



HAL
open science

The b-chromatic number of power graphs

Brice Effantin, Hamamache Kheddouci

► **To cite this version:**

Brice Effantin, Hamamache Kheddouci. The b-chromatic number of power graphs. *Discrete Mathematics and Theoretical Computer Science*, 2003, Vol. 6 no. 1 (1), pp.45-54. 10.46298/dmtcs.333 . hal-00958987

HAL Id: hal-00958987

<https://inria.hal.science/hal-00958987>

Submitted on 13 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The b -chromatic number of some power graphs

Brice Effantin¹ and Hamamache Kheddouci²

LE2I FRE-CNRS 2309, Université de Bourgogne, B.P. 47870, 21078 Dijon Cedex, France

¹brice.effantin@u-bourgogne.fr

²kheddouc@u-bourgogne.fr

received Jul 19, 2002, revised Oct 29, 2002, accepted Apr 22, 2003.

Let G be a graph on vertices v_1, v_2, \dots, v_n . The b -chromatic number of G is defined as the maximum number k of colors that can be used to color the vertices of G , such that we obtain a proper coloring and each color i , with $1 \leq i \leq k$, has at least one representant x_i adjacent to a vertex of every color j , $1 \leq j \neq i \leq k$. In this paper, we give the exact value for the b -chromatic number of power graphs of a path and we determine bounds for the b -chromatic number of power graphs of a cycle.

Keywords: b -chromatic number, coloring, cycle, path, power graphs

1 Introduction

We consider graphs without loops or multiple edges. Let G be a graph with a vertex set V and an edge set E . We denote by $d(x)$ the degree of the vertex x in G , and by $dist_G(x, y)$ the distance between two vertices x and y in G . The p -th power graph G^p is a graph obtained from G by adding an edge between every pair of vertices at distance p or less, with $p \geq 1$. It is easy to see that $G^1 = G$. In the literature, power graphs of several classes have been investigated [2, 3, 8]. In this note we study a vertex coloring of power graphs. The power graph of a path and the power graph of a cycle can be also considered as respectively subclasses of *distance graphs* and *circulant graphs*. The *distance graph* $G(D)$ with distance set $D = \{d_1, d_2, \dots\}$ has the set Z of integers as vertex set, with two vertices $i, j \in Z$ adjacent if and only if $|i - j| \in D$. The *circulant graph* can be defined as follows. Let n be a natural number and let $S = \{k_1, k_2, \dots, k_r\}$ with $k_1 < k_2 < \dots < k_r \leq n/2$. Then the vertex set of the circulant graph $G(n, S)$ is $\{0, 1, \dots, n-1\}$ and the set of neighbors of the vertex i is $\{(i \pm k_j) \bmod n \mid j = 1, 2, \dots, r\}$.

The study of distance graphs was initiated by Eggleton and al. [4]. Recently, the problem of coloring of this class of graphs has attracted considerable attention, see e.g. [12, 13]. Circulant graphs have been extensively studied and have a vast number of applications to multicomputer networks and distributed computation (see [1, 10]). The special cases we consider are the distance graph $G(D)$ with finite distance set $D = \{1, 2, \dots, p\}$ which is isomorphic to the p -th power graph of a path and the circulant graph $G(n, S)$ with $S = \{1, 2, \dots, p\}$ which is isomorphic to the p -th power graph of a cycle.

A k -coloring of G is defined as a function c on $V(G) = \{v_1, v_2, \dots, v_n\}$ into a set of colors $C = \{1, 2, \dots, k\}$ such that for each vertex v_i , with $1 \leq i \leq n$, we have $c_{v_i} \in C$. A *proper k -coloring* is a k -coloring satisfying the condition $c_x \neq c_y$ for each pair of adjacent vertices $x, y \in V(G)$. A *dominating proper k -coloring* is a proper k -coloring satisfying the following property P : for each i , $1 \leq i \leq k$, there exists a vertex x_i of color i such that, for each j , with $1 \leq j \neq i \leq k$, there exists a vertex y_j of color j adjacent to x_i . A set of vertices satisfying the property P is called a *dominating system*. Each vertex of a dominating system is called a *dominating vertex*. The *b-chromatic number* $\phi(G)$ of a graph G is defined as the maximum k such that G admits a dominating proper k -coloring.

The b-chromatic number was introduced in [7]. The motivation, similarly as for the previously studied *achromatic number* (cf. e.g. [5, 6]), comes from algorithmic graph theory. The achromatic number $\psi(G)$ of a graph G is the largest number of colors which can be assigned to the vertices of G such that the coloring is proper and every pair of distinct colors appears on an edge. A proper coloring of a graph G using $k > \chi(G)$ colors could be improved if the vertices of two color classes could be recolored by a single color so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by the achromatic number. A more versatile form of recoloring strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes, so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by $\phi(G)$ (these recolorings are discussed in [7] and [11]). Thus $\phi(G) \leq \psi(G)$ (also given in [7]). From this point of view, both complexity results and tight bounds for the b-chromatic number are interesting. The following bounds of b-chromatic number are already presented in [7].

Proposition 1 *Assume that the vertices x_1, x_2, \dots, x_n of G are ordered such that $d(x_1) \geq d(x_2) \geq \dots \geq d(x_n)$. Then $\phi(G) \leq m(G) \leq \Delta(G) + 1$, where $m(G) = \max\{1 \leq i \leq n : d(x_i) \geq i - 1\}$ and $\Delta(G)$ is the maximum degree of G .*

R. W. Irving and D. F. Manlove [7] proved that finding the b-chromatic number of any graph is a NP-hard problem, and they gave a polynomial-time algorithm for finding the b-chromatic number of trees. Kouider and Mahéo [9] gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. They gave, in particular, a lower bound for the b-chromatic number of the cartesian product of two graphs where each one has a stable dominating system. More recently in [11], the authors characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper k -coloring even for connected bipartite graphs and $k = \Delta(G) + 1$. They also determine the asymptotic behavior for the b-chromatic number of random graphs.

In this paper, we present several exact values and determine bounds for the b-chromatic number of power graphs of paths and cycles.

Let $Diam(G)$ be the diameter of a graph G , defined as the maximum distance between any pair of vertices of G . Let us begin with the following observation.

Fact 2 *For any graph G of order n , if $Diam(G) \leq p$, then $\phi(G^p) = n$, with $p \geq 2$.*

Proof. If $Diam(G) \leq p$, it is trivial to see that G^p is a complete graph. So $\phi(G^p) = n$. □

Let G be a path or a cycle on vertices x_1, x_2, \dots, x_n . We fix an orientation of G (left to right if G is a path and clockwise if G is a cycle). For each $1 \leq i \leq n$, we denote by x_i^+ (resp. x_i^-) the successor (resp.

predecessor) of x_i in G (if any). For $1 \leq i \neq j \leq n$, we define $[x_i, x_j]_G$, $[x_i, x_j)_G$ and $(x_i, x_j)_G$ as the set of consecutive vertices on G from respectively x_i to x_j , x_i to x_j^- and x_i^+ to x_j^- , following the fixed orientation of G . If there is no ambiguity, we denote $[x_i, x_j]_G$, $[x_i, x_j)_G$ and $(x_i, x_j)_G$ by respectively $[x_i, x_j]$, $[x_i, x_j)$ and (x_i, x_j) .

In all figures, the graph G is represented with solid edges. Edges added in a p -th power graph G^p are represented with dashed edges. In some figures, vertices are surrounded and represent a dominating system of the coloring. In any coloring of a graph G , we will say that a vertex x of G is *adjacent* to a color i if there exists a neighbor of x which is colored by i .

2 Power Graph of a Path

In this section, we determine the b-chromatic number of a p -th power graph of a path, with $p \geq 1$. First we give a lemma used in the proof of Theorem 4. Then the b-chromatic number of a p -th power graph of a path is computed.

Lemma 3 *For any $p \geq 1$, and for any $n \geq p + 1$, let P_n be the path on vertices x_1, x_2, \dots, x_n . For each integer k , with $p + 1 \leq k \leq \min(2p + 1, n)$, there exists a proper k -coloring on P_n^p . Moreover each vertex x , such that $x \in \{x_{k-p}, x_{k-p+1}, \dots, x_{n-k+p+1}\}$, is adjacent to each color j , with $1 \leq j \neq c_x \leq k$.*

Proof. As $k \geq p + 1$, it is easy to see that if we put the set of colors $\{1, 2, \dots, k\}$ cyclically on $V(P_n)$, then two adjacent vertices will not have the same color. The coloring is thus a proper k -coloring.

Let $S = \{x_{k-p}, x_{k-p+1}, \dots, x_{n-k+p+1}\}$. First we show that each vertex of S is adjacent to at least $k - 1$ vertices. Observe that the vertex x_{k-p} is adjacent to $(k - p - 1) + p = k - 1$ vertices. And the vertex $x_{n-k+p+1}$ is adjacent to $p + n - (n - k + p + 1) = k - 1$ vertices. Since each vertex x_i , with $k - p + 1 \leq i \leq n - k + p$, has a degree $d(x_i) \geq d(x_{k-p})$, then each vertex of S is adjacent to at least $k - 1$ other vertices.

Next, we can see by the construction that all the colors $\{1, 2, \dots, k\} \setminus \{c_{x_i}\}$ appear between the first and the last neighbor of x_i . Therefore each vertex x_i of S is adjacent to each color j , with $1 \leq j \neq c_{x_i} \leq k$ and $k - p \leq i \leq n - k + p + 1$. \square

The b-chromatic number of a p -th power graph of a path is given by:

Theorem 4 *Let P_n be a path on vertices x_1, x_2, \dots, x_n . The b-chromatic number of P_n^p , with $p \geq 1$, is given by:*

$$\varphi(P_n^p) = \begin{cases} n & \text{if } n \leq p + 1, & (1) \\ p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor & \text{if } p + 2 \leq n \leq 4p + 1, & (2) \\ 2p + 1 & \text{if } n \geq 4p + 2 & (3) \end{cases}$$

Proof.

1. If $n \leq p + 1$, then $\text{Diam}(P_n) \leq p$. So, by Fact 2, $\varphi(P_n^p) = n$.
2. We prove first that $\varphi(P_n^p) \geq p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$ for $p + 2 \leq n \leq 4p + 1$. Let $k = p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$. By Lemma 3, we give a proper k -coloring of P_n^p . For example, Figure 1 shows a dominating proper 5-coloring of P_8^3 .

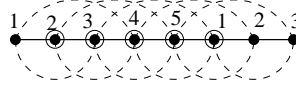


Fig. 1: Coloring of P_8^3

Let S' be the set of vertices $\{x_{k-p}, x_{k-p+1}, \dots, x_{2k-p-1}\}$. Since $2k-p-1 \leq n-k+p+1$, then $S' \subseteq \{x_{k-p}, x_{k-p+1}, \dots, x_{n-k+p+1}\}$. By Lemma 3, S' is a dominating system. As the coloring is proper and has a dominating system, we obtain a dominating proper k -coloring. So, $\varphi(P_n^p) \geq p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.

Next we prove that $\varphi(P_n^p) \leq p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$ for $p+2 \leq n \leq 4p+1$. The proof is by contradiction. Suppose that there exists a dominating proper k' -coloring such that

$$k' > p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor. \quad (1)$$

Let $W = \{w_1, w_2, \dots, w_{k'}\}$ be a dominating system of the coloring on P_n^p (following the orientation of P_n , we meet $w_1, w_2, \dots, w_{k'}$). The vertices w_1 and $w_{k'}$ are adjacent to, at most, p different colors in $[w_1, w_{k'}]$. As w_1 (respectively $w_{k'}$) is a dominating vertex, it must be adjacent to at least $k' - 1$ different colors. Then, there are at least $k' - p - 1$ vertices on $[x_1, w_1]$ (respectively $(w_{k'}, x_n)$). Therefore, $n - k' \geq n - |[w_1, w_{k'}]| \geq 2(k' - p - 1)$.

On the other hand by hypothesis $k' \geq p+2 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$, so that $n - k' \leq n - p - 2 - \left\lfloor \frac{n-p-1}{3} \right\rfloor$. These two results give the following inequality,

$$2(k' - p - 1) \leq n - k' \leq n - p - 2 - \left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

$$k' \leq \frac{1}{2}(n+p - \left\lfloor \frac{n-p-1}{3} \right\rfloor). \quad (2)$$

By (1) and (2), we obtain,

$$\frac{1}{2}(n+p - \left\lfloor \frac{n-p-1}{3} \right\rfloor) \geq k' \geq p+2 + \left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

$$n-p-4 \geq 3 \left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

which is a contradiction. Hence such a coloring does not exist. Therefore, $\varphi(P_n^p) \leq p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.

We deduce from these two parts that $\varphi(P_n^p) = p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.

3. $\Delta(P_n^p) = 2p$, so by Proposition 1, $\phi(P_n^p) \leq 2p + 1$. Lemma 3 gives a proper $(2p + 1)$ -coloring and shows that each vertex x of the set $\{x_{p+1}, x_{p+2}, \dots, x_{3p+1}\}$ is adjacent to each color j with $1 \leq j \neq c_x \leq k$. So this set is a dominating system and $\phi(P_n^p) \geq 2p + 1$. Therefore $\phi(P_n^p) = 2p + 1$. For example, Figure 2 gives a dominating proper 7-coloring of P_{15}^3 .

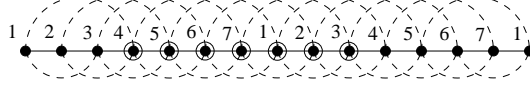


Fig. 2: Coloring of P_{15}^3

3 Power Graph of a Cycle

In this section, we study the b-chromatic number of a p -th power graph of a cycle, with $p \geq 1$. First we give two lemmas used in the proof of Theorem 7. Then we bound the b-chromatic number of a p -th power graph of a cycle.

Lemma 5 *Let C_n^p be a p -th power graph of a cycle C_n , with $p \geq 2$. For any $2p + 3 \leq n \leq 4p$, let $k \geq \min(n - p - 1, p + 1 + \lfloor \frac{n-p-1}{3} \rfloor)$. Then $n \leq 2k$.*

Proof. The proof is by contradiction. Suppose $n \geq 2k + 1$. We consider two cases. Firstly, $k \geq n - p - 1$. So,

$$\begin{aligned} n &\geq 2k + 1 \geq 2(n - p - 1) + 1, \\ n &\leq 2p + 1, \end{aligned}$$

which is a contradiction. Secondly, $k \geq p + 1 + \lfloor \frac{n-p-1}{3} \rfloor$. So,

$$\begin{aligned} n &\geq 2k + 1 \geq 2\left(p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor\right) + 1, \\ n - 2p - 3 &\geq 2 \left\lfloor \frac{n-p-1}{3} \right\rfloor, \end{aligned}$$

which is a contradiction too. \square

Lemma 6 *For any $p \geq 2$, and for any $2p + 3 \leq n \leq 4p$, let C_n be the cycle on vertices x_1, x_2, \dots, x_n . Let $k = \min(n - p - 1, p + 1 + \lfloor \frac{n-p-1}{3} \rfloor)$. So there exists a proper k -coloring on C_n^p . Moreover each vertex x , such that $x \in \{x_{k-p}, x_{k-p+1}, \dots, x_{2k-p-1}\}$, is adjacent to each color j , with $1 \leq j \neq c_x \leq k$.*

Proof. We put the set of colors $\{1, 2, \dots, k\}$ cyclically on $V(C_n)$. As $k \leq p + 1 + \lfloor \frac{n-p-1}{3} \rfloor$ and $n \leq 4p$, then $k \leq 2p + 1$. Moreover, by Lemma 5 we deduce that $2k \geq n \geq 2p + 3 \geq k + 2$. So, the full set of colors $\{1, 2, \dots, k\}$ appears consecutively at least once, and at most twice, in the cyclic coloring of C_n^p . As $2k \geq n \geq 2p + 3$, we have $k \geq p + 1$. Furthermore, by definition of k we have $n - k \geq p + 1$. Thus, as $k \geq p + 1$ and $n - k \geq p + 1$, the coloring is proper.

Let P_n be the subpath of C_n induced by x_1, x_2, \dots, x_n . Let $S = \{x_{k-p}, x_{k-p+1}, \dots, x_{k+(k-p-1)}\}$. As $p+1 \leq k \leq 2p+1$ and $2k-p-1 \leq n-k+p+1$, then by Lemma 3 each vertex x_i of S , with $k-p \leq i \leq 2k-p-1$, is adjacent to each color q , with $1 \leq q \neq c_{x_i} \leq k$, on P_n^p . Therefore each vertex x_i of S is adjacent to each color q , with $1 \leq q \neq c_{x_i} \leq k$, on C_n^p . \square

Theorem 7 *Let C_n be a cycle on vertices x_1, x_2, \dots, x_n . The b -chromatic number of C_n^p , with $p \geq 1$, is*

$$\varphi(C_n^p) = \begin{cases} n & \text{if } n \leq 2p+1, & (1) \\ p+1 & \text{if } n = 2p+2, & (2) \\ (\geq) \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor) & \text{if } 2p+3 \leq n \leq 3p & (3) \\ p+1 + \lfloor \frac{n-p-1}{3} \rfloor & \text{if } 3p+1 \leq n \leq 4p & (4) \\ 2p+1 & \text{if } n \geq 4p+1 & (5) \end{cases}$$

Proof.

1. If $n \leq 2p+1$, then $\text{Diam}(C_n) \leq p$. So, by Fact 2, $\varphi(C_n^p) = n$.
2. To color the graph, we put the set of colors $\{1, 2, \dots, p+1\}$ cyclically twice. One can easily see that this coloring is a proper $(p+1)$ -coloring. Let S be the set of vertices $\{x_1, x_2, \dots, x_{p+1}\}$. Each vertex x_i , with $1 \leq i \leq p+1$, is adjacent to $n-2$ vertices. Since $n-2 \geq p+1$, then each vertex x_i ($1 \leq i \leq p+1$) of S is adjacent to all colors other than c_{x_i} . So the set S is a dominating system. We now show that, in any dominating proper coloring, vertices x_i and x_{i+p+1} must have the same color. For the subgraph induced by vertices x_1, x_2, \dots, x_{p+1} , we have a clique and we can assume without loss of generality that these vertices are colored by $1, 2, \dots, p+1$ respectively. If there exists a dominating vertex of color j , for some $j > p+1$, then this vertex is x_{p+1+i} for some i ($1 \leq i \leq p+1$). Vertex x_{p+1+i} is not adjacent to x_i , but every other vertex is adjacent to x_i , so that x_{p+1+i} cannot be a dominating vertex, a contradiction. Therefore $\varphi(C_n^p) = p+1$ for $n = 2p+2$.
3. Let $k = \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor)$.
By Lemma 6 there exists a dominating proper k -coloring for $2p+3 \leq n \leq 3p$. Therefore $\varphi(C_n^p) \geq \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor)$. For example, in Figure 3, we give a dominating proper 6-coloring of C_{11}^4 .
4. Let $k = p+1 + \lfloor \frac{n-p-1}{3} \rfloor$.
For $3p+1 \leq n \leq 4p$, Lemma 6 gives a dominating proper k -coloring. This proves that $\varphi(C_n^p) \geq \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor)$. For example, Figure 4 shows a dominating proper 6-coloring of C_{11}^3 .
Next, we prove that $\varphi(C_n^p) \leq k$. Suppose there exists a dominating proper k' -coloring for C_n^p , with $k' \geq p+2 + \lfloor \frac{n-p-1}{3} \rfloor$, for the sake of contradiction. Let $W = \{w_1, w_2, \dots, w_{k'}\}$ be a set of dominating vertices on C_n (following the orientation of C_n , we meet $w_1, w_2, \dots, w_{k'}$). We distinguish two cases.

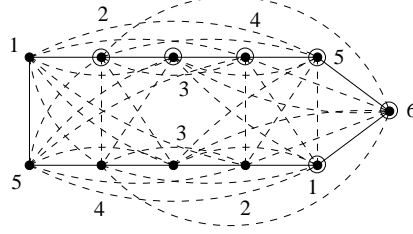


Fig. 3: Coloring of C_{11}^4 ($n - p - 1 = 6$, $p + 1 + \lfloor \frac{n-p-1}{3} \rfloor = 7$ and $\phi(C_{11}^4) \geq 6$)

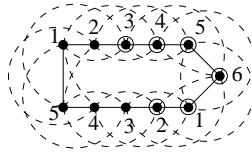


Fig. 4: Coloring of C_{11}^3

Case 1: for each i , with $1 \leq i \leq k'$, $|(w_i, w_{i+1})| \leq p - 1$.

As $k' \geq p + 2 + \lfloor \frac{n-p-1}{3} \rfloor$, by a straightforward modification of the proof of Lemma 5, we have $n < 2k'$. So, there exists at least one color c not repeated in C_n^p (i.e. there are not two distinct vertices with the same color c). Without loss of generality, suppose that c appears on the vertex x , with $x \in V(C_n)$. Therefore x is a dominating vertex and each other dominating vertex is adjacent to x . Then, $|(w_1, x)| \leq p$ and $|(x, w_{k'})| \leq p$. As for each i , with $1 \leq i \leq k'$, we have $|(w_i, w_{i+1})| \leq p - 1$ and since on the cycle the next dominating vertex from $w_{k'}$ is w_1 , then

$$|(w_{k'}, w_1)| \leq p - 1,$$

where

$$|(w_{k'}, w_1)| = n - |[w_1, x]| - |[x, w_{k'}]| - 1.$$

Therefore, we have

$$n - |[w_1, x]| - |[x, w_{k'}]| - 1 \leq p - 1,$$

$$n - 2p - 1 \leq p - 1,$$

$$n \leq 3p,$$

which is a contradiction.

Case 2: There exists r , with $1 \leq r \leq k'$ and r is taken modulo k' , such that $|(w_r, w_{r+1})| \geq p$.

Let X be the set of vertices of $[w_{r+1}, w_r]$ (see Figure 5). Let X_C be the set of colors appearing in X . Let $\Gamma_X(x_i)$ be the set of neighbors of x_i in X and $\Gamma_X^c(x_i)$ the set of colors appearing in $\Gamma_X(x_i)$, with $1 \leq i \leq n$. Let $A = X_C \setminus (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$. Let $B = X_C \setminus (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$. We discuss two subcases.

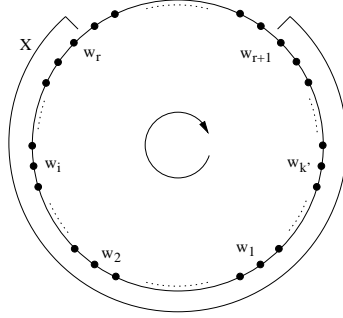


Fig. 5: A dominating system on C_n^p and the set X

Subcase 1: $|X| \leq 2p + 2$. Since all dominating vertices belong to X , we have $|X| \geq k'$. Then, $|(w_r, w_{r+1})| \leq n - k'$ and $|X_C| = k'$. As the vertices of $\Gamma_X(w_r)$ form a clique, then $|\Gamma_X^c(w_r)| = |\Gamma_X(w_r)| = p$. So we have $|A| = |X_C| - |\Gamma_X^c(w_r)| - 1 = k' - p - 1$. In the same way, we deduce that $|B| = k' - p - 1$. As $|X| \leq 2p + 2$, we have $X \subseteq (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 6.a). So, $A \subseteq (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$ and $B \subseteq (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$. Let $q \in \{1, 2, \dots, k'\}$. If $q \in A$ (resp. $q \in B$) then $q \notin (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$ (resp. $q \notin (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$) and so $q \notin B$ (resp. $q \notin A$). Therefore, $A \cap B = \emptyset$. As w_r (resp. w_{r+1}) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of A (resp. B) must be repeated in (w_r, w_{r+1}) . Therefore,

$$\begin{aligned} |A| + |B| &\leq |(w_r, w_{r+1})|, \\ 2(k' - p - 1) &\leq n - k', \\ 3k' &\leq n + 2p + 2, \\ 3 \left\lfloor \frac{n - p - 1}{3} \right\rfloor &\leq n - p - 4, \end{aligned}$$

which is a contradiction.

Subcase 2: $|X| \geq 2p + 3$. As in Subcase 1, we have $|A| = k' - p - 1$ and $|B| = k' - p - 1$. Let $X' = X \setminus (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 6.b). So $|X'| \geq |A \cap B|$. Since w_r (resp. w_{r+1}) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of A (resp. B) must be repeated in (w_r, w_{r+1}) . Then,

$$\begin{aligned} |A| + |B| - |A \cap B| &\leq |(w_r, w_{r+1})| \leq n - 2p - 2 - |X'|, \\ 2(k' - p - 1) - |A \cap B| &\leq n - 2p - 2 - |A \cap B|, \\ 2(p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor) &\leq n, \end{aligned}$$

which is a contradiction. Therefore there does not exist a dominating proper k' -coloring, with $k' \geq p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$.

This completes the proof of $\phi(C_n^p) = p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$.

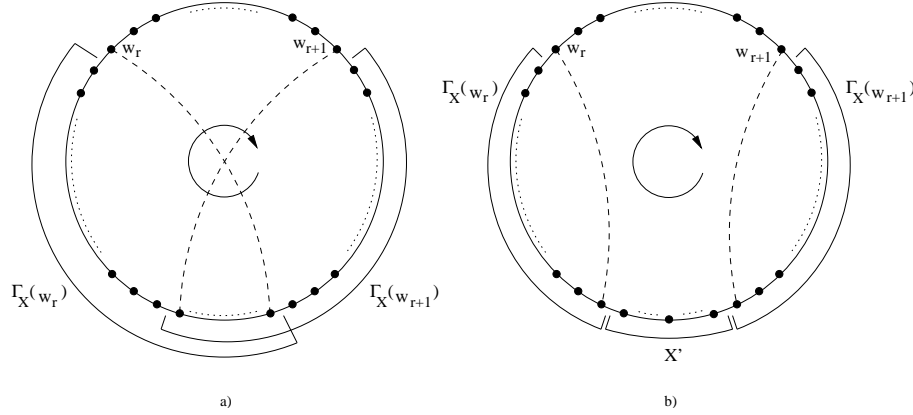


Fig. 6: Neighborhoods of w_r and w_{r+1} on X when a) $|X| \leq 2p+2$ and b) $|X| \geq 2p+3$

5. As $\Delta = 2p$, by Proposition 1, $\varphi(C_n^p) \leq 2p+1$. We then give a proper $(2p+1)$ -coloring. It is constructed in two steps. First, we put $(2p+1)$ different colors on the $(2p+1)$ first vertices ($c_{x_i} := i$ for $1 \leq i \leq 2p+1$). In the second step, we have two cases. If $n = 4p+1$, we color the remaining vertices as follows: $c_{x_i} := c_{x_{i-2p-1}}$ for $2p+2 \leq i \leq n$. If $n \geq 4p+2$, then the remaining vertices are colored as follows: $c_{x_i} := c_{x_{i-2p-1}}$ for $2p+2 \leq i \leq 4p+2$, and $c_{x_i} := c_{x_{i-p-1}}$ for $4p+3 \leq i \leq n$. Then the distance between two vertices colored by the same color c is at least $p+1$. So the coloring is proper. By an analogue proof of Lemma 3, one can prove that each vertex x_i , with $p+1 \leq i \leq 3p+1$, is a dominating vertex. So this coloring is a dominating proper $(2p+1)$ -coloring. This construction shows that $\varphi(C_n^p) \geq 2p+1$. Therefore we have proved that $\varphi(C_n^p) = 2p+1$. For example, Figure 7 gives a dominating proper 7-coloring C_{16}^3 . \square

4 Open Problem

In section 3, we have obtained the exact values of $\varphi(C_n^p)$, except in case $2p+3 \leq n \leq 3p$ where we give a lower bound. We believe that $\min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor)$ is the exact value of $\varphi(C_n^p)$ for $2p+3 \leq n \leq 3p$.

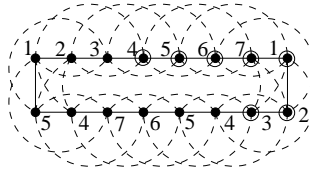


Fig. 7: Coloring of C_{16}^3

Acknowledgments

The authors thank the referees for useful suggestions that led to this improved version.

References

- [1] J.-C. BERMOND, F. CORMELLAS, D. F. HSU, Distributed loop computer networks: a survey, *Journal of Parallel and Distributed Computing* 24 (1995) 2-10.
- [2] A. BRANDSTÄDT, V. D. CHEPOI, F. F. DRAGAN, Perfect elimination orderings of chordal powers of graphs, *Discrete Mathematics* 158 (1996) 273-278.
- [3] E. DAHLHAUS AND P. DUCHET, On strongly chordal graphs, *Ars Combinatoria* 24B (1987) 23-30.
- [4] R.B. EGGLETON, P. ERDŐS AND D.K. SKILTON, Coloring the real line, *Journal of Combinatorial Theory* B39 (1985) 86-100.
- [5] F. HARARY AND S. HEDETNIEMI, The achromatic number of a graph, *Journal of Combinatorial Theory* 8 (1970) 154-161.
- [6] F. HUGHES AND G. MACGILLIVRAY, The achromatic number of graphs: A survey and some new results, *Bull. Inst. Comb. Appl.* 19 (1997) 27-56.
- [7] R. W. IRVING AND D. F. MANLOVE, The b-chromatic number of a graph, *Discrete Applied Mathematics* 91 (1999) 127-141.
- [8] H.KHEDDOUCI, J.F.SACLÉ AND M.WOŹNIAK, Packing of two copies of a tree into its fourth power, *Discrete Mathematics* 213 (1-3) (2000) 169-178.
- [9] M.KOUIDER AND M.MAHEO, Some bounds for the b-chromatic number of a graph, *Discrete Mathematics* 256, Issues 1-2, (2002) 267-277.
- [10] D.E. KNUTH, The Art of Computer Programming, Vol. 3 *Addison-Wesley, Reading, MA.* 1975.
- [11] J. KRATOCHVÍL, Z. TUZA AND M. VOIGT, On the b-chromatic number of graphs, *Proceedings WG'02 - 28th International Workshop on Graph-Theoretic Concepts in Computer Science, Cesky Krumlov, Czech Republic, volume 2573 of Lecture Notes in Computer Science.* Springer Verlag 2002.
- [12] I.Z. RUZSA, Z. TUZA AND M. VOIGT, Distance Graphs with Finite Chromatic Number, *Journal of Combinatorial Theory* B85 (2002) 181-187.
- [13] X. ZHU, Pattern periodic coloring of distance graphs, *Journal of Combinatorial Theory* B73 (1998) 195-206.