_p) \\ &= x_p(k) + \sum_{j \in N_p} w_{jp} L_{jp} - x_p(k) \\ &= \underbrace{w_{sp} L_{sp}}_{-A} + \sum_{j \in N_p - \{s\}} w_{jp} L_{jp}.\end{aligned}$$

Taking the other case, suppose now $p \in X_5$, let B be the increase in $\nabla_s V_k$ of a node s due to its neighbor $p \in X_5$:

$$B = w_{sp} > 0,$$

then this increase is decreased again in $\nabla_p V_k$, but we should consider two cases:

- $p: X_5 \rightarrow X_m$, $m \geq 4$, then

$$\nabla_p V_k = \underbrace{w_{ps} L_{sp}}_{\leq -B} + \sum_{j \in N_p - \{s\}} w_{jp} L_{jp}, \quad (28)$$

- $p: X_5 \rightarrow X_3$, then

$$\begin{aligned}\nabla_p V_k &\leq -1/2 \\ &\leq - \sum_{j \in N_p} w_{pj} \\ &= \underbrace{-w_{ps}}_{-B} - \sum_{j \in N_p - \{s\}} w_{jp},\end{aligned}$$

and p decreases in the same amount that its neighbor s increased.

Remark: For every positive value that increases ∇V_k , there is a unique corresponding negative value that compensates this increase by decreasing ∇V_k . This is because for any link $l \sim (i, j) \in \mathcal{E}$, the increase in $\nabla_i V_k$ due to l forces a decrease in $\nabla_j V_k$ due to the same link, and so there is one to one mapping between the increased values and the decreased ones.

As a result of the discussion we can have the total ∇V_k cannot increase, namely

$$\nabla V_k = \sum_i \nabla_i V_k \leq 0.$$

B Proof of Corollary 1

The state equation of $y_i(k)$ for a node i is give by

$$\begin{aligned} y_i(k) &= \frac{k}{k+1} y_i(k-1) + \frac{1}{k+1} x_i(k) = \frac{1}{k+1} \sum_{t=0}^{t=k} x_i(t) \\ &= \frac{1}{k+1} \left(\sum_{t=0}^{t=T_{conv}-1} x_i(t) \right) + \frac{1}{k+1} \left(\sum_{t=T_{conv}}^{t=k} x_i(t) \right), \end{aligned}$$

where T_{conv} is the finite time iteration when the nodes' values start cycling. As k approaches infinity, the left part in the sum vanishes while the right part converges to the average of the values in a cycle, i.e.

$$\lim_{k \rightarrow \infty} y_i(k) = y_i^* = \frac{1}{P} \sum_{t=T_{conv}}^{t=T_{conv}+P-1} x_i(t),$$

where P is the cycle period. Since for $k \geq T_{conv}$ we have $|x_i(k) - x_{ave}| \leq 1$ from Proposition 4, then $|y_i^* - x_{ave}| \leq 1$.



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