



Overlap-free symmetric D0L words

Anna Frid

► To cite this version:

Anna Frid. Overlap-free symmetric D0L words. Discrete Mathematics and Theoretical Computer Science, 2001, Vol. 4 no. 2 (2), pp.357-362. 10.46298/dmtcs.293 . hal-00958967

HAL Id: hal-00958967

<https://inria.hal.science/hal-00958967v1>

Submitted on 13 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Overlap-Free Symmetric D0L words[†]

Anna E. Frid

Sobolev Institute of Mathematics, pr. Koptyuga, 4, 630090, Novosibirsk, Russia
e-mail: `frid@math.nsc.ru`

received Sep 27, 2000, revised Oct 1, 2001, accepted Oct 15, 2001.

A D0L word on an alphabet $\Sigma = \{0, 1, \dots, q-1\}$ is called symmetric if it is a fixed point $w = \varphi(w)$ of a morphism $\varphi : \Sigma^* \rightarrow \Sigma^*$ defined by $\varphi(i) = \overline{t_1 + i} \overline{t_2 + i} \dots \overline{t_m + i}$ for some word $t_1 t_2 \dots t_m$ (equal to $\varphi(0)$) and every $i \in \Sigma$; here \bar{a} means $a \bmod q$.

We prove a result conjectured by J. Shallit: if all the symbols in $\varphi(0)$ are distinct (i.e., if $t_i \neq t_j$ for $i \neq j$), then the symmetric D0L word w is overlap-free, i.e., contains no factor of the form $axaxa$ for any $x \in \Sigma^*$ and $a \in \Sigma$.

Keywords: overlap-free word, D0L word, symmetric morphism

1 Introduction

In his classical 1912 paper [15] (see also [3]), A. Thue gave the first example of an overlap-free infinite word, i. e., of a word which contains no subword of the form $axaxa$ for any symbol a and word x . Thue's example is known now as the *Thue-Morse word*

$$w_{TM} = 01101001100101101001011001101001\dots$$

It was rediscovered several times, can be constructed in many alternative ways and occurs in various fields of mathematics (see the survey [1]).

The set of all overlap-free words was studied e. g. by E. D. Fife [8] who described all binary overlap-free infinite words and P. Séébold [13] who proved that the Thue-Morse word is essentially the only binary overlap-free word which is a fixed point of a morphism. Nowadays the theory of overlap-free words is a part of a more general theory of pattern avoidance [5].

J.-P. Allouche and J. Shallit [2] asked if the initial Thue's construction of an overlap-free word could be generalized and found a whole family of overlap-free infinite words built by a similar principle. This paper contains a further generalization of that result; its main theorem was conjectured by J. Shallit [14].

Let us give all the necessary definitions and state the main theorem. Consider a finite alphabet $\Sigma = \Sigma_q = \{0, 1, \dots, q-1\}$. For an integer i , let \bar{i} denote the residue of i modulo q . A morphism $\varphi : \Sigma_q^* \rightarrow \Sigma_q^*$ is called *symmetric* if for all $i \in \Sigma_q$ the equality holds

$$\varphi(i) = \overline{t_1 + i} \overline{t_2 + i} \dots \overline{t_m + i},$$

[†]Supported in part by INTAS (grant 97-1001) and RFBR (grant 01-01-06018).

where $t_1 t_2 \dots t_m$ is an arbitrary word (equal to $\varphi(0)$). Clearly, if $t_1 = 0$, then φ has a fixed point, i. e., a (right) infinite word $w = w(\varphi)$ satisfying

$$w = \varphi(w).$$

Without loss of generality we assume that w starts with 0.

A symmetric morphism is *growing* if $|\varphi(0)| \geq 2$. We shall call a fixed point of a growing symmetric morphism a *symmetric DOL word*. For example, the Thue-Morse word w_{TM} is a fixed point of a symmetric morphism φ_{TM} :

$$\begin{cases} \varphi_{TM}(0) = 01, \\ \varphi_{TM}(1) = 10. \end{cases}$$

Symmetric DOL words include also other useful examples, such as the Dejean word [7], the Keränen word [11] and others (see Section 10.5 in [12], where in particular the term “symmetric” is introduced). Note that the class of symmetric DOL words is included in a wider class of uniform marked DOL words whose properties were studied e. g. in [10].

Note that an infinite word $w = w_1 w_2 \dots w_n \dots$, where $w_i \in \Sigma$, is the fixed point of the symmetric morphism φ if and only if

$$\forall k \geq 0 \forall i \in \{1, \dots, m\} \quad w_{km+i} = \overline{w_{k+1} + t_i}. \quad (1)$$

Indeed, this equality means that w_{km+i} is equal to the i th symbol of $\varphi(w_{k+1})$.

For every $m > 1$, let $\varphi_{m,q} : \Sigma_q^* \rightarrow \Sigma_q^*$ be the symmetric morphism defined by $\varphi_{m,q}(0) = 0\bar{1}\bar{2}\dots\overline{m-1}$. Note that $\varphi_{TM} = \varphi_{2,2}$. Let $w_{m,q}$ be the fixed point of $\varphi_{m,q}$ starting with 0; then the i th symbol of $w_{m,q}$ for each i can also be defined as $s_m(i)$, where $s_m(i)$ is the sum of the digits in the base- m representation of i .

J.-P. Allouche and J. Shallit proved the following generalization of Thue’s result:

Theorem 1 ([2]) *The word $w_{m,q}$ is overlap-free if and only if $m \leq q$.*

J. Shallit conjectured also that symmetric DOL words of a much wider class are overlap-free. We turn this conjecture into

Theorem 2 *If $\varphi : \Sigma_q^* \rightarrow \Sigma_q^*$ is a growing symmetric morphism, and if all symbols occurring in $\varphi(0)$ are distinct, then the fixed point $w = w(\varphi)$ is overlap-free.*

The remaining part of the paper is devoted to the proof of this result.

2 Proof of Theorem 2

Let us start with introducing some more notions and citing a result by J. Berstel and L. Boasson [4] which we shall need later.

A *partial word* is a word on the alphabet $\Sigma \cup \{\diamond\}$, where the symbol $\diamond \notin \Sigma$ is called the *hole*[‡]. Each hole means an unknown symbol of Σ . A (partial) word $u = u_1 \dots u_n$, where u_i are symbols, is called (*locally*) *p*-periodic if $u_i = u_{i+p}$ for all $i \in \{1, \dots, n-p\}$ such that $u_i \neq \diamond$ and $u_{i+p} \neq \diamond$.

The following result is a generalization of the classical Fine and Wilf’s theorem [9, 6]:

Theorem 3 ([4]) *Let u be a partial word of length n which is p -periodic and q -periodic. If u contains only one hole, and if $n \geq p + q$, then u is $\gcd(p, q)$ -periodic.*

Now let us start the proof of Theorem 2 and first consider the easiest case:

[‡] This definition slightly differs from the one given in [4].

Lemma 1 *If the symmetric morphism ϕ is defined by $\phi(0) = 0\bar{c} \overline{2c} \dots \overline{(m-1)c}$ for some integer $c > 0$, and if all the symbols of $\phi(0)$ are distinct, then the fixed point w of ϕ is overlap-free.*

Proof. Let $S \subset \Sigma$ be the set of symbols occurring in w and q' be its cardinality. Denote $\Sigma' = \{0, \dots, q' - 1\}$ and define $h : (\Sigma')^* \rightarrow S^*$ as the symbol-to-symbol morphism transforming each symbol $i \in \Sigma'$ to $h(i) = \bar{ci}$. Since the cardinalities of S and Σ' coincide, and since each symbol of S can be represented as \bar{ci} for some i , h is a one-to-one mapping. But it can be easily checked that $\phi h = h\phi_{m,q'}$. Since $w_{m,q'} = \phi_{m,q'}(w_{m,q'})$, we have $h(w_{m,q'}) = h(\phi_{m,q'}(w_{m,q'})) = \phi(h(w_{m,q'}))$, so $h(w_{m,q'})$ is the fixed point of ϕ ; it starts with 0 since $h(0) = 0$. We see that $h(w_{m,q'}) = w$, that is, w is obtained from $w_{m,q'}$ by renaming symbols. It is overlap-free due to Theorem 1. \square

A *block* is an image of symbol under a morphism. Let $S(m)$ denote the class of all symmetric morphisms on Σ of block length m with all the symbols in a block distinct. We assume also that the image of 0 always starts with 0, so that all the morphisms of $S(m)$ admit fixed points. Clearly, the class $S(m)$ is non-empty only if $m \leq q$.

Our goal is to prove that, for any fixed m , all the fixed points of morphisms of $S(m)$ are overlap-free. Suppose the opposite and consider the minimal counter-example, i. e., a morphism $\phi \in S(m)$ and its fixed point w containing an overlap $v = axaxa$ of minimal length (so that overlaps occurring in other fixed points of morphisms of $S(m)$ are not shorter). Here $a \in \Sigma$ and $x \in \Sigma^*$; we denote the length $|ax|$ by l , and thus have $|v| = 2l + 1$. Let us fix an occurrence of v to w and its position with respect to blocks of ϕ : we consider v as a word obtained from $\phi(s)$, where s is a factor of w , by erasing $\alpha - 1$ symbols from the left and $m - \beta$ symbols from the right, where $1 \leq \alpha, \beta \leq m$. So, v starts with the symbol numbered α of a block and ends with the symbol numbered β .

Claim 1 *The inequality $l \geq m$ holds.*

Proof. Suppose that $l < m$. The 1st, $(l + 1)$ th, and $(2l + 1)$ th symbols of v are equal and thus must lie in three different blocks. So, v contains a complete block. But this block must be l -periodic since v is l -periodic; hence it must contain two equal symbols since $l < m$. A contradiction. \square

Claim 2 *The block length m does not divide l .*

Proof. Suppose the opposite: let $l = mk$. Then the length of the “inverse image” s of v is equal to $2k + 1$. Since v is an overlap, its $(mi + 1)$ th symbol is equal to the $(m(i + k) + 1)$ th one for any $i \in \{0, \dots, k\}$; they are symbols numbered α of respectively the $(i + 1)$ th and the $(i + k + 1)$ th blocks of $\phi(s)$. Since the morphism ϕ is symmetric, each block is uniquely determined by its α th symbol, so $(i + 1)$ th and $(i + k + 1)$ th symbols of s are equal. Thus, s is an overlap in w shorter than v , a contradiction. \square

For every word $u = u_1 u_2 \dots u_{n+1} \in \Sigma^{n+1}$, where $u_1, \dots, u_{n+1} \in \Sigma$, let us define the word $r(u) \in \Sigma^n$ as obtained from u by subtraction of consecutive symbols:

$$r(u) = \overline{u_2 - u_1} \overline{u_3 - u_2} \dots \overline{u_{n+1} - u_n}.$$

Clearly, u can be reconstructed from its first symbol u_1 and the word $r(u) = r_1 \dots r_n$, where $r_1, \dots, r_n \in \Sigma$:

$$u = u_1 \overline{u_1 + r_1} \overline{u_1 + r_1 + r_2} \dots \overline{u_1 + r_1 + \dots + r_n}. \quad (2)$$

Let us consider the word $r(v) = r(axaxa)$. Its length is equal to $2l$, and it is l -periodic as well as v . Since ϕ is symmetric, the word $r(\phi(i))$ does not depend on the symbol $i \in \Sigma$; we denote $r(\phi(i)) = b = b_1 \dots b_{m-1}$,

where $b_1, \dots, b_{m-1} \in \Sigma$. Since v starts with the symbol number α of a block and ends with the symbol number β , we have

$$r(v) = b_\alpha \dots b_{m-1} c_1 b c_2 b \dots b c_n b_1 \dots b_{\beta-1},$$

where $|s| = n + 1$ and $c_1 \dots c_n$ are symbols of Σ depending on pairs of consecutive blocks in $\varphi(s)$; if $\alpha = m$, then $r(v)$ just starts with c_1 , and if $\beta = 1$, $r(v)$ just ends with c_n . Let n' be the last number such that $c_{n'}$ is situated in the first occurrence of $r(axa)$ in $r(v)$. Since $r(v)$ is l -periodic, for all $i \in \{1, \dots, n'\}$ the symbol c_i is equal to the symbol of $r(v)$ situated at distance l from it. Due to Claim 2, $l \not\equiv 0 \pmod{m}$, and thus all these symbols are equal to $b_{l'}$, where $l \equiv l' \pmod{m}$. So, the word $r(axa)$ (equal to the prefix of length l of $r(v)$) is m -periodic:

$$r(axa) = b_\alpha \dots b_{m-1} (b_{l'} b)^{n'-1} b_{l'} b_1 \dots b_{\gamma-1},$$

where $\gamma - \alpha \equiv l \pmod{m}$, $\gamma \in \{1, \dots, m\}$.

Let us consider the prefix of $r(v)$ of length $m + l$. It exists due to Claim 1 and is equal to

$$r(axa) b_\gamma \dots b_{m-1} c_{n'+1} b_1 \dots b_{\gamma-1}.$$

Substituting the unknown symbol $c_{n'+1}$ by a hole \diamond , we obtain a partial word

$$b_\alpha \dots b_{m-1} (b_{l'} b)^{n'} \diamond b_1 \dots b_{\gamma-1},$$

which is l -periodic as well as $r(v)$. But at the same time, it is m -periodic; thus, due to Theorem 3 it is p -periodic, where $p = \gcd(l, m)$. Consequently, $b = r(\varphi(0))$ is also p -periodic: $b = \overline{(b_1 \dots b_p)^{m'-1} b_1 \dots b_{p-1}}$, where $m' = m/p$. Let us return to $\varphi(0)$ and denote $g_1 = 0$, $g_k = \overline{b_1 + b_2 + \dots + b_{k-1}}$ for $k \in \{2, \dots, p\}$, and $c = \overline{b_1 + b_2 + \dots + b_p}$; due to (2), we see that $\varphi(0)$ is of the form

$$\varphi(0) = g_1 \dots g_p \overline{g_1 + c} \dots \overline{g_p + c} \dots \overline{g_1 + (m'-1)c} \dots \overline{g_p + (m'-1)c}. \quad (3)$$

Here $g_1 = 0$ since φ has a fixed point, and $m' = m/p$. The words of the form $\overline{g_1 + ic} \dots \overline{g_p + ic}$, where $i \in \{0, \dots, m'-1\}$, will be called *subblocks*. Note that for all $k \in \{1, \dots, p\}$, a subblock is uniquely determined by its k th symbol, and that w consists of consecutive subblocks.

Let w_i denote the i th symbol of the fixed point w of φ , i. e., let $w = w_1 \dots w_n \dots$, where $w_i \in \Sigma$. Consider the arithmetical subsequence

$$w' = w_1 w_{p+1} w_{2p+1} \dots w_{np+1} \dots$$

Claim 3 *The word w' is the fixed point of a morphism $\varphi' \in S(m)$.*

Proof. Let us define the symmetric morphism φ' by

$$\varphi'(0) = g_1 \overline{g_1 + c} \dots \overline{g_1 + (m'-1)c} \overline{g_2 + c} \dots \overline{g_2 + (m'-1)c} \dots \overline{g_p + c} \dots \overline{g_p + (m'-1)c}.$$

Since $\varphi'(0)$ is obtained from $\varphi(0)$ by permuting symbols, and all the symbols of $\varphi(0)$ are distinct, so are the symbols of $\varphi'(0)$. Since $g_1 = 0$, and φ' is symmetric by definition, $\varphi' \in S(m)$. So we must prove only that w' is its fixed point, i. e., that

$$\forall k \geq 0 \forall i \in \{1, \dots, m\} \quad w'_{km+i} \text{ is equal to the } i\text{th symbol of } \varphi'(w'_{k+1}), \quad (4)$$

where w'_k is the k th symbol of $w' = w'_1 w'_2 \dots w'_n \dots$

Clearly, each $i \in \{1, \dots, m\}$ can be uniquely represented as $i = jm' + \delta$, where $j \in \{0, \dots, p-1\}$ and $\delta \in \{1, \dots, m'\}$. Since by definition of w' for all v we have $w'_v = w_{p(v-1)+1}$, for any $k \geq 0$

$$w'_{km+i} = w'_{km+jm'+\delta} = w_{p(km+jm'+\delta-1)+1} = w_{(pk+j)m+p(\delta-1)+1}.$$

By Equality (1), $w_{(pk+j)m+p(\delta-1)+1}$ is equal to the $(p(\delta-1)+1)$ th symbol of $\varphi(w_{pk+j+1})$, that is, to $\overline{(\delta-1)c + w_{pk+j+1}}$ (recall that $g_1 = 0$). In its turn, w_{pk+j+1} is the $(j+1)$ th symbol of the subblock starting with $w_{pk+1} = w'_{k+1}$. It is equal to $\overline{w'_{k+1} + g_{j+1}}$, and thus, $w'_{km+i} = \overline{w'_{k+1} + (\delta-1)c + g_{j+1}}$. By the definition of φ' , it is equal to the symbol numbered $jm' + \delta = i$ of $\varphi'(w'_{k+1})$. We have proved (4) and Claim 3. \square

Claim 4 *The word w' contains an overlap of length $2l' + 1$, where $l' = l/p$.*

Proof. Let our occurrence of the overlap v to w start with the k th symbol of a subblock, i. e., let $\alpha \equiv k \pmod{p}$, where $k \in \{1, \dots, p\}$. It means that $v = w_{jp+k}w_{jp+k+1} \dots w_{(j+2l')p+k}$ for some $j \geq 0$; since v is an overlap, $w_{(j+v)p+1} = w_{(j+v+l')p+1}$ for all $v \in \{1, \dots, l'\}$. But we have also $w_{jp+k} = w_{(j+l')p+k}$, and since a subblock is uniquely determined by its k th symbol, $w_{jp+1} = w_{(j+l')p+1}$. So, the word $w_{jp+1}w_{(j+1)p+1} \dots w_{(j+2l')p+1}$ is l' -periodic, and it is the needed overlap in w' . \square

As it follows from Claims 3 and 4, we have found a fixed point of a morphism of $S(m)$ containing an overlap of length $l' = l/p$. But if $p > 1$, this contradicts to the minimality of our counter-example. On the other hand, if $p = 1$, then it follows from (3) that

$$\varphi(0) = 0\bar{c} \overline{2c} \dots \overline{(m-1)c}.$$

But a fixed point of such a morphism cannot be a counter-example according to Lemma 1. A contradiction. Theorem 2 is proved. \square

Acknowledgements

I thank J. Shallit for introducing me to the problem, J.-P. Allouche for useful comments, and the referee for correcting the proof of Claim 1.

References

- [1] J.-P. Allouche and J. Shallit. The ubiquitous Prouhet-Thue-Morse sequence. In C. Ding, T. Haleseth and H. Niederreiter, eds., *Sequences and Their Applications, Proceedings of SETA'98*, Springer-Verlag, 1999, 1–16.
- [2] J.-P. Allouche and J. Shallit. Sums of digits, overlaps, and palindromes. *Discr. Math. Theoret. Comput. Sci.* **4** (2000), 1–10.
- [3] J. Berstel. Axel Thue's work on repetitions in words. In P. Leroux and C. Reutenauer, eds., *Séries Formelles et Combinatoire Algébrique*, no. 11 in Publications du LACIM, Université du Québec à Montréal, 1992, 65–80.
- [4] J. Berstel and L. Boasson. Partial words and a theorem of Fine and Wilf. *Theoret. Comput. Sci.* **218** (1999), no. 1, 135–141.

- [5] J. Cassaigne. Unavoidable patterns. In M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, to appear.
- [6] C. Choffrut and J. Karhumäki. Combinatorics on words. In G. Rozenberg and A. Salomaa, eds., *Handbook of Formal Languages*, v. 1, chapter 6. Springer-Verlag, 1997.
- [7] F. Dejean. Sur un théorème de Thue. *J. Combin. Theory. Ser. A*. **13** (1972), 25–36.
- [8] E. D. Fife. Binary sequences which contain no *BBb*. *Trans. Amer. Math. Soc.* **261** (1980), 115–136.
- [9] N. J. Fine and H. S. Wilf. Uniqueness theorem for periodic functions. *Proc. Amer. Math. Soc.* **16** (1965), 109–114.
- [10] A. Frid. Applying a uniform marked morphism to a word. *Discr. Math. Theoret. Comput. Sci.* **3** (1999), 125–140.
- [11] V. Keränen. Abelian squares are avoidable on 4 letters. In *Automata, Languages and Programming, Proceedings of ICALP'92 Lecture Notes in Comput. Sci.*; V. 700, Berlin: Springer, 1992, 41–52.
- [12] G. Lallement. *Semigroups and Combinatorial Applications*, Wiley, 1979.
- [13] P. Séébold. Sequences generated by infinitely iterated morphisms. *Discrete Appl. Math.* **11** (1985), 255–264.
- [14] J. Shallit, private communication.
- [15] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912), 1–67. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell, ed., Universitetsforlaget, Oslo, 1977, 413–478.