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Overlap-Free Symmetric D0L words[†]

Anna E. Frid

Sobolev Institute of Mathematics, pr. Koptyuga, 4, 630090, Novosibirsk, Russia
e-mail: frid@math.nsc.ru

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A D0L word on an alphabet $\Sigma = \{0, 1, \dots, q-1\}$ is called symmetric if it is a fixed point $w = \varphi(w)$ of a morphism $\varphi : \Sigma^* \rightarrow \Sigma^*$ defined by $\varphi(i) = \overline{t_1 + i} \overline{t_2 + i} \dots \overline{t_m + i}$ for some word $t_1 t_2 \dots t_m$ (equal to $\varphi(0)$) and every $i \in \Sigma$; here \bar{a} means $a \bmod q$.

We prove a result conjectured by J. Shallit: if all the symbols in $\varphi(0)$ are distinct (i.e., if $t_i \neq t_j$ for $i \neq j$), then the symmetric D0L word w is overlap-free, i.e., contains no factor of the form $axaxa$ for any $x \in \Sigma^*$ and $a \in \Sigma$.

Keywords: overlap-free word, D0L word, symmetric morphism

1 Introduction

In his classical 1912 paper [15] (see also [3]), A. Thue gave the first example of an overlap-free infinite word, i. e., of a word which contains no subword of the form $axaxa$ for any symbol a and word x . Thue's example is known now as the *Thue-Morse word*

$$w_{TM} = 01101001100101101001011001101001\dots$$

It was rediscovered several times, can be constructed in many alternative ways and occurs in various fields of mathematics (see the survey [1]).

The set of all overlap-free words was studied e. g. by E. D. Fife [8] who described all binary overlap-free infinite words and P. Séébold [13] who proved that the Thue-Morse word is essentially the only binary overlap-free word which is a fixed point of a morphism. Nowadays the theory of overlap-free words is a part of a more general theory of pattern avoidance [5].

J.-P. Allouche and J. Shallit [2] asked if the initial Thue's construction of an overlap-free word could be generalized and found a whole family of overlap-free infinite words built by a similar principle. This paper contains a further generalization of that result; its main theorem was conjectured by J. Shallit [14].

Let us give all the necessary definitions and state the main theorem. Consider a finite alphabet $\Sigma = \Sigma_q = \{0, 1, \dots, q-1\}$. For an integer i , let \bar{i} denote the residue of i modulo q . A morphism $\varphi : \Sigma_q^* \rightarrow \Sigma_q^*$ is called *symmetric* if for all $i \in \Sigma_q$ the equality holds

$$\varphi(i) = \overline{t_1 + i} \overline{t_2 + i} \dots \overline{t_m + i},$$

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where $t_1 t_2 \dots t_m$ is an arbitrary word (equal to $\varphi(0)$). Clearly, if $t_1 = 0$, then φ has a fixed point, i. e., a (right) infinite word $w = w(\varphi)$ satisfying

$$w = \varphi(w).$$

Without loss of generality we assume that w starts with 0.

A symmetric morphism is *growing* if $|\varphi(0)| \geq 2$. We shall call a fixed point of a growing symmetric morphism a *symmetric DOL word*. For example, the Thue-Morse word w_{TM} is a fixed point of a symmetric morphism φ_{TM} :

$$\begin{cases} \varphi_{TM}(0) = 01, \\ \varphi_{TM}(1) = 10. \end{cases}$$

Symmetric DOL words include also other useful examples, such as the Dejean word [7], the Keränen word [11] and others (see Section 10.5 in [12], where in particular the term “symmetric” is introduced). Note that the class of symmetric DOL words is included in a wider class of uniform marked DOL words whose properties were studied e. g. in [10].

Note that an infinite word $w = w_1 w_2 \dots w_n \dots$, where $w_i \in \Sigma$, is the fixed point of the symmetric morphism φ if and only if

$$\forall k \geq 0 \forall i \in \{1, \dots, m\} \quad w_{km+i} = \overline{w_{k+1} + t_i}. \quad (1)$$

Indeed, this equality means that w_{km+i} is equal to the i th symbol of $\varphi(w_{k+1})$.

For every $m > 1$, let $\varphi_{m,q} : \Sigma_q^* \rightarrow \Sigma_q^*$ be the symmetric morphism defined by $\varphi_{m,q}(0) = 0\overline{1} \overline{2} \dots \overline{m-1}$. Note that $\varphi_{TM} = \varphi_{2,2}$. Let $w_{m,q}$ be the fixed point of $\varphi_{m,q}$ starting with 0; then the i th symbol of $w_{m,q}$ for each i can also be defined as $s_m(i)$, where $s_m(i)$ is the sum of the digits in the base- m representation of i .

J.-P. Allouche and J. Shallit proved the following generalization of Thue’s result:

Theorem 1 ([2]) *The word $w_{m,q}$ is overlap-free if and only if $m \leq q$.*

J. Shallit conjectured also that symmetric DOL words of a much wider class are overlap-free. We turn this conjecture into

Theorem 2 *If $\varphi : \Sigma_q^* \rightarrow \Sigma_q^*$ is a growing symmetric morphism, and if all symbols occurring in $\varphi(0)$ are distinct, then the fixed point $w = w(\varphi)$ is overlap-free.*

The remaining part of the paper is devoted to the proof of this result.

2 Proof of Theorem 2

Let us start with introducing some more notions and citing a result by J. Berstel and L. Boasson [4] which we shall need later.

A *partial word* is a word on the alphabet $\Sigma \cup \{\diamond\}$, where the symbol $\diamond \notin \Sigma$ is called the *hole*[‡]. Each hole means an unknown symbol of Σ . A (partial) word $u = u_1 \dots u_n$, where u_i are symbols, is called (*locally*) *p*-periodic if $u_i = u_{i+p}$ for all $i \in \{1, \dots, n-p\}$ such that $u_i \neq \diamond$ and $u_{i+p} \neq \diamond$.

The following result is a generalization of the classical Fine and Wilf’s theorem [9, 6]:

Theorem 3 ([4]) *Let u be a partial word of length n which is p -periodic and q -periodic. If u contains only one hole, and if $n \geq p+q$, then u is $\gcd(p, q)$ -periodic.*

Now let us start the proof of Theorem 2 and first consider the easiest case:

[‡] This definition slightly differs from the one given in [4].

Lemma 1 *If the symmetric morphism φ is defined by $\varphi(0) = 0\bar{c}2\bar{c}\dots\overline{(m-1)c}$ for some integer $c > 0$, and if all the symbols of $\varphi(0)$ are distinct, then the fixed point w of φ is overlap-free.*

Proof. Let $S \subset \Sigma$ be the set of symbols occurring in w and q' be its cardinality. Denote $\Sigma' = \{0, \dots, q' - 1\}$ and define $h : (\Sigma')^* \rightarrow S^*$ as the symbol-to symbol morphism transforming each symbol $i \in \Sigma'$ to $h(i) = \bar{c}i$. Since the cardinalities of S and Σ' coincide, and since each symbol of S can be represented as $\bar{c}i$ for some i , h is a one-to-one mapping. But it can be easily checked that $\varphi h = h\varphi_{m,q'}$. Since $w_{m,q'} = \varphi_{m,q'}(w_{m,q'})$, we have $h(w_{m,q'}) = h(\varphi_{m,q'}(w_{m,q'})) = \varphi(h(w_{m,q'}))$, so $h(w_{m,q'})$ is the fixed point of φ ; it starts with 0 since $h(0) = 0$. We see that $h(w_{m,q'}) = w$, that is, w is obtained from $w_{m,q'}$ by renaming symbols. It is overlap-free due to Theorem 1. \square

A *block* is an image of symbol under a morphism. Let $S(m)$ denote the class of all symmetric morphisms on Σ of block length m with all the symbols in a block distinct. We assume also that the image of 0 always starts with 0, so that all the morphisms of $S(m)$ admit fixed points. Clearly, the class $S(m)$ is non-empty only if $m \leq q$.

Our goal is to prove that, for any fixed m , all the fixed points of morphisms of $S(m)$ are overlap-free. Suppose the opposite and consider the minimal counter-example, i. e., a morphism $\varphi \in S(m)$ and its fixed point w containing an overlap $v = axaxa$ of minimal length (so that overlaps occurring in other fixed points of morphisms of $S(m)$ are not shorter). Here $a \in \Sigma$ and $x \in \Sigma^*$; we denote the length $|ax|$ by l , and thus have $|v| = 2l + 1$. Let us fix an occurrence of v to w and its position with respect to blocks of φ : we consider v as a word obtained from $\varphi(s)$, where s is a factor of w , by erasing $\alpha - 1$ symbols from the left and $m - \beta$ symbols from the right, where $1 \leq \alpha, \beta \leq m$. So, v starts with the symbol numbered α of a block and ends with the symbol numbered β .

Claim 1 *The inequality $l \geq m$ holds.*

Proof. Suppose that $l < m$. The 1st, $(l + 1)$ th, and $(2l + 1)$ th symbols of v are equal and thus must lie in three different blocks. So, v contains a complete block. But this block must be l -periodic since v is l -periodic; hence it must contain two equal symbols since $l < m$. A contradiction. \square

Claim 2 *The block length m does not divide l .*

Proof. Suppose the opposite: let $l = mk$. Then the length of the “inverse image” s of v is equal to $2k + 1$. Since v is an overlap, its $(mi + 1)$ th symbol is equal to the $(m(i + k) + 1)$ th one for any $i \in \{0, \dots, k\}$; they are symbols numbered α of respectively the $(i + 1)$ th and the $(i + k + 1)$ th blocks of $\varphi(s)$. Since the morphism φ is symmetric, each block is uniquely determined by its α th symbol, so $(i + 1)$ th and $(i + k + 1)$ th symbols of s are equal. Thus, s is an overlap in w shorter than v , a contradiction. \square

For every word $u = u_1u_2\dots u_{n+1} \in \Sigma^{n+1}$, where $u_1, \dots, u_{n+1} \in \Sigma$, let us define the word $r(u) \in \Sigma^n$ as obtained from u by subtraction of consecutive symbols:

$$r(u) = \overline{u_2 - u_1} \overline{u_3 - u_2} \dots \overline{u_{n+1} - u_n}.$$

Clearly, u can be reconstructed from its first symbol u_1 and the word $r(u) = r_1 \dots r_n$, where $r_1, \dots, r_n \in \Sigma$:

$$u = u_1 \overline{u_1 + r_1} \overline{u_1 + r_1 + r_2} \dots \overline{u_1 + r_1 + \dots + r_n}. \tag{2}$$

Let us consider the word $r(v) = r(axaxa)$. Its length is equal to $2l$, and it is l -periodic as well as v . Since φ is symmetric, the word $r(\varphi(i))$ does not depend on the symbol $i \in \Sigma$; we denote $r(\varphi(i)) = b = b_1 \dots b_{m-1}$,

where $b_1, \dots, b_{m-1} \in \Sigma$. Since v starts with the symbol number α of a block and ends with the symbol number β , we have

$$r(v) = b_\alpha \dots b_{m-1} c_1 b c_2 b \dots b c_n b_1 \dots b_{\beta-1},$$

where $|s| = n + 1$ and $c_1 \dots c_n$ are symbols of Σ depending on pairs of consecutive blocks in $\varphi(s)$; if $\alpha = m$, then $r(v)$ just starts with c_1 , and if $\beta = 1$, $r(v)$ just ends with c_n . Let n' be the last number such that $c_{n'}$ is situated in the first occurrence of $r(axa)$ in $r(v)$. Since $r(v)$ is l -periodic, for all $i \in \{1, \dots, n'\}$ the symbol c_i is equal to the symbol of $r(v)$ situated at distance l from it. Due to Claim 2, $l \not\equiv 0 \pmod{m}$, and thus all these symbols are equal to $b_{l'}$, where $l \equiv l' \pmod{m}$. So, the word $r(axa)$ (equal to the prefix of length l of $r(v)$) is m -periodic:

$$r(axa) = b_\alpha \dots b_{m-1} (b_{l'} b)^{n'-1} b_{l'} b_1 \dots b_{\gamma-1},$$

where $\gamma - \alpha \equiv l \pmod{m}$, $\gamma \in \{1, \dots, m\}$.

Let us consider the prefix of $r(v)$ of length $m + l$. It exists due to Claim 1 and is equal to

$$r(axa) b_\gamma \dots b_{m-1} c_{n'+1} b_1 \dots b_{\gamma-1}.$$

Substituting the unknown symbol $c_{n'+1}$ by a hole \diamond , we obtain a partial word

$$b_\alpha \dots b_{m-1} (b_{l'} b)^{n'} \diamond b_1 \dots b_{\gamma-1},$$

which is l -periodic as well as $r(v)$. But at the same time, it is m -periodic; thus, due to Theorem 3 it is p -periodic, where $p = \gcd(l, m)$. Consequently, $b = r(\varphi(0))$ is also p -periodic: $b = \overline{(b_1 \dots b_p)^{m'-1} b_1 \dots b_{p-1}}$, where $m' = m/p$. Let us return to $\varphi(0)$ and denote $g_1 = 0$, $g_k = \overline{b_1 + b_2 + \dots + b_{k-1}}$ for $k \in \{2, \dots, p\}$, and $c = \overline{b_1 + b_2 + \dots + b_p}$; due to (2), we see that $\varphi(0)$ is of the form

$$\varphi(0) = g_1 \dots g_p \overline{g_1 + c} \dots \overline{g_p + c} \dots \overline{g_1 + (m' - 1)c} \dots \overline{g_p + (m' - 1)c}. \quad (3)$$

Here $g_1 = 0$ since φ has a fixed point, and $m' = m/p$. The words of the form $\overline{g_1 + ic} \dots \overline{g_p + ic}$, where $i \in \{0, \dots, m' - 1\}$, will be called *subblocks*. Note that for all $k \in \{1, \dots, p\}$, a subblock is uniquely determined by its k th symbol, and that w consists of consecutive subblocks.

Let w_i denote the i th symbol of the fixed point w of φ , i. e., let $w = w_1 \dots w_n \dots$, where $w_i \in \Sigma$. Consider the arithmetical subsequence

$$w' = w_1 w_{p+1} w_{2p+1} \dots w_{np+1} \dots$$

Claim 3 *The word w' is the fixed point of a morphism $\varphi' \in S(m)$.*

Proof. Let us define the symmetric morphism φ' by

$$\varphi'(0) = g_1 \overline{g_1 + c} \dots \overline{g_1 + (m' - 1)c} \overline{g_2 g_2 + c} \dots \overline{g_2 + (m' - 1)c} \dots g_p \overline{g_p + c} \dots \overline{g_p + (m' - 1)c}.$$

Since $\varphi'(0)$ is obtained from $\varphi(0)$ by permuting symbols, and all the symbols of $\varphi(0)$ are distinct, so are the symbols of $\varphi'(0)$. Since $g_1 = 0$, and φ' is symmetric by definition, $\varphi' \in S(m)$. So we must prove only that w' is its fixed point, i. e., that

$$\forall k \geq 0 \forall i \in \{1, \dots, m\} \quad w'_{km+i} \text{ is equal to the } i\text{th symbol of } \varphi'(w'_{k+1}), \quad (4)$$

where w'_k is the k th symbol of $w' = w'_1 w'_2 \dots w'_n \dots$

Clearly, each $i \in \{1, \dots, m\}$ can be uniquely represented as $i = jm' + \delta$, where $j \in \{0, \dots, p-1\}$ and $\delta \in \{1, \dots, m'\}$. Since by definition of w' for all v we have $w'_v = w_{p(v-1)+1}$, for any $k \geq 0$

$$w'_{km+i} = w'_{km+jm'+\delta} = w_{p(km+jm'+\delta-1)+1} = w_{(pk+j)m+p(\delta-1)+1}.$$

By Equality (1), $w_{(pk+j)m+p(\delta-1)+1}$ is equal to the $(p(\delta-1)+1)$ th symbol of $\varphi(w_{pk+j+1})$, that is, to $(\delta-1)c + w_{pk+j+1}$ (recall that $g_1 = 0$). In its turn, w_{pk+j+1} is the $(j+1)$ th symbol of the subblock starting with $w_{pk+1} = w'_{k+1}$. It is equal to $w'_{k+1} + g_{j+1}$, and thus, $w'_{km+i} = w'_{k+1} + (\delta-1)c + g_{j+1}$. By the definition of φ' , it is equal to the symbol numbered $jm' + \delta = i$ of $\varphi'(w'_{k+1})$. We have proved (4) and Claim 3. \square

Claim 4 *The word w' contains an overlap of length $2l' + 1$, where $l' = l/p$.*

Proof. Let our occurrence of the overlap v to w start with the k th symbol of a subblock, i. e., let $\alpha \equiv k \pmod{p}$, where $k \in \{1, \dots, p\}$. It means that $v = w_{jp+k}w_{jp+k+1} \dots w_{(j+2l')p+k}$ for some $j \geq 0$; since v is an overlap, $w_{(j+v)p+1} = w_{(j+v+l')p+1}$ for all $v \in \{1, \dots, l'\}$. But we have also $w_{jp+k} = w_{(j+l')p+k}$, and since a subblock is uniquely determined by its k th symbol, $w_{jp+1} = w_{(j+l')p+1}$. So, the word $w_{jp+1}w_{(j+1)p+1} \dots w_{(j+2l')p+1}$ is l' -periodic, and it is the needed overlap in w' . \square

As it follows from Claims 3 and 4, we have found a fixed point of a morphism of $S(m)$ containing an overlap of length $l' = l/p$. But if $p > 1$, this contradicts to the minimality of our counter-example. On the other hand, if $p = 1$, then it follows from (3) that

$$\varphi(0) = 0\bar{c} \overline{2c} \dots \overline{(m-1)c}.$$

But a fixed point of such a morphism cannot be a counter-example according to Lemma 1. A contradiction. Theorem 2 is proved. \square

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