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On the proper orientation number of bipartite graphs

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Abstract: An *orientation* of a graph G is a digraph D obtained from G by replacing each edge by exactly one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the *indegree* of v in D, denoted by $d_D^-(v)$, is the number of arcs with head v in D. An orientation D of G is *proper* if $d_D^-(u) \neq d_D^-(v)$, for all $uv \in E(G)$. The *proper orientation number* of a graph G, denoted by $\overrightarrow{\chi}(G)$, is the minimum of the maximum indegree over all its proper orientations. In this paper, we prove that $\overrightarrow{\chi}(G) \leq \left\lfloor \left(\Delta(G) + \sqrt{\Delta(G)}\right)/2 \right\rfloor + 1$ if G is a bipartite graph, and $\overrightarrow{\chi}(G) \leq 4$ if G is a tree. It is well-known that $\overrightarrow{\chi}(G) \leq \Delta(G)$, for every graph G. However, we prove that deciding whether $\overrightarrow{\chi}(G) \leq \Delta(G) - 1$ is already an $\mathcal{NP}$-complete problem. We also show that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 2$, for planar *subcubic* graphs G. Moreover, we prove that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 3$, for planar bipartite graphs G with maximum degree 5.

Key-words: proper orientation, graph colouring, bipartite graph

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Sur l'indice d'orientation propre des graphes bipartis

Résumé : Une *orientation* d'un graphe G est un digraphe D obtenu à partir de G en remplaçant chaque arête par exactement un des deux arcs possibles avec les mêmes extremités. Pour tout $v \in V(G)$, le *dégré entrant* de v dans D, noté $d_D^-(v)$, est le nombre d'arcs de D ayant v pour tête. Une orientation D de G est *propre* si $d_D^-(u) \neq d_D^-(v)$, pour tout $uv \in E(G)$. L'*indice d'orientation propre* d'un graphe G, noté $\overrightarrow{\chi}(G)$, est le plus petit dégré maximum d'une orientation propre de G. Dans ce papier, nous prouvons que $\overrightarrow{\chi}(G) \leq \left\lfloor \left(\Delta(G) + \sqrt{\Delta(G)} \right) /2 \right\rfloor + 1$ si G est un graphe biparti, et $\overrightarrow{\chi}(G) \leq 4$ si G est un arbre. Il est connu que $\overrightarrow{\chi}(G) \leq \Delta(G)$, pour tout graphe G. En revancche, nous prouvons que décider si $\overrightarrow{\chi}(G) \leq \Delta(G) - 1$ est déjà un problème $\mathcal{NP}$-complet. Nous montrons aussi qu'il est $\mathcal{NP}$-complet de décider si $\overrightarrow{\chi}(G) \leq 2$ pour un graphe planaire subcubique G. Enfin nous montrons qu'il est $\mathcal{NP}$-complet de décider si $\overrightarrow{\chi}(G) \leq 3$ pour un graphe planaire biparti G de dégré maximum 5.

Mots-clés : orientation propre, coloration de graphe, graphe biparti

1 Introduction

In this paper, all graphs are *simple*, that is without loops and multiple edges. We follow standard terminology as used in [3].

An *orientation* D of a graph G is a digraph obtained from G by replacing each edge by just one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the *indegree* of v in D, denoted by $d_D^-(v)$, is the number of arcs with head v in D. We use the notation $d^-(v)$ when the orientation D is clear from the context. The orientation D of G is *proper* if $d^-(u) \neq d^-(v)$, for all $uv \in E(G)$. An orientation with maximum indegree at most k is called a *k-orientation.* The *proper orientation number* of a graph G, denoted by $\overrightarrow{\chi}(G)$, is the minimum integer k such that G admits a proper k-orientation. This graph parameter was introduced by Ahadi and Dehghan [1]. It is well-defined for any graph G since one can always obtain a proper $\Delta(G)$-orientation (see [1]). In other words, $\overrightarrow{\chi}(G) \leq \Delta(G)$. Note that every proper orientation of a graph G induces a proper vertex colouring of G. Thus, $\overrightarrow{\chi}(G) \geq \chi(G) - 1$. Hence, we have the following sequence of inequalities:

$$\omega(G) - 1 \leq \chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G).$$

These inequalities are best possible in the sense that, for a complete graph K, $\omega(K) - 1 = \chi(K) - 1 = \overrightarrow{\chi}(K) = \Delta(K)$. However, one might expect better upper bounds on some parameter by taking a convex combination of two others. Reed [6] showed that there exists $\epsilon_0 > 0$ such that $\chi(G) \leq \epsilon_0 \cdot \omega(G) + (1 - \epsilon_0)\Delta(G)$ for every graph G and conjectured the following.

Conjecture 1 (Reed [6])**.** For every graph G,

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil.$$

If true, this conjecture would be tight. Johannson [4] settled Conjecture 1 for $\omega(G) = 2$ and $\Delta(G)$ sufficiently large.

Likewise, one may wonder if similar upper bounds might be derived for the proper orientation number.

Problem 1.

(a) Does there exist a positive ϵ_1 such that $\overrightarrow{\chi}(G) \leq \epsilon_1 \cdot \omega(G) + (1 - \epsilon_1)\Delta(G)$?

(b) Does there exist a positive ϵ_2 such that $\overrightarrow{\chi}(G) \leq \epsilon_2 \cdot \chi(G) + (1 - \epsilon_2)\Delta(G)$?

Observe that both questions are intimately related. Indeed if the answer to (a) is positive for ϵ_1, then the answer to (b) is also positive for ϵ_1. On the other hand, if the answer to (b) is positive for ϵ_2, then the answer to (a) is also positive for $\epsilon_1 = \epsilon_0 \cdot \epsilon_2$ by the above-mentioned result of Reed.

In Section 2, we answer Problem 1 positively in the case of bipartite graphs by showing that

$$\text{if } G \text{ is bipartite, then } \overrightarrow{\chi}(G) \leq \left\lfloor \frac{\Delta(G) + \sqrt{\Delta(G)}}{2} \right\rfloor + 1.$$

We also present bipartite graphs G such that $\overrightarrow{\chi}(G) = \Delta(G) = k$ for $k \in \{1, 2, 3\}$.

In Section 3, we prove that $\overrightarrow{\chi}(T) \leq 4$, for every tree T. Moreover, we show that $\overrightarrow{\chi}(T) \leq 3$ if $\Delta(T) \leq 6$, and $\overrightarrow{\chi}(T) \leq 2$ if $\Delta(T) \leq 3$. We also argue that all these bounds are tight.

In Section 4, we study the computational complexity of computing the proper orientation number of a bipartite graph. In their seminal paper, Ahadi and Dehghan proved that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) = 2$ for planar graphs G. We first improve their reduction and show that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 2$, for planar *subcubic* graphs G. Moreover, we prove that deciding whether $\overrightarrow{\chi}(G) \leq \Delta(G) - 1$ is an $\mathcal{NP}$-complete problem for general graphs G. Finally, we show that it is also $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 3$ for planar bipartite graphs G with maximum degree 5.

In the final section, we draw some conclusive remarks and discuss some open problems and directions for further research.

2 General upper bound

The aim of this section is to prove the following theorem.

Theorem 1. *Let G be a bipartite graph and let k be a positive integer. If $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$, then $\overrightarrow{\chi}(G) \leq \Delta(G) - k$.*

In order to prove this theorem, we describe an algorithm (see Algorithm 1) that produces a proper $(\Delta(G) - k)$-orientation.

Let $G = (X \cup Y, E)$ be a bipartite graph as in the statement of Theorem 1. The algorithm consists of two phases.

The first phase (lines 1 to 8 in Algorithm 1) produces an orientation, not necessarily proper, of the edges of G in such a way that the indegree of each vertex in X is at most k and the indegree of each vertex in Y is at most $\Delta(G) - k$. It proceeds as follows. We first orient all edges $xy \in E(G)$ from x to y, where $x \in X$ and $y \in Y$. Then we define k matchings as described subsequently.

Let $G_1 = G$, and let M_1 be a matching in G_1 that covers all vertices of maximum degree. For each $i \in \{2, \ldots, k\}$, let G_i be the graph obtained from G_{i-1} by removing the edges in M_{i-1}, that is $G_i = G_{i-1} - M_{i-1}$, and let M_i be a matching in G_i that covers all vertices of degree $\Delta(G_i)$. Such a M_i exists since it is well known that every bipartite graph H has a proper $\Delta(H)$-edge-colouring. Clearly, we have $\Delta(G_i) = \Delta(G_{i-1}) - 1$, for each $i \in \{2, 3, \ldots, k\}$. Let $M := \bigcup_{i=1}^{k} M_i$. Observe that if a vertex has degree $\Delta(G) - k + j$ in G, where $j \in \{1, 2, \ldots, k\}$, then it is incident to at least j edges in M. Hence, for all $j \in \{1, 2, \ldots, k\}$ and for each vertex y in Y of degree $\Delta(G) - k + j$ in G, we reverse the orientation of exactly j edges in M incident to y. This ends the first phase.

The second phase reverses the orientation of some edges in $E(G) \setminus M$, step by step, in order to obtain a $(\Delta(G) - k)$-orientation. It remains to prove that this orientation is proper under the assumption of Theorem 1.

Before proving Theorem 1, let us introduce some notation and state a few properties of Algorithm 1. If $x \in X$ and $y \in Y$ are adjacent and have the same indegree, then there exists a *conflict* between x and y, and that x (or y) *conflicts*. For each $x \in X$ and $\ell \in \{0, 1, \ldots, \Delta(G)-1\}$, let $N_{\leq \ell}(x) = \{y \in Y : xy \in E(G) \text{ and } d^-(y) \leq \ell\}$ and, similarly, let $N_{=\ell}(x) = \{y \in Y : xy \in E(G) \text{ and } d^-(y) = \ell\}$.

Recall that the first phase of this algorithm corresponds to lines 1 to 8 and the second phase to lines 9 to 15. The 'for' loop from line 10 to 15 is called the *third loop*.

Claim 1.1. *At the end of the first phase of Algorithm 1, the following hold:*

(a) $d^-(x) \leq k$ for all $x \in X$, and $d^-(y) \leq \Delta(G) - k$ for all $y \in Y$; and

(b) if $x \in X$ is dominated by some $y \in Y$, then $d^-(y) = \Delta(G) - k$.

Proof. Both statements follow directly from the reversal of the orientation of some edges in M in the first phase. $\square$

In the second phase of the algorithm, we treat step by step the vertices $x \in X$ in such a way that, once they satisfy the condition of the 'while' loop (line 15), the reversal of the orientation of edges in line 14 ensures that:

(i) x does not conflict at the end of the current step; and

(ii) x will never conflict after the current step.

The main argument we use to prove the second fact is the following:

Claim 1.2. *In the second phase of Algorithm 1, the indegree of every vertex in X never decreases and the indegree of every vertex in Y never increases.*

Proof. This is straightforward, since all re-orientations are towards X. □

Moreover, recall that, when decreasing the indegree of vertices in Y in the second phase of the algorithm, we only use vertices that have indegree in $\{\Delta(G) - k - 1, \dots, 2\}$ at the end of the first phase.

Claim 1.3. *During a given iteration ℓ of the third loop of Algorithm 1, only vertices belonging to Y of indegree at most ℓ have their indegree decreased.*

Proof. The only edges that are re-oriented are those incident to vertices of $N_{\leq \ell}(x)$, which, by definition, have indegree at most ℓ. □

Using previous claims, we now prove that all vertices in X that are removed from $\widetilde{X}$ do not conflict at the end of the execution of Algorithm 1.

Claim 1.4. *Suppose that x is removed from $\widetilde{X}$ during a given iteration ℓ of the third loop of Algorithm 1. At the end of the execution of the algorithm, the following hold:*

Algorithm 1: Proper Orientation of Bipartite Graphs

Input: Bipartite graph $G = (X \cup Y, E)$ and $k \in \mathbb{N}$ s.t. $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$.
Output: Proper $(\Delta(G) - k)$-orientation for G.

1 $G_1 \longleftarrow G$
2 Orient all edges in G from X to Y
3 **for** $i = 1, \dots, k$ **do**
4     $M_i \longleftarrow$ matching of G_i saturating all vertices of degree $\Delta(G_i)$
5     $G_{i+1} \longleftarrow G_i - M_i$
6 $M \longleftarrow \bigcup_{i=1}^{k} M_i$
7 **foreach** $y \in Y$ **do**
8   reverse the orientation of $\max\{0; d_G(y) - \Delta(G) + k\}$ edges of M incident to y
9 $\widetilde{X} \longleftarrow X$
10 **for** $\ell = \Delta(G) - k - 1, \dots, 2$ **do**
11   **while** $\exists x \in X$ *s.t.* $|N_{\leq \ell}(x)| \geq \ell - d^-(x)$ *and* $|N_{=\ell}(x)| \leq \ell - d^-(x)$ **do**
12         $\widetilde{Y} \longleftarrow$ set of $\ell - d^-(x)$ vertices of highest indegree in $N_{\leq \ell}(x)$
13   **foreach** $y \in \widetilde{Y}$ **do**
14   Reverse the orientation of xy (i.e. re-orient xy towards x)
15         $\widetilde{X} \longleftarrow \widetilde{X} \setminus \{x\}$

(a) $d^-(x) = \ell$; and

(b) x has no neighbours of indegree ℓ.

Proof. First note that, by Claims 1.1 and 1.3, every vertex in $\widetilde{X}$ is dominated only by vertices of indegree $\Delta(G)-k$ at the beginning of the second phase. Therefore, at the beginning of iteration ℓ of the third loop, that is before re-orienting the edges incident to x, the vertex x dominates all vertices in $N_{\leq\ell}(x)$. Thus, after re-orienting $\ell - d^-(x)$ edges that are incident to x and to vertices of $N_{\leq\ell}(x)$, x has indegree exactly ℓ. Note that, after x is removed from $\widetilde{X}$, the indegree of x does not change in the remainder of the second phase. Therefore, Property (a) holds at the end of the execution of Algorithm 1.

Note also that, when x is removed from $\widetilde{X}$, it has at most $\ell - d^-(x)$ neighbours of indegree ℓ (due to the condition of the 'while' loop in line 15 of Algorithm 1), and all of these vertices belong to $\widetilde{Y}$. Thus, they have their indegree decreased by 1 (lines 14 and 14). Therefore, at the end of this iteration of the 'while' loop, x has no neighbour of indegree ℓ. By Claim 1.3 and since ℓ decreases throughout the algorithm, no vertex in Y of indegree strictly larger than ℓ has its indegree decreased during or after this iteration. Moreover, by Claim 1.2, no vertex in Y has its indegree increased. Therefore, we conclude that, at the end of the execution of Algorithm 1, Property (b) holds. □

With those claims in hands, we are now ready to prove Theorem 1.

Proof of Theorem 1. Assume that $G = (X \cup Y, E)$ is a bipartite graph such that $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$. We shall prove that Algorithm 1 returns a proper $(\Delta(G) - k)$-orientation of G. Note that the above inequality implies that $k < \frac{\Delta(G)}{2}$.

By Claim 1.4, at the end of the execution of Algorithm 1, all vertices in $X \setminus \widetilde{X}$ do not conflict. Thus, it suffices to show is that the vertices in $\widetilde{X}$ do not conflict.

Suppose to the contrary that there exists $x^* \in \widetilde{X}$ that conflicts at the end of the execution of Algorithm 1. Let $y^* \in Y$ be a neighbour of x^* such that $d^-(y^*) = d^-(x^*)$ at the end of the execution.

Firstly, note that, at each iteration of the third loop, the condition of the 'while' loop in line 15 is not satisfied by x^*. Thus, during the execution of the second phase, the indegree of x^* does not change. Moreover, by Claims 1.1 and 1.2, we have $1 \leq d^-(x^*) = d^-(y^*) \leq k$ at the end of the execution.

We claim that if x^* is dominated by some vertex $y \in Y$ at the end of the execution, then there is no conflict between x^* and y. Indeed, no edge is re-oriented towards x^* in the second phase, so y already dominates x at the end of the first phase. By Claim 1.1, y has indegree $\Delta(G) - k$ at the end of the first phase. Observe that the indegree of y does not decrease during the second phase because the third loop starts with the value $\Delta(G) - k - 1$ and, by Claim 1.3, only vertices in Y with degree at most this value have their indegrees decreased. Since $k < \frac{\Delta(G)}{2}$, there is no conflict between y and x^*. Thus, during all of the second phase, x^* is dominated by $d^-(x)$ vertices of indegree $\Delta(G) - k$ and dominates its remaining neighbours, among which is y^*.

We now use the fact that the 'while' condition is never satisfied by x^* to get the contradiction that $\Delta(G) \leq 2k + \frac{\sqrt{1+8k}+1}{2}$. By Claim 1.3, vertex y^* must have its indegree set to $d^-(x^*)$ no later than iteration $\ell_1 := d^-(x^*) + 1$ in the third loop. Thus, at the end of iteration ℓ_1, $|N_{\leq\ell_1}(x^*)| \geq \ell_1 - d^-(x^*) = 1$ holds since $y^* \in N_{\leq\ell_1}(x^*)$. Since the 'while' condition is never satisfied by x^*, we deduce that $|N_{=\ell_1}(x^*)| > \ell_1 - d^-(x^*) = 1$. Hence, x^* has at least two outneighbours, say y_1 and y_2, of indegree exactly ℓ_1. Note that, by Claim 1.3, y_1 and y_2 have indegree ℓ_1 at the end of the execution of the algorithm. In addition, these vertices are distinct from y^*.

During iteration ℓ_1, by Claim 1.3, only vertices of indegree at most ℓ_1 have their indegrees decreased. Thus, at the end of iteration $\ell_2 := d^-(x^*) + 2$, y_1 and y_2 already have indegree exactly ℓ_1, so we have $|N_{\leq \ell_2}(x^*)| \geq \ell_2 - d^-(x^*) = 2$. Since the 'while' condition is not satisfied by x^*, it has at least three outneighbours of indegree exactly ℓ_2 at the end of iteration ℓ_2, that is $|N_{=\ell_2}(x^*)| > 2$. Moreover, those vertices have indegree ℓ_2 at the end of the execution of the algorithm. In the same way, for all $i \in \{3, 4, \ldots, \Delta(G) - k - d^-(x^*) - 1\}$, setting $\ell_i = d^-(x^*) + i$, we show that x^* has at least $i+1$ outneighbours of indegree exactly ℓ_i at the end of iteration ℓ_i of the third loop, and thus at the end of the execution of Algorithm 1.

Hence, x^* dominates at least $\xi := \sum_{i=0}^{\Delta - k - d^-(x^*)-1}(i+1)$ vertices. However, x^* has at most $\Delta(G)$ neighbours, so

$$\xi = \frac{(\Delta(G) - k - d^-(x^*))\,(\Delta(G) - k - d^-(x^*) + 1)}{2} \leq \Delta(G) - d^-(x^*).$$

Solving this inequality, we obtain $\Delta(G) \leq k + d^-(x^*) + \frac{\sqrt{1+8k}+1}{2}$. Since $d^-(x^*) \leq k$, we conclude that $\Delta(G) \leq 2k + \frac{\sqrt{1+8k}+1}{2}$, which contradicts the hypothesis of the theorem. Therefore, no vertex in $\widetilde{X}$ conflicts and thus the orientation produced by Algorithm 1 is a proper $(\Delta(G)-k)$-orientation for G. □

Theorem 2. *If G is a bipartite graph, then*

$$\overrightarrow{\chi}(G) \leq \left\lfloor \frac{\Delta(G) + \sqrt{\Delta(G)}}{2} \right\rfloor + 1.$$

Proof. By Theorem 1, for every $k \in \mathbb{N}$, if $\Delta(G) > 2k + \frac{\sqrt{1+8k}+1}{2}$, then $\overrightarrow{\chi}(G) \leq \Delta(G) - k$. In order to obtain a good upper bound for $\overrightarrow{\chi}(G)$, we must find the largest positive integer k such that the condition of Theorem 1 holds for a given graph G.

Solving the inequality for k, we obtain that $k < \frac{\Delta(G) - \sqrt{\Delta(G)}}{2}$. Since k is integer, the condition holds when

$$k \leq \left\lceil \frac{\Delta(G) - \sqrt{\Delta(G)}}{2} \right\rceil - 1.$$

Therefore, we conclude that

$$\overrightarrow{\chi}(G) \leq \Delta(G) - \left\lceil \frac{\Delta(G) - \sqrt{\Delta(G)}}{2} \right\rceil + 1 = \left\lfloor \frac{\Delta(G) + \sqrt{\Delta(G)}}{2} \right\rfloor + 1.$$

□

Note that if G is bipartite and $\Delta(G) \in \{2, 3, 4\}$, then the bound of Theorem 2 is equal to the trivial upper bound $\overrightarrow{\chi}(G) \leq \Delta(G)$. For $\Delta(G) = 1$ and $\Delta(G) = 2$, this bound is tight due to the paths with 2 and 4 vertices, respectively. In the sequel, we present a tight example for the case $\Delta(G) = 3$.

Proposition 1. *There exists a bipartite graph G with $\Delta(G) = 3$ and $\overrightarrow{\chi}(G) = 3$.*

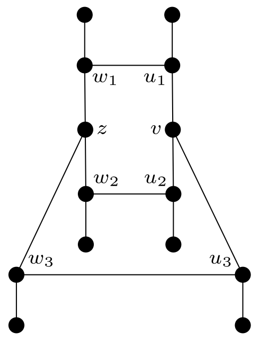


Figure 1: A bipartite graph B_3 for which $\overrightarrow{\chi}(B_3) = \Delta(B_3) = 3$

Proof. Consider the bipartite graph B_3 depicted in Figure 1. Suppose to the contrary that there exists a proper 2-orientation D of B_3. Observe that, if a vertex x has a neighbour of degree 1, say y, then, in any proper orientation, x has indegree 1 only if y dominates x.

If $d^-(v) = 0$, then v dominates u_i, for all $i \in \{1, 2, 3\}$. Note that $d^-(u_i) \neq 1$ for all $i \in \{1, 2, 3\}$ due to their neighbours of degree 1. Thus, we obtain $d^-(u_i) = 2$, for each $i \in \{1, 2, 3\}$. Moreover, w_i dominates z and u_i for all $i \in \{1, 2, 3\}$, because $d^-(w_i) \neq 2$ and w_i has a neighbour of degree 1. Therefore, we have $d^-(z) = 3$, a contradiction.

If $d^-(v) = 1$, then w.l.o.g. v is dominated by u_3, and dominates u_1 and u_2. Similarly to the previous case, we have that $d^-(u_1) = d^-(u_2) = 2$ and z is dominated by w_1 and w_2. Thus, $d^-(z) = 2$ and z dominates w_3, a contradiction, since w_3 has a neighbour of degree 1 and $d^-(w_3) \neq 2$.

Finally, if $d^-(v) = 2$, then v dominates u_i for some $i \in \{1, 2, 3\}$. We again arrive to a contradiction, since u_i has a neighbour of degree 1 and $d^-(u_i) \neq 2$.

Therefore, there exists no proper 2-orientation of B_3. Since $\Delta(B_3) = 3$, we conclude that $\overrightarrow{\chi}(B_3) = 3$. □

Ahadi and Dehghan proved that, for every $r \in \mathbb{N}$, if G is a bipartite r-regular graph, then $\overrightarrow{\chi}(G) = \lceil \frac{r+1}{2} \rceil$ [1]. In this sense, we next show an upper bound tighter than the one of Theorem 2 for bipartite graphs whose minimum degree is very close to its maximum degree.

Proposition 2. *If G is a bipartite graph, then*

$$\overrightarrow{\chi}(G) \leq \Delta(G) - \left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor .$$

Proof. Let (X, Y) be a bipartition of G and let $k = \left\lfloor \frac{\delta(G)-1}{2} \right\rfloor$. The following steps correspond to exactly the first phase of Algorithm 1.

Consider the orientation of G defined as follows. We first orient all edges $xy \in E(G)$ from x to y, where $x \in X$ and $y \in Y$. Then, we define k matchings as described subsequently. Let $G_1 = G$, and let M_1 be a matching in G_1 that covers all vertices of maximum degree. For each $i \in \{2, \ldots, k\}$, let $G_i = G_{i-1} - M_{i-1}$, and let M_i be a matching in G_i that covers all vertices of degree $\Delta(G_i)$. Clearly, we have $\Delta(G_i) = \Delta(G_{i-1}) - 1$, for each $i \in \{2, 3, \ldots, k\}$. Let $M := \bigcup_{i=1}^{k} M_i$. Observe that if a vertex has degree $\Delta(G) - k + j$ in G, where $j \in \{1, 2, \ldots, k\}$, then it is incident to at least j edges in M. Hence, for all $j \in \{1, 2, \ldots, k\}$ and for each vertex y in Y of degree $\Delta(G) - k + j$ in G, we reverse the orientation of exactly j edges in M incident to y.

In this orientation of G, every vertex of X has indegree at most k, and every vertex of Y has indegree at least $\delta(G) - k$, which is larger than k, and at most $\Delta(G) - k$. Therefore, we have a proper $(\Delta(G) - k)$-orientation of G. □

3 Trees

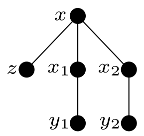
The main goal of this section is to prove the following theorem.

Theorem 3. *If T is a tree, then the following statements hold:*

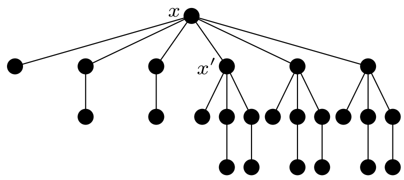
(1) if $\Delta(T) \leq 3$, then $\overrightarrow{\chi}(T) \leq 2$;
(2) if $\Delta(T) \leq 6$, then $\overrightarrow{\chi}(T) \leq 3$;
(3) $\overrightarrow{\chi}(T) \leq 4$.

Before proving Theorem 3, we show that the three statements of the theorem are tight in the following sense: there is a tree with maximum degree 4 and proper orientation number 3, and a tree with maximum degree 7 and proper orientation number 4.

Let R_2 and R_3 be the trees with root x depicted in Figures 2(a) and 2(b), respectively.



(a) Tree R_2 with root x.



(b) Tree R_3 with root x.

Figure 2: Rooted trees R_2 and R_3.

Let H be an induced proper subgraph of G and let $x \in V(H)$. The subgraph H is *x-pendant* if there exist no edges between $V(H) \setminus \{x\}$ and $V(G) \setminus V(H)$.

Lemma 1. *Let G be a graph containing an x-pendant R_2 and let D be a proper orientation of G.*

(i) If $d^-(x) = 1$, then x is dominated by z.
(ii) If $d^-(x) = 2$, then x is dominated by x_1 and x_2.

Moreover, if $d^-(x) \in \{1, 2\}$, then x dominates all its neighbours that are not in R_2.

Proof. (i) If $d^-(x) = 1$, then $d^-(z) \neq 1$ because D is proper. Therefore, we have $d^-(z) = 0$, and thus z dominates x.

(ii) Suppose that $d^-(x) = 2$ and x dominates x_i, for some $i \in \{1, 2\}$. Thus, $d^-(x_i) \geq 1$ and, since D is proper, we deduce that $d^-(x_i) = 1$. It follows that x_i dominates y_i, so $d^-(y_i) = 1$, which is a contradiction since D is proper. Therefore, x is dominated by x_1 and x_2. □

Corollary 1. *Let T_3 be the tree obtained from two copies of R_2 with roots x and $\hat{x}$ by adding the edge $x\hat{x}$. It follows that $\overrightarrow{\chi}(T_3) = 3$.*

Lemma 2. *Let G be a graph with an x-pendant R_3. In any proper 3-orientation of G, x dominates all its neighbours in $G - R_3$.*

Proof. Suppose to the contrary that there exists a proper 3-orientation D of G in which a vertex $y \in V(G-R_3)$ dominates x. It follows that $d^-(x) \geq 1$. By Lemma 1, we obtain $d^-(x) = 3$. Moreover, x dominates at least one of its neighbours of degree 4 in R_3, say x'. Since D is proper, $d^-(x') \in \{1, 2\}$. Therefore, the subtree R_2 rooted at x' contradicts Lemma 1. □

Corollary 2. *Let T_4 be the tree obtained from two copies of R_3 with roots x and $\hat{x}$ by adding the edge $x\hat{x}$. It follows that $\overrightarrow{\chi}(T_4) = 4$.*

In order to prove Theorem 3, we need a few definitions. In a tree T, a vertex x is a *twig-vertex* if it is not a leaf and all its neighbours except possibly one are leaves. A *twig* is a subtree induced by a twig-vertex and its adjacent leaves. The *twig rooted at* x is the twig that contains the twig-vertex x.

A vertex x of T is a *bough-vertex* if it is neither a leaf, nor a twig-vertex, and all its neighbours except possibly one are twig-vertices or leaves. A *bough* is a subtree induced by a bough-vertex x, the leaves adjacent to x, and the vertices of the twigs whose roots are adjacent to x. The *bough rooted at* x is the bough that contains the bough-vertex x.

A vertex x of T is a *branch-vertex* if it is neither a leaf, nor a twig-vertex, nor a bough-vertex, and all its neighbours except possibly one are bough-vertices, twig-vertices or leaves.

We now proceed to prove Theorem 3.

Proof of Theorem 3. We prove the three statements by using similar arguments. For $i \in \{1, 2, 3\}$, we consider a minimal counter-example M_i to statement (i) with respect to the number of vertices, and derive a contradiction that implies that no counter-example exists. Since M_i is a minimal counter-example, we have $\overrightarrow{\chi}(M_i) > i+1$, but $\overrightarrow{\chi}(T) \leq i+1$, for any proper subtree T of M_i. We use the latter fact to derive a proper $(i+1)$-orientation of M_i, which contradicts $\overrightarrow{\chi}(M_i) > i+1$.

Claim 3.1. *For every $i \in \{1, 2, 3\}$, every vertex of M_i is adjacent to at most one leaf.*

Proof. Suppose to the contrary that a vertex v is adjacent to two leaves, u and w. Let D be a proper $(i+1)$-orientation of $M_i - u$. In case $d_D^-(v) \neq 1$, one can obtain a proper $(i+1)$-orientation of M_i from D by orienting vu from v to u. Consequently, we have $d_D^-(v) = 1$. Thus, vw is oriented from w to v, because D is proper. Hence, v dominates all its neighbours except w in D, so $d_D^-(y) \geq 2$ for all $y \in N_{M_i}(v) \setminus \{u, w\}$. In particular, u and w are the only leaves adjacent to v. Therefore, one can obtain a proper $(i+1)$-orientation of M_i from D by reversing the orientation of vw and orienting vu from v to u, a contradiction. □

Claim 3.2. *For every $i \in \{1, 2, 3\}$, every twig-vertex of M_i is adjacent to exactly one leaf.*

Proof. It follows directly from the definition of twig-vertex and Claim 3.1. □

Claim 3.3. *For every $i \in \{1, 2, 3\}$, every vertex of M_i is adjacent to at most two twig-vertices.*

Proof. Suppose to the contrary that a vertex v is adjacent to p twig-vertices $t_1, \ldots, t_p$ with $p \geq 3$. By Claim 3.2, t_j is adjacent to a single leaf u_j, for every $j \in \{1, \ldots, p\}$. By minimality of M_i, the tree $T = M_i - \bigcup_{j=3}^{p} \{t_j, u_j\}$ has a proper $(i+1)$-orientation D. If $d_D^-(v) \neq 2$, then D can be extended to a proper $(i+1)$-orientation of M_i by orienting the edges vt_j and u_jt_j towards t_j, for all $j \in \{3, \ldots, p\}$. Hence, we have $d_D^-(v) = 2$, and so v must be dominated by t_1 and t_2 since D is proper. Thus, every vertex $y \in N_T(v) \setminus \{t_1, t_2\}$ satisfies $d_D^-(y) \geq 1$. Therefore, D can be extended to a proper $(i+1)$-orientation of M_i by orienting all edges vt_j and t_ju_j towards t_j, for every $j \in \{1, \ldots, p\}$, a contradiction. □

Claim 3.4. *For every $i \in \{1,2,3\}$, every bough-vertex of M_i is adjacent to exactly one leaf and exactly two twig-vertices. In other words, every bough is isomorphic to R_2.*

Proof. Let $i \in \{1,2,3\}$ and let v be a bough-vertex of M_i. By definition of bough-vertex, v has at most one neighbour that is neither a twig nor a leaf. Let us denote this vertex by r if it exists. Let $t_1, \dots, t_p$ be the twig-vertices adjacent to v. By definition of bough-vertex and Claim 3.3, we have $p \in \{1,2\}$. Furthermore, by Claim 3.2, t_j is adjacent to a single leaf u_j, for all $j \in \{1, \dots, p\}$.

By Claim 3.1, vertex v is adjacent to at most one leaf. Suppose to the contrary that v is adjacent to no leaves. Thus, $N_{M_i}(v) = \{t_1, \dots, t_p, r\}$. Let $T = M_i - \bigcup_{j=1}^{p}\{t_j, u_j\}$. By the minimality of M_i, T has a proper $(i+1)$-orientation D. Note that $d_D^-(v) \le d_T(v) \le 1$. Consequently, one can obtain a proper $(i+1)$-orientation of M_i by orienting the edges vt_j and t_ju_j towards t_j, for every $j \in \{1, \dots, p\}$, a contradiction. Hence, v is adjacent to exactly one leaf w.

Suppose to the contrary that $p = 1$. Let D be a proper $(i+1)$-orientation of $M_i - \{t_1, u_1\}$. If $d_D^-(v) \ne 2$, D can be extended to a proper $(i+1)$-orientation of M_i by orienting vt_1 and t_1u_1 towards t_1. It follows that $d_D^-(v) = 2$. Thus, the vertex r exists and rv and vw are oriented towards v. In this case, D can be extended to a proper $(i+1)$-orientation of M_i by reversing the orientation of vw, orienting vt_1 towards v, and t_1u_1 towards u_1, a contradiction. Therefore, we conclude that $p = 2$. □

We are now ready to prove (1). By Claim 3.4, the tree M_1 has neither branch-vertices nor two adjacent bough-vertices since $\Delta(M_1) \le 3$. Consequently, by Claims 3.2 and 3.4, the tree M_1 is either the tree R_2 depicted in Figure 2(a), or a path on at most four vertices. It is a simple matter to check that these trees have proper orientation number at most 2. Hence, $\overrightarrow{\chi}(M_1) \le 2$, a contradiction. This proves (1).

An orientation of a tree is *antidirected* if every vertex has indegree 0 or outdegree 0. There are two kinds of antidirected orientations of a rooted tree, one in which the root has indegree 0, called *out-antidirected*, and one in which the root has outdegree 0, called *in-antidirected*. An orientation of a bough in M_i is *nice* if it is obtained from an in-antidirected orientation by reversing the arc between the root and its adjacent leaf.

Claim 3.5. *For every $i \in \{2,3\}$, every vertex v of M_i is adjacent to at most three bough-vertices.*

Proof. The argument is similar to the one in Claim 3.3. Suppose to the contrary that v is adjacent to $p \ge 4$ bough-vertices b_j, $j \in \{1, \dots, p\}$. For $j \in \{1, \dots, p\}$, let B_j be the bough rooted at b_j, and let $T = M_i - \bigcup_{j=4}^{p} B_j$. By the minimality of M_i, T has a proper $(i+1)$-orientation D.

If $d_D^-(v) \ne 3$, then D can extended to a proper $(i+1)$-orientation of M_i by taking the nice orientation of B_j and orienting vb_j towards b_j for all $j \in \{4, \dots, p\}$. This orientation is proper because the indegrees of the vertices of D are unchanged and $d_D^-(b_j) = 3$ for all $j \in \{4, \dots, p\}$. Thus, we have $d_D^-(v) = 3$. Since D is proper, vertex v is dominated by b_1, b_2 and b_3 in D, and so it dominates all its neighbours that are not bough-vertices. Thus, one can obtain a proper $(i+1)$-orientation of M_i by taking the nice orientation of B_j and (re)-orienting vb_j towards b_j for all $j \in \{1, \dots, p\}$. This orientation is proper because the indegrees of the vertices of $V(T) \setminus (\bigcup_{j=1}^{3} V(B_j) \cup \{v\})$ are not changed, the orientation of B_j for $j \in \{1, \dots, p\}$ is proper, $d^-(v) = 0$ and all neighbours of v are dominated by v, and so they have indegree at least 1. □

Claim 3.6. *For every $i \in \{2,3\}$, every branch-vertex is adjacent to exactly one leaf, two twig-vertices and three bough-vertices. In other words, every branch is isomorphic to R_3.*

Proof. The arguments are similar to the ones presented in Claim 3.4.

Let $i \in \{2,3\}$ and let v be a branch-vertex of M_i. By definition of branch-vertex, v has at most one neighbour that is neither a bough, nor a twig, nor a leaf. Let us denote this vertex by r if it exists. Let $b_1,\dots,b_p$ be the bough-vertices that are adjacent to v, and $t_1,\dots,t_q$ be the twig-vertices adjacent to v. By definition of branch-vertex and Claim 3.5 we have $p \in \{1,2,3\}$, and by Claim 3.3 we have $q \in \{0,1,2\}$. For $j \in \{1,\dots,p\}$, let B_j be the bough rooted at b_j. By Claim 3.2, vertex t_k is adjacent to a single leaf u_k, for $k \in \{1,\dots,q\}$.

Let $T = M_i - (\bigcup_{j=1}^{p} B_j \cup \bigcup_{k=1}^{q} t_k)$. By minimality of M_i, the tree T admits a proper $(i+1)$-orientation D. By Claim 3.1, v is adjacent to at most one leaf. Suppose to the contrary that v has no leaves in its neighbourhood. Thus, $d_D^-(v) \le d_T(v) \le 1$. It follows that D can be extended to a proper $(i+1)$-orientation of M_i by taking the nice orientation of B_j and orienting the edge vb_j towards b_j, for every $j \in \{1,\dots,p\}$, and orienting the edges vt_k and t_ku_k towards t_k, for every $k \in \{1,\dots,q\}$. This orientation is indeed proper since $d^-(b_j) = 3$, for all $j \in \{1,\dots,p\}$, and $d^-(t_k) = 2$, for all $k \in \{1,\dots,q\}$. Therefore, v is adjacent to a single leaf w.

Suppose to the contrary that $q \le 1$. Thus, we have $d_D^-(v) \le d_T(v) \le 2$. If $d_D^-(v) \ne 2$, D can be extended to a proper $(i+1)$-orientation of M_i as above, that is by taking the nice orientation of B_j and orienting the edges vb_j towards b_j for $j \in \{1,\dots,p\}$, and orienting the edges vt_1 and t_1u_1 towards t_1 if $q = 1$. It follows that $d_D^-(v) = 2$, and both rv and wv are oriented towards v. Thus, one can obtain a proper $(i+1)$-orientation of M_i as follows. We use the nice orientation of B_j and orient the edges vb_j towards b_j, for each $j \in \{1,\dots,p\}$. In addition, we revert the orientation of vw, and orient vt_1 and t_1u_1 away from t_1 if $q = 1$. This orientation is indeed proper because the degree of every vertex in $T - w$ is not changed, $d_D^-(w) \in \{0,1\}$, $d_D^-(b_j) = 3$ for all $j \in \{1,\dots,p\}$, and $d_D^-(t_1) = 0$ if $q = 1$. Therefore, we obtain $q = 2$.

Suppose to the contrary that $p \le 2$. Let $T' = M_i - \bigcup_{j=1}^{p} B_j$. By minimality of M_i, T' admits a proper $(i+1)$-orientation D'. If $d_{D'}^-(v) \ne 3$, then by taking the nice orientation of B_j and orienting the edge vb_j towards b_j for $j \in \{1,\dots,p\}$, we obtain a proper $(i+1)$-orientation of M_i. If $d_{D'}^-(v) = 3$, then at least two vertices v_1 and v_2 in $\{w, t_1, t_2\}$ dominate v in D'. Reverting the orientations of the arcs incident to p vertices in $\{v_1, v_2\}$, orienting all edges vb_j towards v and taking the out-antidirected orientation of B_j for every $j \in \{1,\dots,p\}$, we obtain a proper $(i+1)$-orientation of M_i. Therefore, we conclude that $p = 3$ by Claim 3.5. □

Claim 3.7. *For every $i \in \{2,3\}$, M_i has a branch-vertex.*

Proof. Suppose to the contrary that M_i has no branch-vertex. By Claims 3.4 and 3.2, M_i is either T_3 (as defined in Corollary 1), or R_2, or a path on at most 4 vertices. By Corollary 1, $\overrightarrow{\chi}(T_3) = 3$, and it is easy to check that R_2 and paths on at most four vertices have proper orientation number at most 2, a contradiction. □

By Claim 3.7, M_2 has a branch-vertex. By Claim 3.6 and since $\Delta(M_2) \le 6$, we deduce that M_2 is isomorphic to R_3. One can check that $\overrightarrow{\chi}(R_3) = 3$, a contradiction. This proves (2).

By Claim 3.7, M_3 has a branch-vertex, say v. By Claim 3.6, v is adjacent to three bough-vertices b_1, b_2, b_3. Let B_j be the bough rooted at b_j for $j \in \{1,2,3\}$, and let $T = M_3 - (B_1 \cup B_2 \cup B_3)$. By minimality of M_3, T has a proper 4-orientation D. If $d_D^-(v) \ne 3$, then D can be extended to a proper 4-orientation of M_3 by taking the nice orientation of B_j and orienting the edge vb_j towards b_j, for every $j \in \{1,2,3\}$. If $d_D^-(v) = 3$, then D can be extended to a proper 4-orientation of M_3 by taking the in-antidirected orientation of B_j and orienting the edge vb_j towards b_j, for every $j \in \{1,2,3\}$. This proves (3). □

We now show a class of trees that have proper orientation number 2. A tree is called *even* if the distance between any two leaves is even.

Proposition 3. *If T is an even tree, then $\overrightarrow{\chi}(T) \leq 2$.*

Proof. Let r be the single neighbour of a leaf of T. Note that all leaves are at odd distance from r. We build a proper orientation D of T with maximum indegree at most 2 as follows:

- Orient all edges from r towards the leaves of T, in other words, orient the edges so that $d^-_D(r) = 0$ and $d^-_D(v) = 1$, for every $v \in V(T) \setminus \{r\}$.
- For each vertex at even distance from r (r excluded), reverse one of its outgoing edges. This is possible since none of these vertices are leaves.

Observe that the vertices with indegree 2 are exactly those at even distance from r (r excluded). Since r has indegree 0 and its neighbours have indegree 1, the obtained orientation is proper. $\square$

4 $\mathcal{NP}$-completeness

In this section, we study the computational complexity of determining the proper orientation number of a graph.

Ahadi and Dehgan [1] showed that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 2$ for planar graphs G by using a reduction from the Planar 3-SAT problem. We first improve this result by showing that it is $\mathcal{NP}$-complete to decide whether the proper orientation number of planar subcubic graphs is at most 2. A graph G is *subcubic* if $\Delta(G) \leq 3$. We also prove the following more general result, losing, however, the planarity property: for every integer $k \geq 3$, it is $\mathcal{NP}$-complete to determine whether $\overrightarrow{\chi}(G) < k$, for graphs G with maximum degree k. In the sequel, we show that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 3$, for planar bipartite graphs G with maximum degree 5.

In the proofs of these results, we present reductions from variants of the Planar 3-SAT problem. A 3-*CNF* formula is a boolean formula $\phi = (X, \mathcal{C})$ in conjunctive normal form in which X is a set of n variables, and $\mathcal{C}$ is a set of m clauses such that every clause has exactly 3 literals and does not contain more than one literal of the same variable. The *formula graph* of a 3-CNF formula $\phi = (X, \mathcal{C})$, denoted by $G(\phi)$, is a bipartite graph with vertex set $X \cup \mathcal{C}$ in which a variable-vertex x is connected by an edge to a clause-vertex C if, and only if, the clause C contains either the literal x or $\neg x$. We say ϕ is *planar* if, and only if, the formula graph $G(\phi)$ is planar. The Planar 3-SAT problem is equivalent to the 3-SAT problem restricted to planar formulas. It is known that the Planar 3-SAT problem is $\mathcal{NP}$-complete [5].

For ease of notation, we denoted by $[t]$ the set of integers $\{1, 2, \ldots, t\}$, for every $t \in \mathbb{N}$. Recall that an induced subgraph H of a graph G is x-pendant if $x \in V(H)$, and all edges between $V(H)$ and $V(G) \setminus V(H)$ are incident to x. In addition, if there exists a single edge xy between x and $V(G) \setminus V(H)$, we say that H is *x-pendant at y*. If H is vertex-transitive, we simply say that H is *pendant* at y.

4.1 Planar subcubic graphs

Recall that $\overrightarrow{\chi}(G) \leq \Delta(G)$, for any graph G. On the other hand, the following theorem shows that, for any integer $k \geq 3$, it is already $\mathcal{NP}$-complete to determine whether $\overrightarrow{\chi}(G) < k$, for graphs G with maximum degree k.

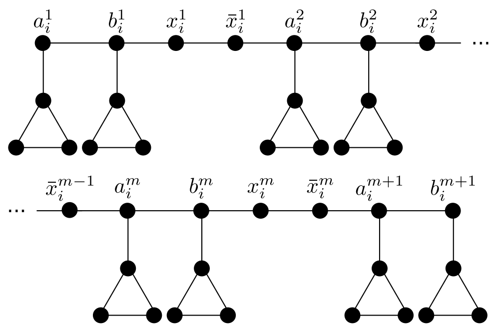
Theorem 4. *Let k be an integer such that $k \geq 3$. The following problem is $\mathcal{NP}$-complete:*
Input : *A graph G with $\Delta(G) = k$ and $\delta(G) = k - 1$.*
Question : *$\overrightarrow{\chi}(G) \leq k - 1$?*

We first prove the case where $k = 3$ and then show how to generalize the proof for larger values of k. Furthermore, the case $k = 3$ has the additional characteristic, which is not true in the general case, that we may assume the graph G to be planar.

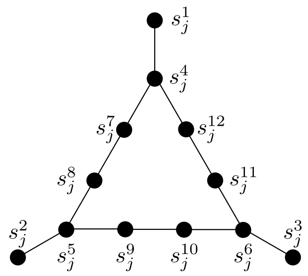
In order to prove that it is $\mathcal{NP}$-complete to decide whether the proper orientation number of a planar subcubic graph is at most 2, we modify the reduction proposed by Ahadi and Dehgan by using a different gadget for the variables of a 3-CNF formula. Let us first prove a lemma that allows us to force the orientation of some edges in any proper k-orientation of a graph.

Lemma 3. *Let k be a positive integer, and let G be a graph containing a clique K of size $k+1$. In any proper k-orientation of G, all edges between $V(K)$ and $V(G) \setminus V(K)$ are oriented from $V(K)$ to $V(G) \setminus V(K)$.*

Proof. Consider a proper k-orientation D of G. Clearly, all vertices of K must have distinct indegrees. Thus, for every $i \in \{0, \dots, k\}$, there exists exactly one vertex in K with indegree i. It follows that $D[V(K)]$ is a transitive tournament, and every vertex in K has all its in-neighbours belonging to K. Therefore, all edges in G between $V(K)$ and $V(G) \setminus V(K)$ are oriented from $V(K)$ to $V(G) \setminus V(K)$. $\square$

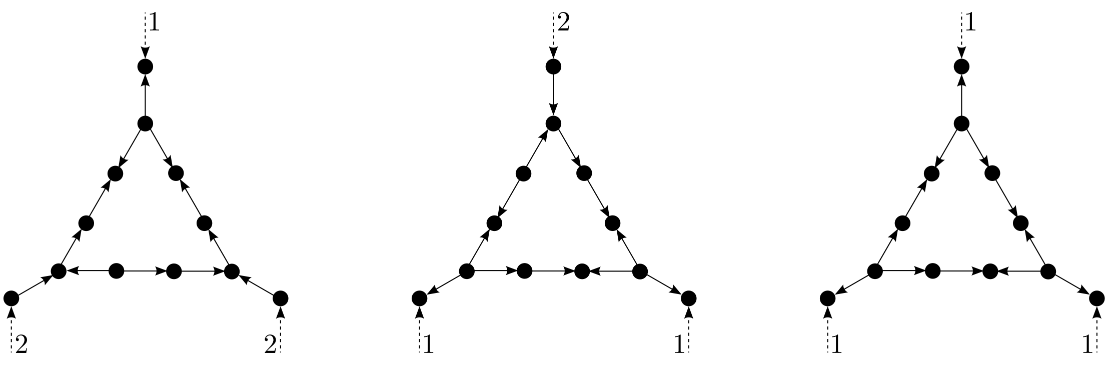


(a) VG_i.



(b) CG_j.

Figure 3: The variable gadget VG_i and the clause gadget CG_j.



(a) (b) (c)

Figure 4: Suitable clause orientations.

Theorem 5. *The following problem is $\mathcal{NP}$-complete:*
INPUT : *A planar graph G with $\Delta(G) = 3$ and $\delta(G) = 2$.*
QUESTION : $\overrightarrow{\chi}(G) \leq 2$*?*

Proof. This problem is trivially in $\mathcal{NP}$ because every proper 2-orientation is a certificate.

Let $\phi = (X, \mathcal{C})$ be an instance of the PLANAR 3-SAT problem where $X = \{x_1, \dots, x_n\}$ and $\mathcal{C} = \{C_1, \dots, C_m\}$, and let $G(\phi)$ be a formula graph of ϕ. In what follows, we show how to construct a planar graph $G'(\phi)$ such that $\overrightarrow{\chi}(G'(\phi)) \leq 2$ if, and only if, ϕ is satisfiable.

Firstly, for every variable x_i of X, we create a variable gadget VG_i (see Figure 3(a)). This gadget has $4m+2$ vertices: a_i^j, b_i^j, for every $j \in [m+1]$, and $x_i^j, \bar{x}_i^j$ for every $j \in [m]$. These vertices form a path P_i, where a_i^j is linked to b_i^j, for every $j \in [m+1]$; x_i^j is linked to $\bar{x}_i^j$ and to b_i^j, and $\bar{x}_i^j$ is linked to a_i^{j+1}, for every $j \in [m]$. In addition, for all $j \in [m+1]$, there is a pendant K_3 at a_i^j and a pendant K_3 at b_i^j.

Next, for every clause C_j in $\mathcal{C}$, we create the clause gadget CG_j depicted in Figure 3(b).

To finish the construction of $G'(\phi)$, we need to link the variable gadgets to the clause gadgets in such a way that the obtained graph is planar. Observe that we just need not to cross the edges linking the gadgets, because each gadget is itself planar and the formula graph $G(\phi)$ is planar. Consider a planar embedding $\tilde{G}(\phi)$ of $G(\phi)$. For each $x \in X$, let $\mathcal{C}_x = \{C \in \mathcal{C} \colon x \in C \text{ or } \neg x \in C\}$ and let $\pi_x \colon [|\mathcal{C}_x|] \to \mathcal{C}_x$ be a permutation of the elements in $\mathcal{C}_x$ in the clockwise order around x in the planar embedding $\tilde{G}(\phi)$. In the graph $G'(\phi)$, for $t \in [|\mathcal{C}_x|]$ and $C_j = \pi_x(t)$, we link the vertex x^t (resp. $\bar{x}^t$) in the gadget associated to the variable x to one of the vertices s_j^1, s_j^2 or s_j^3. We link the vertices of the variable gadgets to the vertices of the clause gadgets in such a way that, for every $j \in [m]$, the vertices s_j^1, s_j^2 and s_j^3 have degree 2 in $G'(\phi)$. Therefore, we have that $G'(\phi)$ is planar, $\delta(G'(\phi)) = 2$ and $\Delta(G'(\phi)) = 3$.

Let us now prove that ϕ is satisfiable if, and only if, $\overrightarrow{\chi}(G'(\phi)) \leq 2$.

Let D be a proper 2-orientation of $G'(\phi)$. Due to Lemma 3 and to the pendant K_3 at vertices of the path P_i, the indegree of each vertex in P_i must be either 1 or 2, for every $i \in [n]$. Consequently, for every fixed $i \in [n]$, we either have $d_D^-(x_i^j) = 1$ and $d_D^-(\bar{x}_i^j) = 2$, or $d_D^-(x_i^j) = 2$ and $d_D^-(\bar{x}_i^j) = 1$, for each $j \in [m]$. Moreover, these indegrees are just due to the orientations of the edges in the corresponding variable gadget. It means that every edge $xs \in E(G'(\phi))$ such that x belongs to a variable gadget and s belongs to a clause gadget is oriented from x to s. Thus, for every $j \in [m]$ and $t \in \{1, 2, 3\}$, the vertex s_j^t has indegree at least 1 in D and it must be different from the indegree of its neighbour in a variable gadget.

Claim 5.1. *In the clause gadget CG_j, at least one of the vertices s_j^1, s_j^2 and s_j^3 has indegree equal to 2.*

Proof. Suppose to the contrary that $d_D^-(s_j^1) = d_D^-(s_j^2) = d_D^-(s_j^3) = 1$. Thus, s_j^t dominates s_j^{t+3} and $d_D^-(s_j^{t+3}) = 2$, for all $t \in \{1, 2, 3\}$. Suppose first that s_j^{12} dominates s_j^4. It follows that s_j^4 dominates s_j^7. Consequently, s_j^7 has indegree 1 and dominates s_j^8, and so the edge $s_j^5 s_j^8$ cannot be properly oriented (recall that $d_D^-(s_j^5) = 2$). Similarly, one get a contradiction if s_j^4 dominates s_j^{12}. Therefore, we obtain that s_j^4 has indegree 1, a contradiction. □

By the construction of $G'(\phi)$, the orientation D induces an assignment $\Gamma \colon X \to \{\text{TRUE}, \text{FALSE}\}$ in which, for every $j \in [m]$, $\Gamma(x) = \text{TRUE}$ if, and only if, $d_D^-(x^j) = 1$. By Claim 5.1, for each $j \in [m]$, there exists $t \in \{1, 2, 3\}$ such that $d_D^-(s_j^t) = 2$. Thus, Γ is a truth assignment that satisfies ϕ.

Reciprocally, suppose that ϕ is satisfiable. Let $\Gamma \colon X \to \{\text{TRUE}, \text{FALSE}\}$ be a truth assignment that satisfies ϕ. Consider a variable $x \in X$. If $\Gamma(x) = \text{TRUE}$, then, orient $a^j b^j$ towards b^j

for every $j \in [m+1]$, and orient $b^j x^j$ towards x^j, $x^j \bar{x}^j$ towards $\bar{x}^j$ and $a^{j+1}\bar{x}^j$ towards $\bar{x}^j$ for every $j \in [m]$. If $\Gamma(x) = $ FALSE, orient $b^j a^j$ towards a^j for every $j \in [m+1]$, and orient $b^j x^j$ towards x^j, $\bar{x}^j x^j$ towards x^j and $a^{j+1}\bar{x}^j$ towards $\bar{x}^j$ for every $j \in [m]$. Thus, for every $j \in [m]$, if $\Gamma(x) = $ TRUE, then $d^-(x^j) = 1$ and $d^-(\bar{x}^j) = 2$, otherwise, $d^-(x^j) = 2$ and $d^-(\bar{x}^j) = 1$. Since Γ satisfies ϕ, no clause has three false literals. Consequently, no clause gadget has all three vertices s^1, s^2 and s^3 forced to have indegree 1 due to neighbours of indegree 2 in the corresponding variable gadgets. In this case, we can extend this proper 2-orientation of the variables gadgets to a proper 2-orientation of $G'(\phi)$ using one of the suitable clause orientations depicted in Figure 4. Therefore, we have $\overrightarrow{\chi}(G'(\phi)) \leq 2$. □

The reduction presented in the proof of Theorem 5 can be easily adapted to prove Theorem 4.

Proof of Theorem 4. Let ϕ be a 3-CNF formula and let $G'(\phi)$ be the graph described in the proof of Theorem 5. Let $k \geq 3$ be an integer and let $G''(\phi)$ be the graph obtained from $G'(\phi)$ by adding $k-3$ pendant K_k at each vertex of $G'(\phi)$, and replacing each pendant K_3 at a vertex by a pendant K_k at the same vertex. By the construction, one can easily check that $\delta(G''(\phi)) = k-1$ and $\Delta(G''(\phi)) = k$.

Note that, in every proper $(k-1)$-orientation, all the vertices of $G''(\phi)$ except those belonging to the pendant K_k's have indegree equal to $k-1$ or $k-2$. Therefore, using similar arguments to those presented in the proof of Theorem 5, we conclude that ϕ is satisfiable if, and only if, $\overrightarrow{\chi}(G''(\phi)) \leq k-1$. □

Theorem 6. *The following problem is* $\mathcal{NP}$*-complete:*
INPUT : *A planar graph* G *with* $\Delta(G) = 4$ *and* $\delta(G) = 3$.
QUESTION : $\overrightarrow{\chi}(G) \leq 3$*?*

Proof. Considering a planar 3-CNF formula and observing that the complete graph on four vertices K_4 is planar, the result follows directly from the proof of Theorem 4. □

4.2 Planar bipartite graphs

In this subsection, we prove that computing the proper orientation number of a graph is still $\mathcal{NP}$-hard for planar bipartite graphs of bounded degree. The idea of our reduction is roughly the same as in Theorems 4 and 5, although we use another class of 3-SAT formulas in order to obtain bipartite instances and the gadgets of the reduction are more complicated.

The gadget we use to replace the pendant cliques of the proofs of Theorems 4 and 5 is the graph B_5, depicted in Figure 5. The following lemma, analogously to Lemma 3, allows us to force the orientation of some edges in any proper 3-orientation of a graph.

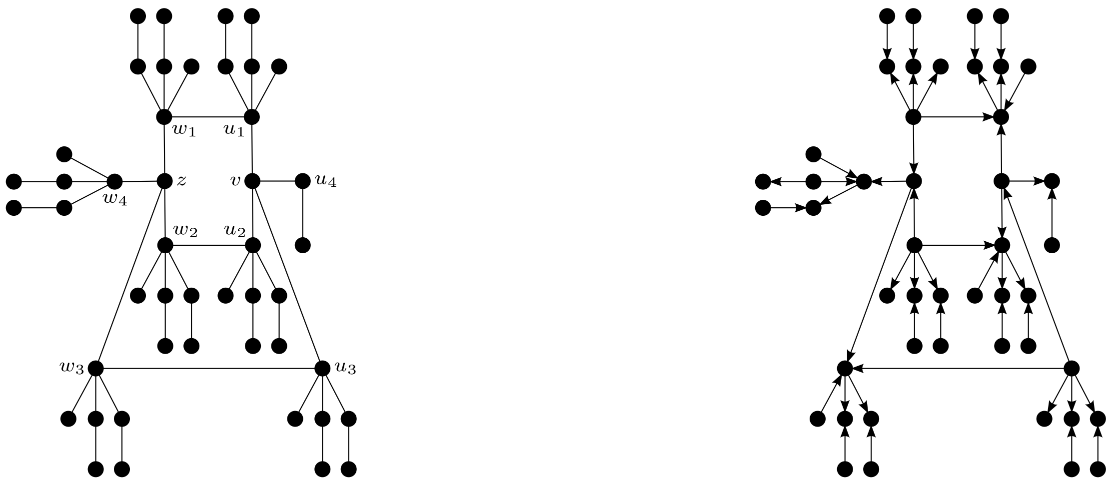
Lemma 4. *Let* G *be a graph with a* v*-pendant* B_5 *at* y*. In any proper* 3*-orientation of* G*,* y *is dominated by* v*.*

Proof. Suppose to the contrary that there exists a proper 3-orientation D of G such that y dominates v.

If $d^-(v) = 1$, then v dominates u_i for all $i \in \{1,2,3,4\}$. Moreover, by Lemma 1, $d^-(u_i) = 3$ for $i \in \{1,2,3\}$. Thus, for $i \in \{1,2,3\}$, $d^-(w_i) \leq 2$. Hence, again by Lemma 1, w_i dominates u_i and z. It follows that $d^-(z) = 3$ and z dominates w_4. This contradicts Lemma 1.

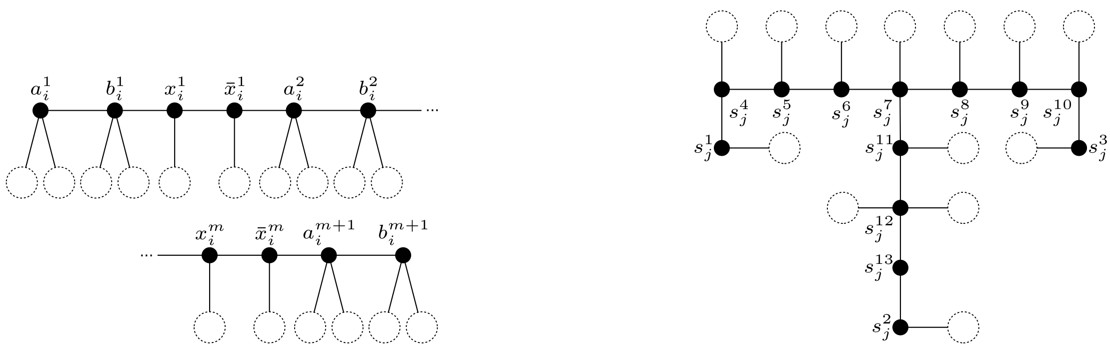
If $d^-(v) = 2$, then v is dominated by u_4, otherwise u_4 has the same indegree as one of its two neighbours. Thus, v dominates u_i for $i \in \{1,2,3\}$. Similarly to the previous case, we obtain a contradiction.

If $d^-(v) = 3$, then v dominates u_i, for some $i \in \{1,2,3\}$. Therefore, $d^-(u_i) \in \{1,2\}$, which contradicts Lemma 1. □



(a) (b)

Figure 5: The bipartite graph B_5 (a) and one of its proper 3-orientations (b).



(a) VG'_i. (b) CG'_j.

Figure 6: The variable gadget VG'_i and the clause gadget CG'_j.

By Theorem 1, all planar bipartite graphs with maximum degree 5 have proper orientation at most 4 . On the other hand, we next show that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) \leq 3$ for such graphs.

Theorem 7. *The following problem is $\mathcal{NP}$-complete:*
INPUT : *A planar bipartite graph G with $\Delta(G) = 5$.*
QUESTION : *$\overrightarrow{\chi}(G) \leq 3$?*

Proof. Again, this problem is in $\mathcal{NP}$ because any proper 3-orientation of G is a certificate.

A 3-SAT formula ϕ is said to be *monotone* if each clause has only positive literals (a positive clause) or only negative literals (a negative clause). Recall that a 3-CNF formula ϕ is planar if the formula graph $G(\phi)$ is planar. The problem of deciding whether a planar monotone 3-SAT formula is satisfiable was recently shown to be $\mathcal{NP}$-complete [2]. Let ϕ be a planar monotone 3-CNF formula with clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$ and variables $X = \{x_1, \ldots, x_n\}$ and let $G(\phi)$ be the formula graph graph corresponding to ϕ. In what follows, we show how to construct a planar bipartite graph $G'(\phi)$ with $\Delta(G) = 5$ such that $\overrightarrow{\chi}(G'(\phi)) \leq 3$ if, and only if, ϕ is satisfiable.

Firstly, for each variable x_i, we create a variable gadget VG'_i depicted in Figure 6(a). This gadget has $4m + 2$ vertices: a_i^j, b_i^j for every $j \in [m + 1]$, and $x_i^j, \bar{x}_i^j$ for every $j \in [m]$. These vertices form a path P_i, where a_i^j is linked to b_i^j, for every $j \in [m+1]$; x_i^j is linked to $\bar{x}_i^j$ and to b_i^j, and $\bar{x}_i^j$ is linked to a_i^{j+1}, for every $j \in [m]$. The gadget VG'_i also contains $6m + 4$ v-pendant B_5: two at a_i^j, two at b_i^j, for every $j \in [m+1]$, one at x_i^j and another at $\bar{x}_i^j$ for every $j \in [m]$ (these B_5 graphs are represented by the dashed white circles in Figure 6(a)).

For every clause C_j of C, we create a clause gadget CG'_j as depicted in Figure 6(b). It is obtained from two paths $s_j^1 s_j^4 \ldots s_j^{10} s_j^3$ and $s_j^{11} s_j^{12} s_j^{13} s_j^2$ linked by the edge $s_j^7 s_j^{11}$. Moreover, there exists a v-pendant B_5 at all these vertices, except at s_j^{13}, and we have an additional v-pendant B_5 at s_j^{12} (these B_5 graphs are represented by the dashed white circles in Figure 6(b)).

To finish the construction of $G'(\phi)$, we need to link the variable gadgets to the clause gadgets in such a way that the obtained graph is planar. Observe that we just need not to cross the edges linking the gadgets since each gadget is itself planar and the formula graph $G(\phi)$ is planar. We can accomplish this in the same way as described in the proof of Theorem 5. Since each clause has exactly three literals, we add these edges so that, for every $j \in [m]$, the vertices s_j^1, s_j^2 and s_j^3 have degree 3 in $G'(\phi)$. Thus, $G'(\phi)$ is planar, and $\Delta(G'(\phi)) = 5$ due to the graph B_5.

Let us now prove that $G'(\phi)$ is bipartite. Clearly, we can properly colour the vertices of each variable gadget with colours A and B. Suppose that we assign the colour A to the vertex x_i^j, for each $i \in [n]$ and $j \in [m]$. For each $j \in [m]$, we can assign the same colour (A or B) to the vertices s_j^1, s_j^2 and s_j^3 since ϕ is monotone. Note that we can easily extend this partial colouring to the other vertices in each clause and variable gadget in $G'(\phi)$. Thus, $G'(\phi)$ is bipartite.

Let us now prove that $\overrightarrow{\chi}(G'(\phi)) \leq 3$ if, and only if, ϕ is satisfiable.

Suppose first that $\overrightarrow{\chi}(G'(\phi)) \leq 3$. Let D be a proper 3-orientation of $G'(\phi)$. By Lemma 4 and due to the v-pendant B_5 graphs that are attached to the vertices in each path P_i, the indegrees of the vertices in this path are either 2 or 3, for every $i \in [n]$. Consequently, for a given $i \in [n]$, we either have $d_D^-(x_i^j) = 2$ and $d_D^-(\bar{x}_i^j) = 3$, or $d_D^-(x_i^j) = 3$ and $d_D^-(\bar{x}_i^j) = 2$, for all $j \in [m]$. Moreover, these indegrees are just due to the orientations of the edges belonging to the corresponding variable gadget. Thus, every edge $xs \in E(G'(\phi))$ such that x belongs to a variable gadget and s belongs to a clause gadget is oriented from x to s. Consequently, for each $j \in [m]$ and $t \in \{1, 2, 3\}$, the vertex s_j^t has indegree at least 2 in D since it has a v-pendant B_5 attached to it.

Claim 7.1. *In every proper 3-orientation of $G'(\phi)$, for every $j \in [m]$, one of the vertices s_j^1, s_j^2 and s_j^3 has indegree 3.*

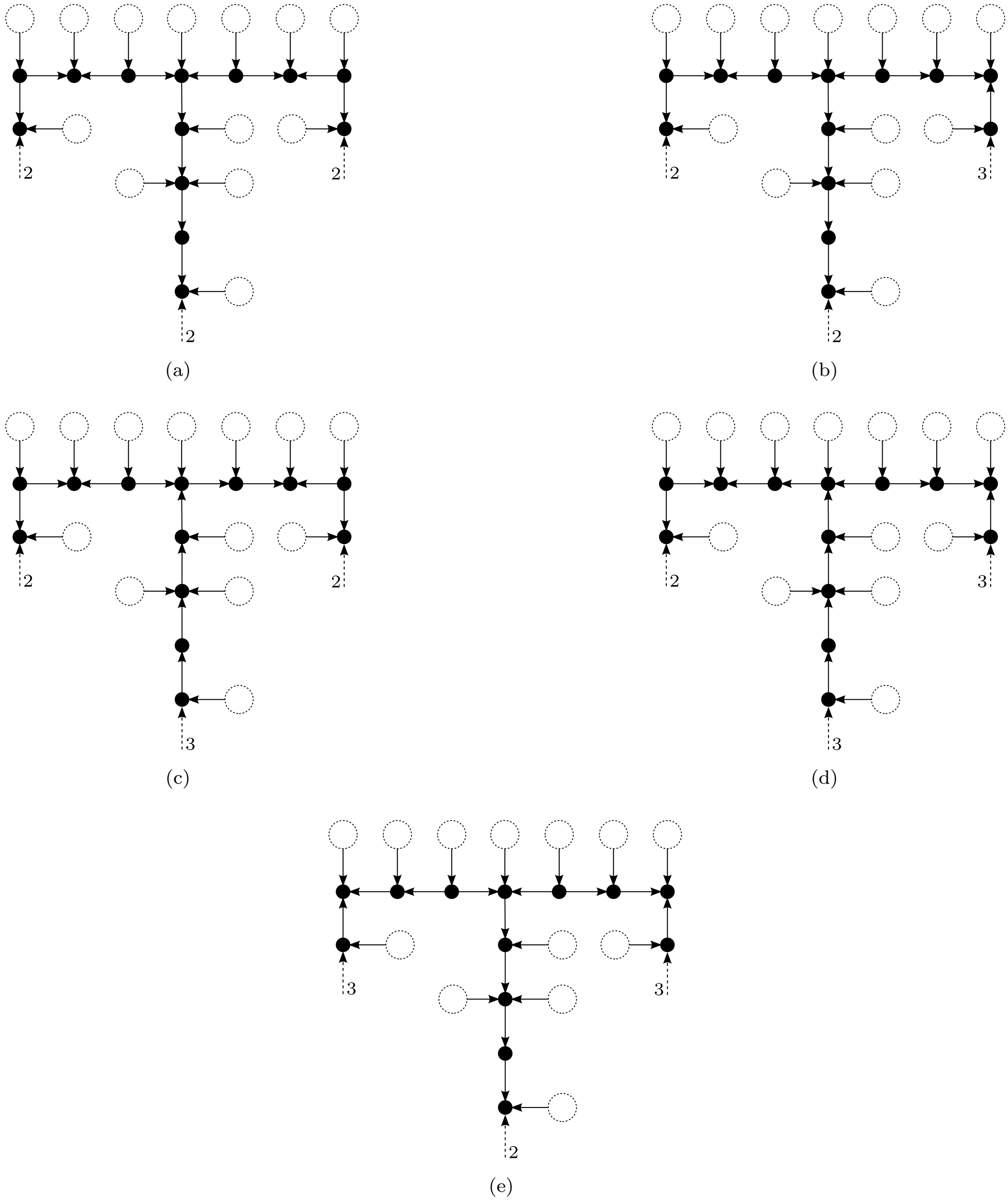


Figure 7: Suitable 3-orientations of the clauses.

Proof. Suppose to the contrary that there exists a proper 3-orientation D of $G'(\phi)$ such that $d^-_D(s^1_j) = d^-_D(s^2_j) = d^-_D(s^3_j) = 2$, for some $j \in [m]$. Thus, the edges $s^2_j s^{13}_j$, $s^{13}_j s^{12}_j$, $s^{12}_j s^{11}_j$ and $s^{11}_j s^7_j$ are oriented from s^2_j to s^{13}_j, s^{13}_j to s^{12}_j, s^{12}_j to s^{11}_j, and from s^{11}_j to s^7_j in D. Since s^7_j is adjacent to a vertex with indegree 2 and it has indegree at least 2 in this orientation, it is dominated by either s^6_j or s^8_j. If s^6_j dominates s^7_j, then the edges $s^7_j s^8_j$, $s^8_j s^9_j$, and $s^9_j s^{10}_j$ are oriented from s^7_j to s^8_j, s^8_j to s^9_j, and from s^{10}_j to s^9_j. Therefore, we have $d^-_{D'}(s^3_j) = d^-_{D'}(s^{10}_j) = 2$, a contradiction. By symmetry, we obtain a similar contradiction if s^8_j dominates s^7_j. □

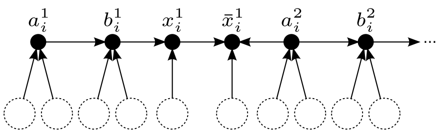
Let $\Gamma\colon X \to \{\textsc{true}, \textsc{false}\}$ be the truth assignment defined as $\Gamma(x) = \textsc{true}$ if, and only if, $d^-_D(x^j) = 2$ for some $j \in [m]$. By Claim 7.1, Γ satisfies ϕ.

Reciprocally, suppose that ϕ is satisfiable. Let $\Gamma\colon X \to \{\textsc{true}, \textsc{false}\}$ be a truth assignment that satisfies ϕ.

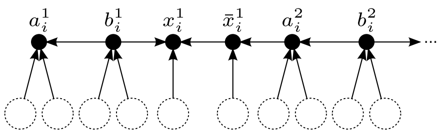
First, orient each v-pendant B_5 according to Figure 5(b). Moreover, for each v-pendant B_5, orient the edge between v and y, the only neighbour of v in $V(G'(\phi)) \setminus V(B_5)$, towards y. Note that the indegree of v is equal to 1 in every v-pendant B_5. Consider a variable $x \in X$. We define an orientation of $G'(\phi)$ as follows:

- If $\Gamma(x) = \textsc{true}$, orient $a^j b^j$ towards b^j for every $j \in [m+1]$, and orient $b^j x^j$ towards x^j, $x^j \bar{x}^j$ towards $\bar{x}^j$ and $a^{j+1}\bar{x}^j$ towards $\bar{x}^j$ for every $j \in [m]$ (see Figure 8(a)).
- If $\Gamma(x) = \textsc{false}$, orient $b^j a^j$ towards a^j for every $j \in [m+1]$, and orient $b^j x^j$ towards x^j, $\bar{x}^j x^j$ towards x^j and $a^{j+1}\bar{x}^j$ towards $\bar{x}^j$ for every $j \in [m]$ (see Figure 8(b)).

Thus, if $\Gamma(x) = \textsc{true}$, then $d^-_D(x^j) = 2$ and $d^-_D(\bar{x}^j) = 3$, otherwise $d^-_D(x^j) = 3$ and $d^-_D(\bar{x}^j) = 2$, for every $j \in [m]$. Since Γ satisfies ϕ, no clause has three false literals. Consequently, in every clause gadget, at least one the vertices s^1, s^2 and s^3 has indegree different from 2 due to its neighbours in the corresponding variable gadgets. Thus, we can extend this proper 3-orientation of the variables gadgets to a proper 3-orientation of $G'(\phi)$ using one of the suitable clause orientations depicted in Figure 7. Note that the orientation of $G'(\phi)$ described above is proper and each vertex has indegree at most 3. Therefore, $\overrightarrow{\chi}(G'(\phi)) \le 3$. □



(a) Orientation TRUE.



(b) Orientation FALSE.

Figure 8: Suitable orientations of the gadget x_i.

5 Conclusive remarks and further research

5.1 Around Problem 1

The upper bound given by Theorem 2 is likely not tight. Hence, it is natural to ask for the maximum proper orientation number $f(k)$ over all bipartite graphs with maximum degree k.

Since a k-regular graph on n vertices has $nk/2$ edges, all its orientations must have maximum indegree at least $\lceil \frac{k+1}{2} \rceil$. Together with Theorem 2, this implies

$$\left\lceil \frac{k+1}{2} \right\rceil \leq f(k) \leq \left\lfloor \frac{k+\sqrt{k}}{2} \right\rfloor + 1.$$

We have shown that $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$. The next question is to determine whether $f(4)$ equals 3 or 4. More generally, one may ask the following.

Problem 2. Does there exists a constant C such that $\overrightarrow{\chi}(G) \leq \frac{\Delta(G)}{2} + C$ for all bipartite graph G? In other words, does $f(k) \leq \frac{k}{2} + C$ hold for all $k \in \mathbb{N}$?

This is true for bipartite graphs whose minimum degree is very close from its maximum degree as shown by Proposition 2.

Problem 1(a) may be seen as determining the largest ϵ_0 such that $\overrightarrow{\chi}(G) \leq \epsilon_0 \cdot \omega(G) + (1 - \epsilon_0)\Delta(G)$ for all graph G. Regular graphs show that $\epsilon_0 \leq 1/2$. Such graphs might be extremal graphs with respect to Problem 1(a).

Problem 3. Is the following relaxation of Conjecture 1 true?

$$\text{For every graph } G,\ \overrightarrow{\chi}(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil.$$

A natural approach towards this problem and Problem 1(a) would be to study triangle-free graphs, that are the graphs with clique number 2. A first step would be to improve the upper bound on the proper orientation number for $\Delta(G)$ large.

5.2 Graphs with bounded treewidth or bounded maximum average degree

Let us denote by $\mathrm{tw}(G)$ the treewidth of G. It is well-known that $\chi(G) \leq \mathrm{tw}(G) + 1$. Theorem 3 states that the proper orientation number of graphs with treewidth 1 is bounded. It is then natural to ask if the same holds for larger values of the treewidth.

Problem 4. Can $\overrightarrow{\chi}(G)$ be bounded by a function of $\mathrm{tw}(G)$?

One can observe that, for fixed integers t and k, determining whether $\overrightarrow{\chi}(G) \leq k$ in a graph G of treewidth at most t can be done in polynomial time using a standard dynamic programming approach.

It is well known that graphs with bounded treewidth have bounded maximum average degree. Recall that the *maximum average degree* $\mathrm{Mad}(G)$ of a graph G is defined as

$$\mathrm{Mad}(G) = \max\left\{ \frac{2|E(H)|}{|V(H)|} : H \text{ is a subgraph of } G \right\}.$$

It is well-known [7] that every graph G admits an orientation with maximum outdegree at most $\lceil \mathrm{Mad}(G)/2 \rceil$. Moreover, $\chi(G) \leq \mathrm{Mad}(G)+1$ because every graph G is $\lfloor \mathrm{Mad}(G) \rfloor$-degenerate. It is thus very natural to generalize Problem 4 and to look for an upper-bound on $\overrightarrow{\chi}(G)$ depending only on $\mathrm{Mad}(G)$.

Problem 5. Can $\overrightarrow{\chi}(G)$ be bounded by a function of $\mathrm{Mad}(G)$?

Theorem 3 settles Problem 5 for graphs G with $\mathrm{Mad}(G) < 2$, i.e. forests. A next step towards Problem 5 would be to prove that the proper orientation number is bounded for some classes of graphs in which the maximum average degree is bounded. All minor-closed families (except the one of all graphs) are such classes. The most famous of these families and perhaps a natural direction for future research is the class of planar graphs.

Problem 6. Does there exist a constant k such that $\overrightarrow{\chi}(G) \leq k$ for all planar graphs G?

5.3 Split graphs

A *split graph* is a graph whose vertex set may be partitioned into a clique K and a stable set S. We assume, without loss of generality, that K is maximal, that is no vertex in S is adjacent to all vertices in K. The pair (K, S) is then called a *canonical partition* of G. For such a partition, we have $\omega(G) = |K|$. It is well-known that split graphs can be recognized in polynomial time, and that finding a canonical partition of a split graph can also be found in polynomial time.

Proposition 4. *If G is a split graph, then one can decide in polynomial-time whether $\overrightarrow{\chi}(G) = \omega(G) - 1$.*

Proof. Let (K, S) be a canonical partition of G. Let H be the following bipartite graph obtained from G: the vertex set of H is $K \cup \{w_0, \dots, w_{|K|-1}\}$ and, for every $v \in K$ and $i \in \{0, \dots, |K|-1\}$, there exists an edge between v and w_i if, and only if, v has no neighbour (in S) with degree i in G.

Let us now prove that $\overrightarrow{\chi}(G) = \omega(G) - 1$ if, and only if, H has a perfect matching.

Suppose that there exists a proper orientation D of G with maximum indegree $|K| - 1$. By Lemma 3, all edges between K and S must be oriented from K to S, so $d^-_D(u) = d_G(u)$ for all $u \in S$. It follows that, for each $v \in K$, $d^-_D(v) \notin \{d_G(u) \colon u \in S \cap N(v)\}$. Moreover, there exists exactly one vertex in K of indegree i, for each $i \in \{0, \dots, |K| - 1\}$. Therefore, the edge set $\{vw_i \colon v \in K \text{ and } i = d^-_D(v)\}$ is a perfect matching in H.

Suppose that H has a perfect matching M. Consider the orientation of G defined as follows. We orient the edges in the clique K so that $v \in K$ has indegree i if, and only if, $vw_i \in M$, where $i \in \{0, \dots, |K| - 1\}$. In addition, we orient all edges between K and S towards S in the graph G. Note that $d_G(u) < |K| - 1$ for all $u \in S$ since (K, S) is a canonical partition. By the construction of H, the orientation described above is a proper $(\omega(G) - 1)$-orientation of G. $\square$

However, one further step does not seem trivial.

Problem 7. Can one decide in polynomial time whether $\overrightarrow{\chi}(G) \leq \omega(G)$ for every split graph G?

Problem 8. Can one decide in polynomial time whether $\overrightarrow{\chi}(G) = \omega(G) - 1$ for every complement of bipartite graph G?

Problem 9.

(a) Does there exist a function h such that $\overrightarrow{\chi}(G) \leq h(\omega(G))$ for every split graph G ?

(b) Does there exist a function h such that $\overrightarrow{\chi}(G) \leq h(\omega(G))$ for every complement of bipartite graph G ?

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