# Well-spread sequences and edge-labellings with constant Hamilton-weight

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A sequence $(a_i)$ of integers is *well-spread* if the sums $a_i + a_j$, for $i < j$, are all different. For a fixed positive integer $r$, let $W_r(N)$ denote the maximum integer $n$ for which there exists a well-spread sequence $0 \leq a_1 < \cdots < a_n \leq N$ with $a_i \equiv a_j \pmod{r}$ for all $i$, $j$. We give a new proof that $W_r(N) < (N/r)^{1/2} + O((N/r)^{1/4})$; our approach improves a bound of Ruzsa [*Acta. Arith.* **65** (1993), 259–283] by decreasing the implicit constant, essentially from 4 to $\sqrt{3}$. We apply this result to verify a conjecture of Jones et al. from [*Discuss. Math. Graph Theory* **23** (2003), 287–307]. The application concerns the growth-rate of the maximum label $\Lambda(n)$ in a 'most-efficient' metric, injective edge-labelling of $K_n$ with the property that every Hamilton cycle has the same length; we prove that $2n^2 - O(n^{3/2}) < \Lambda(n) < 2n^2 + O(n^{61/40})$.

**Keywords:** Well-spread, weak Sidon, graph labelling, Hamilton cycle

## 1   Introduction

Ostensibly our purpose is to prove a conjecture from [JKMW03] concerning the growth-rate of the maximum label in a certain edge-labelling of $K_n$. The essential ingredient in the proof, Theorem 4, determines asymptotically the maximum 'density' of a finite, well-spread sequence of nonnegative integers. This result was first proved (explicitly) by Ruzsa [Ruz93]; our proof improves upon his bound and as such may be of independent interest.

### *Sets and sequences*

We write $\mathbb{Z}^+$ and $\mathbb{N}$, respectively, for the sets of positive and nonnegative integers. Kotzig [Kot72] called a sequence $(a_i)$ of integers *well-spread* if the sums $a_i + a_j$, for $i < j$, are all different; *weak Sidon* is used synonymously, e.g., in [Ruz93]. He studied finite, well-spread sequences in part due to their relationship with 'magic valuations'—now called 'edge-magic total labellings'—of graphs; see [PW99] for further details. If we strengthen the condition and require that all the sums $a_i + a_j$, for $i \leq j$, be distinct, then $(a_i)$ is called a *Sidon sequence*. In connection with his studies in Fourier theory, Sidon [Sid32, Sid35] considered these sequences under the name $B_2$-*sequence*; see [HR83] for a basic reference. Every Sidon sequence is well-spread, but it is easy to construct examples to show that the converse is false: e.g., $(1, 2, 3)$. We shall fix a modulus $r \in \mathbb{Z}^+$ and consider *constant-residue* integral sequences $(a_i)$, i.e., ones

for which $a_i \equiv a_j \pmod{r}$ for all $i$, $j$; our application depends on the case $r = 2$, viz., the *constant-parity* sequences.

Our main number-theoretic contribution (Theorem 4) concerns the asymptotic behaviour of the following functions from $\mathbb{N}$ onto $\mathbb{Z}^+$:

$$W(N) \quad := \quad \max\{n : \text{there is a well-spread sequence } 0 \le a_1 < \cdots < a_n \le N\};$$
$$W_r(N) \quad := \quad \max\{n : \text{there is a constant-residue, well-spread sequence } 0 \le a_1 < \cdots < a_n \le N\}.$$

We use $S$, $S_r$, respectively, for the functions defined by replacing 'well-spread' by 'Sidon' in these definitions. Several basic inequalities follow at once:

$$S_r(N) \le W_r(N), S(N) \le W(N) \quad \text{for each } N \in \mathbb{N}. \tag{1}$$

Since the well-spread and Sidon properties are invariant under (integral) affine transformations, the maximum length of either type of sequence contained in an $(N+1)$-term arithmetic progression is the same as among an initial segment of $N+1$ nonnegative integers. Thus,

$$S_r(N) = S\left(\left\lfloor \frac{N}{r} \right\rfloor\right) \quad \text{and} \quad W_r(N) = W\left(\left\lfloor \frac{N}{r} \right\rfloor\right). \tag{2}$$

Though we need only $W_2$ for our graph labelling application, we shall state our number-theoretic results in terms of $W_r$ since we prefer to display explicitly the dependence on the modulus $r$.

## Graphs and labellings

Since we employ standard graph-theoretic notation, we refer the reader to any basic text—e.g. [Wes01]—for omitted definitions. We use $[n] := \{1, \ldots, n\}$ for the vertex set of a complete graph $K_n$. If $A$ is an edge with ends $i$, $j$, then we write $A = ij$. An *edge-labelling* of $K_n$ is a function $\lambda : E(K_n) \to \mathbb{Z}^+$. We say that $\lambda$ has *constant Hamilton-weight* whenever the value of $\sum_{A \in E(H)} \lambda(A)$ is independent of the Hamilton cycle $H$, and is *metric* if it satisfies the triangle-inequality: $\lambda(ik) \le \lambda(ij) + \lambda(jk)$ for every triple $i, j, k \in [n]$.

Our main graph-theoretic contribution (Theorem 6) verifies a conjecture from [JKMW03] by determining the asymptotic growth-rate of the following function from $\mathbb{Z}^+$ into $\mathbb{Z}^+$:

$$\Lambda(n) := \min_{\lambda} \max_{A \in E(K_n)} \lambda(A),$$

the minimum being taken over all metric, injective edge-labellings $\lambda$ of $K_n$ having constant Hamilton-weight.

## Background

Let us begin with a celebrated result of Erdős and others on the 'density' of finite Sidon sequences. Here and throughout this paper, all asymptotic assertions are contingent on the relevant parameter ($N$ or $n$) tending to infinity.

**Theorem 1** $S(N) \sim N^{1/2}$; *i.e.,*

$$\left(1 - o(1)\right)N^{1/2} < S(N) < \left(1 + o(1)\right)N^{1/2}. \tag{3}$$

The upper bound in (3)—in the form $N^{1/2} + O(N^{1/4})$—was proved by Erdős and Turán [ET41], who also established the lower bound $(1/\sqrt{2} - o(1))N^{1/2}$; later Chowla [Cho44a, Cho44b] and independently Erdős (1944, unpublished) applied a result of Singer [Sin38] (Theorem 5 below) to improve the lower bound to that in (3). Bose and Chowla [BC63] proved a generalization of (3) to '$B_r$-sequences'; this reference also provides a more accessible discussion of Chowla's result. Eventually Lindström [Lin69] improved the upper bound to $N^{1/2} + N^{1/4} + O(1)$. It remains open—and was given a price tag by Erdős— to decide whether, for every $\varepsilon > 0$, the inequality $S(N) < N^{1/2} + o(N^\varepsilon)$ holds. See [BS85, Sós91] for further discussion and references. See [AKS81, Guy94, Ruz98, Sid32, Sid35] for a precise statement and related progress on the corresponding infinite problem.

The following theorem from [JKMW03] provides a connection between sequences and labellings; see also [KP03] and the references therein for antecedents of this result.

**Theorem 2** *For $n \geq 3$, a metric, injective edge-labelling $\lambda$ of $K_n$ has constant Hamilton-weight if and only if there is a constant-parity, well-spread $\mathbb{N}$-sequence $(a_i)_{i=1}^n$ such that*

$$\lambda(ij) = \frac{a_i + a_j}{2} \text{ for each edge } ij \text{ of } K_n.$$

*The sequence $(a_i)$ is uniquely determined by $\lambda$.*

Theorem 2 shows that if we define $\psi_{cp} : \mathbb{Z}^+ \to \mathbb{N}$ by

$$\psi_{cp}(n) := \min\{a_{n-1} + a_n : \text{ there exists a constant-parity, well-spread } \mathbb{N}\text{-sequence } a_1 < \cdots < a_n\},$$

then

$$\Lambda(n) = \frac{\psi_{cp}(n)}{2} \quad \text{for every } n \geq 3. \tag{4}$$

We note in passing that for finite Sidon sequences $(a_i)$, similar 'sum-sets' $\{a_i + a_j \mid i \leq j\}$ have been investigated considerably; see [Ruz96] for recent results and further references. For our study of $\Lambda$, we additionally introduce the function $\sigma_{cp} : \mathbb{Z}^+ \to \mathbb{N}$, defined by

$$\sigma_{cp}(n) := \min\{a_n : \text{ there exists a constant-parity, well-spread } \mathbb{N}\text{-sequence } a_1 < \cdots < a_n\}.$$

## Packing with 2-sums

The definition of $\psi_{cp}$ exhibits a 'packing flavour'; indeed, a variant of $\psi_{cp}$ using this terminology was studied by Graham and Sloane [GS80]. They defined $v_\alpha(n)$ to be the smallest nonnegative integer $N$ such that there exists an integral sequence $0 = a_1 < \cdots < a_n$ with the property that the sums $a_i + a_j$, for $i < j$, belong to $[0, N]$ and represent each element of this set at most once. If $\psi$ denotes our $\psi_{cp}$ without the constant-parity condition, then $\psi = v_\alpha$. Graham and Sloane tabulated the values $v_\alpha(n)$ for $n \leq 10$, gave exemplary sequences, and outlined a proof of

$$2n^2 - O(n^{3/2}) < v_\alpha(n) < 2n^2 + O(n^{36/23}). \tag{5}$$

They also considered the three functions that arise when $i < j$ is changed to $i \leq j$ (giving the Sidon version of $v_\alpha$) or when the arithmetic is done modulo $N$, and the four functions resulting from changing *smallest* to *largest* and *at most* to *at least* (giving the covering analogues of the four packing functions). By now these eight functions enjoy a vast literature, much of which was cited already in [GS80].

After proving our main graph-theoretic result (Theorem 6), we shall indicate a slight improvement to the upper bound in (5). Similar improvements are possible in the bounds for the other packing functions.

## 2   Well-spread sequences

Theorem 1 and (2) show that $S_r(N) \sim (N/r)^{1/2}$. The functions $W_r$ exhibit the same asymptotic behaviour, since Ruzsa [Ruz93] proved that a well-spread sequence contained in the set $\{1, \ldots, N\}$ contains at most $N^{1/2} + 4N^{1/4} + 11$ terms. An upper bound for $W(N)$ of the form $N^{1/2} + O(N^{1/4})$ is also implicit in the work of Graham and Sloane [GS80] and was probably known to these authors. Presently, we shall derive this result again, in terms of $W_r$.

To get started, we need a cruder estimate:

**Lemma 3** *If N is sufficiently large, then* $W_r(N) < 2.001(N/r)^{1/2}$.

*Proof.* Let $n = W_r(N)$ and $0 \le a_1 < \cdots < a_n \le N$ be a well-spread sequence with each $a_i \equiv k \pmod{r}$, for some $0 \le k < r$. The sums $a_i + a_j$, for $i < j$, are distinct, at most $2N - r$, congruent modulo $r$ to $2k$, and hence lie in the set $\{2k + r, 2k + 2r, \ldots, 2k + \ell r\}$, where $\ell := \lfloor (2N - r - 2k)/r \rfloor$. Thus $\binom{n}{2} \le \ell$, from which $n < (2\ell)^{1/2} + 1$, and hence the assertion, follow easily.                                                 □

**Theorem 4** $W_r(N) < (N/r)^{1/2} + O((N/r)^{1/4})$.

*Proof.* Let $N$ be large enough to invoke Lemma 3, and set $n := W_r(N)$. Then there exists a constant-residue, well-spread sequence $0 \le a_1 < \cdots < a_n \le N$.

For $1 \le i < j \le n$, Lindström [Lin69] called $j - i$ the *order* of the difference $a_j - a_i$. He observed that the differences of order $\nu > 0$ can be arranged into sequences of the form

$$a_\alpha - a_\beta, a_\beta - a_\gamma, a_\gamma - a_\delta, \ldots,$$

where $\alpha - \beta = \beta - \gamma = \gamma - \delta = \cdots = \nu$. Because of 'telescoping', the sum of all these differences is at most $\nu N$ (and less than $\nu N$ for $\nu > 1$). Thus, for $m \ge 2$, the sum $\mathcal{S}$ of all the positive differences of order at most $m$ is less than $m(m+1)N/2$.

Let us call $a_i$ a *mean-point* if $2a_i = a_j + a_k$ for some $j, k \in [n]$; notice that in this case $a_i - a_k = a_j - a_i$. Except for the values $a_j - a_i$, for mean points $a_i$ (or $a_j$), the differences $a_k - a_\ell$, for $1 \le \ell < k \le n$, are all different since $(a_i)$ is well-spread. As the only candidates for mean-points are $a_2, \ldots, a_{n-1}$, we have at most $t := n - 2$ differences occurring with higher multiplicity, and the well-spread property implies that this multiplicity is 2. Since $(a_i)$ has constant-residue, the differences are all multiples of $r$. If $1 \le m < n$ and $s := n - (m+1)/2$, then the number of positive differences of order at most $m$ is $mn - m(m+1)/2 = ms$. Therefore,

$$\mathcal{S} \ge \sum_{i=1}^{t} (ri + ri) + \sum_{j=1}^{ms-2t} (rt + rj) = \frac{rms(ms+1)}{2} - rt(ms - t).$$

For $1 < m < n$, it follows that

$$\frac{rms(ms+1)}{2} - rt(ms - t) < \frac{m(m+1)}{2}N,$$

so that

$$\frac{r(ms)^2}{2} < \frac{m(m+1)}{2}N + rmst.$$

Since $s,t < n$, the second term on the right side is less than $rmn^2$, which by Lemma 3 is at most $(2.001)^2 mN < 4.5mN$. Thus, $s^2 < N(1+10/m)/r$, and since $(1+x)^{1/2} < 1+x/2$ for $x = 10/m$, we have

$$n = \frac{m+1}{2} + s < \frac{m+1}{2} + \left(\frac{N}{r}\right)^{1/2}\left(1+\frac{5}{m}\right). \tag{6}$$

With $m := \lceil (N/r)^{1/4}\rceil$, this gives the bound in the statement of the theorem. □

**Remarks** Our proof of Theorem 4 adapts the main idea of Lindström [Lin69] to well-spread, constant-residue sequences. Ruzsa [Ruz93] also based his proof on the idea of studying the 'small' differences $a_j - a_i$, though in a "somewhat hidden" fashion (quote from [Ruz93]). Here we compare the resulting implicit constants.

To optimize ours, we iterate the proof once again. Instead of applying Lemma 3 (to bound $rmn^2$ from above), we apply Theorem 4 itself. This allows us to replace '10' by '$3 + O((N/r)^{-1/4})$'. To minimize the right side of (the adjusted) inequality (6), we now choose $m$ to be $\lceil \sqrt{3}(N/r)^{1/4}\rceil$. These modifications replace the big-oh term in Theorem 4 by $\sqrt{3}(N/r)^{1/4} + O(1)$. Ruzsa's proof essentially produces the value 4 in place of our $\sqrt{3}$.

While we're comparing bounds, we should mention that the upper bound for $S_r(N)$ implied by (1) and Theorem 4 does *not* improve on earlier results. For example, Lindström's bound [Lin69] together with (2) gives the implied constant 1 in place of our $\sqrt{3}$ □

# 3 Edge-labellings with constant Hamilton-weight

We turn to verifying the main conjecture from [JKMW03]. Proofs of the following basic connections are left to the reader (or see [JKMW03]):

$$W_2(N) \geq \sigma_{cp}^{-1}(N) \quad \text{for every } N \in \text{range}(\sigma_{cp}); \tag{7}$$

$$\psi_{cp}(n) \geq \sigma_{cp}(n) + \sigma_{cp}(n-1) \quad \text{for every } n \geq 2. \tag{8}$$

We also need a simple upper bound on $\sigma_{cp}(n)$, a theorem on the density of primes, and Singer's theorem on difference sets. The first of these follows immediately from our work in [JKMW03]:

$$\sigma_{cp}(n) < 2n^2(1+o(1)). \tag{9}$$

For the second, we opt for the present state-of-the-art, due to Baker et al. [BHP01]: if $x$ is sufficiently large, then there is a prime $p$ with

$$x < p \leq x + x^{21/40}. \tag{10}$$

For the third, we have

**Theorem 5 ([Sin38])** *If $q$ is a prime power, then there are integers $b_0, b_1, \ldots, b_q \in [0, q^2+q]$ such that the differences $b_i - b_j$, for $i \neq j$, are congruent, modulo $q^2 + q + 1$, to the integers $1, 2, \ldots, q^2 + q$. In particular, $(b_i)_{i=0}^q$ forms a Sidon sequence, hence is well-spread.*

Finally, we state and prove our main graph-theoretic result:

**Theorem 6** $\Lambda(n) \sim 2n^2$; *more precisely,*

$$2n^2 - O(n^{3/2}) < \Lambda(n) < 2n^2 + O(n^{61/40}). \tag{11}$$

*Proof.* For the upper bound, consider an integer $n$, large enough to apply (10) with $x = n - 1$; then we can find a prime $p$ so that

$$n - 1 < p < n + n^{21/40}.$$

Theorem 5 delivers a well-spread sequence $0 \le b_0 < b_1 < \cdots < b_p \le p^2 + p$. Now $a_i := 2b_{i-1}$, for $i = 1, 2, \ldots, n$, defines a constant-parity, well-spread sequence with

$$a_{n-1} + a_n = 2(b_{n-2} + b_{n-1}) \le 4p^2 + 4p - 6 < 4n^2 + O(n^{61/40}).$$

By definition, $\psi_{cp}(n) \le a_{n-1} + a_n$, and since $\Lambda(n) = \psi_{cp}(n)/2$ (see (4)), the upper bound in (11) follows.

For the lower bound, let $n \in \mathbb{N}$ and $N = \sigma_{cp}(n)$. Then (7) and Theorem 4 imply that

$$n = \sigma_{cp}^{-1}(N) \le W_2(N) < \left(\frac{N}{2}\right)^{1/2} + O\left(\left(\frac{N}{2}\right)^{1/4}\right),$$

so that

$$2n^2 < N + O(N^{3/4}).$$

Now (9) shows that

$$\sigma_{cp}(n) = N > 2n^2 - O(n^{3/2}).$$

Thus (8) gives $\psi_{cp}(n) > 4n^2 - O(n^{3/2})$, and again applying (4) yields the desired bound. $\square$

### Closing remarks

We first elaborate on the lower bound in (3). The idea in the proof of the upper bound in Theorem 6 can be used to show that $S_r(N) > (N/r)^{1/2}$ for infinitely many integers $N$ and that $S_r(N) > (N/r)^{1/2} - (N/r)^{21/80}$ for sufficiently large $N$. Absent the modulus $r$, these observations have been made elsewhere; cf. [PS95]. The slight improvement here over previously published bounds—e.g., in [PS95], the fraction $5/16$ replaces $21/80$—results from our use of a more recent prime density theorem.

Baker and Harman [BH96] sketch the history of such theorems, i.e., those of the form

$$[x, x + x^\vartheta] \text{ contains a prime whenever } x \text{ is sufficiently large}$$

for a specified constant $\vartheta$; cf. (10).

An alternate approach to Theorem 6 is to reduce the problem to one considered in [GS80]. It is not difficult to see that $\psi_{cp}(n)$ is achieved when $a_1 = 0$, so that the constant parity is even. Then $\Lambda(n)$ can be identified with Graham and Sloane's $v_\alpha(n)$, so that (5) also gives $\Lambda(n) \sim 2n^2$.

Turning this observation around shows that our (11) improves (5). This stems from the decrease in the minimum $\vartheta$ since [GS80] appeared. The present value $\vartheta = 21/40$ (cf. $13/23$ available to Graham and Sloane) improves not only (5), but also the upper bounds for the other three packing functions considered in [GS80].

## Acknowledgements

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