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Discrete Mathematics and Theoretical Computer Science **4**, 2001, 123–132

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received Sep 12, 2000, revised Jun 11, 2001, accepted Jun 15, 2001.

In this paper we apply Pólya's Theorem to the problem of enumerating Cayley graphs on permutation groups up to isomorphisms induced by conjugacy in the symmetric group. We report the results of a search of all three-regular Cayley graphs on permutation groups of degree at most nine for small diameter graphs. We explore several methods of constructing covering graphs of these Cayley graphs. Examples of large graphs with small diameter are obtained.

Keywords: Cayley graph, cubic graph, diameter, Pólya's Theorem, permutation group.

1 Introduction

The (Δ, D) problem asks for the largest value n such that a graph on n vertices exists with diameter D and maximum vertex degree Δ. The Moore bound for the diameter D of a graph with n vertices and maximum vertex degree $\Delta \geq 3$ gives $n \leq 1+\Delta+(\Delta-1)\Delta+(\Delta-1)^2\Delta+\cdots+(\Delta-1)^{D-1}\Delta$. Very few graphs satisfy equality in the Moore bound. The evaluation of $n(\Delta, D)$, the largest integer such that a graph on $n(\Delta, D)$ vertices with maximum vertex degree Δ and diameter D exists, appears to be an intractable problem in general. Even the more modest goal of proving the existence of a family of graphs satisfying $D \leq \log_{\Delta-1}(n)+O(1)$ seems difficult in the case $\Delta = 3$. For random regular graphs $D = \log_{\Delta-1}(n\log n)+O(1)$, and similar results have been obtained for constructions with random components. For details see [BC66] and [BV82]. Jerrum and Skyum [JS84] have obtained the best non-random constructive results known for an infinite family of cubic graphs. Their constructions yield graphs which satisfy approximately $D = 1.47\log_2 n$. Kantor [Kan92] has shown the existence of cubic Cayley graphs with diameter $O(\log n)$, however the constant c such that $D \leq c\log_2 n$ implicit in his estimate is very large. In addition to the random graph results and the constructions of infinite families, there is much interest in the construction of specific large graphs with small diameters. For a table of the largest known graphs for $\Delta \leq 10$ and $D \leq 10$, see [Exc00]. These graphs establish lower bounds for $n(\Delta, D)$.

We examine all three-regular Cayley graphs on permutation groups of degree at most nine. While the graphs obtained with $D \leq 10$ are not as large as those in the table, these Cayley graphs form the building block for constructing larger graphs which can easily be analyzed. By forming covering graphs of the Cayley graphs in various ways we obtain graphs of $D \leq 21$ which give good lower bounds for $n(3, D)$ for $11 \leq D \leq 21$. Moreover several examples of bipartite graphs with low degree and diameter are obtained.

We consider only undirected graphs, so the generator set A of our Cayley graphs will have three elements, and $A = A^{-1}$ where $A^{-1} = \{\alpha^{-1} \mid \alpha \in A\}$. We refer to such sets as Cayley sets. For three-regular graphs the Cayley sets will have the form $A = \{\sigma, \sigma^{-1}, \tau\}$ for some σ and τ where τ has order two and σ has order at least three, or $A = \{\tau_1, \tau_2, \tau_3\}$ where τ_i has order two for each i. If $\alpha \in S_n$ let $\alpha^\beta = \beta^{-1}\alpha\beta$ be the conjugate of α by β, and if $A \subset S_n$ let $A^\beta = \{\alpha^\beta \,|\, \alpha \in A\}$. If A and B are Cayley sets in S_n and $A^\beta = B$ for some $\beta \in S_n$, then the Cayley graphs generated by A and B are isomorphic. Isomorphism of Cayley graphs from non-conjugate Cayley sets is difficult to determine from the Cayley sets alone. Thus in searching three-regular Cayley graphs we first address the problem of enumerating the possible Cayley sets up to conjugation. Although we are primarily interested in the sets generating cubic graphs, we tackle this problem in the more general setting.

2 Enumeration of Cayley Sets up to Conjugacy

Any Cayley graph of vertex degree m has a generator set of the form

$$S = \{\sigma_1, \ldots, \sigma_k, \sigma_1^{-1}, \ldots, \sigma_k^{-1}, \tau_1, \ldots, \tau_l\}$$

where each σ_i has order at least three, each τ_j is an involution, and $2k + l = m$. We will call such a set a Cayley set of type (k,l). Let X_n be the set of equivalence classes in S_n under the equivalence relation $x \cong y$ if and only if $x = y$ or $x = y^{-1}$. We will denote these equivalence classes by any appropriate representative. There is a one-to-one correspondence between Cayley sets of type (k,l) and the subsets of X_n the form $\{\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l\}$. We will call these subsets of X_n Cayley sets of type (k,l) also.

The action of S_n on S_n by conjugation induces an action of S_n on X_n. Enumerating the Cayley graphs of vertex degree m on S_n up to isomorphisms induced by conjugation in S_n is equivalent to enumerating the sets of type (k,l) in X_n with $2k + l = m$ up to the action of S_n by conjugation. To accomplish this we compute the cycle index of the action of S_n by conjugation on X_n. We first recall some notation and basic facts. We refer to [PR73] for background on cycle indices and Pólya's Theorem.

If G is a permutation group acting on a set X with n elements and $g \in G$ then the we denote the cycle type of g by the monomial $m(g) = \prod_1^n x_i^{c_i}$, where g has c_i cycles of length i in its disjoint cycle decomposition. The cycle index of G is defined as $Z(G) = \frac{1}{|G|}\sum_{g \in G} m(g)$ and the cycle sum by $ZS(G) = \sum_{g \in G} m(g)$. We use $Z(G; w_1, w_2 \ldots, w_n)$ to denote $Z(G)$ with w_i substituted for x_i for each i. We denote $Z(G; x_r, x_{2r} \ldots, x_{nr})$ by $Z_r(G)$.

Let P_n be the set of partitions of n. We also denote the elements of P_n by monomials $\prod_1^n x_i^{c_i}$ where $\sum_1^n ic_i = n$. Let $h(a)$ be the number of permutations on S_n with cycle type a.

$$h\Big(\prod_{i=1}^{n} x_i^{c_i}\Big) = \frac{n!}{\prod_{i=1}^{n} c_i!\, i^{c_i}} \quad \text{and} \quad Z(S_n) = \frac{1}{n!}\sum_{a \in P_n} h(a)\, a. \tag{1}$$

Let C_n and D_n be the cyclic and dihedral groups of degree n. Then regarding C_n and D_n as permutation groups on $\{1, 2, \ldots, n\}$ in the usual manner we have

$$Z(C_n) = \frac{1}{n}\sum_{d|n} \phi(d)\, x_d^{n/d} \tag{2}$$

and

$$Z(D_n) = \frac{1}{2}Z(C_n) + \begin{cases} \frac{1}{2}x_1 x_2^{(n-1)/2} & \text{if n is odd} \\ \frac{1}{4}(x_2^{n/2} + x_1^2 x_2^{(n-2)/2}) & \text{if n is even} \end{cases} \tag{3}$$

where ϕ denotes the Euler Phi function.

For each permutation $\alpha \in S_n$ we will need the cycle index of the centralizer $C(\alpha) = \{\beta \in S_n \,|\, \alpha^\beta = \alpha\}$ and the cycle index of the pseudo-centralizer $PC(\alpha) = C(\alpha) \cup F(\alpha)$, where $F(\alpha) = \{\beta \in S_n \,|\, \alpha^\beta = \alpha^{-1}\}$. Since these cycle indices will depend only on the cycle type $m(\alpha)$ we will sometimes use the notations $Z(C(m(\alpha)))$ and $Z(PC(m(\alpha)))$. If α is a permutation of cycle type x_k^j in S_{kj} then $C(\alpha) \cong S_k[C_j]$, the wreath product of S_k with C_j. Pólya [Pól37] showed that cycle index of the wreath product $A[B]$ is given by $Z(A[B]) = Z(A; Z_1(B), Z_2(B), \ldots)$ and thus

$$Z(C(\alpha)) = Z(C(x_j^k)) = Z(S_k; Z_1(C_j), Z_2(C_j), \ldots). \quad (4)$$

The centralizer of a permutation α of cycle type $\prod_1^n x_i^{c_i}$ is isomorphic to the product of the centralizers of permutations α_i of type $x_i^{c_i}$ in S_{ic_i}, hence

$$Z(C(\prod_{i=1}^n x_i^{c_i})) = \prod_{i=1}^n Z(S_{c_i}; Z_1(C_i), Z_2(C_i), \ldots). \quad (5)$$

To obtain the cycle index of the pseudo-centralizer $PC(\alpha)$, we need to consider separately the set $F(\alpha)$. Let $F_n = D_n - C_n$ for $n \geq 3$ and $F_n = C_n$ for $n \leq 2$. We identify F_n with the set of permutations in S_n mapping an n-cycle to its inverse under conjugation. Although F_n is not a group for $n \geq 3$ we may consider its cycle index, and $Z(F_n) = Z(C_n)$ for $n \leq 2$ and $Z(F_n) = 2Z(D_n) - Z(C_n)$ for $n \geq 3$. By equations 2 and 3 we obtain explicitly that

$$Z(F_n) = \begin{cases} x_1 x_2^{(n-1)/2} & \text{if n is odd} \\ \frac{1}{2}(x_2^{n/2} + x_1^2 x_2^{(n-2)/2}) & \text{if n is even.} \end{cases} \quad (6)$$

Now consider $F(\alpha)$ where $\alpha \in S_{kj}$ has cycle type x_j^k. Denote the cycle-index of $F(\alpha)$ by $Z(F(x_j^k))$ as it depends only on the cycle type. Note that if $g \in F(\alpha)$ then g determines a permutation $\gamma(g) \in S_k$ of the k j-cycles $w_1, w_2, \ldots, w_k$ of α. If $(w_{t_1}, w_{t_2}, \ldots, w_{t_r})$ is in an r-cycle of $\gamma(g)$ then g^r determines a permutation of the elements of each w_{t_s}. Thus g^r determines an element of F_j (r odd) or C_j (r even) when we consider its action on the elements of w_{t_1}. Each l cycle of F_j or C_j determined by g^r determines an rl cycle of g. Moreover there are j^{r-1} possible permutations of the rj elements of the r j-cycles $w_{t_1}, w_{t_2}, \ldots, w_{t_r}$ determining the same r-cycle $(w_{t_1}, w_{t_2}, \ldots, w_{t_r})$ and β in F_j or C_j. Therefore the contribution of each x_r in the cycle-sum $ZS(S_k)$ to the cycle sum $ZS(F(x_k^j))$ is given by $j^{r-1}ZS_r(F_j) = j^r Z_r(F_j)$ (r odd) or $j^{r-1}ZS_r(C_j) = j^r Z_r(C_j)$ (r even). Thus $ZS(F(x_k^j)) = ZS(S_k; jZ_1(F_j), j^2 Z_2(C_j), j^3 Z_2(F_j), j^4 Z_4(C_j), \ldots)$, and dividing both sides by $j^k k!$ yields

$$Z(F(x_k^j)) = Z(S_k; Z_1(F_j), Z_2(C_j), Z_3(F_j), Z_4(C_j), \ldots). \quad (7)$$

Now if α has cycle type $\prod_1^n x_i^{c_i}$, there is an isomorphism between $\prod_1^n PC(\alpha_i)$ and a subgroup of S_n mapping the product $\prod_1^n F(\alpha_i)$ to $F(\alpha)$, where $\alpha_i \in S_{ic_i}$ has type $x_i^{c_i}$. This gives $Z(F(\prod_1^n x_i^{c_i})) = \prod_1^n Z(F(x_i^{c_i}))$, so by 5 and 7 we obtain the cycle index of the pseudo-centralizer:

$$Z(PC(\prod_{i=1}^n x_i^{c_i})) = \frac{1}{2}\prod_{i=1}^n Z(S_{c_i}; Z_1(C_i), Z_2(C_i), \ldots) + \frac{1}{2}\prod_{i=1}^n Z(S_{c_i}; Z_1(F_i), Z_2(C_i), Z_3(F_i), Z_4(C_i), \ldots). \quad (8)$$

If G is any subgroup of S_n and b is a cycle type, we use $G.b$ to denote G regarded as a permutation group acting on the elements of X_n of cycle type b by conjugation. We will identify a cycle type b with the set of permutations in S_n having that cycle type. Recall from equation 1 that $h(b)$ is the number of permutations in S_n with cycle type b. We use $o(\beta)$ for the order of a permutation $\beta \in S_n$, and as this depends only on cycle type we use $o(b)$ for the order of any element of cycle type b. We use $[t]Q$ to denote the coefficient of a monomial t in a polynomial Q. Also if α has cycle type $a = \prod_1^n x_i^{c_i}$ then the cycle type of α^d is given by $\prod_1^n x_{i/(i,d)}^{c_i(i,d)}$, where (i,d) is the greatest common divisor of i and d. We may denote this cycle type as a^d, as it depends only on a and d. We denote the Möbius function by μ. With these preliminaries, we are now in a position to calculate the cycle index of the action of S_n by conjugation on X_n.

Enumeration Theorem:

$$Z(G.b) = \frac{1}{|G|} \sum_{g \in G} \prod_{k|o(g)} x_k^{c(m(g),k,b)}$$

where

$$c(a,k,b) = \frac{1}{k} \sum_{d|k} \mu(k/d)\, w(a^d,b), \text{ and } w(a,b) = \frac{n!}{h(a)} [a]Z(PC(b)).$$

Proof:

If $\alpha_1 \in S_n$ and b is a cycle-type, let $f(\alpha_1,b) = |\{\beta \in b | \beta^{\alpha_1} = \beta \text{ or } \beta^{-1}\}|$. Now $f(\alpha_1,b)$ depends only on the cycle type a of α_1, thus for any $\beta_1 \in b$

$$\begin{aligned} h(a)\, f(\alpha_1,b) &= |\{(\alpha,\beta) \in a \times b | \beta^{\alpha} = \beta \text{ or } \beta^{-1}\}| \\ &= |\{(\alpha,\beta) \in a \times b | \alpha \in PC(\beta)\}| \\ &= h(b)\, |a \cap PC(\beta_1)| \\ &= h(b)\, [a] ZS(PC(\beta_1)). \end{aligned} \tag{9}$$

Now let $w(\alpha,b)$ denote the number of elements of X_n of cycle type b fixed under conjugation by α. If $o(b) \le 2$ then $w(\alpha,b) = f(\alpha,b)$ and $|PC(\beta_1)| = |C(\beta_1)| = n!/h(b)$ for any β_1 of type b. If $o(b) \ge 3$ then $w(\alpha,b) = (1/2)\, f(\alpha,b)$ and $|PC(\beta_1)| = 2|C(\beta_1)| = 2(n!)/h(b)$ for any β_1 of cycle type b. In either case substitution into equation 9 yields

$$w(\alpha,b) = \frac{n!}{h(a)} [a]Z(PC(b)). \tag{10}$$

Note that $w(\alpha,b)$ depends only on the cycle type a of α, so we define $w(a,b)$ by equation 10 for cycle-types a and b. If $c(a,k,b)$ is the number of k-cycles of the action of any α of cycle type a by conjugation on the elements of X_n of cycle type b, then

$$w(\alpha^k,b) = w(a^k,b) = \sum_{d|k} d\, c(a,d,b) \quad \text{and} \quad c(a,k,b) = \frac{1}{k} \sum_{d|k} \mu(k/d)\, w(a^d,b)) \tag{11}$$

The terms on the right hand side of equation 11 can therefore be evaluated using equations 8 and 10. Thus we obtain the monomial $m(a,b)$ for the action of any α of cycle type a by conjugation on b by $m(a,b) = \prod_{k|o(a)} x_k^{c(a,k,b)}$. Therefore given $Z(G) = \frac{1}{|G|} \sum_{g \in G} m(g)$ we obtain

$$Z(G.b) = \frac{1}{|G|} \sum_{g \in G} \prod_{k|o(g)} x_k^{c(m(g),k,b)} \tag{12}$$

as claimed.□

In considering the cycle index of the action of subgroups of S_n on X_n by conjugation we use a different set of variables for each transitivity set b. We let $x_{(b,i)}$ denote an i-cycle of elements of type b, so that now $m(a,b) = \prod_{k|o(a)} x_{(b,k)}^{c(a,k,b)}$, and $m(a,X_n) = \prod_{b \in P_n} m(a,b)$ is the monomial of the action of a permutation of cycle type a on X_n. Now if G is any subgroup of S_n we obtain the cycle-index $Z(G.X_n)$ for the action of G on X_n by conjugation by replacing each monomial a by $m(a,X_n)$. In particular for $G = S_n$ we obtain $Z(S_n.X_n) = \frac{1}{n!}\sum_{a \in P_n} h(a)\, m(a,X_n)$. Let $F_n(x,y)$ be the polynomial obtained from $Z(S_n.X_n)$ by substituting $(1+x^i)$, $(1+y^i)$ or 1 for each $x_{(b,i)}$ according to whenever $o(b) \geq 3$, $o(b) = 2$ or $o(b) = 1$. By Pólya's Theorem the coefficient of $x^k y^l$ in $F_n(x,y)$ will give the number of inequivalent (k,l) subsets of X_n. Thus these polynomials are the generating functions for the number of (k,l) subsets of X_n. We give the terms of $F_n(x,y)$ in Table 1 below up to $O(y^7)$ for $3 \leq n \leq 10$, considering x as $O(y^2)$. We have investigated the three-regular Cayley graphs on subgroups of S_9. We see from $F_9(x,y)$ that there are 2641 of these with generating set of type (1,1) and 12022 of type (0,3) for a total of 14,663 Cayley sets. When generating a complete list of 14,663 inequivalent Cayley sets it is desirable to know how many there are with each possible set of cycle-types for the generators. This information is contained in the polynomial $Z(S_n.X_n)$. For example the substitution of $(1+y^i)$ for $x_{(x_1^3 x_2^3, i)}$, $(1+z^i)$ for $x_{(x_1 x_2^4, i)}$ and 1 for every other variable in $Z(S_9.X_9)$ enables us to read the number of inequivalent Cayley sets with three generators of type $x_1^3 x_2^3$ or $x_1 x_2^4$ in S_9. Alternatively we could use the above techniques to compute the action of $C(\alpha)$ on the set of permutations of types $x_1^3 x_2^3$ or $x_1 x_2^4$ in X_9 by conjugation, where α has type $x_1^3 x_2^3$.

One further approach to enumeration of the Cayley sets with a specified set of cycle types for the generators is worthy of note. Let us consider the case of counting the number of Cayley sets in S_9 with three permutations of type $x_1 x_2^4$. The type $x_1 x_2^4$ may be identified with an unlabeled graph Γ on 9 points with 4 disjoint edges. Any permutation α of type $x_1 x_2^4$ may be identified with a labeling of the graph Γ with numbers $\{1,2,\ldots,9\}$. Equivalence up to relabeling in the graph context corresponds to equivalence up to conjugation with permutations. The number of unlabeled superpositions of n differently colored copies of Γ gives the number of inequivalent ordered n-tuples of permutations of type $x_1 x_2^4$. If we regard the colors as interchangeable then the number of unlabeled superpositions of n differently colored copies of Γ gives the number of inequivalent n-multisets of permutations of type $x_1 x_2^4$. We wish to count the number of three-sets. The three-multisets which do not have three distinct elements are in one-to-one correspondence with ordered pairs. Palmer and Robertson [PR73] show how to count superpositions of colored graphs. Their techniques cover the cases of interchangeable and non-interchangeable colors. There are 548 superpositions of three differently colored copies of Γ where the colors are interchangeable, and 12 superpositions of two copies of Γ where the colors are not interchangeable. Therefore we obtain $548 - 12 = 536$ inequivalent Cayley sets with three permutations of type $x_1 x_2^4$ in S_9. The graph analogy needs some modification for permutations which are not involutions, however the same technique remains applicable.

Table 1: Generating Functions for Number of (k,l)-Subsets of X_n.

$$
\begin{aligned}
F_3(x,y) = {} & 1+x+y+xy+y^2+xy^2+y^3+xy^3 \\
F_4(x,y) = {} & 1+2x+3x^2+5x^3+2y+7xy+15x^2y+5y^2+20xy^2+47x^2y^2+10y^3+41xy^3 \\
& +12y^4+56xy^4+12y^5+10y^6+O(y^7) \\
F_5(x,y) = {} & 1+4x+26x^2+215x^3+2y+24xy+315x^2y+8y^2+173xy^2+3070x^2y^2+37y^3 \\
& +1077xy^3+149y^4+5404xy^4+535y^5+1658y^6+O(y^7) \\
F_6(x,y) = {} & 1+7x+166x^2+9090x^3+3y+95xy+6682x^2y+20y^2+1781xy^2+211359x^2y^2 \\
& +197y^3+33957xy^3+2245y^4+564974xy^4+26616y^5+290929y^6+O(y^7) \\
F_7(x,y) = {} & 1+11x+1011x^2+480924x^3+3y+267xy+143355x^2y+29y^2+15316xy^2 \\
& +15432773x^2y^2+676y^3+1004405xy^3+25948y^4+55582020xy^4 \\
& +1071459y^5+39494992y^6+O(y^7) \\
F_8(x,y) = {} & 1+17x+7032x^2+32374554x^3+4y+909xy+3841393x^2y+60y^2+163651xy^2 \\
& +1416913393x^2y^2+3094y^3+36907736xy^3+380762y^4+6895794512xy^4 \\
& +53589180y^5+6683796440y^6+O(y^7) \\
F_9(x,y) = {} & 1+25x+54952x^2+2692273145x^3+4y+2641xy+118640929x^2y+83y^2+1825707xy^2 \\
& +153502228335x^2y^2+12022y^3+1497261258xy^3+5667310y^4 \\
& +972152058485xy^4+2840588522y^5+1229693537151y^6+O(y^7) \\
F_{10}(x,y) = {} & 1+36x+505742x^2+272445118869x^3+5y+8969xy+4311730098x^2y+151y^2 \\
& +23565356xy^2+20353301123666x^2y^2+55912y^3+71501074475xy^3+96948583y^4 \\
& +168911533776760xy^4+177992264581y^5+280252256218298y^6+O(y^7)
\end{aligned}
$$

3 Graphs of Small Diameter.

We refer to a Cayley graph with generator set of type $(0,3)$ or type $(1,1)$ in X_9 as a $G(3,9)$ graph. We examined every possible generating set for $G(3,9)$ up to equivalence under conjugation, using the enumeration results as a check on the correctness of the lists obtained. We performed an exhaustive examination of these graphs and tabulated below examples of graphs of diameter D such that no larger $G(3,9)$ graph of diameter D exists. For most values of D the largest $G(3,9)$ graph of diameter D is not unique, and for each such D we have selected one example rather that listing them all. While the $G(3,9)$ graphs do not yield any new lower bounds for $n(3,D)$ with $D \leq 10$, it is natural to explore further with the examples which have relatively low diameter. Given a generator set $\{\sigma_1, \tau_1\}$ for a $G(3,9)$ we may consider generator sets $\{\sigma_1\sigma_2, \tau_1\tau_2\}$ where σ_2 and τ_2 are permutations fixing $\{1,2,\ldots,9\}$, and τ_2 has order at most 2. The resulting graph will be a covering graph of the original graph. Taking σ_2 to be a k-cycle and τ_2 to be the identity yields a d-fold cover of the initial graph for some d dividing k. There are many examples where this leads to a larger graph with little or no corresponding increase in the diameter. Taking $\sigma_2 = \tau_2 = (n+1\ n+2)$ yields a bipartite double cover of the initial graph, assuming it was not bipartite to begin with. The girth calculations were performed by comparing the sequence of sphere sizes about the identity in the initial graph and comparing it to the corresponding sequence for the bipartite double cover. We also pursued taking combinations $\{\sigma_1\sigma_2, \tau_1\tau_2\}$ where both $\{\sigma_1, \tau_1\}$ and $\{\sigma_2, \tau_2\}$ generated low diameter graphs.

Likewise we investigated covering graphs of $G(3,9)$ graphs with generating sets $\{\tau_1, \tau_2, \tau_3\}$ where each τ_i is an involution for $1 \leq i \leq 3$ by graphs with generating sets $\{\tau_1\tau_4, \tau_2\tau_5, \tau_3\tau_6\}$, where each τ_j is an involution or the identity for $4 \leq j \leq 6$. The best examples obtained are in Table 3 below. We include only examples which improve over the degree nine results.

Delorme [Del85] defines $b(\Delta,D)$ as the largest integer such that a bipartite graph on $b(\Delta,D)$ vertices with maximum vertex degree Δ and diameter D exists. The diameter 7 graph on 168 vertices and the diameter 12 graph on 2160 vertices in Table 3 are bipartite. Other large bipartite graphs found include a diameter 10 graph on 672 vertices of girth 8 and generator set $\{(1\,2)(3\,4)(5\,6)(9\,10)$, $(1\,3)(5\,7)(6\,8)(9\,10)$, $(1\,4)(2\,5)(3\,8)(6\,7)(9\,10)\}$ and a diameter 9 graph on 360 vertices with girth 10 and generator set $\{(1\,2)$, $(2\,6\,7\,8)(3\,4\,5)\}$. The resulting bounds $b(3,9) \geq 360$ and $b(3,10) \geq 672$ match those obtained in [BD88], and the bound $b(3,7) \geq 168$ improves the bound given there. However the cubic symmetric graphs F364E and F720C of the Foster Census [Roy01] improve these bounds to $b(3,9) \geq 364$ and $b(3,10) \geq 720$. The Foster Census includes a complete listing of the cubic symmetric graphs of up to 768 vertices, so all of our smaller graphs have isomorphic copies on this list. None of the examples supersede the records for the smallest cubic graphs of given girth. The most notable example for girth was the bipartite graph with generator set $\{(1\,2)(3\,4)(5\,6)$, $(1\,3\,2\,5\,4\,6\,7\,8)(9\,10\,11)\}$, which has 1008 points, diameter 12 and girth 16. The smallest known graph of girth 16 has 990 points. See [Big98] for a survey of girth results. In Tables 2 and 3 below diameter is denoted by D, the order of the graph by n, and the girth by g. We indicate in the b column whether or not each graph is bipartite.

The calculations pertaining to this paper were performed using *Mathematica* [Wol99]. Searches of the GAP [Gro99] small group and transitive permutation group libraries did not yield any larger cubic graphs of given diameter. However we emphasize that these searches were not exhaustive.

Table 2. Largest $\mathbf{G(3,9)}$ Graphs of Given Diameter.

D	n	g	b	Generators
1	4	3	n	$\{(12),(34),(12)(34)\}$
2	8	4	n	$\{(13572468),(12)(34)(56)(78)\}$
3	14	6	y	$\{(12)(34)(56),(13)(25)(47),(14)(36)(57)\}$
4	24	6	y	$\{(12),(13),(14)\}$
5	60	9	n	$\{(12)(34),(2348)(567)\}$
6	72	8	n	$\{(12),(13)(45),(14)(25)(36)\}$
7	144	8	y	$\{(12345678),(12)(36)(49)\}$
8	240	12	n	$\{(12),(13)(45),(24)(67)\}$
9	504	12	n	$\{(12)(34)(56)(78),(12)(35)(47)(69),(14)(38)(57)(69)\}$
10	720	8	n	$\{(12)(34),(13)(56),(14)(35)(78)\}$
11	1080	12	n	$\{(12345)(678),(12)(39)\}$
12	1512	9	n	$\{(123456789),(12)(36)(49)(58)\}$
13	2880	16	n	$\{(12)(34)(56)(78),(13)(29)(57),(19)(58)\}$
14	5040	10	n	$\{(12)(34),(13)(25),(14)(26)(37)\}$
15	10080	12	n	$\{(12)(34)(56)(78),(12)(34)(59),(17)(29)(58)\}$
16	10080	10	y	$\{(12)(34)(56)(78),(12)(34)(57)(69),(12)(36)(58)\}$
17	20160	14	n	$\{(12)(34),(156)(23478)\}$
18	40320	10	n	$\{(12)(34),(35)(24678)\}$
19	40320	8	y	$\{(12345678),(12)(35)(46)\}$
20	181440	9	n	$\{(123456789),(13)(26)(49)(58)\}$
21	362880	14	n	$\{(12)(34)(56)(78),(12)(35)(47)(69),(13)(26)(57)\}$

Table 3. Large Covers of $\mathbf{G(3,9)}$ Graphs with Given Diameter.

D	n	g	b	Generators
7	168	12	y	$\{(12)(34)(56)(89),(13)(25)(47)(8\,10),(14)(36)(57)(8\,11)\}$
10	864	12	n	$\{(12)(34)(56)(78),(23)(468)(57)(9\,10\,11\,12\,13\,14\,15\,16\,17)\}$
11	1344	12	n	$\{(1234567)(89)(10\,11\,12\,13\,14\,15\,16\,17),(12)(36)\}$
12	2160	14	y	$\{(12345678)(10\,11\,12\,13\,14),(12)(35)(89)\}$
13	4032	12	n	$\{(12)(34)(56)(9\,10),(13)(25)(47)(9\,11),$ $(16)(27)(35)(48)(9\,10)(11\,12)\}$
14	6048	15	n	$\{(123456789)(10\,11\,12\,13),(12)(36)(49)(58)\}$
15	12096	18	n	$\{(12)(34)(56)(78)(10\,11)(12\,13),(12)(35)(47)(69)(10\,12),$ $(13)(28)(49)(56)(10\,11)\}$
16	20160	16	n	$\{(1234)(567)(89)(10\,11\,12\,13\,14\,15\,16\,17),(12)(35)\}$
17	35280	16	n	$\{(12345)(67)(89)(10\,11\,12\,13\,14\,15\,16),(12)(36)$
18	60480	15	n	$\{(12345)(678)(9\,10\,11),(16)(78)\}$
19	120960	15	n	$\{(12345)(678)(9\,10\,11),(12)(36)(48)\}$

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