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# A characterization for all interval doubling schemes of the lattice of permutations

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The lattice  $\mathcal{S}_n$  of all permutations on a  $n$ -element set has been shown to be *bounded* [5], which is a strong constructive property characterized by the fact that  $\mathcal{S}_n$  admits what we call an *interval doubling scheme*. In this paper we characterize all interval doubling schemes of the lattice  $\mathcal{S}_n$ , a result that gives a nice precision on the bounded nature of the lattice of permutations. This theorem is a direct corollary of two strong properties (Proposition 3 and Theorem 3) that are also given with their proofs.

**Keywords:** Permutations, lattice, bounded lattice, interval doubling schemes, arrow relations, linear extension, tableaux.

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## 1 Introduction

Permutations on a  $n$ -element set play a fundamental role in numerous fields, especially in computer science and pure or applied mathematics. The main reason for the importance of permutations in applied mathematics is that they appear like a fundamental ordinal model since the set  $\mathcal{S}_n$  of all permutations on a  $n$ -element set is clearly in a bijective correspondence with the set of all linear orders on a set with the same cardinality. Guilbaud and Rosenstiehl have proved in 1963 that the set of all permutations on a finite set is a lattice [15]: indeed the transpositions which reverse neighbours in a permutation (and which give rise, as a generating system, to the symmetric group  $\mathcal{S}_n$ ) define the cover relation of an order relation  $\leq_{\mathcal{S}_n}$ , the so-called *weak Bruhat order* on  $\mathcal{S}_n$ , which is a lattice (Björner [4]). The lattice  $(\mathcal{S}_n, \leq_{\mathcal{S}_n})$  of permutations, also called *Permutoedron*, has been studied by several authors (see e.g. Barbut and Monjardet [1], Chameni-Nembua [6], Le Conte de Poly-Barbut [16, 17], Edelman and Greene [11], Markowsky [19]). Recently, Duquenne and Cherfouh [10] (and, independently and in a more general context, Le Conte de Poly-Barbut [18]) have proved that this lattice is *semidistributive*. This property has been reinforced in [5] where it is proved that the lattice of permutations is *bounded* (see Day [8] for the origins of this concept and for a number of characterizations), which means that the Permutoedron can be obtained from the two-element lattice  $\mathcal{Q} = (\{0, 1\}, \leq)$  by a (finite) sequence of *doublings of convex sets* (the definition of this constructive operation can be found for instance in [2] or in [14]). Another characterization of bounded lattices is expressed in terms of *A-table*, a (particularly rich lattice encoding) concept defined from the *arrow relations*. A lattice is bounded if and only if its *A-table* admits what we call an *interval doubling*

*scheme* [14]. In [2], the authors have shown how all interval doubling schemes of a bounded lattice  $L$  are in a 1-1 correspondence with all different ways to construct  $L$  from the two-element lattice by the doubling operation mentioned above. These interval doubling schemes are also in a 1-1 correspondence with all linear extensions of the poset of join-irreducible congruences and with all linear extensions of the poset  $J_L$  with respect to the  $D$  – *rank* defined in [12] (see this reference and also [9] for more definitions and details on this subject). The aim of this paper is to provide a simple characterization of all interval doubling schemes of the lattice of permutations (Corollary 2). This characterization is obtained as a corollary of two results: Proposition 3 which characterizes the order relation between all join-irreducible permutations (and dually between all meet-irreducible permutations) in terms of arrow relations, and Theorem 3 that generalizes our characterization result to a larger class of lattices.

In the following section, we give all necessary notions on lattices and on the arrow relations, and provide some useful preliminaries about permutations. Section 3 presents the results with their proofs.

## 2 Preliminaries

A partially ordered set (or poset)  $(L, \leq)$  is a *lattice* if any pair  $\{x, y\}$  of elements of  $L$  has a least *upper bound* (also called *join*) denoted by  $x \vee y$  and a greatest *lower bound* (also called *meet*) denoted by  $x \wedge y$ . The order relation  $\leq$  on  $L$  is the transitive closure of the *cover relation*  $\prec$  of  $L$ , which is defined by  $x \prec y$  if  $x < y$  and there exists no  $z \in L$  such that  $x < z < y$ . We then say that  $x$  is *covered* by  $y$  (or  $y$  *covers*  $x$ ). An element  $j$  (resp.  $m$ ) of  $L$  is a *join-* (resp. *meet-*) *irreducible* of  $L$  if it cannot be obtained as the join (resp. meet) of elements of  $L$  distinct from  $j$  (resp. from  $m$ ). Equivalently, an element  $j$  (resp.  $m$ ) of  $L$  is a non-zero (resp. non-unit) join- (resp. meet-) irreducible if it covers (resp. is covered by) a unique element in  $L$ , which is then denoted by  $j^-$  (resp.  $m^+$ ). The set of non-zero join- (resp. non-unit meet-) irreducibles of a lattice  $L$  is denoted by  $J_L$  — or  $J$  if no confusion arises — (resp.  $M_L$  or  $M$ ). We will also identify any lattice  $(L, \leq)$  with its underlying set  $L$  and all lattices will be represented by their *diagram*, i.e. by the transitive reduction of their cover relation, directed from bottom to top.

For other definitions about lattices not recalled here, see for instance the books by Barbut and Monjardet [1], Birkhoff [3] or Davey and Priestley [7].

Let  $N = \{1, 2, \dots, n\}$  be a set with  $n$  elements. A *permutation* on  $N$  is a bijection  $\alpha$  on  $N$  that we denote  $\alpha = \alpha_1 \dots \alpha_i \dots \alpha_n$ . For  $i = 1, \dots, n-1$ ,  $\alpha_i$  and  $\alpha_{i+1}$  are said *adjacent* in the permutation  $\alpha$ . We denote by  $\mathcal{S}_n$  the set of all permutations on  $N$ . It is clear that  $\mathcal{S}_n$  is in a bijective correspondence with the set of all *linear orders* on a set with cardinality  $n$ . According to this bijection, we will freely mix these two notions, considering a permutation  $\alpha$  as the linear order  $\alpha = \{(\alpha_i, \alpha_j) : 1 \leq i < j \leq n\}$  and applying all ordinal notations and concepts to permutations. In the following, an ordered pair  $(\alpha_i, \alpha_j)$  will rather be written  $\alpha_i \alpha_j$ .

For a permutation  $\alpha$ , we note  $A(\alpha)$  the set of all *agreements* of  $\alpha$ , i.e. the set of all ordered pairs  $(\alpha_i, \alpha_j)$  of  $\alpha$  satisfying  $\alpha_i < \alpha_j$ . The set  $D(\alpha)$  of all *disagreements* of  $\alpha$  is defined as the set of all ordered pairs  $(\alpha_i, \alpha_j)$  of  $\alpha$  satisfying  $\alpha_j < \alpha_i$ . An order relation between the permutations of  $\mathcal{S}_n$  is defined by  $\alpha \leq \beta$  if  $A(\beta) \subseteq A(\alpha)$  (or equivalently if  $D(\alpha) \subseteq D(\beta)$ ). For this order relation (which equals the weak Bruhat order on the set of permutations), the set  $\mathcal{S}_n$  of permutations is a lattice with the natural order  $0_{\mathcal{S}_n} = 1 \dots k(k+1) \dots n$  as least element and the dual order  $1_{\mathcal{S}_n} = n \dots k(k-1) \dots 1$  as greatest element. The cover relation  $\prec$  on  $\mathcal{S}_n$  is defined by  $\alpha \prec \beta$  if there exists a unique ordered pair  $(\alpha_i, \alpha_{i+1})$  of elements of  $N$  that are adjacent in  $\alpha$  and  $\beta$  and that satisfy  $\beta = [\alpha \setminus (\alpha_i, \alpha_{i+1})] \cup (\alpha_{i+1}, \alpha_i)$ . If a permutation

$\alpha$  admits an agreement made of two adjacent elements  $\alpha_i$  and  $\alpha_{i+1}$  (i.e.  $\alpha = \alpha_1 \dots \alpha_i \alpha_{i+1} \dots \alpha_n$  with  $\alpha_i < \alpha_{i+1}$ ), we say that  $\alpha$  has a *increasing* (in  $i$ ) and dually, if  $\alpha$  admits a disagreement made of two adjacent elements  $\alpha_i$  and  $\alpha_{i+1}$  (i.e.  $\alpha = \alpha_1 \dots \alpha_i \alpha_{i+1} \dots \alpha_n$  with  $\alpha_{i+1} < \alpha_i$ ), we say that  $\alpha$  has an *decreasing* (in  $i$ ).

The irreducible permutations are characterized by means of the increasings and decreasings as follows: a permutation is join-irreducible (resp. meet-irreducible) if and only if it admits a unique decreasing (resp. increasing).

**Remark 1** Any join-irreducible permutation  $\gamma$  can be denoted by  $\gamma = A|\overline{A} = Bv|u\overline{B}$  ( $u, v \in N$ ), where  $A = Bv$  and  $\overline{A} = u\overline{B}$  are the left and right factors of  $\gamma$  compatible with the order  $0_{S_n}$ , and where  $vu$  is the unique decreasing of  $\gamma$ . Dually, any meet-irreducible permutation  $\mu$  will be written  $\mu = D|\overline{D} = Cl|p\overline{C}$ , with  $D = Cl$  and  $\overline{D} = p\overline{C}$  the left and right factors of  $\mu$  compatible with the order  $1_{S_n}$  and where  $lp$  is the unique increasing of  $\mu$ . It is clear that a join-irreducible (resp. meet-irreducible) permutation is completely determined by  $A$  or by  $\overline{A}$  (resp. by  $D$  or by  $\overline{D}$ ).

**Remark 2** The unique lower cover of a join-irreducible permutation  $\gamma = A|\overline{A} = Bv|u\overline{B}$  is the permutation  $\gamma^- = Bw|\overline{B}$ , obtained by changing the unique decreasing  $vu$  of  $\gamma$  into the increasing  $wv$ . Dually, the unique upper cover of the meet-irreducible permutation  $\mu = D|\overline{D} = Cl|p\overline{C}$  is the permutation  $\mu^+ = Cpl\overline{C}$ , obtained by changing the unique increasing  $lp$  of  $\mu$  into the decreasing  $pl$ .

The following fact is well known:

**Remark 3**  $|J| = |M| (= 2^n - (n + 1))$ .

The original definition of bounded lattices can be found for instance in [12]. There exists a simple characterization of these lattices due to Day [8] and expressed in [14] in terms of *arrow relations* or equivalently, in terms of *A-table*, a very useful tool for the description of numerous properties on lattices. We start with the definition of these notions, introduced by Wille and the "Darmstadt School" (see [13] and [20]).

**Definition 1** Let  $j$  be a join-irreducible and  $m$  a meet-irreducible of a lattice  $L$ . The three arrow relations  $\uparrow, \downarrow$  and  $\Downarrow$  are defined as follows:

- (a)  $j \uparrow m$  if  $m$  is maximal in  $\{t \in L : j \not\leq t\}$ .
- (b)  $j \downarrow m$  if  $j$  is minimal in  $\{t \in L : t \not\leq m\}$ .
- (c)  $j \Downarrow m$  if  $j \uparrow m$  and  $j \downarrow m$ .

Equivalently,  $j \uparrow m$  if and only if  $j \not\leq m$  and  $j \leq m^+$  and, dually,  $j \downarrow m$  if and only if  $j \not\leq m$  and  $j^- \leq m$ .

Moreover it is useful to define the following "strict" arrow relations:

**Definition 2** Let  $L, j$  and  $m$  be as above.

- (d)  $j \uparrow \cdot m$  if  $j \uparrow m$  and  $j \not\Downarrow m$ .
- (e)  $j \downarrow \cdot m$  if  $j \downarrow m$  and  $j \not\Downarrow m$ .

It is obvious that relations  $\leq, \Downarrow, \uparrow$ , and  $\downarrow$ . do not intersect. This allows us to define a fundamental concept of lattice theory, the *A-table* ( $A$  for *Arrow*) of a lattice (Wille and the "Darmstadt School", 1983).

**Definition 3** Let  $(L, \leq) = L$  be a lattice and  $J$  and  $M$  its sets of join-irreducibles and of meet-irreducibles respectively. The  $A$ -table of  $L$  — denoted by  $A_L$  — is equal to the tuple  $A_L = (J, M, \leq, \Downarrow, \Uparrow, \Downarrow)$ .

According to its definition, the  $A$ -table of a lattice  $L$  can always be described by a two-dimensional table, whose rows are indexed by the join-irreducibles of  $L$ , and whose columns are indexed by its meet-irreducibles. Each cell  $(j, m)$  in the  $A$ -table contains a cross  $\times$  if  $j \leq m$ , and an arrow  $\Uparrow$ ,  $\Downarrow$ , or  $\Downarrow$  according to the adequate case. The cell  $(j, m)$  contains the symbol  $\circ$  if  $(j, m)$  does not satisfy any of the previous conditions.

Obviously there exists many ways to present the  $A$ -table of a lattice  $L$ , according to the orders chosen on  $J$  and  $M$  for the indexing of rows and columns of the  $A$ -table. This naturally leads to the simple and fundamental notion of *tableau*, that we define below:

**Definition 4** A tableau  $T$  of lattice  $L$  is a triple  $T = (A_L, \mathcal{L}_J, \mathcal{L}_M)$  where  $A_L$  is the  $A$ -table of  $L$  and  $\mathcal{L}_J$  and  $\mathcal{L}_M$  two linear orders respectively defined on  $J$  and  $M$  and defining (from the top to the bottom and from the left to the right) the two orders of the "rows" and of the "columns" of tableau  $T$ .

So the  $A$ -table of  $L$  can be described by  $(|J|! \times |M|!)$  equivalent such tableaux and a tableau is completely determined by the two orders  $\mathcal{L}_J$  and  $\mathcal{L}_M$ . Figure 1 shows (the diagram of) the lattice of permutations  $\mathcal{S}_4$  and its  $A$ -table, given by a tableau.

**Remark 4** Each row and each column of this tableau contains at least one  $\Downarrow$ .

In the introduction, we have recalled that the Permutoedron is semidistributive. Let us recall that a lattice  $L$  is said *semidistributive* if for all elements  $x, y, z \in L$ ,  $x \wedge y = x \wedge z$  implies  $x \wedge y = x \wedge (y \vee z)$  and  $x \vee y = x \vee z$  implies  $x \vee y = x \vee (y \wedge z)$ . The following characterization of these lattices is well known:

**Proposition 1** A finite lattice  $L$  is semidistributive if and only if the two following conditions are satisfied:

1. For any  $j \in J$ , there exists a unique  $m \in M$  such that  $j \Downarrow m$ .
2. For any  $m \in M$ , there exists a unique  $j \in J$  such that  $j \Downarrow m$ .

This result means that  $L$  is semidistributive if and only if the  $A$ -table of  $L$  contains a unique  $\Downarrow$  on each line and on each column. According to Remark 4, this fact implies that  $J$  and  $M$  have the same cardinality and that these two sets are in a bijective correspondence induced by the relation  $\Downarrow$ . Theorem 2 gives the expression of this bijection in the case of the semidistributive lattice of permutations.

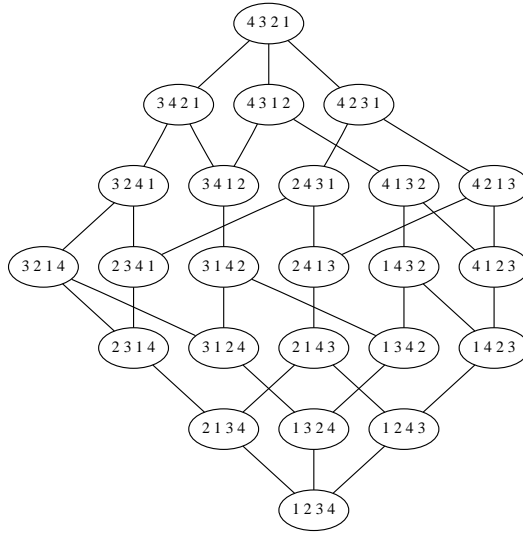
The characterization of bounded lattices given in Theorem 1 clearly shows that a bounded lattice is semidistributive. We begin with one definition.

**Definition 5** Let  $L$  be a semidistributive lattice and  $T = (A_L, \mathcal{L}_J, \mathcal{L}_M)$  a tableau of its  $A$ -table. We say that  $T$  is an interval doubling scheme of  $L$  if it satisfies the two following conditions:

1. The  $|J|$  double arrows  $\Downarrow$  are on the principal diagonal of  $T$ .
2. All arrows  $\Uparrow$  are below this diagonal and all arrows  $\Downarrow$  are above.

For an illustration of that definition see Figure 2.

**Theorem 1 (Geyer, 1994)** A lattice  $L$  is bounded if and only if its  $A$ -table admits at least one interval doubling scheme.



$J \setminus M$	3421	4231	3241	2431	4312	4213	3214	2413	4132	3142	1432
1243	↕	×	↓.	×	×	×	↓.	×	×	↓.	×
1324	×	↕	×	↓.	×	↓.	×	↓.	×	×	×
1342	×	↑.	↕	○	×	○	↓.	○	×	×	×
1423	↑.	×	○	↑	×	×	○	↓.	×	○	×
2134	×	×	×	×	↕	×	×	×	↓.	↓.	↓.
2314	×	×	×	×	↑.	↕	×	↓.	○	○	○
2341	×	×	×	×	↑.	↑.	↕	○	○	○	○
2413	↑.	×	○	×	↑.	×	○	↕	○	○	○
3124	×	↑.	×	○	×	○	×	○	↕	×	↓.
3412	×	↑.	↑.	○	×	○	○	○	↑.	↕	○
4123	↑.	×	○	↑.	×	×	○	○	×	○	↕

Fig. 1: The lattice  $\mathcal{S}_4$  of permutations and (a tableau of) its  $A$ -table.

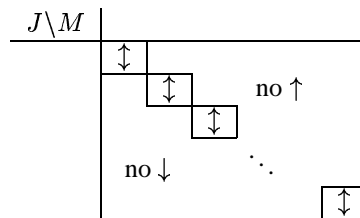


Fig. 2: An interval doubling scheme

Clearly, the tableau in Figure 1 is an interval doubling scheme of the lattice  $\mathcal{S}_4$ . It is completely determined by the chosen linear order on  $J$  (which is the *lexicographical* order on  $J$ ) since, for this linear order, there exists a unique linear order on  $M$  allowing to get all  $\Downarrow$  of the  $A$ -table on its principal diagonal. In [5], the author has shown the bounded nature of the Permutoedron by showing that the unique tableau given by the lexicographical order on  $J$  and allowing the alignment of the  $\Downarrow$  on the diagonal is an interval doubling scheme of  $\mathcal{S}_n$ . Our aim here is to characterize all interval doubling schemes of the Permutoedron. To do so, we use the following characterization of the arrow relations in the lattice  $\mathcal{S}_n$  (whose proof, easy, is left to the reader):

**Proposition 2** *Let  $\gamma = Bv|u\overline{B}$  be a join-irreducible and  $\mu = Cl|p\overline{C}$  a meet-irreducible of the lattice  $\mathcal{S}_n$ .*

1.  $\gamma \leq \mu \iff D(\gamma) \subseteq D(\mu) \iff A(\mu) \subseteq A(\gamma)$ .
2.  $\gamma \uparrow \mu \iff pl \in D(\gamma) \text{ and } D(\gamma) \subseteq D(\mu^+)$ .
3.  $\gamma \downarrow \mu \iff uv \in A(\mu) \text{ and } A(\mu) \subseteq A(\gamma^-)$ .
4.  $\gamma \Downarrow \mu \iff pl \in D(\gamma), uv \in A(\mu), D(\gamma) \subseteq D(\mu^+) \text{ and } A(\mu) \subseteq A(\gamma^-)$ .

### 3 The results

We have seen that a semidistributive lattice is characterized by the existence of a unique  $\Downarrow$  on each line and on each column of its  $A$ -table, this relation  $\Downarrow$  inducing in the lattice a bijective correspondence between the sets  $J$  and  $M$ . Without proof, we give the following theorem and corollary which make this bijection explicit in the case of the Permutoedron. For the proof see [5].

**Theorem 2** 1. *Let  $\gamma = Bv|u\overline{B}$  be a join-irreducible and  $\mu$  a meet-irreducible of lattice  $\mathcal{S}_n$ . Then*

$$\gamma \Downarrow \mu \iff \mu = Cu|v\overline{C} \quad \text{with} \quad \begin{cases} C = (\{x \in B : u < x\} \cup \{x \in \overline{B} : v < x\}, >) \\ \overline{C} = (\{x \in B : x < u\} \cup \{x \in \overline{B} : x < v\}, >) \end{cases}$$

2. *Let  $\mu = Cl|p\overline{C}$  be a meet-irreducible and  $\gamma$  a join-irreducible of lattice  $\mathcal{S}_n$ . Then*

$$\gamma \Downarrow \mu \iff \gamma = Bp|l\overline{B} \quad \text{with} \quad \begin{cases} B = (\{x \in C : x < p\} \cup \{x \in \overline{C} : x < l\}, <) \\ \overline{B} = (\{x \in C : p < x\} \cup \{x \in \overline{C} : l < x\}, <) \end{cases}$$

**Corollary 1** *The relation  $\Downarrow$  induces a bijection between the sets of join-irreducibles and meet-irreducibles of  $\mathcal{S}_n$  (semidistributivity of  $\mathcal{S}_n$ ).*

**Definition 6** *In a semidistributive lattice, the meet-irreducible image of the join-irreducible element  $j$  in the bijection  $\Downarrow$  will be denoted by  $m_j$  and conversely, the join-irreducible image of the meet-irreducible element  $m$  will be denoted by  $j_m$ .*

We shall now prove a strong property of the arrow relations in the lattice of permutations, that will be used to characterize all interval doubling schemes of  $\mathcal{S}_n$ .

**Proposition 3** *Let  $\gamma$  be a join-irreducible of  $\mathcal{S}_n$  and  $\mu_\gamma$  its image in the bijection  $\Downarrow$ . The following assertions are satisfied:*

1.  $\forall \gamma' \in J, (\gamma' \uparrow \cdot \mu_\gamma \iff \gamma <_{S_n} \gamma')$ .
2.  $\forall \mu' \in M, (\gamma_\mu \downarrow \cdot \mu' \iff \mu' <_{S_n} \mu)$ .

*Proof:* To prove point 1, we set:

$$\begin{cases} \gamma = Bv|u\overline{B} \\ \mu_\gamma = Cu|v\overline{C} \\ \mu_\gamma^+ = Cvu\overline{C} \\ \gamma' = B's|r\overline{B}' \end{cases}$$

We then consider the following conditions:

$$\begin{cases} (a) \, vu \in D(\gamma') \\ (b) \, \forall x \in B, (u < x \implies x \in B') \\ (c) \, \forall x \in \overline{B}, (x < v \implies x \in \overline{B}') \end{cases}$$

and we prove the equivalence of the three following conditions:

$$\begin{cases} (i) \, \gamma <_{S_n} \gamma' \\ (ii) \, \text{conditions (a), (b) and (c) are satisfied} \\ (iii) \, \gamma' \uparrow \cdot \mu_\gamma \end{cases}$$

(i)  $\implies$  (ii) :  $\gamma <_{S_n} \gamma'$  implies  $vu \in D(\gamma')$ . Condition (a) is satisfied and implies  $v \in B's$  and  $u \in r\overline{B}'$ .

Let  $x \in B$  be such that  $u < x$ . So we have  $xu \in D(\gamma)$ , hence  $xu \in D(\gamma')$ . Then if  $x \in r\overline{B}'$ , we have  $ux \in A(\gamma')$ , which is impossible. If  $x = s$ , then  $v \in B's$  implies  $v \leq s$ , and so  $x = s \in vu\overline{B}$ , which contradicts  $x \in B$ . Finally,  $x \in B'$  and condition (b) is satisfied. Now for condition (c), let us consider  $x \in \overline{B}$  such that  $x < v$ . We have  $vx \in D(\gamma)$  and so  $vx \in D(\gamma')$ . If  $x \in B's$  then  $xv \in A(\gamma')$ , which is impossible. If  $x = r$  then  $u \in r\overline{B}'$  implies  $r \leq u$ , hence  $x = r \in Bvu$ , which contradicts  $x \in \overline{B}$ . Finally,  $x \in \overline{B}'$  and condition (c) is satisfied.

(ii)  $\implies$  (iii) : Let us suppose that all conditions (a), (b) and (c) are satisfied. We prove that this implies  $(B's|r\overline{B}')\gamma' \uparrow \mu_\gamma (= Cu|v\overline{C})$ , i.e. (Proposition 2) that we have  $vu \in D(\gamma')$  and  $A(\mu_\gamma^+) \subseteq A(\gamma')$ . Condition (a) gives  $vu \in D(\gamma')$ . It remains to prove that  $A(\mu_\gamma^+) \subseteq A(\gamma')$ . Let us compute the set  $A(\mu_\gamma^+)$  of all agreements of  $\mu_\gamma^+ = Cvu\overline{C}$ :

$A(\mu_\gamma^+) = \{xv : x \in C \text{ and } x < v\} \cup \{ux : x \in \overline{C} \text{ and } u < x\} \cup \{xy : x \in C, y \in \overline{C} \text{ and } x < y\}$ . We now prove that  $A(\mu_\gamma^+) \subseteq A(\gamma')$ .

- $\{xv : x \in C \text{ and } x < v\}$ : we have  $u < x$  since  $x \in C$  and on the other hand,  $x \in C$  and  $x < v$  imply (Theorem 2)  $x \in B$ . Thus condition (b) implies  $x \in B'$ . Since  $v \in B's$  (by condition (a)) and  $x < v$ , we obtain  $xv \in A(\gamma')$ .
- $\{ux : x \in \overline{C} \text{ and } u < x\}$ : we have  $x < v$  since  $x \in \overline{C}$  and on the other hand,  $x \in \overline{C}$  and  $u < x$  imply  $x \in \overline{B}$  (Theorem 2). Thus condition (c) gives  $x \in \overline{B}'$  and, since  $u \in r\overline{B}'$  by condition (a), we have  $ux \in A(\gamma')$ .



- $\{xy : x \in C, y \in \overline{C} \text{ and } x < y\}$ :  $x \in C, y \in \overline{C} \text{ and } x < y$  imply  $u < x < y < v$ . So we have on one hand  $x \in C$  and  $x < v$  that lead to  $x \in B$  (Theorem 2) and since  $u < x$ , we obtain  $x \in B'$  according to (b). On the other hand, we have  $y \in \overline{C}$  and  $u < y$  that lead to  $y \in \overline{B}$  and since  $y < v$ , we obtain  $y \in \overline{B}'$  by condition (c). Finally,  $x \in B', y \in \overline{B}'$  and  $x < y$  and so  $xy \in A(\gamma')$ .

(iii)  $\implies$  (i) : This fact trivially holds in any semidistributive lattice.

We have just given the proof of point 1. of the proposition. The proof of point 2. is dual.  $\blacksquare$

Before we establish the characterization result for all interval doubling schemes of the Permutoedron (Corollary 2) describing all possible constructions of  $\mathcal{S}_n$  by doublings of convex sets starting from the two-element lattice (details on this subject are given in [2]), we first prove the more general theorem below.

For a semidistributive lattice  $L$  and a linear order  $\mathcal{L}_J$  on  $J$ , we denote by  $\mathcal{L}_M^*$  the unique linear order on  $M$  such that the tableau  $T = (A_L, \mathcal{L}_J, \mathcal{L}_M^*)$  has all  $\uparrow$  on the principal diagonal.

**Theorem 3** *Let  $L$  be a semidistributive lattice satisfying conditions 1. and 2. of Proposition 3 and let  $\mathcal{L}_J$  be a linear order on the join-irreducible elements of  $L$ . The following conditions are equivalent:*

1.  $T = (A_L, \mathcal{L}_J, \mathcal{L}_M^*)$  is an interval doubling scheme of lattice  $L$ .
2.  $\mathcal{L}_J$  is a linear extension of  $(J, \leq_L)$  and  $\mathcal{L}_M^*$  is a linear extension of  $(M, \geq_L)$ .

*Proof:* 1.  $\implies$  2: Let  $\mathcal{L}_J$  be a linear order on  $J$ . The tableau  $T = (A_L, \mathcal{L}_J, \mathcal{L}_M^*)$  satisfies the alignment of the  $\uparrow$  on the principal diagonal. If  $\mathcal{L}_J$  is not a linear extension of  $(J, \leq_L)$ , there exists  $j, j' \in J$  such that  $j \mathcal{L}_J j'$  and  $j' <_L j$ . But  $j' <_L j$  implies  $j \uparrow m_{j'}$  (Proposition 3). Thus,  $j$  and  $j'$  satisfy  $j \mathcal{L}_J j'$  and  $j \uparrow m_{j'}$ , and  $T$  is not an interval doubling scheme.

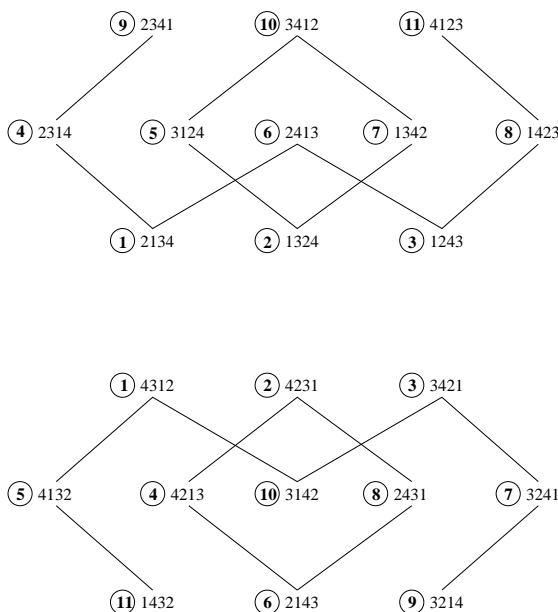
Dually, we would show that a tableau  $T = (A_L, \mathcal{L}_J, \mathcal{L}_M^*)$  is not an interval doubling scheme as soon as  $\mathcal{L}_M^*$  is not a linear extension of  $(M, \geq_L)$ .

2.  $\implies$  1: Let  $\mathcal{L}_J$  be a linear extension of  $(J, \leq_L)$ . Let us assume that the tableau  $T = (A_L, \mathcal{L}_J, \mathcal{L}_M^*)$  is not an interval doubling scheme. So there exists  $j, j' \in J$  such that  $j \mathcal{L}_J j'$  and  $(j \uparrow m_{j'} \text{ or } j' \downarrow m_j)$ . In this case,  $\mathcal{L}_M^*$  is not a linear extension of  $(M, \geq_L)$ . Indeed,  $j \mathcal{L}_J j'$  implies  $j' \not\leq_L j$  by hypothesis. But this last condition implies  $j \not\uparrow m_{j'}$  (Proposition 3). So we have  $j' \downarrow m_j$ , which, according to the same proposition, implies  $m_j <_L m_{j'}$  and, since  $j \mathcal{L}_J j'$  implies  $m_j \mathcal{L}_M^* m_{j'}$ ,  $\mathcal{L}_M^*$  is not a linear extension of  $(M, \geq_L)$ .  $\blacksquare$

The announced result is now a direct corollary of Proposition 3 and Theorem 3.

**Corollary 2 (Characterization)** *Let  $\mathcal{L}_J$  be a linear order on the join-irreducible permutations of the Permutoedron. The following conditions are equivalent:*

1.  $T = (A_{\mathcal{S}_n}, \mathcal{L}_J, \mathcal{L}_M^*)$  is an interval doubling scheme of lattice  $\mathcal{S}_n$ .
2.  $\mathcal{L}_J$  is a linear extension of  $(J, \leq_{\mathcal{S}_n})$  and  $\mathcal{L}_M^*$  is a linear extension of  $(M, \geq_{\mathcal{S}_n})$ .



**Fig. 3:** A linear extension  $\mathcal{L}_J$  of  $(J, \leq_{S_4})$  for which the linear order  $\mathcal{L}_M^*$  on  $M$  is not a linear extension of  $(M, \geq_{S_4})$ .

Figure 3 shows that for a given linear extension  $\mathcal{L}_J$  of the join-irreducibles of lattice  $\mathcal{S}_4$ , the linear order  $\mathcal{L}_M^*$  on the meet-irreducibles is not necessarily a linear extension of  $(M, \geq_{S_n})$  and so the theorem is not trivial. Indeed, if we numerate the join-irreducible permutations from 1 to 11 as shown on the figure, a linear extension of  $(J, \leq_{S_4})$  is then:  $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < 11$ , whereas the corresponding linear order  $\mathcal{L}_M^*$  on  $M$  is not a linear extension of  $(M, \geq_{S_n})$  (observe permutations 6 and 8).

## 4 Conclusion

We have recalled that all interval doubling schemes of a bounded lattice  $L$  are in bijection with all different ways to construct  $L$  starting from the two-element lattice by doublings of convex sets. The result of characterization of all interval doubling schemes of the Permutoedron appears therefore like a significant step towards a better understanding of the constructive properties of this lattice. Moreover, in [5], the author has conjectured that all (finite) *Coxeter* lattices (containing the infinite family of lattices  $(\mathcal{S}_n)_{n \geq 2}$ ) could themselves be bounded lattices. In this case, the question would arise to know if there exists a similar characterization result about the interval doubling schemes of these lattices.

At last, we have proved Proposition 3 in a particular class of lattices (semidistributive lattices satisfying both conditions of Proposition 3) and we are interested to determine if this class is maximal for this property on the arrow relations.

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## References

- [1] M. Barbut and B. Monjardet, *Ordre et classification*, vols. 1 and 2, Hachette, Paris, France (1970).
- [2] K. Bertet and N. Caspard, Doubling convex sets in lattices: characterizations and recognition algorithms, preprint. Submitted.
- [3] G. Birkhoff, *Lattice Theory* (first ed.), Amer. Math.Soc., Providence (1940).
- [4] A. Björner, Orderings of Coxeter groups, *Contemp. Math.*, Amer. Math. Soc. Providence, R. I., (1984), vol. 34, 175-195.
- [5] N. Caspard, The lattice of permutations is bounded, preprint. Submitted.
- [6] C. Chameni-Nembua, Permutoèdre et choix social, Thèse de Mathématiques Appliquées aux Sciences Sociales, Université Paris 5 (1989).
- [7] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press (1992).
- [8] A. Day, Characterizations of finite lattices that are bounded-homomorphic images or sublattices of free lattices, *Canadian J. Math.* **31** (1979), 69-78.
- [9] A. Day, Congruence normality: the characterization of the doubling class of convex sets, *Algebra Universalis* **31** (1994), 397-406.
- [10] V. Duquenne and A. Cherfouh, On permutation lattices, *Math. Soc. Sciences* **27** (1994), 73-89.
- [11] P. Edelman and C. Greene, Combinatorial correspondences for Young tableaux, balanced tableaux and maximal chains in the weak Bruhat order of  $S_n$ , *Combinatorics and Algebra*, C. Greene (ed.), Contemporary Mathematics **34** American Mathematical Society, Providence, R.I. (1984), 155-162.
- [12] R. Freese, J. Jezek and J.B. Nation, *Free lattices*, Mathematical Surveys and Monographs, vol. 42, American Mathematical Society, (1991).
- [13] B. Ganter and R. Wille, *Formale Begriffsanalyse: Mathematische Grundlagen* (1997) Springer Verlag, Berlin (english translation 1999).
- [14] W. Geyer, The generalized doubling construction and formal concept analysis, *Algebra Universalis* **32** (1994), 341-367.
- [15] G. Guilbaud and P. Rosenstiehl, Analyse algébrique d'un scrutin, *Math. Sci. Hum.* **4** (1963), 9-33.
- [16] C. Le Conte de Poly-Barbut, Automorphismes du permutoèdre et votes de Condorcet, *Math. Inf. Sci. hum.* **111** (1990), 73-82.

- [17] C. Le Conte de Poly-Barbut, Le diagramme du treillis permutoèdre est intersection des diagrammes de 2 produits directs d'ordres totaux, *Math. Inf. Sci. Hum.* **112** (1990), 49-53.
- [18] C. Le Conte de Poly-Barbut, Sur les treillis de Coxeter finis, *Math. Inf. Sci. Hum.*, **125** (1994), 41-57.
- [19] G. Markowsky, Permutation Lattices Revisited, *Math. Soc. Sciences* **27** (1994), 59-72.
- [20] R. Wille, Subdirect decomposition of concept lattices, *Algebra Universalis* **17** (1983), 275-287.

