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# Paths of specified length in a random $k$ -partite graph

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Fix positive integers  $k$  and  $l$ . Consider a random  $k$ -partite graph on  $n$  vertices obtained by partitioning the vertex set into  $V_i$ , ( $i = 1, \dots, k$ ) each having size  $\Omega(n)$  and choosing each possible edge with probability  $p$ . Consider any vertex  $x$  in any  $V_i$  and any vertex  $y$ . We show that the expected number of simple paths of even length  $l$  between  $x$  and  $y$  differ significantly depending on whether  $y$  belongs to the same  $V_i$  (as  $x$  does) or not. A similar phenomenon occurs when  $l$  is odd. This result holds even when  $k, l$  vary slowly with  $n$ . This fact has implications to coloring random graphs. The proof is based on establishing bijections between sets of paths.

**Keywords:** random graphs, paths, bijections

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## 1 Motivation

This problem arose in the analysis of algorithms for coloring random  $k$ -colorable graphs [2, 3]. Consider a random graph drawn as explained in the abstract. To separate a color class, we fix a vertex  $x$  in the largest (or smallest)  $V_i$  and compute the number of  $l$ -paths (paths of length  $l$ ),  $n(x, y, l)$ , between  $x$  and an arbitrary vertex  $y$ . Depending on whether  $y$  belongs to the same class as  $x$  belongs to, the expectation of this quantity differs significantly. If we can show that  $n(x, y, l)$  is close to its expected value almost surely, this gives us a way of separating the class containing  $x$ . Repeating this  $k - 2$  times, one gets a  $k$ -coloring. The expectation of  $n(x, y, l)$  is  $N(x, y, l)p^l$ , where  $N(x, y, l)$  is the total number of  $l$ -paths in the complete  $k$ -partite graph formed by  $V_i$ s. The result stated in the abstract shows that the expectations differ significantly as required.

We do not discuss the algorithmic issues here since they have been outlined in [2]. We only prove the results stated in the abstract using only counting arguments. Even though the results are obviously true for bipartite graphs, for  $k \geq 3$ , it is not so straightforward. We believe the arguments used here would be of interest to know. The basic idea is to partition (for each pair of start-end vertices) the corresponding set of  $l$ -paths into groups (based on the color classes of intermediate vertices). Then, for two different

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pairs, we establish an (almost) bijection between the corresponding groups. For any such mapped pair of groups, we also establish an (almost) bijection between the  $l$ -paths in them. This establishes the required statement.

## 2 Paths of specified length

**Definition 2.1** By an  $l$ -path between two vertices  $x$  and  $y$ , we mean a simple path of length  $l$  between  $x$  and  $y$ . A simple path is one in which no vertex appears more than once. An  $l$ -path is represented as a  $(l+1)$ -tuple  $\langle x, v_1, \dots, v_{l-1}, y \rangle$  of vertices such that successive vertices in this sequence belong to different partite sets  $V_i$ .

**Notations :**  $G$  is a complete  $k$ -partite graph on the partite sets  $V_1, \dots, V_k$  with each  $|V_i| \geq n/C$  for some constant  $C \geq k$ . For each  $i$ ,  $n_i$  denotes the size  $|V_i|$ . For each  $i$ , let  $W_i = V_i \cup \dots \cup V_k$ . For all  $i$  ( $1 \leq i \leq k-1$ ), for all  $x \in V_i$ , for all  $y \in W_i$  such that  $y \neq x$ , let  $N(x, y, l, i)$  denote the number of  $l$ -paths between  $x$  and  $y$  involving only vertices from  $W_i$ . Given a tuple  $\sigma$  with integral component values and an integer  $j$ ,  $c(\sigma, j)$  denotes the number of times  $j$  appears in  $\sigma$ .

We obtain the following results.

**Theorem 2.1** Assume that  $n_1 \leq \dots \leq n_k$ . Let  $l$  be any fixed even integer  $\geq 2$ . For all  $i$ ,  $1 \leq i \leq k-1$ , for all  $x, y \in V_i$ , for all  $z \in W_i - V_i$ , we have

$$\begin{aligned} N(x, z, l, i) &= \Theta(n^{l-1}) \text{ if } i \leq k-2 \\ N(x, y, l, i), N(x, y, l, i) - N(x, z, l, i) &= \Theta(n^{l-1}) \end{aligned}$$

**Proof:** Consider any  $i$  ( $i = 1, \dots, k-1$ ) and any  $x, y \in V_i$  and  $z \in V_r, r > i$  and fix these parameters. We use the factorial functions defined as follows :  $(n)_0 = 1$ .  $(n)_l = n(n-1)\dots(n-l+1)$ ,  $l \geq 1$ . Let  $P(x, y)$  denote the set of all  $l$ -paths between  $x$  and  $y$  involving only vertices from the  $k-i+1$  partite sets  $V_j$  ( $i \leq j \leq k$ ).  $P(x, z)$  is defined similarly. That is,

$$P(x, y) = \{ \langle x, v_1, \dots, v_{l-1}, y \rangle \mid \text{the sequence is an } l\text{-path between } x \text{ and } y \}.$$

$$P(x, z) = \{ \langle x, v_1, \dots, v_{l-1}, z \rangle \mid \text{the sequence is an } l\text{-path between } x \text{ and } z \}.$$

Clearly, we have  $|P(x, y)| = O(n^{l-1})$  and  $|P(x, z)| = O(n^{l-1})$ . Also if  $i = k-1$ , then there are only two partite sets, namely,  $V_{k-1}$  and  $V_k$  and hence  $P(x, z) = \emptyset$  and  $N(x, z, l, i) = 0$ . Define

$$B_i^s = \{ \langle \sigma_1, \dots, \sigma_{l-1} \rangle \mid \sigma_1 \neq i, \sigma_{l-1} \neq i, i \leq \sigma_j \leq k, \sigma_j \neq \sigma_{j+1} \forall j \}$$

$$B_i^d = \{ \langle \sigma_1, \dots, \sigma_{l-1} \rangle \mid \sigma_1 \neq i, \sigma_{l-1} \neq r, i \leq \sigma_j \leq k, \sigma_j \neq \sigma_{j+1} \forall j \}$$

In the above, the superscript  $s$  (or  $d$ ) is a short notation for the word ‘‘same’’ (or ‘‘different’’). We have  $|B_i^s|, |B_i^d| \leq k^{l-1}$ .

Now  $f : P(x, y) \rightarrow B_i^s$  is a mapping which identifies each  $l$ -path  $\langle x, v_1, \dots, v_{l-1}, y \rangle$  with the unique  $(l-1)$ -tuple  $\langle \sigma_1, \dots, \sigma_{l-1} \rangle$  in  $B_i^s$  where if  $v_m \in V_j$  then  $\sigma_m = j$ . Similarly, we can define a mapping  $g : P(x, z) \rightarrow B_i^d$  which identifies each  $l$ -path in  $P(x, z)$  with a unique  $(l-1)$ -tuple in  $B_i^d$ . We use the elements of  $B_i^s$  ( or  $B_i^d$  ) to partition the set  $P(x, y)$  ( or  $P(x, z)$  ) as follows.

$$P(x, y) = \bigcup_{\sigma \in B_i^s} P_\sigma \text{ where } P_\sigma = \{ \tau \in P(x, y) \mid f(\tau) = \sigma \}.$$

$$P(x, z) = \bigcup_{\sigma \in B_l^d} P_\sigma \text{ where } P_\sigma = \{\tau \in P(x, z) \mid g(\tau) = \sigma\}.$$

Now, for each  $\sigma \in B_l^s \cup B_l^d$ ,  $|P_\sigma| = (\prod_{i \leq j \leq k} (n_j)_{c(\sigma, j)}) = (\prod_{i \leq j \leq k} (n_j)^{c(\sigma, j)}) \cdot [1 - o(1)]$ . As a result, for each  $\sigma \in B_l^s \cup B_l^d$ ,  $|P_\sigma| = \Theta(n^{l-1})$ . The  $[1 - o(1)]$  factor arises not only because of factorials, but also because  $x, y$  and  $z$  have to be excluded from consideration.

Also  $B_l^s$  is non-empty and it contains at least one element, namely, the tuple  $\langle r, i, r, i, \dots, r \rangle$ . Hence  $N(x, y, l, i) = |P(x, y)| = \Theta(n^{l-1})$ . Also, if  $i \leq k - 2$ , then there are at least 3 partite sets to be considered and hence  $B_l^d$  is non-empty. Hence  $N(x, z, l, i) = |P(x, z)| = \Theta(n^{l-1})$  if  $i \leq k - 2$ .

We need to prove that  $|P(x, y)| - |P(x, z)| = \Theta(n^{l-1})$ . In order to prove this, it is enough to prove that the following two assertions are true.

1.  $|B_l^s| \geq |B_l^d| + 1$  and
2. There exists a *one-to-one* mapping  $h : B_l^d \rightarrow B_l^s$  such that for each  $\tau \in B_l^d$ , we have  $|P_{h(\tau)}| \geq |P_\tau| [1 - o(1)]$ .

We prove that the two assertions are true as follows. Now, partition  $B_l^s, B_l^d$  into

$$B_l^s = B_{l,1}^s \cup \dots \cup B_{l,l-1}^s \cup B_{l,l}^s$$

$$B_l^d = B_{l,2}^d \cup \dots \cup B_{l,l-1}^d \cup B_{l,l}^d$$

where

$$B_{l,l}^s = \{\sigma \in B_l^s \mid \sigma_{l-1} \neq i, \sigma_{l-1} \neq r\}.$$

$$B_{l,l}^d = \{\sigma \in B_l^d \mid \sigma_{l-1} \neq i, \sigma_{l-1} \neq r\}.$$

$$B_{l,j}^s = \{\sigma \in B_l^s \mid \sigma_{j-1} \neq i, \sigma_{j-1} \neq r, \sigma_m = i, r \text{ for } m \geq j\}, \text{ for } 2 \leq j \leq l-1.$$

$$B_{l,j}^d = \{\sigma \in B_l^d \mid \sigma_{j-1} \neq i, \sigma_{j-1} \neq r, \sigma_m = i, r \text{ for } m \geq j\}, \text{ for } 2 \leq j \leq l-1.$$

$$B_{l,1}^s = \{\langle r, i, r, i, \dots, r \rangle\}$$

Now  $B_{l,1}^d$  cannot be defined similarly since  $l$  is even. It is easy to see that the definitions form a well-defined partition of  $B_l^s$  and  $B_l^d$ . In other words, for each  $\sigma \in B_l^s$ , there exists a unique value of  $j$  between 1 and  $l$  such that  $\sigma \in B_{l,j}^s$ . Similarly, for each  $\tau \in B_l^d$ , there exists a unique value of  $j$  between 2 and  $l$  such that  $\tau \in B_{l,j}^d$ .

Now we claim that for all  $j$  such that  $2 \leq j \leq l$ ,  $|B_{l,j}^s| = |B_{l,j}^d|$ . For  $j = l$ , this follows from  $B_{l,l}^s = B_{l,l}^d$ . For  $j < l$ , consider the mapping  $h_j : B_{l,j}^d \rightarrow B_{l,j}^s$  defined as follows. Let  $\tau \in B_{l,j}^d$  be any tuple. Then,  $h_j(\tau) = \sigma$  where  $\sigma$  is defined as

- For all  $m$  ( $1 \leq m \leq j-1$ ),  $\sigma_m = \tau_m$ .
- For all  $m$  such that  $j \leq m \leq l-1$ ,  $\sigma_m = i$  if  $\tau_m = r$  and  $\sigma_m = r$  if  $\tau_m = i$ .

Clearly  $\sigma \in B_{l,j}^s$ . Also it can be verified that  $h_j$  is a one-to-one and onto mapping. Since  $B_{l,j}^s$  and  $B_{l,j}^d$  are finite sets, it follows that  $|B_{l,j}^s| = |B_{l,j}^d|$ .

Thus, we have  $|B_l^s| \geq |B_l^d| + 1$  and the first assertion is true.

To prove the second assertion, define the mapping  $h : B_l^d \rightarrow B_l^s$  to be as follows. For each  $\tau \in B_l^d$ , define  $h(\tau) = h_j(\tau)$  where  $j$  is such that  $\tau \in B_{l,j}^d$ . Clearly,  $h$  is a one-to-one mapping since each  $h_j$  is a one-to-one mapping.

We prove that for each  $\tau \in B_l^d$ , we have  $|P_{h(\tau)}| \geq |P_\tau|[1 - o(1)]$ . Let  $\tau \in B_l^d$  be any tuple and let  $\sigma$  denote the tuple  $h(\tau)$ . We know  $\tau \in B_{l,j}^d$  for some  $j$ ,  $2 \leq j \leq l$ .

If  $j = l$ , then we have  $\sigma = \tau$  and hence  $|P_\sigma| \geq |P_\tau|[1 - o(1)]$ .

If  $j = l-2, l-4, \dots, 2$ , then clearly,  $c(\tau, m) = c(\sigma, m)$  for all values of  $m$  ( $i \leq m \leq k$ ) and hence  $|P_\sigma| \geq |P_\tau|[1 - o(1)]$ .

If  $j = l-1, l-3, \dots, 3$ , then clearly,  $c(\tau, m) = c(\sigma, m)$  for all values of  $m$  ( $i \leq m \leq k$ ) such that  $m \neq i$ ,  $m \neq r$ . Also,  $c(\sigma, r) = c(\tau, r) + 1$  and  $c(\tau, i) = c(\sigma, i) + 1$ . Since  $n_i \leq n_r$  ( $r > i$ ) by assumption, we have  $|P_\sigma| \geq |P_\tau|[1 - o(1)]$ .

Thus, we have

$$\begin{aligned}
N(x, y, l, i) - N(x, z, l, i) &= |P(x, y)| - |P(x, z)| \\
&= \left| \bigcup_{\sigma \in B_l^s} P_\sigma \right| - \left| \bigcup_{\tau \in B_l^d} P_\tau \right| \\
&= \sum_{j=l-1, \dots, 3} \left( \sum_{\sigma \in B_{l,j}^s} |P_\sigma| \right) + \sum_{j=l, l-2, \dots, 2} \left( \sum_{\sigma \in B_{l,j}^s} |P_\sigma| \right) + \sum_{\sigma = \langle r, i, \dots, r \rangle} |P_\sigma| \\
&\quad - \sum_{j=l-1, \dots, 3} \left( \sum_{\tau \in B_{l,j}^d} |P_\tau| \right) - \sum_{j=l, l-2, \dots, 2} \left( \sum_{\tau \in B_{l,j}^d} |P_\tau| \right) \\
&\geq |P_\sigma| - o(|P_\sigma|) \text{ where } \sigma = \langle r, i, \dots, r \rangle.
\end{aligned} \tag{1}$$

Thus,

$$\begin{aligned}
N(x, y, l, i) - N(x, z, l, i) &= \Theta((n_r)_{l/2} (n_i)_{l/2-1}) \\
&= \Theta(n^{l-1})
\end{aligned}$$

Hence,

$$\begin{aligned}
N(x, z, l, i) &= \Theta(n^{l-1}) \text{ if } i \leq k-2 \\
N(x, y, l, i), N(x, y, l, i) - N(x, z, l, i) &= \Theta(n^{l-1}).
\end{aligned}$$

This completes the proof of the theorem. ■

Using similar arguments, we can prove the following theorem also.

**Theorem 2.2** Assume that  $n_1 \geq \dots \geq n_k$ . Let  $l$  be any fixed odd integer  $\geq 3$ . For all  $i$  ( $1 \leq i \leq k-1$ ), for all  $x, y \in V_i$ , for all  $z \in W_i - V_i$ , we have

$$\begin{aligned} N(x, y, l, i) &= \Theta(n^{l-1}) \text{ if } i \leq k-2 \\ N(x, z, l, i), N(x, z, l, i) - N(x, y, l, i) &= \Theta(n^{l-1}) \end{aligned}$$

### 3 Conclusions

1. The main result of the paper is that the number of  $l$ -paths joining a vertex  $x$  (in the largest or smallest  $V_i$  depending on the parity of  $l$ ) and a vertex  $y$  differs significantly depending on where  $y$  comes from. A close look at the proof (particularly, derivation of (1)) shows that this holds even if we allow  $k, l$  and  $C$  to vary with  $n$ , provided  $lk^l C^l = o(n)$ .

2. It would be interesting to extend these results to structures other than simple paths. Such results can be applied to the design and analysis of efficient algorithms for random graphs (see [1] for a survey).

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