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$3x + 1$ Minus the +

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We use Conway's *Fractran* language to derive a function $R : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ of the form

$$R(n) = r_i n \text{ if } n \equiv i \pmod{d}$$

where d is a positive integer, $0 \leq i < d$ and r_0, r_1, \dots, r_{d-1} are rational numbers, such that the famous $3x + 1$ conjecture holds if and only if the R -orbit of 2^n contains 2 for all positive integers n . We then show that the R -orbit of an arbitrary positive integer is a constant multiple of an orbit that contains a power of 2. Finally we apply our main result to show that any cycle $\{x_0, \dots, x_{m-1}\}$ of positive integers for the $3x + 1$ function must satisfy

$$\sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in \mathcal{O}} \left\lfloor \frac{x_i}{2} \right\rfloor + k.$$

where $\mathcal{O} = \{i : x_i \text{ is odd}\}$, $\mathcal{E} = \{i : x_i \text{ is even}\}$, and $k = |\mathcal{O}|$. The method used illustrates a general mechanism for deriving mathematical results about the iterative dynamics of arbitrary integer functions from *Fractran* algorithms.

Keywords: Collatz conjecture, $3x + 1$ problem, *Fractran*, discrete dynamical systems

1 Introduction and Main Results

The famous $3x + 1$ conjecture (cf. [3],[4]) states that for every $n \in \mathbb{Z}^+$ there exists $k \in \mathbb{Z}^+$ such that $T^k(n) = 1$ where

$$T(n) = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{3}{2}n + \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

and $T^k = \underbrace{T \circ T \circ \dots \circ T}_k$ denotes the k -fold composition of T with itself. If we let $T_0(x) = \frac{x}{2}$ and $T_1(x) = \frac{3}{2}x + \frac{1}{2}$, then for any n and k , $T^k(n) = T_{v_{k-1}} \circ T_{v_{k-2}} \circ \dots \circ T_{v_0}(n)$ for some $v_0, \dots, v_{k-1} \in \{0, 1\}$ and $v_i \equiv T^i(n) \pmod{2}$. Several authors (cf. [3]) have given explicit formulas for this composition, e.g.

$$T_{v_{k-1}} \circ T_{v_{k-2}} \circ \dots \circ T_{v_0}(n) = \frac{3^m}{2^k} n + \sum_{i=0}^{k-1} v_i \frac{3^{v_{i+1} + \dots + v_{k-1}}}{2^{k-i}} \text{ where } m = \sum_{i=0}^{k-1} v_i.$$

Compare this somewhat unwieldy expression with the much simpler one

$$R_{v_{k-1}} \circ R_{v_{k-2}} \circ \cdots \circ R_{v_0}(n) = \frac{3^m}{2^k} n$$

when $R_0(n) = \frac{1}{2}n$ and $R_1(n) = \frac{3}{2}n$. With this example in mind, it is natural to ask if there is some function of the form

$$R(n) = \begin{cases} r_0 n & \text{if } n \equiv 0 \pmod{d} \\ r_1 n & \text{if } n \equiv 1 \pmod{d} \\ \vdots & \vdots \\ r_{d-1} n & \text{if } n \equiv d-1 \pmod{d} \end{cases} \quad (1.1)$$

where r_1, \dots, r_{d-1} are rational numbers and $d \geq 2$ such that knowledge of certain R -orbits would settle the $3x+1$ problem, i.e. is there an addition-free variant of the $3x+1$ function whose dynamics encode the conjecture? We answer this question in the affirmative with the following result

Theorem 1 *There are infinitely many functions R of the form (1.1) having the property that the $3x+1$ conjecture is true if and only if for all positive integers n the R -orbit of 2^n contains 2. In particular,*

$$R(n) = \begin{cases} \frac{1}{11}n & \text{if } 11 \mid n \\ \frac{136}{15}n & \text{if } 15 \mid n \text{ and NOTA} \\ \frac{5}{17}n & \text{if } 17 \mid n \text{ and NOTA} \\ \frac{4}{5}n & \text{if } 5 \mid n \text{ and NOTA} \\ \frac{26}{21}n & \text{if } 21 \mid n \text{ and NOTA} \\ \frac{7}{13}n & \text{if } 13 \mid n \text{ and NOTA} \\ \frac{1}{7}n & \text{if } 7 \mid n \text{ and NOTA} \\ \frac{33}{4}n & \text{if } 4 \mid n \text{ and NOTA} \\ \frac{5}{2}n & \text{if } 2 \mid n \text{ and NOTA} \\ 7n & \text{otherwise} \end{cases} \quad (1.2)$$

(where NOTA means “None of the Above” conditions hold) is one such function. Furthermore, for any nonnegative integer n the R -orbit of 2^n contains the subsequence

$$2^n, 2^{T(n)}, 2^{T^2(n)}, 2^{T^3(n)} \dots$$

and these are the only powers of two that occur.

Note that the function R given in the theorem is of the form (1.1) if we take

$$d = \text{lcm}(11, 15, 17, 5, 21, 13, 7, 4, 2) = 1021020$$

since the first condition satisfied by n will also be the first condition satisfied by $n + dj$ for any j .

Proof: The proof is a straightforward application of Conway’s *Fractran* language and its mathematical consequences. We refer the reader to [2] for details. A *Fractran* program consists of a finite list of positive rational numbers, $[r_1, \dots, r_t]$. The state of a *Fractran* machine consists of a single positive integer

S . The exponents of the primes in the prime factorization of S are used as registers for storing nonnegative integers. The program is executed by multiplying S by the first rational number in the list for which the product is a nonnegative integer (and halts if no such integer exists). Thus, each *Fractran* program corresponds to a function of the form (1.1) where execution of the program corresponds to iteration of the function.

The *Fractran* program

$$\left[\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7 \right] \quad (1.3)$$

when started with $S = 2^n$, will produce $S = 2^{T(n)}$ as the next S power of 2 in the orbit. To see this, consider the flowchart for this program indicated in Figure 1. (In what follows we will only be concerned with an initial state that is a power of 2, as required.)

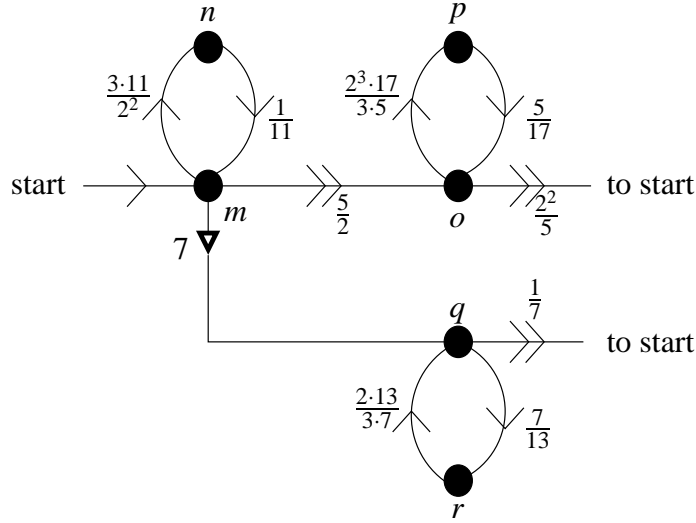


Fig. 1: A Fractran program for T

The edges of the flowchart are labeled in order of decreasing priority using a single arrow, double arrow, and triangle respectively. At a given node, the current state S is multiplied by the fraction labeling the edge of highest priority for which the product is a positive integer. The powers of the primes 5, 7, 11, 13, 17 in S correspond to the nodes o, q, n, r, p respectively, a positive exponent of one of the primes indicating the program is at that node (and it is at node m if it is at no other node). The exponents of 2 and 3 in S are used as registers to compute T . We will refer to these exponents as α and β respectively.

When the program is started with $S = 2^n$ at node m , it will execute the loop between nodes m and n exactly $q = \lfloor \frac{n}{2} \rfloor$ times, each time decreasing α by 2 and incrementing β . This results in $S = 2^{n \bmod 2} 3^q$.

If n is odd then $n = 2q + 1$ for some positive integer q and execution proceeds to node o where the state becomes $S = 3^q 5$. The loop between nodes o and p then produces $S = 2^{3q} 5$ which is then multiplied by

$\frac{2^2}{5}$ to produce

$$S = 2^{3q+2} = 2^{(6q+4)/2} = 2^{(6q+3+1)/2} = 2^{(3(2q+1)+1)/2} = 2^{(3n+1)/2} = 2^{T(n)}$$

as required.

If n is even, then upon completion of the mn loop S is multiplied by 7 moving execution to node q . The loop between nodes q and r produces $S = 2^q 7$ which is then multiplied by $1/7$ to produce

$$S = 2^q = 2^{n/2} = 2^{T(n)}$$

as required.

Iteration of the function R given in the theorem starting with seed 2^n corresponds exactly to execution of this *Fractran* program (the sequence of states being the R -orbit of 2^n). Since the choice of primes and algorithm used in this program was arbitrary, there are infinitely many such programs, and thus infinitely many such functions. This completes the proof. \square

Theorem 1 shows the relationship between the R -orbits of two powers and the $3x + 1$ problem. One might ask for its own sake[†] how the iterates of R behave for arbitrary positive integer inputs. We answer this question with the following result.

Theorem 2 *Let R be defined as in (1.2). Then for all $a, b, c, d, e, f, g, h \in \mathbb{N}$*

1. *for all $m \in \mathbb{Z}^+$ with $\gcd(m, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17) = 1$,*

$$R\left(2^a 3^b 5^c 7^d 11^e 13^f 17^g m\right) = m \cdot R\left(2^a 3^b 5^c 7^d 11^e 13^f 17^g\right)$$

and

2. *there exists $k \in \mathbb{N}$ such that $R^k\left(2^a 3^b 5^c 7^d 11^e 13^f 17^g\right) = 2^j$ for some j .*

Thus if we iterate R starting with an arbitrary positive integer n , the prime factors of n that are greater than 17 are left unchanged, and the iterates of the remaining factor eventually reach a two power (after which the behavior proceeds as indicated in Theorem 1).

Proof: The proof of part (1) follows immediately from the definition of R , since prime factors greater than 17 are not affected when a positive integer is multiplied by any of the rational numbers listed in (1.3).

To prove part (2), let S be the set of positive integers that are not divisible by a prime greater than 17. Since no prime greater than 17 is a factor of the numerator of any fraction in (1.3), R maps elements of S to elements of S .

Let S' be the subset of S consisting of integers of the form $2^a 3^b$ for some $a, b \in \mathbb{N}$. Let $a, b \in \mathbb{N}$. By the definition of R , $R^2(2^{a+2} 3^b) = 2^a 3^{b+1}$ so that $R^{2b}(2^{a+2b}) = 2^a 3^b$. Thus any element of S' is in the R -orbit of a power of two. Since the R -orbit of 2^{a+2b} contains infinitely many terms that are powers of two by Theorem 1, so does the R -orbit of $2^a 3^b$ for any $a, b \in \mathbb{N}$. Thus it suffices to show that the R -orbit of any element of S contains an element of S' .

Define $\alpha : S \rightarrow \mathbb{N}$ by $\alpha(2^{e_1} 3^{e_2} 5^{e_3} 7^{e_4} 11^{e_5} 13^{e_6} 17^{e_7}) = \sum_{i=2}^7 e_i$. We argue by contradiction, and suppose that we have an element n of S so that all iterates $R^k(n) \notin S'$. Then all terms in the R -orbit of n are divisible

[†] Thanks to the anonymous referee of an earlier draft of this paper for suggesting this line of inquiry.

by some prime in $\{5, 7, 11, 13, 17\}$. Thus by the definition of R , for all $k \geq 1$, $R^k(n) = r_k R^{k-1}(n)$ for some $r_k \in \{\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}\}$. For any $k \in \mathbb{N}$, if $r_{k+1} \in \{\frac{1}{11}, \frac{136}{15}, \frac{4}{5}, \frac{26}{21}, \frac{1}{7}\}$ then

$$\alpha(R^{k+1}(n)) = \alpha(r_{k+1}R^k(n)) < \alpha(R^k(n))$$

and if $r_{k+1} \in \{\frac{5}{17}, \frac{7}{13}\}$ then

$$\alpha(R^{k+1}(n)) = \alpha(r_{k+1}R^k(n)) = \alpha(R^k(n)).$$

So the R -orbit of n has nonincreasing values of α , i.e. the sequence

$$\alpha(n), \alpha(R(n)), \alpha(R^2(n)), \dots \quad (1.4)$$

is a nonincreasing. Since none of the terms are a two power (by our assumption), (1.4) is a nonincreasing sequence of positive integers whose terms are all less than or equal to $\alpha(n)$. Thus there must be some $h \geq 0$ such that $\alpha(R^k(n)) = \alpha(R^h(n))$ for all $k \geq h$. So $r_k \in \{\frac{5}{17}, \frac{7}{13}\}$ for all $k \geq h$. But multiplication by these values of r_k decreases the exponent of either 13 or 17 in the prime factorization of an integer, so that repeated multiplication by these fractions eventually produces a non-integer value. This contradicts our assumption and completes the proof. \square

Conway [1] used an argument similar to the proof of Theorem 1 to show that there exist functions of the form (1.1) for which the fate of the orbit of an arbitrary positive integer is algorithmically undecidable. In Theorem 1 we turn this method around to obtain a positive result, and now illustrate how this result can be used to obtain mathematical results about the conjecture itself.

2 An Application

Let x_0, \dots, x_{n-1} be positive integers such that $x_i = T(x_{i-1})$ for $0 < i < n$ and $x_0 = T(x_{n-1})$. In this situation we say $\{x_0, \dots, x_{n-1}\}$ is a T -cycle. If the $3x+1$ conjecture is true, then the only T -cycle of positive integers is $\{1, 2\}$ (the existence of any other positive integer in a T -cycle being a counterexample). Thus it is of interest to study the properties of positive integer T -cycles.

Suppose $\{x_0, \dots, x_{n-1}\}$ is a T -cycle of positive integers with $x_i = T(x_{i-1})$ for $0 < i < n$ and $x_0 = T(x_{n-1})$. Then by Theorem 1 the R -orbit of 2^{x_0} is also cyclic and contains $\{2^{x_0}, \dots, 2^{x_{n-1}}\}$ as a subset. Thus there exists some positive integer t such that $R^t(x_0) = x_0$. But each application of R is simply multiplication by one of the rational numbers in $\{\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7\}$ so that we must have

$$x_0 = R^t(x_0) = \left(\frac{1}{11}\right)^a \left(\frac{136}{15}\right)^b \left(\frac{5}{17}\right)^c \left(\frac{4}{5}\right)^d \left(\frac{26}{21}\right)^e \left(\frac{7}{13}\right)^f \left(\frac{1}{7}\right)^g \left(\frac{33}{4}\right)^h \left(\frac{5}{2}\right)^i 7^j x_0$$

for some nonnegative integers $a, b, c, d, e, f, g, h, i, j$ with $a + b + c + d + e + f + g + h + i + j = t$. Collecting prime factors on the right hand side and dividing by x_0 gives us

$$2^{3b+2d+e-2h-i} 3^{-b-e+h} 5^{-b+c-d+i} 7^{-e+f-g+j} 11^{-a+h} 13^{e-f} 17^{b-c} = 1.$$

This yields the system of linear equations

$$\begin{aligned}
 3b + 2d + e - 2h - i &= 0 \\
 -b - e + h &= 0 \\
 -b + c - d + i &= 0 \\
 -e + f - g + j &= 0 \\
 -a + h &= 0 \\
 e - f &= 0 \\
 b - c &= 0
 \end{aligned}$$

which is equivalent to the system

$$\begin{aligned}
 a &= 2c + i \\
 b &= c \\
 d &= i \\
 e &= c + i \\
 f &= c + i \\
 g &= j \\
 h &= 2c + i.
 \end{aligned} \tag{2.1}$$

Now define $O = \{i : x_i \text{ is odd}\}$ and $\mathcal{E} = \{i : x_i \text{ is even}\}$ and let $k = |O|$ so that $|\mathcal{E}| = n - k$. Then as explained in the proof of Theorem 1 we see that

$$\begin{aligned}
 i &= k \\
 j &= n - k \\
 c &= \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor \\
 a &= \sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor
 \end{aligned} \tag{2.2}$$

Substituting (2.2) into $a = 2c + i$ from (2.1) we obtain

$$\sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor = 2 \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor + k. \tag{2.3}$$

But $\sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor + \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor$. Substituting this into (2.3) and simplifying proves

Corollary 1 *If $\{x_0, \dots, x_{n-1}\}$ is a T -cycle of positive integers and*

$$O = \{i : x_i \text{ is odd}\} \text{ and } \mathcal{E} = \{i : x_i \text{ is even}\}$$

then

$$\sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor + k.$$

It should be noted that this formula can be proven directly from the known relationship

$$\sum_{i \in \mathcal{E}} x_i = \sum_{i \in \mathcal{O}} x_i + k \quad (2.4)$$

(obtained by noticing that $\{x_0, \dots, x_{n-1}\} = \{T(x_0), \dots, T(x_{n-1})\}$ so that $\sum x_i = \sum T(x_i)$ and thus $\sum_{i \in \mathcal{E}} x_i + \sum_{i \in \mathcal{O}} x_i = \sum_{i \in \mathcal{O}} \frac{3x_i+1}{2} + \sum_{i \in \mathcal{E}} \frac{x_i}{2}$ which can be solved to obtain (2.4)). However, the method used here reveals the results of the Corollary without specifically searching for those results. Thus this method provides a general approach for discovering new mathematical results by simply coding different algorithms for computing T (or any other computable integer function) and solving a simple linear system.

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