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$3x + 1$ Minus the +

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We use Conway's *Fractran* language to derive a function $R : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ of the form

$$R(n) = r_i n \text{ if } n \equiv i \pmod{d}$$

where d is a positive integer, $0 \leq i < d$ and $r_0, r_1, \dots, r_{d-1}$ are rational numbers, such that the famous $3x + 1$ conjecture holds if and only if the R -orbit of 2^n contains 2 for all positive integers n . We then show that the R -orbit of an arbitrary positive integer is a constant multiple of an orbit that contains a power of 2. Finally we apply our main result to show that any cycle $\{x_0, \dots, x_{m-1}\}$ of positive integers for the $3x + 1$ function must satisfy

$$\sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in \mathcal{O}} \left\lfloor \frac{x_i}{2} \right\rfloor + k.$$

where $\mathcal{O} = \{i : x_i \text{ is odd}\}$, $\mathcal{E} = \{i : x_i \text{ is even}\}$, and $k = |\mathcal{O}|$. The method used illustrates a general mechanism for deriving mathematical results about the iterative dynamics of arbitrary integer functions from *Fractran* algorithms.

Keywords: Collatz conjecture, $3x + 1$ problem, *Fractran*, discrete dynamical systems

1 Introduction and Main Results

The famous $3x + 1$ conjecture (cf. [3],[4]) states that for every $n \in \mathbb{Z}^+$ there exists $k \in \mathbb{Z}^+$ such that $T^k(n) = 1$ where

$$T(n) = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{3}{2}n + \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

and $T^k = \underbrace{T \circ T \circ \dots \circ T}_k$ denotes the k -fold composition of T with itself. If we let $T_0(x) = \frac{x}{2}$ and $T_1(x) = \frac{3}{2}x + \frac{1}{2}$, then for any n and k , $T^k(n) = T_{v_{k-1}} \circ T_{v_{k-2}} \circ \dots \circ T_{v_0}(n)$ for some $v_0, \dots, v_{k-1} \in \{0, 1\}$ and $v_i \equiv T^i(n) \pmod{2}$. Several authors (cf. [3]) have given explicit formulas for this composition, e.g.

$$T_{v_{k-1}} \circ T_{v_{k-2}} \circ \dots \circ T_{v_0}(n) = \frac{3^m}{2^k} n + \sum_{i=0}^{k-1} v_i \frac{3^{v_{i+1} + \dots + v_{k-1}}}{2^{k-i}} \text{ where } m = \sum_{i=0}^{k-1} v_i.$$

Compare this somewhat unwieldy expression with the much simpler one

$$R_{v_{k-1}} \circ R_{v_{k-2}} \circ \cdots \circ R_{v_0}(n) = \frac{3^m}{2^k} n$$

when $R_0(n) = \frac{1}{2}n$ and $R_1(n) = \frac{3}{2}n$. With this example in mind, it is natural to ask if there is some function of the form

$$R(n) = \begin{cases} r_0 n & \text{if } n \equiv 0 \pmod{d} \\ r_1 n & \text{if } n \equiv 1 \pmod{d} \\ \vdots & \vdots \\ r_{d-1} n & \text{if } n \equiv d-1 \pmod{d} \end{cases} \quad (1.1)$$

where $r_1, \dots, r_{d-1}$ are rational numbers and $d \geq 2$ such that knowledge of certain R -orbits would settle the $3x+1$ problem, i.e. is there an addition-free variant of the $3x+1$ function whose dynamics encode the conjecture? We answer this question in the affirmative with the following result

Theorem 1 *There are infinitely many functions R of the form (1.1) having the property that the $3x+1$ conjecture is true if and only if for all positive integers n the R -orbit of 2^n contains 2. In particular,*

$$R(n) = \begin{cases} \frac{1}{11}n & \text{if } 11 \mid n \\ \frac{136}{15}n & \text{if } 15 \mid n \text{ and NOTA} \\ \frac{5}{17}n & \text{if } 17 \mid n \text{ and NOTA} \\ \frac{4}{5}n & \text{if } 5 \mid n \text{ and NOTA} \\ \frac{26}{21}n & \text{if } 21 \mid n \text{ and NOTA} \\ \frac{7}{13}n & \text{if } 13 \mid n \text{ and NOTA} \\ \frac{1}{7}n & \text{if } 7 \mid n \text{ and NOTA} \\ \frac{33}{4}n & \text{if } 4 \mid n \text{ and NOTA} \\ \frac{5}{2}n & \text{if } 2 \mid n \text{ and NOTA} \\ 7n & \text{otherwise} \end{cases} \quad (1.2)$$

(where NOTA means “None of the Above” conditions hold) is one such function. Furthermore, for any nonnegative integer n the R -orbit of 2^n contains the subsequence

$$2^n, 2^{T(n)}, 2^{T^2(n)}, 2^{T^3(n)} \dots$$

and these are the only powers of two that occur.

Note that the function R given in the theorem is of the form (1.1) if we take

$$d = \text{lcm}(11, 15, 17, 5, 21, 13, 7, 4, 2) = 1021020$$

since the first condition satisfied by n will also be the first condition satisfied by $n + dj$ for any j .

Proof: The proof is a straightforward application of Conway’s *Fractran* language and its mathematical consequences. We refer the reader to [2] for details. A *Fractran* program consists of a finite list of positive rational numbers, $[r_1, \dots, r_t]$. The state of a *Fractran* machine consists of a single positive integer

S . The exponents of the primes in the prime factorization of S are used as registers for storing nonnegative integers. The program is executed by multiplying S by the first rational number in the list for which the product is a nonnegative integer (and halts if no such integer exists). Thus, each *Fractran* program corresponds to a function of the form (1.1) where execution of the program corresponds to iteration of the function.

The *Fractran* program

$$\left[\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7 \right] \quad (1.3)$$

when started with $S = 2^n$, will produce $S = 2^{T(n)}$ as the next S power of 2 in the orbit. To see this, consider the flowchart for this program indicated in Figure 1. (In what follows we will only be concerned with an initial state that is a power of 2, as required.)

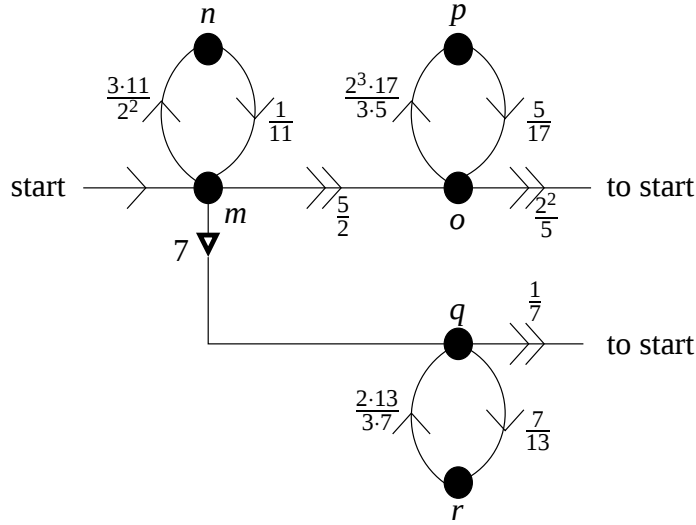


Fig. 1: A Fractran program for T

The edges of the flowchart are labeled in order of decreasing priority using a single arrow, double arrow, and triangle respectively. At a given node, the current state S is multiplied by the fraction labeling the edge of highest priority for which the product is a positive integer. The powers of the primes 5, 7, 11, 13, 17 in S correspond to the nodes o, q, n, r, p respectively, a positive exponent of one of the primes indicating the program is at that node (and it is at node m if it is at no other node). The exponents of 2 and 3 in S are used as registers to compute T . We will refer to these exponents as α and β respectively.

When the program is started with $S = 2^n$ at node m , it will execute the loop between nodes m and n exactly $q = \lfloor \frac{n}{2} \rfloor$ times, each time decreasing α by 2 and incrementing β . This results in $S = 2^{n \bmod 2} 3^q$.

If n is odd then $n = 2q + 1$ for some positive integer q and execution proceeds to node o where the state becomes $S = 3^q 5$. The loop between nodes o and p then produces $S = 2^{3q} 5$ which is then multiplied by

$\frac{2^2}{5}$ to produce

$$S = 2^{3q+2} = 2^{(6q+4)/2} = 2^{(6q+3+1)/2} = 2^{(3(2q+1)+1)/2} = 2^{(3n+1)/2} = 2^{T(n)}$$

as required.

If n is even, then upon completion of the mn loop S is multiplied by 7 moving execution to node q . The loop between nodes q and r produces $S = 2^q 7$ which is then multiplied by $1/7$ to produce

$$S = 2^q = 2^{n/2} = 2^{T(n)}$$

as required.

Iteration of the function R given in the theorem starting with seed 2^n corresponds exactly to execution of this *Fractran* program (the sequence of states being the R -orbit of 2^n). Since the choice of primes and algorithm used in this program was arbitrary, there are infinitely many such programs, and thus infinitely many such functions. This completes the proof. $\square$

Theorem 1 shows the relationship between the R -orbits of two powers and the $3x + 1$ problem. One might ask for its own sake[†] how the iterates of R behave for arbitrary positive integer inputs. We answer this question with the following result.

Theorem 2 *Let R be defined as in (1.2). Then for all $a, b, c, d, e, f, g, h \in \mathbb{N}$*

1. *for all $m \in \mathbb{Z}^+$ with $\gcd(m, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17) = 1$,*

$$R\left(2^a 3^b 5^c 7^d 11^e 13^f 17^g m\right) = m \cdot R\left(2^a 3^b 5^c 7^d 11^e 13^f 17^g\right)$$

and

2. *there exists $k \in \mathbb{N}$ such that $R^k(2^a 3^b 5^c 7^d 11^e 13^f 17^g) = 2^j$ for some j .*

Thus if we iterate R starting with an arbitrary positive integer n , the prime factors of n that are greater than 17 are left unchanged, and the iterates of the remaining factor eventually reach a two power (after which the behavior proceeds as indicated in Theorem 1).

Proof: The proof of part (1) follows immediately from the definition of R , since prime factors greater than 17 are not affected when a positive integer is multiplied by any of the rational numbers listed in (1.3).

To prove part (2), let S be the set of positive integers that are not divisible by a prime greater than 17. Since no prime greater than 17 is a factor of the numerator of any fraction in (1.3), R maps elements of S to elements of S .

Let S' be the subset of S consisting of integers of the form $2^a 3^b$ for some $a, b \in \mathbb{N}$. Let $a, b \in \mathbb{N}$. By the definition of R , $R^2(2^{a+2} 3^b) = 2^a 3^{b+1}$ so that $R^{2b}(2^{a+2b}) = 2^a 3^b$. Thus any element of S' is in the R -orbit of a power of two. Since the R -orbit of 2^{a+2b} contains infinitely many terms that are powers of two by Theorem 1, so does the R -orbit of $2^a 3^b$ for any $a, b \in \mathbb{N}$. Thus it suffices to show that the R -orbit of any element of S contains an element of S' .

Define $\alpha : S \rightarrow \mathbb{N}$ by $\alpha(2^{e_1} 3^{e_2} 5^{e_3} 7^{e_4} 11^{e_5} 13^{e_6} 17^{e_7}) = \sum_{i=2}^7 e_i$. We argue by contradiction, and suppose that we have an element n of S so that all iterates $R^k(n) \notin S'$. Then all terms in the R -orbit of n are divisible

[†] Thanks to the anonymous referee of an earlier draft of this paper for suggesting this line of inquiry.

by some prime in $\{5, 7, 11, 13, 17\}$. Thus by the definition of R , for all $k \geq 1$, $R^k(n) = r_k R^{k-1}(n)$ for some $r_k \in \{\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}\}$. For any $k \in \mathbb{N}$, if $r_{k+1} \in \{\frac{1}{11}, \frac{136}{15}, \frac{4}{5}, \frac{26}{21}, \frac{1}{7}\}$ then

$$\alpha(R^{k+1}(n)) = \alpha(r_{k+1}R^k(n)) < \alpha(R^k(n))$$

and if $r_{k+1} \in \{\frac{5}{17}, \frac{7}{13}\}$ then

$$\alpha(R^{k+1}(n)) = \alpha(r_{k+1}R^k(n)) = \alpha(R^k(n)).$$

So the R -orbit of n has nonincreasing values of α , i.e. the sequence

$$\alpha(n), \alpha(R(n)), \alpha(R^2(n)), \dots \quad (1.4)$$

is a nonincreasing. Since none of the terms are a two power (by our assumption), (1.4) is a nonincreasing sequence of positive integers whose terms are all less than or equal to $\alpha(n)$. Thus there must be some $h \geq 0$ such that $\alpha(R^k(n)) = \alpha(R^h(n))$ for all $k \geq h$. So $r_k \in \{\frac{5}{17}, \frac{7}{13}\}$ for all $k \geq h$. But multiplication by these values of r_k decreases the exponent of either 13 or 17 in the prime factorization of an integer, so that repeated multiplication by these fractions eventually produces a non-integer value. This contradicts our assumption and completes the proof. $\square$

Conway [1] used an argument similar to the proof of Theorem 1 to show that there exist functions of the form (1.1) for which the fate of the orbit of an arbitrary positive integer is algorithmically undecidable. In Theorem 1 we turn this method around to obtain a positive result, and now illustrate how this result can be used to obtain mathematical results about the conjecture itself.

2 An Application

Let $x_0, \dots, x_{n-1}$ be positive integers such that $x_i = T(x_{i-1})$ for $0 < i < n$ and $x_0 = T(x_{n-1})$. In this situation we say $\{x_0, \dots, x_{n-1}\}$ is a T -cycle. If the $3x+1$ conjecture is true, then the only T -cycle of positive integers is $\{1, 2\}$ (the existence of any other positive integer in a T -cycle being a counterexample). Thus it is of interest to study the properties of positive integer T -cycles.

Suppose $\{x_0, \dots, x_{n-1}\}$ is a T -cycle of positive integers with $x_i = T(x_{i-1})$ for $0 < i < n$ and $x_0 = T(x_{n-1})$. Then by Theorem 1 the R -orbit of 2^{x_0} is also cyclic and contains $\{2^{x_0}, \dots, 2^{x_{n-1}}\}$ as a subset. Thus there exists some positive integer t such that $R^t(x_0) = x_0$. But each application of R is simply multiplication by one of the rational numbers in $\{\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7\}$ so that we must have

$$x_0 = R^t(x_0) = \left(\frac{1}{11}\right)^a \left(\frac{136}{15}\right)^b \left(\frac{5}{17}\right)^c \left(\frac{4}{5}\right)^d \left(\frac{26}{21}\right)^e \left(\frac{7}{13}\right)^f \left(\frac{1}{7}\right)^g \left(\frac{33}{4}\right)^h \left(\frac{5}{2}\right)^i 7^j x_0$$

for some nonnegative integers $a, b, c, d, e, f, g, h, i, j$ with $a + b + c + d + e + f + g + h + i + j = t$. Collecting prime factors on the right hand side and dividing by x_0 gives us

$$2^{3b+2d+e-2h-i} 3^{-b-e+h} 5^{-b+c-d+i} 7^{-e+f-g+j} 11^{-a+h} 13^{e-f} 17^{b-c} = 1.$$

This yields the system of linear equations

$$\begin{aligned}
 3b + 2d + e - 2h - i &= 0 \\
 -b - e + h &= 0 \\
 -b + c - d + i &= 0 \\
 -e + f - g + j &= 0 \\
 -a + h &= 0 \\
 e - f &= 0 \\
 b - c &= 0
 \end{aligned}$$

which is equivalent to the system

$$\begin{aligned}
 a &= 2c + i \\
 b &= c \\
 d &= i \\
 e &= c + i \\
 f &= c + i \\
 g &= j \\
 h &= 2c + i.
 \end{aligned} \tag{2.1}$$

Now define $O = \{i : x_i \text{ is odd}\}$ and $\mathcal{E} = \{i : x_i \text{ is even}\}$ and let $k = |O|$ so that $|\mathcal{E}| = n - k$. Then as explained in the proof of Theorem 1 we see that

$$\begin{aligned}
 i &= k \\
 j &= n - k \\
 c &= \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor \\
 a &= \sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor
 \end{aligned} \tag{2.2}$$

Substituting (2.2) into $a = 2c + i$ from (2.1) we obtain

$$\sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor = 2 \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor + k. \tag{2.3}$$

But $\sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor + \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor$. Substituting this into (2.3) and simplifying proves

Corollary 1 *If $\{x_0, \dots, x_{n-1}\}$ is a T -cycle of positive integers and*

$$O = \{i : x_i \text{ is odd}\} \text{ and } \mathcal{E} = \{i : x_i \text{ is even}\}$$

then

$$\sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor + k.$$

It should be noted that this formula can be proven directly from the known relationship

$$\sum_{i \in \mathcal{E}} x_i = \sum_{i \in \mathcal{O}} x_i + k \quad (2.4)$$

(obtained by noticing that $\{x_0, \dots, x_{n-1}\} = \{T(x_0), \dots, T(x_{n-1})\}$ so that $\sum x_i = \sum T(x_i)$ and thus $\sum_{i \in \mathcal{E}} x_i + \sum_{i \in \mathcal{O}} x_i = \sum_{i \in \mathcal{O}} \frac{3x_i+1}{2} + \sum_{i \in \mathcal{E}} \frac{x_i}{2}$ which can be solved to obtain (2.4)). However, the method used here reveals the results of the Corollary without specifically searching for those results. Thus this method provides a general approach for discovering new mathematical results by simply coding different algorithms for computing T (or any other computable integer function) and solving a simple linear system.

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