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# *$P_4$ -Free Colorings and $P_4$ -Bipartite Graphs*

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A vertex partition of a graph into disjoint subsets  $V_i$ s is said to be a  $P_4$ -free coloring if each color class  $V_i$  induces a subgraph without a chordless path on four vertices (denoted by  $P_4$ ). Examples of  $P_4$ -free 2-colorable graphs (also called  $P_4$ -bipartite graphs) include parity graphs and graphs with “few”  $P_4$ s like  $P_4$ -reducible and  $P_4$ -sparse graphs. We prove that, given  $k \geq 2$ ,  $P_4$ -FREE  $k$ -COLORABILITY is NP-complete even for comparability graphs, and for  $P_5$ -free graphs. We then discuss the recognition, perfection and the Strong Perfect Graph Conjecture (SPGC) for  $P_4$ -bipartite graphs with special  $P_4$ -structure. In particular, we show that the SPGC is true for  $P_4$ -bipartite graphs with one  $P_3$ -free color class meeting every  $P_4$  at a midpoint.

**Keywords:** Perfect graph, the Strong Perfect Graph Conjecture, graph partition, cograph, NP-completeness

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## 1 Introduction

A graph  $G$  is *perfect* if, for each induced subgraph  $H$  of  $G$ , the chromatic number of  $H$  is equal to the clique number of  $H$ . Claude Berge introduced perfect graphs and conjectured around 1960's that a graph is perfect if and only if it has no induced cycle of odd length at least five or the complement of such a cycle. Nowadays this conjecture is known as the Strong Perfect Graph Conjecture (SPGC) and is still open. We refer to [4] for more information on perfect graphs.

A measure of a graph's imperfection has been considered by Brown and Corneil [8] as follows. Given a graph  $G$  and a positive integer  $k$ , a map  $\pi : V(G) \rightarrow \{1, \dots, k\}$  is a *perfect  $k$ -coloring* of  $G$  if the subgraphs induced by each color class  $\pi^{-1}(i)$  is perfect. Thus, a graph is perfect if and only if it is perfect 1-colorable. Note also that, by the Perfect Graph Theorem [33], a graph  $G$  is perfect  $k$ -colorable if and only if its complement  $\overline{G}$  is perfect  $k$ -colorable. In this paper we consider a particular example of perfect colorings. Our discussion is motivated by the fact that the perfection of a graph depends only on the structure of its induced paths on four vertices (denoted by  $P_4$ ); see [36]. In this sense, graphs with empty  $P_4$ -structure ( $P_4$ -free graphs) form a somewhat based graph class in discussing graph's perfection; they are indeed perfect by a result due to Seinsche [38] (see also Jung [31]).

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Now, we call a perfect  $k$ -coloring of a graph  $P_4$ -free  $k$ -coloring if the subgraphs of that graph induced by the color classes are  $P_4$ -free. Note that the  $P_4$  is self-complementary, hence  $G$  is  $P_4$ -free  $k$ -colorable if and only if  $\overline{G}$  is  $P_4$ -free  $k$ -colorable. For general graphs, Brown [6] proved that  $P_4$ -FREE  $k$ -COLORABILITY is NP-complete for  $k \geq 3$ , and in [1], Achlioptas proved a more general result implying the NP-completeness of  $P_4$ -FREE  $k$ -COLORABILITY for  $k \geq 2$ . In the next section we shall prove that, for any integer  $k \geq 2$ ,  $P_4$ -FREE  $k$ -COLORABILITY is NP-complete even for (particular) perfect graphs, and for  $P_5$ -free graphs. In Section 3 we shall give some examples of  $P_4$ -free 2-colorable graphs, which we also call  $P_4$ -bipartite graphs. Many well understood classes of perfect graphs consists of  $P_4$ -bipartite graphs only. In Sections 4 and 5, perfect  $P_4$ -bipartite graphs and the SPGC for  $P_4$ -bipartite graphs with special  $P_4$ -structure will be discussed.

The complement of a graph  $G$  is denoted by  $\overline{G}$ . Graphs having no induced subgraphs isomorphic to a given graph  $H$  are called  $H$ -free. If  $X$  is a set of vertices in  $G$ ,  $G[X]$  is the subgraph of  $G$  induced by  $X$ , and  $N_G(X)$  is the *neighborhood* of  $X$  in  $G$ ; that is, the set of all vertices outside  $X$  adjacent to some vertex in  $X$ . If the context is clear, we simply write  $N(X)$ . The path on  $m$  vertices  $v_1, v_2, \dots, v_m$  with edges  $v_i v_{i+1}$  ( $1 \leq i < m$ ) is denoted by  $P_m = v_1 v_2 \cdots v_m$ . The vertices  $v_1$  and  $v_m$  are the *endpoints* of that path, the other vertices are the *midpoints*. The cycle on  $m$  vertices  $v_1, v_2, \dots, v_m$  with edges  $v_i v_{i+1}$  ( $1 \leq i < m$ ) and  $v_1 v_m$  is denoted by  $C_m = v_1 v_2 \cdots v_m$ .  $C_{2k+1}$  and  $\overline{C_{2k+1}}$ ,  $k \geq 2$ , are also called *odd holes*, respectively, *odd antiholes*. Graphs without odd holes and odd antiholes are called *Berge graphs*.

## 2 NP-completeness results

We now consider the following problem for fixed positive integer  $k$ .

$P_4$ -FREE  $k$ -COLORABILITY *Is a given graph  $P_4$ -free  $k$ -colorable?*

We show in this section that, for fixed  $k \geq 2$ ,  $P_4$ -FREE  $k$ -COLORABILITY is NP-complete for perfect graphs. Notice that  $P_4$ -free 1-colorability (that is, recognizing  $P_4$ -free graphs) is solvable in linear time [14]. We shall reduce the following NP-complete problem ([37], see also [16]) to  $P_4$ -FREE  $k$ -COLORABILITY.

NOT-ALL-EQUAL 3SAT *Given a collection  $C$  of clauses over set  $V$  of Boolean variables such that each clause has exactly three literals. Is there a truth assignment for  $V$  such that each clause in  $C$  has at least one true literal and at least one false literal?*

A comparability graph  $G$  is one which admits a transitive orientation  $\vec{G}$ : If  $(x, y)$  and  $(y, z)$  are arcs of  $\vec{G}$ , then  $(x, z)$  is also an arc of  $\vec{G}$ . It is well known that comparability graphs are perfect. A typical example of comparability graphs are  $P_4$ -free graphs, as proved by Jung [31].

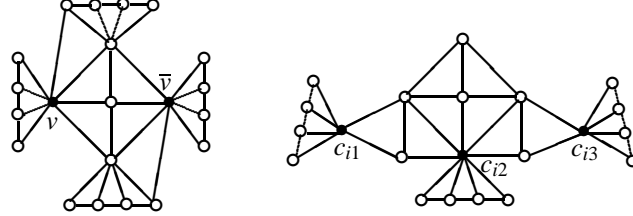
**Lemma 1** *Given a comparability graph  $G$ , it is NP-complete to decide whether  $G$  is  $P_4$ -bipartite.*

**Proof.** The problem is clearly in NP. We shall reduce NOT-ALL-EQUAL 3SAT to our problem. Let  $C = \{C_1, C_2, \dots, C_m\}$  be any set of clauses  $C_i = (c_{i1}, c_{i2}, c_{i3})$  given as input for NOT-ALL-EQUAL 3SAT, where the literals  $c_{ik}$  ( $1 \leq i \leq m, 1 \leq k \leq 3$ ) are taken from the set of variables  $V$ . We shall construct a comparability graph  $G$  which has a partition into two  $P_4$ -free graphs if and only if  $C$  is satisfiable. For convenience, we call a vertex partition of a graph into two  $P_4$ -free graphs a *good partition* of that graph. For each variable  $v \in V$  let  $G(v, \bar{v})$  be the graph shown in Figure 1 (left).

**Observation 1**  *$G(v, \bar{v})$  has a good partition. Every good partition of  $G(v, \bar{v})$  must contain the labelled vertex  $v$  in one part and the labelled vertex  $\bar{v}$  in the other part.  $\diamond$*

For each clause  $C_i$ , let  $G(C_i)$  be the graph shown in Figure 1 (right).

**Observation 2**  $G(C_i)$  has a good partition. Every good partition of  $G(C_i)$  must contain two of the labelled vertices  $c_{i1}, c_{i2}, c_{i3}$  in one part and the other labelled vertex in the other part. Moreover, every partition of  $\{c_{i1}, c_{i2}, c_{i3}\}$  into two non-empty subsets can be extended to a good partition of  $G(C_i)$ .  $\diamond$



**Fig. 1:** The graphs  $G(v, \bar{v})$  (left) and  $G(C_i)$  (right)

The proofs of the observations will follow by inspection, hence are omitted. We now create the graph  $G = G(C)$  from the graphs  $G(v, \bar{v})$  ( $v \in V$ ) and the graphs  $G(C_i)$  ( $1 \leq i \leq m$ ) as follows: For each  $v \in V$  and each  $1 \leq i \leq m$ , connect the vertex  $x \in \{v, \bar{v}\}$  in  $G(v, \bar{v})$  with the vertex  $c_{ik}$  in  $G(C_i)$  by an edge if, and only if,  $x$  is the literal  $c_{ik}$  in the clause  $C_i$ . Thus, in  $G$ , every  $c_{ik}$  ( $1 \leq k \leq 3$ ) has exactly one neighbor outside  $G(C_i)$  which is one of the labelled vertices  $v, \bar{v}$  in a graph  $G(v, \bar{v})$  (with  $c_{ik} \in \{v, \bar{v}\}$  in the given NOT-ALL-EQUAL 3SAT instance).

Suppose that  $G$  has a good partition into two  $P_4$ -free graphs  $A$  and  $B$ . Then it is easy to see that, for all  $v \in V$ , if  $x \in \{v, \bar{v}\}$  is adjacent to  $c_{ik}$ , then  $x$  and  $c_{ik}$  are in different parts  $A, B$ . We define a truth assignment for NOT-ALL-EQUAL 3SAT as follows:

$v$  is true if and only if the labelled vertex  $v$  in  $G(v, \bar{v})$  belongs to  $A$ .

By Observation 1, this assignment is well-defined. By Observation 2, it is clear that each clause  $C_i$  has at least one but not all true literals.

Conversely, suppose that there is a truth assignment satisfying NOT-ALL-EQUAL 3SAT. Then let  $A(v, \bar{v}), B(v, \bar{v})$  be a good partition of  $G(v, \bar{v})$  such that  $A(v, \bar{v})$  contains the true vertex in  $\{v, \bar{v}\}$  and  $B(v, \bar{v})$  contains the false vertex of them. Such a good partition exists by Observation 1. Let  $A_i, B_i$  be a good partition of  $G(C_i)$  such that  $A_i$  contains the false literals vertices in  $\{c_{i1}, c_{i2}, c_{i3}\}$  and  $B_i$  contains the true vertices of them. Such a good partition exists by Observation 2, and the fact that every  $C_i$  has at least one but not all true literals. Set

$$A = \bigcup_{v \in V} A(v, \bar{v}) \cup \bigcup_{1 \leq i \leq m} A_i, \quad B = \bigcup_{v \in V} B(v, \bar{v}) \cup \bigcup_{1 \leq i \leq m} B_i.$$

Clearly,  $V(G) = A \cup B$ . Now, each  $A(v, \bar{v})$  and each  $A_i$  is a  $P_4$ -free graph, and no edge exists between two parts of the  $A(v, \bar{v})$ 's and  $A_i$ 's, hence  $A$  is a  $P_4$ -free subgraph of  $G$ . Similarly,  $B$  is  $P_4$ -free. Thus,  $G$  is  $P_4$ -bipartite.

To complete the proof, note that each  $G(v, \bar{v})$  and each  $G(C_i)$  admits a transitive orientation such that the labelled vertices  $v, \bar{v}$  are sinks and the labelled vertices  $c_{i1}, c_{i2}, c_{i3}$  are sources. To obtain a transitive orientation of  $G$ , direct the edges  $xy$ ,  $x \in \{v, \bar{v}\}$  and  $y \in \{c_{i1}, c_{i2}, c_{i3}\}$  with  $x = y$  in the given instance of NOT-ALL-EQUAL 3SAT, from  $y$  to  $x$ .  $\square$

**Theorem 1** *Given a comparability graph  $G$  and an integer  $k \geq 2$ , it is NP-complete to decide whether  $G$  is  $P_4$ -free  $k$ -colorable.*

**Proof.** The case  $k = 2$  is settled by Lemma 1. We shall make use of a construction for vertex-critical  $P_4$ -free  $k$ -colorable graphs in [7] to reduce the case  $k = 2$  to the case  $k \geq 3$ . Let  $H$  be a comparability graph, and let  $G$  be the graph obtained from an induced  $P_4$  by substituting three (arbitrary) vertices by the graph  $H$ . Then  $G$  is clearly a comparability graph, and it can easily be seen that  $G$  is  $P_4$ -free  $k$ -colorable if and only if  $H$  is  $P_4$ -free  $(k - 1)$ -colorable.  $\square$

We shall remark that Brown [6] and Achlioptas [1] showed the NP-completeness of  $P_4$ -FREE  $k$ -COLORABILITY for fixed  $k \geq 3$  by reducing  $k$ -COLORABILITY to  $P_4$ -FREE  $k$ -COLORABILITY. Since  $k$ -COLORABILITY can be decided in polynomial time when considering perfect graphs (see [17]), Brown's and Achlioptas's reduction cannot be used in proving NP-completeness of  $P_4$ -FREE  $k$ -COLORABILITY for perfect graphs.

Since a graph is  $P_4$ -free  $k$ -colorable if and only if its complement is,  $P_4$ -FREE  $k$ -COLORABILITY is NP-complete for cocomparability graphs as well. Graphs which are both comparability graphs and cocomparability graphs are called *permutation graphs*. We do not know the complexity of  $P_4$ -FREE COLORABILITY on permutation graphs.

**Problem 1** *Find a polynomial time algorithm for solving  $P_4$ -FREE  $k$ -COLORABILITY on permutation graphs, or prove that the problem is NP-complete for the class of permutation graphs.*

Notice that, using the construction mentioned in the proof of Theorem 1, one can show that for every fixed  $k \geq 1$  there are  $P_4$ -free  $k$ -colorable permutation graphs which are not  $P_4$ -free  $(k - 1)$ -colorable.

We now are going to show that  $P_4$ -FREE  $k$ -COLORABILITY is NP-complete for  $(C_4, C_5)$ -free graphs. As a consequence,  $P_4$ -FREE  $k$ -COLORABILITY is also NP-complete for  $P_5$ -free graphs. This is best possible in the sense that the problem is trivial for  $P_4$ -free graphs.

**Lemma 2** *Given a  $(C_4, C_5)$ -free graph  $G$ , it is NP-complete to decide whether  $G$  is  $P_4$ -bipartite.*

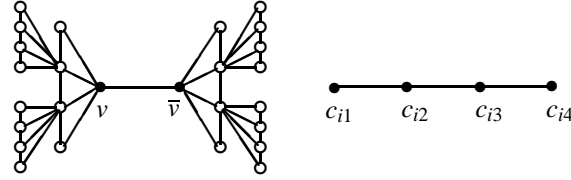
**Proof.** We shall reduce NOT-ALL-EQUAL 4SAT to our problem (the NP-completeness of NOT-ALL-EQUAL 4SAT follows easily from that of NOT-ALL-EQUAL 3SAT). Let  $C = \{C_1, C_2, \dots, C_m\}$  be any set of clauses  $C_i = (c_{i1}, c_{i2}, c_{i3}, c_{i4})$  given as input for NOT-ALL-EQUAL 4SAT, where the literals  $c_{ik}$  ( $1 \leq i \leq m, 1 \leq k \leq 4$ ) are taken from the set of variables  $V$ . We may assume that,

$$\text{for every } v \in V, \text{ no clause } C_i \text{ contains both } v \text{ and } \bar{v}. \quad (1)$$

We now construct a  $(C_4, C_5)$ -free graph  $G$  which has a partition into two  $P_4$ -free graphs if and only if  $C$  is satisfiable. For each variable  $v \in V$  let  $G(v, \bar{v})$  be the graph shown in Figure 2 (left). For each clause  $C_i$ , let  $G(C_i)$  be the  $P_4$  shown in Figure 2 (right). We create the graph  $G = G(C)$  from the graphs  $G(v, \bar{v})$  ( $v \in V$ ) and the graphs  $G(C_i)$  ( $1 \leq i \leq m$ ) as follows: For each  $v \in V$  and each  $1 \leq i \leq m$ , connect the vertex  $x \in \{v, \bar{v}\}$  in  $G(v, \bar{v})$  with the vertex  $c_{ik}$  in  $G(C_i)$  by an edge if, and only if,  $x$  is the literal  $c_{ik}$  in the clause  $C_i$ . Clearly, the construction and assumption (1) guarantee that  $G$  cannot contain an induced  $C_4$  or  $C_5$ .

Now, we can show, similar to Lemma 1, that  $G$  is  $P_4$ -bipartite if and only if  $C$  is satisfiable.  $\square$

**Theorem 2** *Given a  $(C_4, C_5)$ -free graph  $G$  and an integer  $k \geq 2$ , it is NP-complete to decide whether  $G$  is  $P_4$ -free  $k$ -colorable.*



**Fig. 2:** The graphs  $G(v, \bar{v})$  (left) and  $G(C_i)$  (right)

**Proof.** The case  $k = 2$  is settled by Lemma 2. Let  $k \geq 3$ . Let  $H$  be a  $(C_4, C_5)$ -free graph. Construct a graph  $G$  as follows: Take  $k + 2$  disjoint copies  $G_1, \dots, G_{k+2}$  of  $H$  and  $k + 2$  new vertices  $v_1, \dots, v_{k+2}$ , and connect every pair  $v_i, v_j$  ( $1 \leq i \neq j \leq k + 2$ ) by an edge and connect every vertex in  $G_i$  with  $v_i$  ( $1 \leq i \leq k + 2$ ) by an edge. Clearly,  $G$  is also  $(C_4, C_5)$ -free.

Suppose that  $H$  is  $P_4$ -free  $k$ -colorable. Then  $G$  is  $P_4$ -free  $(k + 1)$ -colorable by coloring the vertices  $v_i$ 's with one new color.

Suppose, conversely, that  $G$  is  $P_4$ -free  $(k + 1)$ -colorable. Then  $H$  is  $P_4$ -free  $k$ -colorable. If not, consider two distinct vertices  $v_i, v_j \in \{v_1, \dots, v_{k+2}\}$  with the same color  $c$  in a  $P_4$ -free  $(k + 1)$ -coloring of  $G$ . Since  $H$  is not  $P_4$ -free  $k$ -colorable, the color  $c$  must appear in every copy of  $H$ . Say, for some  $i \neq j$ ,  $x \in G_i$  and  $y \in G_j$  are colored by  $c$ . But then  $xv_iv_jy$  is a monochromatic  $P_4$  in  $G$ , a contradiction. Thus,  $H$  must be  $P_4$ -free  $k$ -colorable, as claimed.  $\square$

Since  $C_4$ -free graphs are  $\overline{P_5}$ -free, Theorem 2 implies that  $P_4$ -FREE  $k$ -COLORABILITY is NP-complete for  $\overline{P_5}$ -free graphs, and, by considering complementation, for  $P_5$ -free graphs as well. This is best possible in the sense that  $P_4$ -FREE  $k$ -COLORABILITY is trivial for  $P_4$ -free graphs.

Also, Theorem 2 implies that  $P_4$ -FREE  $k$ -COLORABILITY is NP-complete for  $(C_5, \overline{C_4})$ -free graphs as well. Notice that graphs which are both  $(C_5, C_4)$ -free and  $(C_5, \overline{C_4})$ -free, i.e., split graphs, are  $P_4$ -free 2-colorable.

### 3 Examples of $P_4$ -bipartite graphs

$P_4$ -bipartite graphs generalize in a very natural way the well understood bipartite graphs, split graphs and cographs. Below we are going to list other well structured (perfect) graph classes that contain  $P_4$ -bipartite graphs only. See [5] for a survey on these and related graph classes.

**PROPER INTERVAL GRAPHS.** Interval graphs without induced  $K_{1,3}$  are called proper interval graphs. In [2], it was shown that every proper interval graph can be partitioned into two  $P_3$ -free subgraphs. In particular, proper interval graphs are  $P_4$ -bipartite. Notice that, for every  $k$ , there exists an interval graph that is  $P_4$ -free  $k$ -colorable, but not  $P_4$ -free  $(k - 1)$ -colorable.

**DISTANCE-HEREDITARY AND PARITY GRAPHS.** Distance-hereditary graphs are those graphs in which for all vertices  $u, v$ , all induced paths connecting  $u$  and  $v$  have equal length [24]. In [9], Burlet and Uhry introduced the bigger class of parity graphs; these graphs are defined by the condition that all induced paths connecting  $u$  and  $v$  have equal parity. Let  $G$  be a parity graph, and let  $v$  be a vertex in  $G$ . In [9, Lemma 4] (see also [35]) it was shown that, for each  $i$ , the set  $N^i(v)$  of vertices at distance exactly  $i$  from  $v$  induces a  $P_4$ -free subgraph in  $G$ . Thus,  $\bigcup N^{2i}(v)$  and  $\bigcup N^{2i+1}(v)$  is a  $P_4$ -free bipartition of  $G$ . We thank Stephan Olariu and Luitpold Babel for their hint to this fact on parity graphs.

In order to give other well known classes that consist of  $P_4$ -bipartite graphs only we need the term of  $p$ -connectedness introduced by Jamison and Olariu [30]. A graph is called  $p$ -connected if, for every partition of its vertex set into two nonempty, disjoint subsets, there is an induced  $P_4$  with vertices in both parts. A  $p$ -component of a graph is a maximal  $p$ -connected subgraph of that graph. Clearly, a graph is a  $P_4$ -bipartite graph if and only if each of its  $p$ -components is a  $P_4$ -bipartite graph.

$P_4$ -REDUCIBLE AND  $P_4$ -SPARSE GRAPHS.  $P_4$ -reducible graphs are those graphs in which each vertex belongs to at most one induced  $P_4$  [26]. In [20], Hoàng introduced the bigger class of  $P_4$ -sparse graphs; these are defined by the condition that each set of at most five vertices induces at most one  $P_4$ . It was shown in [29] that every  $p$ -component of a  $P_4$ -sparse graph is a split graph. Since split graphs are  $P_4$ -bipartite, all  $P_4$ -sparse graphs are  $P_4$ -bipartite.

$P_4$ -EXTENDIBLE AND  $P_4$ -LITE GRAPHS.  $P_4$ -extendible graphs [28] are those graphs in which each  $p$ -component has at most five vertices.  $P_4$ -lite graphs [27] are those graphs in which every induced subgraph with at most six vertices either has at most two  $P_4$ s or is a (special) split graph. It was shown in [3] that every  $p$ -component of a  $P_4$ -lite graph is a split graph or has at most six vertices. Notice that all graphs with at most six vertices are  $P_4$ -bipartite, hence  $P_4$ -lite and  $P_4$ -extendible graphs are  $P_4$ -bipartite.

COGRAPH CONTRACTIONS. In [25] Hujter and Tuza introduced the graphs called *cograph contractions*. These are graphs obtained from a cograph by contracting some pairwise disjoint stable sets and then making the ‘contracted vertices’ pairwise adjacent. It was shown in [32] that a graph is a cograph contraction if and only if it admits a clique meeting each  $P_4$  in a midpoint and each  $\overline{P}_5$  in both endpoints of the  $P_5$ . In particular, cograph contractions are  $P_4$ -bipartite graphs.

Notice that the complements of the graphs mentioned above are also  $P_4$ -bipartite graphs.

## 4 Which $P_4$ -bipartite graphs are perfect?

Let  $G$  be a graph whose vertices are colored red and white (each vertex receives only one color). A  $P_4$   $abcd$  of  $G$  is said to be of type

- 1 (or RRRR) if  $a, b, c, d$  are red,
- 2 (or WRRR) if  $a$  is white and  $b, c, d$  are red,
- 3 (or RWRR) if  $a, c, d$  are red and  $b$  is white,
- 4 (or RRWW) if  $a, b$  are red and  $c, d$  are white,
- 5 (or RWRW) if  $a, c$  are red and  $b, d$  are white,
- 6 (or RWWR) if  $a, d$  are red and  $b, c$  are white,
- 7 (or WRRW) if  $a, d$  are white and  $b, c$  are red,
- 8 (or RWWW) if  $a$  is red and  $b, c, d$  are white,
- 9 (or WRWW) if  $a, c, d$  are white and  $b$  is red,
- 10 (or WWWW) if  $a, b, c, d$  are white.

Clearly,  $G$  is  $P_4$ -bipartite if and only if its vertices can be colored red and white in such a way that no  $P_4$  is of type 1 or 10. We also write  $G = (R, W, E)$  for  $P_4$ -bipartite graph  $G = (V, E)$  with partition  $V = R \cup W$  such that  $G[R]$  and  $G[W]$  are  $P_4$ -free subgraphs in  $G$ .

For non-empty subset  $S \subseteq \{2, 3, \dots, 9\}$ , we call a graph  $G$  a  $S$ -graph if the vertices of  $G$  can be colored red and white such that every  $P_4$  of  $G$  is of type  $t \in S$ . Thus  $S$ -graphs are  $P_4$ -bipartite. Bipartite graphs (respectively, complements of bipartite graphs) are, for instance,  $\{5\}$ -graphs (respectively,  $\{4\}$ -graphs).

Many classes of perfect  $P_4$ -bipartite graphs have been described in terms of types of  $P_4$ s. In [21], Hoàng proved that “odd  $P_4$ -bipartite graphs” are perfect; here the  $P_4$ -bipartite graph  $G = (R, W, E)$  is *odd* if every  $P_4$  of  $G$  has odd number of vertices in  $R$  (hence in  $W$ ). Thus, odd  $P_4$ -bipartite graphs are exactly the  $\{2, 3, 8, 9\}$ -graphs. Chvátal, Lenhart and Sbihi [13, Theorem 2], and independently Gurvich [19] extended odd  $P_4$ -bipartite graphs to a larger class of perfect  $P_4$ -bipartite graphs; they proved that all  $\{2, 3, 4, 5, 8, 9\}$ -graphs are perfect. These results and more related results in [12, 13] motivate the following question:

What are the maximal subsets  $S \subseteq \{2, 3, \dots, 9\}$  with the property that all  $S$ -graphs are perfect?

We shall point out that the complete answer to this question already follows by the results in [12, 13].

**Theorem 3** *Let  $S$  be a maximal subset of  $\{2, 3, \dots, 9\}$  such that all  $S$ -graphs are perfect. Then  $S$  is exactly one of the following sets:  $S_1 = \{4, 5, 6, 7\}$ ,  $S_2 = \{2, 3, 4, 5, 8, 9\}$ ,  $S_3 = \{3, 4, 5, 6, 8\}$ , and  $S_4 = \{2, 4, 5, 7, 9\}$ .*

**Proof.** First, color the odd hole  $C_9$  in the way RRWRRWRRW. Then every  $P_4$  of this  $C_9$  is of type 3 or 7, and every  $P_4$  of the complement of this  $C_9$  is of type 2 or 6. Second, color the odd hole  $C_9$  in the way WWRWWRWR. Then every  $P_4$  of this  $C_9$  is of type 6 or 9, and every  $P_4$  of the complement of this  $C_9$  is of type 7 or 8. Therefore, as odd holes and odd antiholes are imperfect,

none of  $\{3, 7\}$ ,  $\{2, 6\}$ ,  $\{6, 9\}$  and  $\{7, 8\}$  is a subset of  $S$ .

Now, it is straightforward to show that  $S$  must be contained in one of the sets  $S_1, S_2, S_3$ , or  $S_4$ .

Finally, all  $S_1$ -graphs are perfect [12], all  $S_2$ -graphs are perfect [13, Theorem 2] (see also [19]), all  $S_3$ -graphs and all  $S_4$ -graphs are perfect [13, Theorem 6]. □

We now turn to the recognition problem for  $P_4$ -bipartite graphs addressed in Theorem 3. Given a graph  $G$ , we consider the system of linear equations

$$w + x + y + z = 2 \quad (w, x, y, z \text{ induce a } P_4 \text{ in } G).$$

It is easy to see the  $G$  is a  $S_1$ -graph if and only if this system of linear equations has a 0/1-solution. Thus,  $S_1$ -graphs can be recognized in polynomial time. Also,  $S_3$ -graphs can be recognized in polynomial time; the task reduces to the 2SAT problem as follows.

For each  $P_4$   $wxyz$  in  $G$ , let  $(x \vee y) \wedge (\bar{w} \vee \bar{z})$  be a Boolean formula.

The 2SAT formula for  $G$  is the product of such all formulas corresponding to the  $P_4$ s in  $G$ . Now, if  $G$  is a  $S_3$ -graph with a  $P_4$ -free coloring  $V(G) = R \cup W$ , then the truth assignment  $v := \text{true} \Leftrightarrow v \in W$  satisfies our 2SAT formula. If, conversely, our 2SAT formula is satisfied, then  $W := \{v : v \text{ is true}\}$ ,  $R := \{v : v \text{ is false}\}$  is a  $P_4$ -free 2-coloring of  $G$  such that every  $P_4$  of  $G$  is of type  $t \in S_3$ . Since a graph is a  $S_4$ -graph if and only if its complement is a  $S_3$ -graph,  $S_4$ -graphs can be recognized in polynomial time, too.

The recognition problem of  $S_2$ -graphs remains open; see also [10].



**Problem 2** Given a graph  $G$ . Can you find in polynomial time a  $P_4$ -free 2-coloring of  $G$  such that every  $P_4$  of  $G$  is of type  $t \in S_2$ , or prove that such a coloring does not exist?

We remark that it can be shown that Problem 2 is NP-complete if  $S_2$  is replaced by  $S_2 \cup \{6\}$ , or replaced by  $S_2 \cup \{7\}$ .

## 5 $P_4$ -bipartite graphs and the SPGC

The results in [21, 13, 19] mentioned in previous section will be implied by the truth of the following

**Conjecture 1** *The SPGC is true for  $P_4$ -bipartite graphs.*

Conjecture 1 has been proved for some particular cases. The following theorem is a consequence of previously known results (see also [23]). It proves Conjecture 1 for  $P_4$ -free graphs with one color class being a stable set or a clique.

**Theorem 4** *Let  $G$  have a stable set (or a clique)  $T$  such that  $T$  meets every  $P_4$  of  $G$ . If  $G$  has no odd hole (respectively, no odd antihole), then  $G$  is perfect.  $\square$*

Also, in [23], Conjecture 1 is proved for  $P_4$ -bipartite graphs with one color class inducing a  $(P_4, C_4, \overline{C_4})$ -free graph and meeting every  $P_4$  in certain way as follows:

**Theorem 5** *Let  $G$  have a subset  $T \subseteq V(G)$  such that*

- (i)  $T$  induces a threshold graph,
- (ii)  $T$  meets every  $P_4$  in an endpoint, or meets every  $P_4$  in a midpoint.

*If  $G$  is Berge, then  $G$  is perfect.  $\square$*

Theorem 4 suggests the following weaker conjecture for  $P_4$ -bipartite graphs with one color class consisting of vertex-disjoint cliques.

**Conjecture 2** *The SPGC is true for  $P_4$ -bipartite graphs with one  $P_3$ -free color class.*

The main result of this section is the following theorem which is related to Theorem 5 and proves Conjecture 2 for the case when the  $P_3$ -free color class meets the  $P_4$ s in a certain way.

**Theorem 6** *Let  $G$  have a subset  $T \subseteq V(G)$  such that*

- (i)  $T$  induces a  $P_3$ -free graph,
- (ii)  $T$  meets every  $P_4$  in an endpoint, or meets every  $P_4$  in a midpoint.

*If  $G$  is Berge, then  $G$  is perfect.*

The proof of Theorem 6 relies on several known results on  $P_4$ -free graphs and minimal imperfect graphs. First, Seinsche [38] proved that

$$\text{a } P_4\text{-free graph or its complement is disconnected.} \quad (2)$$

Two vertices  $x, y$  are *twins* if, for all other vertices  $z$ ,  $z$  is adjacent to  $x$  if and only if  $z$  is adjacent to  $y$ . The next property of  $P_4$ -free graphs is well known and can be derived from (2).

$$\text{Every } P_4\text{-free graph with at least two vertices has a pair of twins.} \quad (3)$$

A graph is *minimal imperfect* if it is not perfect but each of its proper induced subgraphs is. The well known Perfect Graph Theorem due to Lovász implies that

the complement of a minimal imperfect graph is also minimal imperfect. (4)

Two (nonadjacent) vertices  $x$  and  $y$  form an *even-pair* if every induced path connecting  $x$  to  $y$  has even length. Meyniel [34] showed that

no minimal imperfect graph has an even-pair. (5)

In particular, no minimal imperfect graph has a *two-pair* which is a pair of vertices  $x, y$  such that every induced path connecting  $x$  to  $y$  has exactly two edges.

A cutset  $S$  of  $G$  is called a *star-cutset*, respectively, a *stable-cutset*, respectively, a *complete multipartite-cutset* if  $G[S]$  has a universal vertex, respectively, has no edge, respectively, is a complete multipartite graph (a complete multipartite graph is one whose vertex set can be partitioned into stable sets  $S_1, \dots, S_m$  such that, for  $i \neq j$ , every vertex in  $S_i$  is adjacent to every vertex in  $S_j$ ). Chvátal [11] showed that

no minimal imperfect graph has a star-cutset. (6)

In particular,

no minimal imperfect graph has a clique-cutset, (7)

and

in a minimal imperfect graph, no vertex dominates another vertex. (8)

Here, the vertex  $x$  *dominates* the vertex  $y$  if  $N(y) \subseteq N(x) \cup x$ . The next property of minimal imperfect graphs was found by Tucker [39] saying that

no minimal imperfect graph has a stable-cutset, unless it is an odd hole. (9)

Finally, Cornuéjols and Reed [15] showed that

no minimal imperfect graph has a complete multipartite-cutset. (10)

*Proof of Theorem 6.* Suppose that  $T$  meets every  $P_4$  in an endpoint. Color the vertices in  $T$  with color red and vertices outside  $T$  with color white. Then  $G$  has only  $P_4$ s of types 3, 4, 5, 6, or 8. In particular,  $G$  is an  $S_3$ -graph, hence perfect (see Theorem 3).

Let us consider the case when  $T$  meets every  $P_4$  in a midpoint, and assume that  $G$  is a minimal imperfect Berge graph. Further, we may assume that

$G - T$  is disconnected.

Otherwise, by (2),  $\overline{G} - T$  is disconnected and so  $T$  would be a stable-cutset or a complete multipartite-cutset of  $\overline{G}$ , contradicting (4) and (9) or (10). In particular, by (7),

$T$  consists of  $m \geq 2$  vertex-disjoint cliques.

For convenience, we call a  $P_4$  *bad* if its both midpoints are outside  $T$ . By our hypothesis, no  $P_4$  in  $G$  is bad.

CASE 1.  $G - T$  has two adjacent twins  $x, y$ .

In this case, we claim that

$x, y$  form an even-pair in  $\overline{G}$ .

To see this, consider an induced path  $P = xx_1 \cdots x_k y$ ,  $k \geq 2$ , in  $\overline{G}$  connecting  $x$  and  $y$ . As  $x, y$  are twins in  $\overline{G} - T$ ,  $x_1$  must belong to  $T$ . Furthermore,

$P$  has no edge in  $\overline{G}[T]$ .

For if  $P$  has an edge in  $\overline{G}[T]$ , then, since  $\overline{G}[T]$  is a complete multipartite graph and  $x_1 \in T$ , this edge must be  $x_1 x_2$ , and  $P$  is the  $P_4$   $xx_1 x_2 y$ . But then  $x_1 y x_2$  is a bad  $P_4$  in  $G$ , a contradiction.

$P$  has no edge in  $\overline{G} - T$ .

Otherwise, let  $i$  be minimal such that  $x_i x_{i+1}$  is an edge in  $\overline{G} - T$ . Note that  $i > 1$ . Set  $x_0 := x$ . Then  $x_{i-1} \in T$  and  $x_{i-2} \in \overline{G} - T$ . But then  $x_{i-1} x_{i+1} x_{i-2} x_i$  is a bad  $P_4$  in  $G$ , a contradiction.

Thus,  $P$  has even number of edges, as claimed. This contradicts (4) and (5), and Case 1 is settled.

CASE 2.  $G - T$  has no adjacent twins.

Write  $G[T] = C_1 \cup C_2 \cup \cdots \cup C_m$  with vertex-disjoint cliques  $C_1, C_2, \dots, C_m$ . Recall that  $m \geq 2$ .

**Observation 3** For all cliques  $C = C_i$ ,  $1 \leq i \leq m$ , and all component  $H$  of  $G - T$ , if  $N(C) \cap H \neq \emptyset$ , then  $H \subseteq N(C)$ .

*Proof of Observation 3* Assume the contrary, and let  $H$  be a component of  $G - T$  and let  $C$  be a clique of  $T$  such that  $N(C) \cap H \neq \emptyset$  and  $H - N(C) \neq \emptyset$ . Let  $x \in N(C) \cap H$  having a neighbor  $y$  in  $H - N(C)$ , and let  $v \in C$  be a neighbor of  $x$ .

By (8), there exists a vertex  $z$  adjacent to  $y$  but not to  $x$ .  $z \in N(C) \cap H$ , otherwise  $zyxv$  would be a bad  $P_4$ . The same argument shows that  $x$  and  $z$  have the same neighbors in  $C$ . Moreover,

$$\text{for all } u \in T - C, \text{ if } u \text{ is adjacent to } y, \text{ then } u \text{ is adjacent to both } x \text{ and } z. \quad (11)$$

(Else  $uyxv$  or  $uyzv$  would be a bad  $P_4$ ), and

$$\text{for all } u \in T, u \text{ is adjacent to } x \text{ if and only if } u \text{ is adjacent to } z. \quad (12)$$

This is clear for  $u \in C$ . Suppose  $u \in T - C$  is adjacent to  $x$  but not to  $z$ , then (11) implies that  $u$  is nonadjacent to  $y$  and so  $uxyz$  is a bad  $P_4$ , a contradiction. The case when  $u$  is adjacent to  $z$  but not  $x$  can be settled in a similar manner. Thus, (12) holds.

We now show that  $x, z$  form a two-pair. Let  $P = xx_1 x_2 \cdots x_k z$  be a chordless path connecting  $x$  and  $z$ , and assume that  $k \geq 2$ . By (12),  $x_1 \in H$ , hence  $x_1$  is adjacent to  $y$  (because  $H$  is  $P_4$ -free).  $x_2$  also belongs to  $H$ , otherwise, by (11),  $x_2$  and  $y$  are nonadjacent and, by (12),  $x_2$  and  $z$  are nonadjacent. But then  $x_2 x_1 y z$  is a bad  $P_4$ .

Thus,  $x_1, x_2 \in H$ . But then  $x_3 x_2 x_1 x$  (or  $zx_2 x_1 x$  if  $k = 2$ ) is a bad  $P_4$ . This contradiction proves Observation 3.  $\diamond$

By (9) and  $m \geq 2$ ,  $G - T$  has a nontrivial component  $H$ . By (3),  $H$  has twins  $x, y$  which are nonadjacent by the hypothesis in this case. Write

$$N = N_H(x) = N_H(y), R = H - N - \{x, y\}.$$

Since  $H$  is connected,  $N$  is nonempty.

**Observation 4** For all vertices  $v \in T$ , if  $v$  is adjacent to  $x$  or  $y$  but not both, then  $v$  is adjacent to all vertices in  $N$ .

*Proof of Observation 4* Otherwise, there would be a bad  $P_4$ .  $\diamond$

By (8), there exists a vertex  $x'$  adjacent to  $x$  but nonadjacent to  $y$ , and a vertex  $y'$  adjacent to  $y$  but nonadjacent to  $x$ . As  $x, y$  are twins in  $G - T$ ,  $x'$  and  $y'$  belong to  $T$ .

**Observation 5** Such vertices  $x'$  and  $y'$  can be chosen in different cliques  $C_i, C_j$ .

*Proof of Observation 5* Assume that there are vertices  $a, b$  in a clique  $C$  of  $T$  such that  $a$  is adjacent to  $x$  but not to  $y$ , and  $b$  is adjacent to  $y$  but not to  $x$ . As  $C$  is not a clique-cutset of  $G$  (see (7)), some vertex of  $H$  has a neighbor in another clique  $C' \neq C$  of  $T$ . By Observation 3,  $x$  has a neighbor  $c \in C'$ .  $c$  cannot be adjacent to  $y$ , otherwise  $cxabc$  would be a  $C_5$ , contradicting the minimality of  $G$ . Now, Observation 5 follows by setting  $C_i = C', C_j = C, x' = c$ , and  $y' = b$ .  $\diamond$

From now on, let  $x' \in C_i, y' \in C_j$  with  $i \neq j$ . By Observation 4,  $x'$  and  $y'$  are adjacent to all vertices in  $N$ .

**Observation 6** For all  $C \in \{C_1, C_2, \dots, C_m\}$ ,  $C \neq C_i$  or  $C \neq C_j$ , and for all  $z \in N$ ,  $N_C(x) \subseteq N_C(z)$  and  $N_C(y) \subseteq N_C(z)$ .

*Proof of Observation 6* If there is a vertex  $v \in N_C(x) - N_C(z)$ , then, by Observation 4,  $v$  must be adjacent to  $y$ . But then  $vyzx'$  (if  $C \neq C_i$ ) or  $vxyz'$  (if  $C \neq C_j$ ) is a bad  $P_4$ . Thus,  $N_C(x) \subseteq N_C(z)$ . By symmetry,  $N_C(y) \subseteq N_C(z)$ .  $\diamond$

**Observation 7**  $N$  cannot have a vertex  $z^*$  that is adjacent to all vertices in  $N - z^*$ .

*Proof of Observation 7* Such a vertex  $z^*$  would dominate  $x$  (contradicting (8)): If  $v$  is a neighbor of  $x$  in  $T$ , and  $v \in C$  for a clique  $C \in \{C_1, C_2, \dots, C_m\}$ , then, as  $C_i$  and  $C_j$  are different cliques,  $C \neq C_i$  or  $C \neq C_j$ , hence, by Observation 6,  $v$  must be adjacent to  $z^*$ .  $\diamond$

By Observation 7, there exist two nonadjacent vertices  $z_1, z_2$  in  $N$ . We are going to show that  $z_1, z_2$  form a two-pair. This contradiction to (5) settles Case 2.

Consider an induced path  $P = z_1 t_1 t_2 \cdots t_k z_2$  in  $G$ , and assume that  $k \geq 2$ . Then

$t_1$  must belong to  $N$ .

For, if  $t_1 \in R$ , then  $t_1 z_1 x z_2$  would be a bad  $P_4$ ; if  $t_1 \in C$ , say  $C \neq C_j$ , then  $t_1 z_1 x z_2$  (if  $t_1$  is not adjacent to  $x$ ) or  $t_1 x z_2 y'$  (if  $t_1$  is adjacent to  $x$ ) is a bad  $P_4$ , a contradiction. The case  $t_1 \in C_j$  is similar. Now,

$t_2$  must belong to  $T$ ,

otherwise,  $z_1 t_1 t_2 z_3$  would be a bad  $P_4$  (set  $t_{k+1} := z_2$ ). Thus,  $t_2 \in C$  for a clique  $C$  of  $T$ , say  $C \neq C_j$ . Moreover,

$t_2$  is adjacent to  $x$  and  $y$ ,

otherwise,  $t_2z_2xz_1$  or  $t_2z_2yz_1$  (if  $t_2$  and  $z_2$  are adjacent), or  $t_2t_1xz_2$  or  $t_2t_1yz_2$  (if  $t_2$  and  $z_2$  are nonadjacent) would be a bad  $P_4$ , a contradiction.

But then  $t_2xz_1y'$  is a bad  $P_4$ . The case  $t_2 \in C_j$  is similar. Thus, there is no induced path of length  $> 2$  connecting  $x$  and  $y$ , and so  $x, y$  form a two-pair. The proof of Theorem 6 is complete.  $\square$

The class of perfect graphs described in Theorem 6 contains all  $P_4$ -free graphs, split graphs, cograph contractions, complements of cograph contractions, strongly  $P_4$ -stable graphs, complements of strongly  $P_4$ -stable graphs ([23]), bipartite graphs, and complements of bipartite graphs. In particular, this new class is not contained in BIP\* ([11]), not in the class of strict-quasi parity graphs ([34]). We do not know whether there is a perfect graph described in Theorem 6 that is not quasi-parity ([34]). Also, we shall remark that these new perfect  $P_4$ -bipartite graphs do not belong to any class of the classes of  $S_i$ -graphs,  $i = 1, \dots, 4$ , described in Theorem 3. This can be seen as follows. Let  $G$  be the graph obtained from the  $\overline{C_6}$  by subdividing the three edges not belonging to a triangle (thus  $G$  has nine vertices). Then  $G$  satisfies the conditions of Theorem 6 with  $T$  consisting of the two disjoint triangles, but  $G$  is not an  $S_i$ -graph, for any  $i = 1, \dots, 4$ .

To conclude the paper, we remark that Fonlupt (see [22]) conjectures that no minimal imperfect Berge graph contains a cutset that induces a  $P_3$ -free graph. Clearly, Conjecture 2 is implied by Fonlupt's conjecture together with (2) and (10).

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