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# A Note on Set Systems with no Union of Cardinality 0 Modulo $m$

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Alon, Kleitman, Lipton, Meshulam, Rabin and Spencer (Graphs. Combin. 7 (1991), no. 2, 97-99) proved that for any hypergraph  $\mathcal{F} = \{F_1, F_2, \dots, F_{d(q-1)+1}\}$ , where  $q$  is a prime-power, and  $d$  denotes the maximum degree of the hypergraph, there exists an  $\mathcal{F}_0 \subset \mathcal{F}$ , such that  $|\bigcup_{F \in \mathcal{F}_0} F| \equiv 0 \pmod{q}$ . The main tool of the proof was a one-to-one correspondence between hypergraphs and polynomials. We give a direct, alternative proof to this correspondence, and also review its implications for set-systems following from the result of Barrington, Beigel and Rudich (Comput. Complexity, 4 (1994), 367-382) for certain mod 6 polynomials.

**Keywords:** Set systems, composite modulus, polynomials over rings

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## 1 Introduction

Alon, Kleitman, Lipton, Meshulam, Rabin and Spencer [1] gave the following definition:

**Definition 1 ([1])** For integers  $d, m \geq 1$ , let  $f_d(m)$  denote the smallest  $t$  such that for any hypergraph  $\mathcal{F} = \{F_1, F_2, \dots, F_t\}$  with maximum degree  $d$  there exists a non-empty  $\mathcal{F}_0 \subset \mathcal{F}$ , such that  $|\bigcup_{F \in \mathcal{F}_0} F| \equiv 0 \pmod{m}$

Baker and Schmidt [2] defined the following quantity:

**Definition 2** For integers  $d, m \geq 1$ , let  $g_d(m)$  denote the smallest  $t$  such that for any polynomial  $h \in \mathbb{Z}[x_1, x_2, \dots, x_t]$  of degree  $d$ , satisfying  $h(\mathbf{0}) = 0$ , there exists an  $\mathbf{0} \neq \varepsilon \in \{0, 1\}^n$ , such that  $h(\varepsilon) \equiv 0 \pmod{m}$ .

The following theorem was proven in [1]:

**Theorem 3 ([1])**

$$f_d(m) = g_d(m)$$

In the next section we give a natural one-to-one correspondence between polynomials and hypergraphs, proving Theorem 3.

For  $p$  prime, and  $\alpha$  positive integer it is known ([1], [2], [4]) that  $g_d(p^\alpha) = d(p^\alpha - 1) + 1$ , so we obtain

**Corollary 4 ([1])** For  $\mathcal{F} = \{F_1, F_2, \dots, F_{d(q-1)+1}\}$ , where  $q$  is a prime-power, and  $d$  denotes the maximum degree of the hypergraph, there exists an  $\emptyset \neq \mathcal{F}_0 \subset \mathcal{F}$ , such that  $|\bigcup_{F \in \mathcal{F}_0} F| \equiv 0 \pmod{q}$ .

This corollary is a generalization of the undergraduate exercise that from arbitrary  $m$  integers, one can choose a non-empty subset, which adds up to 0 modulo  $m$  (the  $d = 1$  case).

In 1991, *Barrington, Beigel and Rudich* [3] gave an explicit construction for polynomials modulo  $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ , showing that

$$g_d(m) = \Omega(d^r).$$

Since the proof of Theorem 3 (both the original and ours in the next section) gives explicit constructions for hypergraphs from polynomials, the following corollary holds:

**Corollary 5** Let  $m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ . Then there exists an explicitly constructible hypergraph  $\mathcal{F}$  of maximum degree  $d$ , such that  $|\mathcal{F}| = \Omega(d^r)$  and for each  $\emptyset \neq \mathcal{F}_0 \subset \mathcal{F}$  it is satisfied that  $|\bigcup_{F \in \mathcal{F}_0} F| \not\equiv 0 \pmod{m}$ .

The authors of [1] gave a doubly-exponential upper bound on  $f_d(m)$ , which was based on a Ramsey-theoretic bound of [2]. More recently, *Tardos and Barrington* [4] showed that

$$f_d(m) = \exp(O(d^{r-1})).$$

## 2 Correspondence between polynomials and hypergraphs

We give here a short and direct proof for Theorem 3. Let  $\mathcal{Q}$  denote the set of rationals. It is well known that the set of functions  $\{f : \{0, 1\}^t \rightarrow \mathcal{Q}\}$  forms a  $2^t$ -dimensional vector space over the rationals. One useful basis of this vector space is the set of OR-functions  $\{\bigvee_{i \in I} x_i : I \subset \{1, 2, \dots, t\}\}$ , where

$$\bigvee_{i \in I} x_i = 1 - \prod_{i \in I} (1 - x_i).$$

It is easy to see that any integer-valued function on the hypercube can be written as the integer-coefficient linear combination of these OR-functions. Moreover, if the function is a degree- $d$  polynomial, then it is enough to use OR functions with  $|I| \leq d$ . If we consider modulo  $m$  polynomials, then the coefficients can be restricted to the set  $\{0, 1, 2, \dots, m-1\}$ . It will be convenient to view modulo  $m$  polynomials as the sum of several OR functions with coefficient 1; instead of multiplying an OR function with a coefficient  $a$  we will add it up exactly  $a$  times.

Consequently, our degree- $d$  modulo  $m$  polynomial has the following form:

$$h = S_1 + S_2 + \dots + S_\ell, \tag{1}$$

where  $S_i$  is an OR-function of degree at most  $d$ .

Now we are ready to define the one-to-one correspondence between degree- $d$  modulo  $m$  polynomials without non-trivial zeroes on the hypercube and hypergraphs, without non-empty subhypergraphs of modulo- $m$  union-size 0. Let  $h$  be a degree- $d$  polynomial written in form (1), and define hypergraph  $\mathcal{F} = \{F_1, F_2, \dots, F_\ell\}$ , where  $F_i = \{S_j : x_i \text{ appears as a variable in } S_j\}$ . Clearly, the degree of this hypergraph is at most the degree of  $h$  that is,  $d$ .

On the other hand, for a hypergraph  $\mathcal{F} = \{F_1, F_2, \dots, F_t\}$  on the ground-set  $\{v_1, v_2, \dots, v_\ell\}$ , let us define  $h(x_1, x_2, \dots, x_t) = S_1 + S_2 + \dots + S_t$ , where

$$S_j = \bigvee_{i: v_j \in F_i} x_i.$$

Obviously, the degree of  $h$  is at most the degree of  $\mathcal{F}$ .

Now we state that  $\mathcal{F}$  has a non-empty subhypergraph with union-size 0 modulo  $m$  if and only if there exists a  $\mathbf{0} \neq \mathbf{x} : h(\mathbf{x}) \equiv 0 \pmod{m}$ . The proof is as follows: For  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n$  let us denote  $I(\mathbf{x}) = \{i : x_i = 1\}$ . Then  $S_j(\mathbf{x}) = 1$  if  $S_j \in \bigcup_{i \in I(\mathbf{x})} F_i$ , and  $S_j(\mathbf{x}) = 0$  otherwise. Thus  $h(\mathbf{x}) = |\bigcup_{i \in I(\mathbf{x})} F_i|$  holds for all  $\mathbf{x} \in \{0, 1\}^n$ . In particular, evaluations of  $h$  and union-sizes of subhypergraphs in  $\mathcal{F}$  become divisible by  $m$  simultaneously.  $\square$

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