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The b-chromatic number of some power graphs

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Let *G* be a graph on vertices $v_1, v_2, ..., v_n$. The b-chromatic number of *G* is defined as the maximum number *k* of colors that can be used to color the vertices of *G*, such that we obtain a proper coloring and each color *i*, with $1 \le i \le k$, has at least one representant x_i adjacent to a vertex of every color *j*, $1 \le j \ne i \le k$. In this paper, we give the exact value for the b-chromatic number of power graphs of a path and we determine bounds for the b-chromatic number of power graphs of a cycle.

Keywords: b-chromatic number, coloring, cycle, path, power graphs

1 Introduction

We consider graphs without loops or multiple edges. Let *G* be a graph with a vertex set *V* and an edge set *E*. We denote by d(x) the degree of the vertex *x* in *G*, and by $dist_G(x, y)$ the distance between two vertices *x* and *y* in *G*. The *p*-th power graph G^p is a graph obtained from *G* by adding an edge between every pair of vertices at distance *p* or less, with $p \ge 1$. It is easy to see that $G^1 = G$. In the literature, power graphs of several classes have been investigated [2, 3, 8]. In this note we study a vertex coloring of power graphs. The power graph of a path and the power graph of a cycle can be also considered as respectively subclasses of *distance graphs* and *circulant graphs*. The *distance graph* G(D) with distance set $D = \{d_1, d_2, \ldots\}$ has the set *Z* of integers as vertex set, with two vertices $i, j \in Z$ adjacent if and only if $|i - j| \in D$. The *circulant graph* can be defined as follows. Let *n* be a natural number and let $S = \{k_1, k_2, \ldots, k_r\}$ with $k_1 < k_2 < \ldots < k_r \le n/2$. Then the vertex set of the circulant graph G(n, S) is $\{0, 1, \ldots, n-1\}$ and the set of neighbors of the vertex *i* is $\{(i \pm k_j)mod \ n \ j = 1, 2 \ldots, r\}$.

The study of distance graphs was initiated by Eggleton and al. [4]. Recently, the problem of coloring of this class of graphs has attracted considerable attention, see e.g. [12, 13]. Circulant graphs have been extensively studied and have a vast number of applications to multicomputer networks and distributed computation (see [1, 10]). The special cases we consider are the distance graph G(D) with finite distance set $D = \{1, 2, ..., p\}$ which is isomorphic to the *p*-th power graph of a path and the circulant graph G(n, S) with $S = \{1, 2, ..., p\}$ which is isomorphic to the *p*-th power graph of a cycle.

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A *k*-coloring of *G* is defined as a function *c* on $V(G) = \{v_1, v_2, ..., v_n\}$ into a set of colors $C = \{1, 2, ..., k\}$ such that for each vertex v_i , with $1 \le i \le n$, we have $c_{v_i} \in C$. A proper *k*-coloring is a *k*-coloring satisfying the condition $c_x \ne c_y$ for each pair of adjacent vertices $x, y \in V(G)$. A dominating proper *k*-coloring is a proper *k*-coloring satisfying the following property *P*: for each *i*, $1 \le i \le k$, there exists a vertex x_i of color *i* such that, for each *j*, with $1 \le j \ne i \le k$, there exists a vertex y_j of color *j* adjacent to x_i . A set of vertices satisfying the property *P* is called a *dominating system*. Each vertex of a dominating system is called a *dominating vertex*. The *b*-chromatic number $\varphi(G)$ of a graph *G* is defined as the maximum *k* such that *G* admits a dominating proper *k*-coloring.

The b-chromatic number was introduced in [7]. The motivation, similarly as for the previously studied *achromatic number* (cf. e.g. [5, 6]), comes from algorithmic graph theory. The achromatic number $\psi(G)$ of a graph *G* is the largest number of colors which can be assigned to the vertices of *G* such that the coloring is proper and every pair of distinct colors appears on an edge. A proper coloring of a graph *G* using $k > \chi(G)$ colors could be improved if the vertices of two color classes could be recolored by a single color so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by the achromatic number. A more versatile form of recoloring strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes, so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by $\varphi(G)$ (these recolorings are discussed in [7] and [11]). Thus $\varphi(G) \leq \psi(G)$ (also given in [7]). From this point of view, both complexity results and tight bounds for the b-chromatic number are interesting. The following bounds of b-chromatic number are already presented in [7].

Proposition 1 Assume that the vertices $x_1, x_2, ..., x_n$ of G are ordered such that $d(x_1) \ge d(x_2) \ge ... \ge d(x_n)$. Then $\varphi(G) \le m(G) \le \Delta(G) + 1$, where $m(G) = max\{1 \le i \le n : d(x_i) \ge i - 1\}$ and $\Delta(G)$ is the maximum degree of G.

R. W. Irving and D. F. Manlove [7] proved that finding the b-chromatic number of any graph is a NP-hard problem, and they gave a polynomial-time algorithm for finding the b-chromatic number of trees. Kouider and Mahéo [9] gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. They gave, in particular, a lower bound for the b-chromatic number of the cartesian product of two graphs where each one has a stable dominating system. More recently in [11], the authors characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper *k*-coloring even for connected bipartite graphs and $k = \Delta(G) + 1$. They also determine the asymptotic behavior for the b-chromatic number of random graphs.

In this paper, we present several exact values and determine bounds for the b-chromatic number of power graphs of paths and cycles.

Let Diam(G) be the diameter of a graph G, defined as the maximum distance between any pair of vertices of G. Let us begin with the following observation.

Fact 2 For any graph G of order n, if $Diam(G) \le p$, then $\varphi(G^p) = n$, with $p \ge 2$.

Proof. If $Diam(G) \le p$, it is trivial to see that G^p is a complete graph. So $\varphi(G^p) = n$.

Let *G* be a path or a cycle on vertices $x_1, x_2, ..., x_n$. We fix an orientation of *G* (left to right if *G* is a path and clockwise if *G* is a cycle). For each $1 \le i \le n$, we denote by x_i^+ (resp. x_i^-) the successor (resp.

predecessor) of x_i in G (if any). For $1 \le i \ne j \le n$, we define $[x_i, x_j]_G$, $[x_i, x_j)_G$ and $(x_i, x_j)_G$ as the set of consecutive vertices on G from respectively x_i to x_j , x_i to x_j^- and x_i^+ to x_j^- , following the fixed orientation of G. If there is no ambiguity, we denote $[x_i, x_j]_G$, $[x_i, x_j)_G$ and $(x_i, x_j)_G$ by respectively $[x_i, x_j]$, $[x_i, x_j)$ and $(x_i, x_j)_G$.

In all figures, the graph G is represented with solid edges. Edges added in a *p*-th power graph G^p are represented with dashed edges. In some figures, vertices are surrounded and represent a dominating system of the coloring. In any coloring of a graph G, we will say that a vertex x of G is *adjacent* to a color *i* if there exists a neighbor of x which is colored by *i*.

2 Power Graph of a Path

In this section, we determine the b-chromatic number of a *p*-th power graph of a path, with $p \ge 1$. First we give a lemma used in the proof of Theorem 4. Then the b-chromatic number of a *p*-th power graph of a path is computed.

Lemma 3 For any $p \ge 1$, and for any $n \ge p+1$, let P_n be the path on vertices $x_1, x_2, ..., x_n$. For each integer k, with $p+1 \le k \le \min(2p+1,n)$, there exists a proper k-coloring on P_n^p . Moreover each vertex x, such that $x \in \{x_{k-p}, x_{k-p+1}, ..., x_{n-k+p+1}\}$, is adjacent to each color j, with $1 \le j \ne c_x \le k$.

Proof. As $k \ge p+1$, it is easy to see that if we put the set of colors $\{1, 2, ..., k\}$ cyclically on $V(P_n)$, then two adjacent vertices will not have the same color. The coloring is thus a proper *k*-coloring.

Let $S = \{x_{k-p}, x_{k-p+1}, \dots, x_{n-k+p+1}\}$. First we show that each vertex of *S* is adjacent to at least k-1 vertices. Observe that the vertex x_{k-p} is adjacent to (k-p-1)+p=k-1 vertices. And the vertex $x_{n-k+p+1}$ is adjacent to p+n-(n-k+p+1)=k-1 vertices. Since each vertex x_i , with $k-p+1 \le i \le n-k+p$, has a degree $d(x_i) \ge d(x_{k-p})$, then each vertex of *S* is adjacent to at least k-1 other vertices.

Next, we can see by the construction that all the colors $\{1, 2, ..., k\} \setminus \{c_{x_i}\}$ appear between the first and the last neighbor of x_i . Therefore each vertex x_i of *S* is adjacent to each color *j*, with $1 \le j \ne c_{x_i} \le k$ and $k - p \le i \le n - k + p + 1$.

The b-chromatic number of a *p-th* power graph of a path is given by:

Theorem 4 Let P_n be a path on vertices $x_1, x_2, ..., x_n$. The b-chromatic number of P_n^p , with $p \ge 1$, is given by:

$$\varphi(P_n^p) = \begin{cases} n \text{ if } n \le p+1, & (1) \\ p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor \text{ if } p+2 \le n \le 4p+1, & (2) \\ 2p+1 \text{ if } n \ge 4p+2 & (3) \end{cases}$$

Proof.

- 1. If $n \le p+1$, then $Diam(P_n) \le p$. So, by Fact 2, $\varphi(P_n^p) = n$.
- 2. We prove first that $\varphi(P_n^p) \ge p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$ for $p+2 \le n \le 4p+1$. Let $k=p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$. By Lemma 3, we give a proper *k*-coloring of P_n^p . For example, Figure 1 shows a dominating proper 5-coloring of P_8^3 .



Fig. 1: Coloring of P_8^3

Let S' be the set of vertices $\{x_{k-p}, x_{k-p+1}, \dots, x_{2k-p-1}\}$. Since $2k - p - 1 \le n - k + p + 1$, then $S' \subseteq \{x_{k-p}, x_{k-p+1}, \dots, x_{n-k+p+1}\}$. By Lemma 3, S' is a dominating system. As the coloring is proper and has a dominating system, we obtain a dominating proper k-coloring. So, $\varphi(P_n^p) \ge p + 1 + \lfloor \frac{n-p-1}{3} \rfloor$.

Next we prove that $\varphi(P_n^p) \le p+1 + \lfloor \frac{n-p-1}{3} \rfloor$ for $p+2 \le n \le 4p+1$. The proof is by contradiction. Suppose that there exists a dominating proper *k*'-coloring such that

$$k' > p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor.$$
⁽¹⁾

Let $W = \{w_1, w_2, \dots, w_{k'}\}$ be a dominating system of the coloring on P_n^p (following the orientation of P_n , we meet $w_1, w_2, \dots, w_{k'}$). The vertices w_1 and $w_{k'}$ are adjacent to, at most, p different colors in $[w_1, w_{k'}]$. As w_1 (respectively $w_{k'}$) is a dominating vertex, it must be adjacent to at least k' - 1 different colors. Then, there are at least k' - p - 1 vertices on $[x_1, w_1)$ (respectively $(w_{k'}, x_n]$). Therefore, $n - k' \ge n - |[w_1, w_{k'}]| \ge 2(k' - p - 1)$.

On the other hand by hypothesis $k' \ge p + 2 + \lfloor \frac{n-p-1}{3} \rfloor$, so that $n - k' \le n - p - 2 - \lfloor \frac{n-p-1}{3} \rfloor$. These two results give the following inequality,

$$2(k'-p-1) \le n-k' \le n-p-2 - \left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

$$k' \le \frac{1}{2}(n+p - \left\lfloor \frac{n-p-1}{3} \right\rfloor).$$
 (2)

By (1) and (2), we obtain,

$$\begin{split} \frac{1}{2}(n+p-\left\lfloor\frac{n-p-1}{3}\right\rfloor) &\geq k' \geq p+2+\left\lfloor\frac{n-p-1}{3}\right\rfloor,\\ n-p-4 &\geq 3\left\lfloor\frac{n-p-1}{3}\right\rfloor, \end{split}$$

which is a contradiction. Hence such a coloring does not exist. Therefore, $\varphi(P_n^p) \leq p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.

We deduce from these two parts that $\varphi(P_n^p) = p + 1 + \lfloor \frac{n-p-1}{3} \rfloor$.

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3. $\Delta(P_n^p) = 2p$, so by Proposition 1, $\varphi(P_n^p) \le 2p + 1$. Lemma 3 gives a proper (2p+1)-coloring and shows that each vertex *x* of the set $\{x_{p+1}, x_{p+2}, \dots, x_{3p+1}\}$ is adjacent to each color *j* with $1 \le j \ne c_x \le k$. So this set is a dominating system and $\varphi(P_n^p) \ge 2p + 1$. Therefore $\varphi(P_n^p) = 2p + 1$. For example, Figure 2 gives a dominating proper 7-coloring of P_{15}^3 .





3 Power Graph of a Cycle

In this section, we study the b-chromatic number of a *p*-th power graph of a cycle, with $p \ge 1$. First we give two lemmas used in the proof of Theorem 7. Then we bound the b-chromatic number of a *p*-th power graph of a cycle.

Lemma 5 Let C_n^p be a p-th power graph of a cycle C_n , with $p \ge 2$. For any $2p+3 \le n \le 4p$, let $k \ge \min(n-p-1, p+1+\left|\frac{n-p-1}{3}\right|)$. Then $n \le 2k$.

Proof. The proof is by contradiction. Suppose $n \ge 2k + 1$. We consider two cases. Firstly, $k \ge n - p - 1$. So,

$$n \ge 2k + 1 \ge 2(n - p - 1) + 1,$$

 $n \le 2p + 1,$

which is a contradiction. Secondly, $k \ge p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$. So,

$$n \ge 2k+1 \ge 2(p+1+\left\lfloor\frac{n-p-1}{3}\right\rfloor)+1,$$
$$n-2p-3 \ge 2\left\lfloor\frac{n-p-1}{3}\right\rfloor,$$

which is a contradiction too.

Lemma 6 For any $p \ge 2$, and for any $2p+3 \le n \le 4p$, let C_n be the cycle on vertices x_1, x_2, \ldots, x_n . Let $k = \min(n-p-1, p+1+\lfloor \frac{n-p-1}{3} \rfloor)$. So there exists a proper k-coloring on C_n^p . Moreover each vertex x, such that $x \in \{x_{k-p}, x_{k-p+1}, \ldots, x_{2k-p-1}\}$, is adjacent to each color j, with $1 \le j \ne c_x \le k$.

Proof. We put the set of colors $\{1, 2, ..., k\}$ cyclically on $V(C_n)$. As $k \le p + 1 + \lfloor \frac{n-p-1}{3} \rfloor$ and $n \le 4p$, then $k \le 2p + 1$. Moreover, by Lemma 5 we deduce that $2k \ge n \ge 2p + 3 \ge k + 2$. So, the full set of colors $\{1, 2, ..., k\}$ appears consecutively at least once, and at most twice, in the cyclic coloring of C_n^p . As $2k \ge n \ge 2p + 3$, we have $k \ge p + 1$. Furthermore, by definition of k we have $n - k \ge p + 1$. Thus, as $k \ge p + 1$ and $n - k \ge p + 1$, the coloring is proper.

Let P_n be the subpath of C_n induced by $x_1, x_2, ..., x_n$. Let $S = \{x_{k-p}, x_{k-p+1}, ..., x_{k+(k-p-1)}\}$. As $p+1 \le k \le 2p+1$ and $2k-p-1 \le n-k+p+1$, then by Lemma 3 each vertex x_i of S, with $k-p \le i \le 2k-p-1$, is adjacent to each color q, with $1 \le q \ne c_{x_i} \le k$, on P_n^p . Therefore each vertex x_i of S is adjacent to each color q, with $1 \le q \ne c_{x_i} \le k$, on C_n^p .

Theorem 7 Let C_n be a cycle on vertices $x_1, x_2, ..., x_n$. The b-chromatic number of C_n^p , with $p \ge 1$, is

$$\varphi(C_n^p) = \begin{cases} n \text{ if } n \le 2p+1, & (1) \\ p+1 \text{ if } n = 2p+2, & (2) \\ (\ge)\min(n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor) \text{ if } 2p+3 \le n \le 3p & (3) \\ p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor \text{ if } 3p+1 \le n \le 4p & (4) \\ 2p+1 \text{ if } n \ge 4p+1 & (5) \end{cases}$$

Proof.

1. If $n \leq 2p+1$, then $Diam(C_n) \leq p$. So, by Fact 2, $\varphi(C_n^p) = n$.

- 2. To color the graph, we put the set of colors $\{1, 2, ..., p+1\}$ cyclically twice. One can easily see that this coloring is a proper (p+1)-coloring. Let *S* be the set of vertices $\{x_1, x_2, ..., x_{p+1}\}$. Each vertex x_i , with $1 \le i \le p+1$, is adjacent to n-2 vertices. Since $n-2 \ge p+1$, then each vertex x_i $(1 \le i \le p+1)$ of *S* is adjacent to all colors other than c_{x_i} . So the set *S* is a dominating system. We now show that, in any dominating proper coloring, vertices x_i and x_{i+p+1} must have the same color. For the subgraph induced by vertices $x_1, x_2, ..., x_{p+1}$, we have a clique and we can assume without loss of generality that these vertices are colored by 1, 2, ..., p+1 respectively. If there exists a dominating vertex of color *j*, for some j > p+1, then this vertex is x_{p+1+i} for some i $(1 \le i \le p+1)$. Vertex x_{p+1+i} is not adjacent to x_i , but every other vertex is adjacent to x_i , so that x_{p+1+i} cannot be a dominating vertex, a contradiction. Therefore $\varphi(C_p^n) = p+1$ for n = 2p+2.
- 3. Let $k = \min(n p 1, p + 1 + \lfloor \frac{n p 1}{3} \rfloor)$.

By Lemma 6 there exists a dominating proper *k*-coloring for $2p + 3 \le n \le 3p$. Therefore $\varphi(C_n^p) \ge \min(n-p-1, p+1+\lfloor \frac{n-p-1}{3} \rfloor)$. For example, in Figure 3, we give a dominating proper 6-coloring of C_{11}^4 .

4. Let $k = p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.

For $3p + 1 \le n \le 4p$, Lemma 6 gives a dominating proper *k*-coloring. This proves that $\varphi(C_n^p) \ge \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor)$. For example, Figure 4 shows a dominating proper 6-coloring of C_{11}^3 .

Next, we prove that $\varphi(C_n^p) \le k$. Suppose there exists a dominating proper k'-coloring for C_n^p , with $k' \ge p + 2 + \lfloor \frac{n-p-1}{3} \rfloor$, for the sake of contradiction. Let $W = \{w_1, w_2, \dots, w_{k'}\}$ be a set of dominating vertices on C_n (following the orientation of C_n , we meet $w_1, w_2, \dots, w_{k'}$). We distinguish two cases.



Fig. 3: Coloring of C_{11}^4 $(n - p - 1 = 6, p + 1 + \lfloor \frac{n - p - 1}{3} \rfloor = 7$ and $\varphi(C_{11}^4) \ge 6)$



Fig. 4: Coloring of C_{11}^3

Case 1: for each *i*, with $1 \le i \le k'$, $|(w_i, w_{i+1})| \le p-1$. As $k' \ge p+2+\left\lfloor \frac{n-p-1}{3} \right\rfloor$, by a straightforward modification of the proof of Lemma 5, we have n < 2k'. So, there exists at least one color *c* not repeated in C_n^p (i.e. there are not two distinct vertices with the same color *c*). Without loss of generality, suppose that *c* appears on the vertex *x*, with $x \in V(C_n)$. Therefore *x* is a dominating vertex and each other dominating vertex is adjacent to *x*. Then, $|[w_1, x)| \le p$ and $|[x, w_{k'})| \le p$. As for each *i*, with $1 \le i \le k'$, we have $|(w_i, w_{i+1})| \le p-1$ and since on the cycle the next dominating vertex from $w_{k'}$ is w_1 , then

$$|(w_{k'}, w_1)| \le p - 1,$$

where

$$|(w_{k'}, w_1)| = n - |[w_1, x)| - |[x, w_{k'})| - 1.$$

Therefore, we have

$$n - |[w_1, x)| - |[x, w_{k'})| - 1 \le p - 1,$$

$$n - 2p - 1 \le p - 1,$$

$$n \le 3p,$$

which is a contradiction.

Case 2: There exists *r*, with $1 \le r \le k'$ and *r* is taken modulo k', such that $|(w_r, w_{r+1})| \ge p$. Let *X* be the set of vertices of $[w_{r+1}, w_r]$ (see Figure 5). Let X_C be the set of colors appearing in *X*. Let $\Gamma_X(x_i)$ be the set of neighbors of x_i in *X* and $\Gamma_X^c(x_i)$ the set of colors appearing in $\Gamma_X(x_i)$, with $1 \le i \le n$. Let $A = X_C \setminus (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$. Let $B = X_C \setminus (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$. We discuss two subcases.



Fig. 5: A dominating system on C_n^p and the set X

Subcase 1: $|X| \leq 2p+2$. Since all dominating vertices belong to X, we have $|X| \geq k'$. Then, $|(w_r, w_{r+1})| \leq n-k'$ and $|X_c| = k'$. As the vertices of $\Gamma_X(w_r)$ form a clique, then $|\Gamma_X^c(w_r)| = |\Gamma_X(w_r)| = p$. So we have $|A| = |X_c| - |\Gamma_X^c(w_r)| - 1 = k' - p - 1$. In the same way, we deduce that |B| = k' - p - 1. As $|X| \leq 2p+2$, we have $X \subseteq (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 6.a). So, $A \subseteq (\Gamma_X^c(w_{r+1}) \cup \{c_{w_r+1}\})$ and $B \subseteq (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$. Let $q \in \{1, 2, \dots, k'\}$. If $q \in A$ (resp. $q \in B$) then $q \notin (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$ (resp. $q \notin (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$) and so $q \notin B$ (resp. $q \notin A$). Therefore, $A \cap B = \emptyset$. As w_r (resp. w_{r+1}) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of A (resp. B) must be repeated in (w_r, w_{r+1}) . Therefore,

$$\begin{aligned} |A| + |B| &\leq |(w_r, w_{r+1})|, \\ 2(k' - p - 1) &\leq n - k', \\ 3k' &\leq n + 2p + 2, \\ 3\left\lfloor \frac{n - p - 1}{3} \right\rfloor &\leq n - p - 4, \end{aligned}$$

which is a contradiction.

Subcase 2: $|X| \ge 2p+3$. As in Subcase 1, we have |A| = k'-p-1 and |B| = k'-p-1. Let $X' = X \setminus (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 6.b). So $|X'| \ge |A \cap B|$. Since w_r (resp. w_{r+1}) is a dominating vertex and $|(w_r, w_{r+1})| \ge p$, the colors of A (resp. B) must be repeated in (w_r, w_{r+1}) . Then,

$$\begin{split} |A| + |B| - |A \cap B| &\leq |(w_r, w_{r+1})| \leq n - 2p - 2 - |X'|, \\ 2(k' - p - 1) - |A \cap B| \leq n - 2p - 2 - |A \cap B|, \\ 2(p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor) \leq n, \end{split}$$

which is a contradiction. Therefore there does not exist a dominating proper k'-coloring, with $k' \ge p + 2 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$. This completes the proof of $\varphi(C_n^p) = p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.



Fig. 6: Neighborhoods of w_r and w_{r+1} on X when a) $|X| \le 2p+2$ and b) $|X| \ge 2p+3$

5. As $\Delta = 2p$, by Proposition 1, $\varphi(C_n^p) \le 2p + 1$.

We then give a proper (2p + 1)-coloring. It is constructed in two steps. First, we put (2p + 1) different colors on the (2p + 1) first vertices $(c_{x_i} := i \text{ for } 1 \le i \le 2p + 1)$. In the second step, we have two cases. If n = 4p + 1, we color the remaining vertices as follows: $c_{x_i} := c_{x_{i-2p-1}}$ for $2p + 2 \le i \le n$. If $n \ge 4p + 2$, then the remaining vertices are colored as follows: $c_{x_i} := c_{x_{i-2p-1}}$ for $2p + 2 \le i \le 4p + 2$, and $c_{x_i} := c_{x_{i-p-1}}$ for $4p + 3 \le i \le n$. Then the distance between two vertices colored by the same color c is at least p + 1. So the coloring is proper. By an analogue proof of Lemma 3, one can prove that each vertex x_i , with $p + 1 \le i \le 3p + 1$, is a dominating vertex. So this coloring is a dominating proper (2p + 1)-coloring. This construction shows that $\varphi(C_n^p) \ge 2p + 1$. Therefore we have proved that $\varphi(C_n^p) = 2p + 1$. For example, Figure 7 gives a dominating proper 7-coloring C_{16}^3 .

4 Open Problem

In section 3, we have obtained the exact values of $\varphi(C_n^p)$, except in case $2p + 3 \le n \le 3p$ where we give a lower bound. We believe that $\min(n - p - 1, p + 1 + \lfloor \frac{n - p - 1}{3} \rfloor)$ is the exact value of $\varphi(C_n^p)$ for $2p + 3 \le n \le 3p$.



Fig. 7: Coloring of C_{16}^3

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