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# (Circular) backbone colouring: forest backbones in planar graphs 

Frédéric Havet* Andrew D. King ${ }^{\dagger} \quad$ Mathieu Liedloff ${ }^{\ddagger} \quad$ Ioan Todinca ${ }^{\ddagger}$

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#### Abstract

Consider an undirected graph $G$ and a subgraph $H$ of $G$, on the same vertex set. The $q$ backbone chromatic number $\mathrm{BBC}_{q}(G, H)$ is the minimum $k$ such that $G$ can be properly coloured with colours from $\{1, \ldots, k\}$, and moreover for each edge of $H$, the colours of its ends differ by at least $q$. In this paper we focus on the case when $G$ is planar and $H$ is a forest. We give a series of NP-hardness results as well as upper bounds for $\mathrm{BBC}_{q}(G, H)$, depending on the type of the forest (matching, galaxy, spanning tree). Eventually, we discuss a circular version of the problem.


## 1 Introduction

All the graphs considered in this paper are simple. Let $G=(V, E)$ be a graph, and let $H=(V, E(H))$ be a spanning subgraph of $G$, called the backbone. A $k$-colouring of $G$ is a mapping $f: V \rightarrow$ $\{1,2, \ldots, k\}$. Let $f$ be a $k$-colouring of $G$. It is a proper colouring if $|f(u)-f(v)| \geq 1$ for all edges $u v \in E(G)$. It is a $q$-backbone colouring for $(G, H)$ if $f$ is a proper colouring of $G$ and $|f(u)-f(v)| \geq q$ for all edges $u v \in E(H)$. The chromatic number $\chi(G)$ is the smallest integer $k$ for which there exists a proper $k$-colouring of $G$. The $q$-backbone chromatic number $\mathrm{BBC}_{q}(G, H)$ is the smallest integer $k$ for which there exists a $q$-backbone $k$-colouring of $(G, H)$.

If $f$ is a proper $k$-colouring of $G$, then $g$ defined by $g(v)=q \cdot f(v)-q+1$ is a $q$-backbone colouring of $(G, H)$ for any spanning subgraph $H$ of $G$. Moreover it is well-known that if $G=H$, this $q$-backbone colouring of $(G, H)$ is optimal. Therefore, since $\mathrm{BBC}_{q}(H, H) \leq \mathrm{BBC}_{q}(G, H) \leq$ $\mathrm{BBC}_{q}(G, G)$, we have

$$
\begin{equation*}
q \cdot \chi(H)-q+1 \leq \mathrm{BBC}_{q}(G, H) \leq q \cdot \chi(G)-q+1 \tag{1}
\end{equation*}
$$

If $H$ is empty (i.e. $E(H)=\emptyset$ ), then $\operatorname{BBC}_{q}(G, H)=\chi(G)$. Hence for any $k \geq 3$, deciding if $\mathrm{BBC}_{q}(G, H) \leq k$ is NP-complete because deciding if a graph is $k$-colourable is NP-complete (See [7]). However, when we impose $G$ or $H$ to belong to certain graph classes, the problem sometimes become polynomial-time solvable. A trivial example is when we consider $H$ with chromatic number at least $r>(k+q-1) / q$. Then $\mathrm{BBC}_{q}(G, H) \geq r q-q+1$, and so deciding if $B B C_{q}(G, H) \leq k$

[^0]can be done instantly by always returning 'no'. A less trivial example is when we impose $H$ to have minimum degree 1 . For such an $H$, deciding if $B B C_{q}(G, H) \leq q+1$ is also polynomial-time solvable, because $\mathrm{BBC}_{q}(G, H)=q+1$ if and only if $G$ is bipartite. This simple observation was already made by Broersma et al. [5] when $H$ is a 1 -factor (a spanning subgraph in which every vertex has degree exactly 1). Furthermore, if we also impose $H$ to be connected, we show in Theorem 17 that deciding if $B B C_{q}(G, H) \leq q+2$ can be done in polynomial time. In contrast, if the condition of $H$ being connected is removed, then it is NP-complete (Theorem 18).

In this paper, we will focus on the particular case when $G$ is a planar graph and $H$ is a forest (i.e. an acyclic graph). Inequality (1) and the Four-Colour Theorem imply that for any planar graph $G$ and spanning subgraph $H, \mathrm{BBC}_{q}(G, H) \leq 3 q+1$. However, for $q=2$, Broersma et al. [4] conjectured that this is not best possible if the backbone is a forest.

Conjecture 1. If $G$ is a planar graph and $F$ a forest in $G$, then $\mathrm{BBC}_{2}(G, F) \leq 6$.
If true, Conjecture 1 would be best possible. Broersma et al. [4] gave an example of a graph $\hat{G}$ with a forest $\hat{F}$ such that $\mathrm{BBC}_{2}(\hat{G}, \hat{F})=6$. See Figure 1. It is then natural to ask how large $\mathrm{BBC}_{q}(G, F)$


Figure 1: A planar graph $\hat{G}$ with a forest $\hat{F}$ (bold edges) such that $\mathrm{BBC}_{q}(\hat{G}, \hat{F})=q+4$.
could be when $G$ is planar and $F$ is a forest for larger values of $q$. We prove the following.
Theorem 2. If $G$ is a planar graph and $F$ a forest in $G$, then $\operatorname{BBC}_{q}(G, F) \leq q+6$.
In fact, we prove a more general result in Proposition 13 : for any pair $(G, H)$ with $H$ a subgraph of $G$,

$$
\mathrm{BBC}_{q}(G, H) \leq(\chi(G)+q-2) \chi(H)-q+2 .
$$

For $q \geq 4$, Theorem 2 is best possible. Indeed, we show a planar graph $G^{*}$ together with a spanning tree $T^{*}$ such that $\mathrm{BBC}_{q}\left(G^{*}, T^{*}\right)=q+6$ for all $q \geq 4$. See Figure 2 and Proposition 15 .

Furthermore, we show in Theorem 31, that for any fixed $q \geq 4$, given a planar graph $G$ and a spanning tree $T$ of $G$, it is NP-complete to decide if $\mathrm{BBC}_{q}(G, T) \leq q+5$.

On the other hand, we believe that if $q=3$, Theorem 2 is not best possible.
Conjecture 3. If $G$ is a planar graph and $F$ a forest in $G$, then $\mathrm{BBC}_{3}(G, F) \leq 8$.
If true, Conjecture 3 would be tight. The pair $\left(G^{*}, F^{*}\right)$ of Figure 2 satisfies $\mathrm{BBC}_{3}\left(G^{*}, F^{*}\right)=8$. We show in Proposition 16 that Conjecture 1 implies Conjecture 3.


Figure 2: A planar graph $G^{*}$ and a tree $T^{*}$ (bold edges) such that $\mathrm{BBC}_{q}\left(G^{*}, T^{*}\right)=q+6$ for $q \geq 4$.

A star is a tree in which a vertex $v$, called the center is adjacent to every other. A galaxy is a forest of stars. As evidence in support of Conjectures 1 and 3, Broersma et al. [5] showed that if $F$ is a galaxy in a planar graph $G$, then $\operatorname{BBC}_{q}(G, F) \leq q+4$. This result is best possible even if $F$ has maximum degree 3 as shown by the example of Figure 1. Furthermore, we show in Theorems 21 and 28 that, for any $q \geq 2$, it is NP-complete to decide if $\mathrm{BBC}_{q}(G, F) \leq q+3$ given a planar graph $G$ and a galaxy of maximum degree 3 .

However, if the backbone is a matching, i.e. a galaxy with maximum degree 1 , then fewer colours are needed. Indeed, Broersma et al. [5] showed that if $M$ is a matching in a planar graph $G$, then for any $q \geq 3, \mathrm{BBC}_{q}(G, M) \leq q+3$. They conjectured that the same holds for $q=2$.

Conjecture 4 (Broersma et al. [5]). If $G$ is a planar graph $G$ and $M$ a matching in $G$, then $\mathrm{BBC}_{2}(G, M) \leq$ 5.

It is natural to ask the same question for galaxies with maximum degree at least 2 . When $q=2$, we answer in the negative by showing that there are pairs of planar graphs and spanning forests of maximum degree 2 whose 2 -backbone chromatic number is 6 . Furthermore, we show that given a planar graph $G$ and a spanning forest $F$ of maximum degree 2, it is NP-complete to decide whether $B B C_{2}(G, F) \leq 5$ (Theorem 22). We also show that given a planar graph $G$ with a hamiltonian path $P$, it is NP-complete to decide whether $B B C_{2}(G, F) \leq 5$. This result refines a result of Broersma et al. $[3,4]$ who proved it for a general graph $G$.

For $q=3$, the problem remains open.
Problem 5. If $G$ is a planar graph $G$ and $F$ a galaxy of maximum degree 2 , is is true that $\mathrm{BBC}_{q}(G, F) \leq$ $q+3$, for all $q \geq 3$ ?

Broersma et al. [5] proved that deciding if $\mathrm{BBC}_{q}(G, M) \leq q+2$ for a given graph $G$ and matching $M$ is NP-complete. We prove in Subection 2.2 that it remains NP-complete even if we impose $G$ to be

|  | $G$ planar |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $H$ forest | $H$ spanning tree | H 1-factor | $H$ Hamilton path |
| $B B C_{q}(G, H) \leq q+1$ ? | NP-C | poly | poly | poly |
| $B B C_{q}(G, H) \leq q+2$ ? | NP-C | poly | NP-C | poly |
| $B B C_{q}(G, H) \leq q+3$ ? | NP-C | NP-C | $\begin{array}{ll} q \geq 3: & \text { Yes } \\ q=2: & \text { ?Yes? } \\ \hline \end{array}$ | $q=2: \quad$ NP-C |
| $B B C_{q}(G, H) \leq q+5$ ? |  | $\begin{aligned} & q \geq 4: \\ & q=3: \\ & q=2: \end{aligned}$ | $\begin{aligned} & \text { NP-C } \\ & \text { ?Yes? } \\ & \text { Yes } \end{aligned}$ |  |
| $B B C_{q}(G, H) \leq q+6$ ? |  |  | Yes |  |

Table 1: Complexity of deciding if $B B C_{q}(G, H) \leq q+k$ for $k \in\{1, \ldots, 6\}$, when $G$ is a planar graph and $H$ a forest of some prescribed classes. NP-C:= NP-Complete; poly:= polynomial-time decidable; Yes: always true; ?Yes? conjectured to be always true.
planar. In contrast, we prove that deciding if $\mathrm{BBC}_{q}(G, T) \leq q+2$ for a given graph $G$ and spanning tree $T$ is polynomial-time solvable.

Many of the complexity results on backbone colouring of planar graphs with a forest backbone are summarized in Table 1.

One can generalize the notion of backbone colouring by allowing a more complicated structure of the frequency space. The most natural one is to consider a circular metric. A circular $k$-colouring of $G$ or $\mathbb{Z}_{k}$-colouring is a mapping $f: V \rightarrow \mathbb{Z}_{k}$. The notions of circular $q$-backbone colouring and circular $q$-backbone chromatic number are defined similarly to those of $q$-backbone colouring and circular $q$-backbone chromatic number by replacing colouring by circular colouring. The circular $q$-backbone chromatic number of a graph pair $(G, H)$ is denoted $\mathrm{CBC}_{q}(G, H)$.

If $f$ is a circular $q$-backbone $k$-colouring, then the mapping $f^{*}$ defined by $f(v)=f(v)+1$ for all vertex $v$ is trivially a $q$-backbone $k$-colouring. On the other hand, a $q$-backbone $k$-colouring yields a circular $q$-backbone $(k+q-1)$-colouring. Hence for every graph pair $(G, H)$, where $H$ is a spanning subgraph of $G$, we have

$$
\begin{equation*}
\operatorname{BBC}_{q}(G, H) \leq \mathrm{CBC}_{q}(G, H) \leq \mathrm{BBC}_{q}(G, H)+q-1 \tag{2}
\end{equation*}
$$

Also,

$$
\begin{equation*}
q \cdot \chi(H) \leq \mathrm{CBC}_{q}(G, H) \leq q \cdot \chi(G) . \tag{3}
\end{equation*}
$$

Observe that if $G$ is bipartite and $H$ is non-empty, Equation (3) implies that $C B C_{q}(G, H)=2 q$. More generally, if $\chi(G)=\chi(H)$, then $\mathrm{CBC}_{q}(G, H)=q \cdot \chi(G)$. However if $2 \leq \chi(H)<\chi(G)$, one can improve the upper bound. We show in Proposition 32 that, for any pair $(G, H)$ with $H$ a subgraph of $G$,

$$
\begin{equation*}
\mathrm{CBC}_{q}(G, H) \leq(\chi(G)+q-2) \chi(H) . \tag{4}
\end{equation*}
$$

Since $\operatorname{CBC}_{q}(G, H)=\chi(G)$ when $H$ is empty and $k$-Colourability is NP-complete, for any fixed $k \geq 3$, given a graph $G$ and a subgraph $H$ it is NP-complete to decide if $\mathrm{CBC}_{q}(G, H) \leq k$. But if insist that $H$ is not empty, then $\operatorname{CBC}_{q}(G, H) \geq 2 q$ by Proposition 3. Hence deciding if $\mathrm{CBC}_{q}(G, H)$ is at most $k$ with $k \leq 2 q-1$ can be done instantly by always returning 'no'. Less
trivially, Proposition 35 shows that if $H$ is a connected spanning subgraph of $G$, then $\mathrm{CBC}_{q}(G, H)=$ $2 q$ if and only if $G$ is bipartite. Hence deciding if $\mathrm{CBC}_{q}(G, H)=2 q$ can be done in polynomial time.

Inequality (4) implies that $\mathrm{CBC}_{q}(G, F) \leq 2 q+4$ for any planar graph $G$ and forest $F$ in $G$. We believe that this upper bound can be reduced by at least one.

Conjecture 6. If $G$ is a planar graph and $F$ a spanning forest of $G$, then $\mathrm{CBC}_{q}(G, F) \leq 2 q+3$.
A natural question is to ask whether this conjecture would be best possible.
Problem 7. For any $q \geq 2$, does there exist a planar graph $G_{q}$ and a spanning forest $F_{q}$ of $G_{q}$ such that $\mathrm{CBC}_{q}\left(G_{q}, F_{q}\right)=2 q+3$ ?

Conjecture 6 holds if the backbone $F$ is a galaxy. It follows directly from (2) and the fact that $\mathrm{BBC}_{q}(G, F) \leq q+4$ in such a case, as mentioned earlier. We believe however that one can use one colour less.

Conjecture 8. Let $G$ be a planar graph and $F$ a galaxy in $G$, then $\mathrm{CBC}_{q}(G, F) \leq 2 q+2$.
If true, this conjecture would be tight, since the circular $q$-backbone chromatic number of a $K_{4}$ with backbone $K_{1,3}$ is $2 q+2$. As evidence in support of Conjecture 8, Broersma et al. [5] deduced from the Four-Colour Theorem that if $G$ is a planar graph and $M$ a matching in $G$ then $\mathrm{CBC}_{q}(G, M) \leq 2 q+2$.

Broersma et al. also give an example of a planar graph $G$ and a matching $M$ such that ( $G, M$ ) has no 2 -backbone $\mathbb{Z}_{5}$-colouring. We show in Theorems 36 and 40 that for any fixed $k \in\{4,5\}$, it is NP-complete to decide if $B B C_{2}(G, M) \leq k$ for given planar graph $G$ and matching $M$. For larger values of $q$, the following questions are still open.

Problem 9. Let $G$ be a planar graph and let $M$ be a matching $M$ in $G$. For any $q \geq 3$, is it true that $\mathrm{CBC}_{q}(G, M) \leq 2 q+1$ ?

Problem 10. Is it NP-complete to decide if $\mathrm{CBC}_{2}(G, F) \leq 6$ for a planar graph $G$ and spanning forest $F$ ?

Problem 11. For any $g \geq 5$, is it NP-complete to decide if $\mathrm{CBC}_{2}(G, M) \leq 4$ for a planar graph $G$ of girth at least $g$ and matching $M$ ?

We prove in Theorem 33 that if $G$ has girth at least 5 , then $\mathrm{CBC}_{q}(G, M) \leq 2 q+1$. We wonder if the same holds for planar graph of girth 4.

Problem 12. Let $G$ be a planar graph of girth 4 and let $M$ be a matching in $G$. Is it true that $\mathrm{CBC}_{q}(G, M) \leq 2 q+1$ ?

Many of the complexity results on circular backbone colouring of planar graphs with a forest backbone are summarized in Table 2.

## 2 Backbone colouring

### 2.1 About Conjectures 1 and 3

Proposition 13. Let $G$ be a graph and let $H$ be a subgraph of $G$. Then $\operatorname{BBC}_{q}(G, H) \leq(\chi(G)+q-$ 2) $\chi(H)-q+2$.


Table 2: Complexity of deciding if $C B C_{q}(G, H) \leq 2 q+k$ for $k \in\{0, \ldots, 4\}$, when $G$ is a planar graph and $H$ a forest of some prescribed classes. NP-C:= NP-Complete; poly:= polynomial-time decidable; Yes: always true; ?Yes? conjectured to be always true.

Proof. Let $g$ be a proper $\chi(G)$-colouring of $G$ and $h$ a proper $\chi(H)$-colouring of $H$. Let $f$ be the colouring defined by:

$$
f(v)= \begin{cases}(h(v)-1)(q-2+\chi(G))+g(v), & \text { if } h(v) \text { is odd } \\ (h(v)-1)(q-2+\chi(G))+\chi(G)+1-g(v) & \text { if } h(v) \text { is even. }\end{cases}
$$

Let us check that $f$ is a $q$-backbone $((\chi(G)+q-2) \chi(H)-q+2)$-colouring of $(G, H)$.
Let $u v \in E(G)$. Without loss of generality, $h(u) \geq h(v)$. If $h(u)=h(v)$, then $u v \notin E(H)$ and $|f(u)-f(v)|=|g(u)-g(v)| \neq 0$.

Assume now that $h(u)>h(v)$. If $h(u)$ and $h(v)$ are both odd or both even, then $f(u)-f(v) \geq$ $2(q-2+\chi(G))-\mid(g(u)-g(v) \mid \geq q$. If $h(u)$ is odd and $h(v)$ is even, then $f(u)-f(v) \geq$ $q-3+g(u)+g(v)$, which is at least $q$ because $g(u)+g(v) \geq 3$ for $g(u) \neq g(v)$. If $h(u)$ is even and $h(v)$ is odd, then $f(u)-f(v) \geq q-2+2 \chi(G)+1-g(u)-g(v)$, which is at least $q$ because $g(u)+g(v) \leq 2 \chi(G)-1$ for $g(u) \neq g(v)$.

A parachute on $v$ or a parachute with harness $v$ is a complete graph on four vertices whose three edges incident to $v$ are in the backbone.

Proposition 14. (i) For $q \geq 2$, in a $q$-backbone ( $q+3$ )-colouring of a parachute, the harness is coloured in $\{1, q+3\}$.
(ii) For $q \geq 3$, in a $q$-backbone $(q+4)$-colouring of a parachute, the harness is coloured in $\{1,2, q+3, q+4\}$.
(iii) For $q \geq 4$, in a $q$-backbone $(q+5)$-colouring of a parachute, the harness is coloured in $\{1,2,3, q+3, q+4, q+5\}$.

Proof. Let $y$ be the harness.
(ii) If $3 \leq \phi(y) \leq q+2$, then at most two colours can appear on its neighbours. Because those three vertices form a clique, they have three different colours and so $\phi(y) \in\{1,2, q+3, q+4\}$.
(iii) If $4 \leq \phi(y) \leq q+2$, then at most two colours can appear on its neighbours. Because those three vertices form a clique, they have three different colours and so $\phi(y) \in\{1,2,3, q+3, q+4, q+$ $5\}$.

Proposition 15. Let $G^{*}$ and $T^{*}$ be the graph and its spanning tree depicted in Figure 2. For any $q \geq 4, B B C_{q}\left(G^{*}, T^{*}\right) \geq q+6$.

Proof. Assume for a contradiction that there is a $q$-backbone $(q+5)$-colouring $\phi$ of $\left(G^{*}, T^{*}\right)$. By Proposition 14-(iii), the vertices $y_{1}, y_{2}, y_{3}, y_{4}, z_{1}, z_{2}$ are coloured in $\{1,2,3, q+3, q+4, q+5\}$. Without loss of generaltiy, we may assume that $\phi\left(y_{2}\right) \in\{1,2,3\}$. But then $\phi\left(z_{1}\right)$ and $\phi\left(z_{2}\right)$ must be in $\{q+3, q+4, q+5\}$, because $y_{2} z_{1}$ and $y_{2} z_{2}$ are in $E\left(T^{*}\right)$. And $\phi\left(y_{1}\right), \phi\left(y_{2}\right)$ and $\phi\left(y_{3}\right)$ are in $\{1,2,3\}$ because $y_{3} z_{1}$ and $y_{1} z_{2}$ and $y_{4} z_{2}$ are in $E\left(T^{*}\right)$. But $\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}$ is a clique in $G^{*}$, so they must all get different colours, a contradiction.

Proposition 16. Conjecture 1 implies Conjecture 3.
Proof. Assume that Conjecture 1 holds. Let $G$ be a planar graph and $F$ a forest in $G$. Then $(G, F)$ admits a 2 -backbone 6 -colouring $\phi$. Let $\psi$ be defined by $\psi(v)=\phi(v)$ if $\phi(v) \in\{1,2\}, \psi(v)=$ $\phi(v)+1$ if $\phi(v) \in\{3,4\}$, and $\psi(v)=\phi(v)+2$ if $\phi(v) \in\{5,6\}$. One easily check that $\psi$ is a 3 -backbone 8 -colouring of $(G, F)$.

## $2.2 q$-backbone $(q+2)$-colouring

Theorem 17. Given a connected graph $G$ and a spanning connected subgraph $H$, one can decide in polynomial time if $\mathrm{BBC}_{q}(G, H) \leq q+2$.

Proof. Observe first that if $H$ is not bipartite, then $\operatorname{BBC}(H, H) \geq 2 q+1$ by $(1)$, and so $\mathrm{BBC}_{q}(G, H) \geq$ $q+3$. So we first check if $H$ is bipartite. If not, we return 'no'. If it is, we get a bipartition $(A, B)$ of $H$.

Observe that if $(G, H)$ has a $q$-backbone $(q+2)$-colouring, then (free to rename $A$ and $B$ ) all the vertices of $A$ are coloured in $\{1,2\}$ and all the vertices of $B$ in $\{q+1, q+2\}$, because $H$ is connected. We then can transform our instance into an instance $I(G, H)$ of 2SAT as follows. For each vertex $v$, we create a variable $x_{v}$. Intuitively, for a vertex $x \in A$ (resp. $x \in B$ ), the variable $x_{v}$ will be true if and only if $v$ is coloured 1 (resp. $q+2$ ) and false if and only if $v$ is coloured 2 (resp. $q+1$ ). Now for each edge $u v$, we create the following clauses.

- If $u$ and $v$ are both in $A$ or both in $B$, we create the clauses $x_{u} \vee x_{v}$ and $\bar{x}_{u} \vee \bar{x}_{v}$;
- if $u \in A$ and $v \in B$, we create the clause $x_{u} \vee x_{v}$.

It is easy to check that $(G, H)$ has a $q$-backbone $(q+2)$-colouring if and only if $I(G, H)$ is satisfiable.
Since 2SAT is well-known to be polynomial-time solvable, we can decide in polynomial time if $\mathrm{BBC}_{q}(G, H) \leq q+2$.

Theorem 18. For any $q \geq 2$, the following problem is NP-complete.
Input: A planar graph $G$ and a 1-factor $F$ of $G$.
Question: $\mathrm{BBC}_{q}(G, F) \leq q+2$ ?

Proof. The problem is trivially in NP since a $q$-backbone $(q+2)$-colouring of $(G, F)$ is clearly a certificate.

Reduction from Not-All-EQUAL 3SAT, which is defined as follows:
Input: A set of clauses each having three literals.
Question: Does there exists a suitable truth assignment, that is such that each clause has at least one true and at least one false literal?

This problem was shown NP-complete by Schaefer [11].

Let $\mathcal{C}=\left\{C_{1}, \ldots, C_{n}\right\}$ be a collection of clauses of size three over a set $U$ of variables. We will construct a graph pair $(G, F)$ such that $F$ is a 1 -factor of $G$. Since $V(F)=V(G)$, we only precise which edges are in $E(F)$.

The following gadget will be useful. A forcing gadget at $v$ or a forcing gadget with head $v$ is the graph depicted in Figure 3.


Figure 3: A forcing gadget with head $v$ (left) and its symbol (right) (Edges of $E(F)$ are in bold.)

A key point in the reduction will be the following claim.
Claim 19. In any $q$-backbone $(q+2)$-colouring of a forcing gadget, its head is coloured in $\{1, q+2\}$.
Proof. Consider a forcing gadget, whose vertices are named as in Figure 3, and $\phi$ a $q$-backbone ( $q+2$ )colouring of it. Since all the vertices are matched in $F$, there all must be coloured in $\{1,2, q+1, q+2\}$.

Assume for a contradiction that $\phi(v)=2$. Then $\phi\left(v_{1}\right)=q+2$. Thus $\phi\left(v_{2}\right) \in\{1, q+1\}$. Now if $\phi\left(v_{2}\right)=q+1$, then necessarily $\phi\left(v_{3}\right)=1$. Therefore, whatever the colouring may be, $v_{4}$ and $v_{5}$ are both adjacent to a vertex coloured 1 . Hence $\left\{\phi\left(v_{4}\right), \phi\left(v_{5}\right)\right\}=\{2, q+2\}$. Therefore $\left\{\phi\left(v_{2}\right), \phi\left(v_{3}\right)\right\}=\{1, q+1\}$. But then $v_{6}$ cannot be coloured.

Similarly, we get a contradiction if $\phi(v)=q+1$.
For every variable $u \in U$, create a variable subgraph $P_{u}$ which is obtained from the path $\left(a_{1}(u), b_{1}(u), a_{2}(u), b_{2}(u), \ldots, a_{n}(u), b_{n}(u)\right)$ by adding a forcing gadget on each of its vertex.

For every clause $C_{i}=\ell_{1} \vee \ell_{2} \vee \ell_{3}$, create a clause gadget $D_{i}$ as shown Figure 4.


Figure 4: The clause $D_{i}$. (Edges of $E(F)$ are in bold, forcing gadgets are represented by their symbols.)

Then for each clause $C_{i}$ and each literal $\ell$ of $C_{i}$, we add a path of length three $\left(c_{i}(l), c_{i}^{\prime}(l), c_{i}^{\prime \prime}(l), a_{i}(u)\right)$ if $\ell$ is the non-negated variable $u$, and $\left(c_{i}(l), c_{i}^{\prime}(l), c_{i}^{\prime \prime}(l), b_{i}(u)\right)$ if $\ell$ is the negated variable $\bar{u}$. We also add two forcing gadgets with heads $c_{i}^{\prime}(l)$ and $c_{i}^{\prime \prime}(l)$.

It is easy to see that the resulting graph $G^{\prime}$ may be drawn in the plane such that the crossed edges are those of type $c_{i}^{\prime}(\ell) c_{i}^{\prime \prime}(\ell)$ for some literal $\ell$. In particular, the two endvertices of a crossed edge are heads of forcing gadgets.

As long as there is a crossing $C$ between two edges $t(C) u(C)$ and $v(C) w(C)$, we replace these two edges by the crossing gadget $C G(C)$ depicted in Figure 5, so that the only edges that are possibly crossed (if there were several crossings on $t u$ or $u v$ ) are $t(C) t^{\prime}(C), u(C) u^{\prime}(C), v(C) v^{\prime}(C)$ and $w(C) w^{\prime}(C)$. After this process, there is no more crossing so the resulting graph $G$ is planar.


Figure 5: The crossing gadget $C G(C)$. (Edges of $E(F)$ are in bold, forcing gadgets are represented by their symbols.)

Claim 20. Let $\phi$ be a $q$-backbone $(q+2)$-colouring of $(G, F)$. For every crossing $C$ in $G^{\prime}$, we have $\{\phi(t(C)), \phi(u(C))\}=\{1, q+2\}$ and $\{\phi(v(C)), \phi(w(C))\}=\{1, q+2\}$.

Subproof. By induction on the reverse order of creation of the crossing gadget.
By construction, $t(C), u(C), v(C), w(C), t^{\prime}(C), u^{\prime}(C), v^{\prime}(C)$, and $w^{\prime}(C)$ are heads of forcing gadgets. So, by Claim 19, they are coloured 1 or $q+2$. Without loss of generality, we may assume that $\phi(t(C))=1$.

If the edge $t(C) t^{\prime}(C)$ was crossed and then replaced by a series of crossing gadget, by induction, $\phi\left(t^{\prime}(C)\right)=q+2$. It is also trivially the case if $t(C) t^{\prime}(C)$ still exists. Hence $\{\phi(a(C)), \phi(b(C))\}=$ $\{1, q+1\}$.

Assume for a contradiction that $\phi(u(C)) \neq q+2$. Then, as above, $\{\phi(c(C)), \phi(d(C))\}=\{1, q+$ $1\}$. This is a contradiction, because $a(C) c(C)$ and $a(C) d(C)$ are edges. Hence $\phi(u(C))=q+2$,
and so $\phi\left(u^{\prime}(C)\right)=1$ and $\{\phi(c(C)), \phi(d(C))\}=\{2, q+2\}$.
In particular, one vertex of $\{a(C), b(C), c(C), d(C)\}$ is coloured 1 and another is coloured $q+2$. Now assume for a contradiction that $\{\phi(v(C)), \phi(w(C))\} \neq\{1, q+2\}$. Then $v(C)$ and $w(C)$ are both coloured 1 or both coloured $q+2$, and so $v^{\prime}(C)$ and $w^{\prime}(C)$ are both coloured $q+2$ or both coloured 1, respectively. This is a contradiction, as all vertices of $\{a(C), b(C), c(C), d(C)\}$ are adjacent to some vertex in $\left\{v^{\prime}(C), w^{\prime}(C)\right\}$.

Let us now prove that $(G, F)$ admits a $q$-backbone $(q+2)$-colouring if and only if $\mathcal{C}$ has a suitable truth assignment.

Assume first that $(G, F)$ admits a $q$-backbone $(q+2)$-colouring $\phi$. Let $u$ be a variable. Since there are heads of forcing gadgets, by Claim 19, all the $a_{i}(u)$ and $b_{i}(u)$ are coloured in $\{1, q+2\}$. Moreover, since they form a path, all the $a_{i}(u)$ are coloured with the same colour and all the $b_{i}(u)$ are coloured with the other. Hence one can define the truth assignment $\psi$ by $\psi(u)=\operatorname{true}$ if $\phi\left(a_{i}(u)\right)=1$ for $1 \leq i \leq n$, and $\psi(u)=$ false if $\phi\left(a_{i}(u)\right)=q+2$ for $1 \leq i \leq n$.

We shall prove that $\psi$ is suitable.
Let $C_{i}=\ell_{1} \vee \ell_{2} \vee \ell_{3}$ be a clause. Claim 20 implies that for $j \in\{1,2,3\}, \phi\left(c_{i}\left(\ell_{j}\right)\right)=1$ if $\psi\left(\ell_{j}\right)=$ false and $\phi\left(c_{i}\left(\ell_{j}\right)\right)=q+2$ if $\psi\left(\ell_{j}\right)=$ true. Now the three $c_{i}\left(\ell_{j}\right), 1 \leq j \leq 3$, cannot be all coloured 1 (resp. $q+2$ ), for otherwise $\left\{\phi\left(d_{i}\left(\ell_{2}\right)\right), \phi\left(d_{i}\left(\ell_{3}\right)\right)\right\}$ must be $\{2, q+2\}$ (resp. $\{1, q+1\}$ ) and so $d_{i}\left(\ell_{1}\right)$ cannot be coloured, because it must be coloured in $\{1, q+2\}$ as head of a forcing gadget. Thus at least one of the $c_{i}\left(l_{j}\right)$ is coloured 1 and at least one is coloured $q+2$, and so $C_{i}$ has at least one true and at least one false literal.

Hence $\psi$ is suitable.

Reciprocally, assume that $\mathcal{C}$ has a suitable truth assignment $\psi$. For all $u \in U$ and all $1 \leq i \leq n$, let us define $\phi\left(a_{i}(u)\right)=1$ and $\phi\left(b_{i}(u)\right)=q+2$ if $\psi(u)=\operatorname{true}$, and $\phi\left(a_{i}(u)\right)=q+2$ and $\phi\left(b_{i}(u)\right)=1$ if $\psi(u)=$ false. Similarly, for every literal $\ell$, we set $\phi\left(c_{i}(\ell)\right)=1, \phi\left(c_{i}^{\prime}(\ell)\right)=q+2, \phi\left(c_{i}^{\prime \prime}(\ell)\right)=1$, if $\ell$ is false, and $\phi\left(c_{i}(\ell)\right)=q+2, \phi\left(c_{i}^{\prime}(\ell)\right)=1, \phi\left(c_{i}^{\prime \prime}(\ell)\right)=q+2$, if $\ell$ is true.

One can extend $\phi$ into a $q$-backbone $(q+2)$-colouring of $(G, F)$. Indeed, it is sufficient to show that we can extend it to forcing, clause and crossing gadgets.

If $v$ is the head of a forcing gadget and $\phi(v)=1$, we can set $\phi\left(v_{1}\right)=q+2, \phi\left(v_{2}\right)=q+1$, $\phi\left(v_{3}\right)=1, \phi\left(v_{4}\right)=q+2, \phi\left(v_{5}\right)=2, \phi\left(v_{6}\right)=2$, and $\phi\left(v_{7}\right)=q+2$. Similarly, we can extend the colouring to the forcing gadget if $\phi(v)=q+2$.

Consider a clause gadget $D_{i}$. Since $C_{i}$ has at least one true and at least one false literal, at least one vertex of $c_{i}\left(\ell_{1}\right), c_{i}\left(\ell_{2}\right) c_{i}\left(\ell_{3}\right)$ is coloured 1 and at least one is coloured $q+2$. If $c_{i}\left(\ell_{1}\right)$ is coloured $q+2$, and $c_{i}\left(\ell_{2}\right)$ and $c_{i}\left(\ell_{2}\right)$ are assigned 1 , then we can set $\phi\left(d_{i}\left(\ell_{1}\right)\right)=1, \phi\left(d_{i}\left(\ell_{2}\right)\right)=2$, and $\phi\left(d_{i}\left(\ell_{3}\right)\right)=q+2$. If $c_{i}\left(\ell_{1}\right)$ and $c_{i}\left(\ell_{2}\right)$ are coloured 1 , and $c_{i}\left(\ell_{3}\right)$ is assigned $q+2$, then we can set $\phi\left(d_{i}\left(\ell_{1}\right)\right)=q+2, \phi\left(d_{i}\left(\ell_{2}\right)\right)=q+1$, and $\phi\left(d_{i}\left(\ell_{3}\right)\right)=1$.

Finally consider a crossing gadget such that $\{\phi(t(C)), \phi(u(C))\}=\{\phi(v(C)), \phi(w(C))\}=$ $\{1, q+2\}$. By symmetry, we may assume that $\phi(t(C))=\phi(v(C))=1$ and $\phi(u(C))=\phi(w(C))=$ $q+2$. Then we can set $\phi\left(t^{\prime}(C)\right)=\phi\left(v^{\prime}(C)\right)=q+2, \phi\left(u^{\prime}(C)\right)=\phi\left(w^{\prime}(C)\right)=2, \phi(a(C))=1$, $\phi(b(C)=q+1, \phi(c(C))=q+2$, and $\phi(d(C))=2$.

### 2.3 2-backbone 5 -colouring

### 2.3.1 Galaxy backbone

Theorem 21. The following problem is $N P$-complete.
Input: A planar graph $G$ and a galaxy $F$ in $G$ with maximum degree 3 .
Question: Is $\mathrm{BBC}_{2}(G, F) \leq 5$ ?

Proof. Reduction from Planar 3-Colourability, which consists of deciding if a given connected planar graph is 3 -colourable. This problem was shown to be NP-complete by Stockmeyer [13]. Clearly, it remains NP-complete when restricted to 2-connected planar graphs.

Let $H$ be a 2 -connected planar graph. We shall construct a planar graph $G$ and a galaxy $F$ with maximum degree 3 in $G$ such that $\mathrm{BBC}_{2}(G, F) \leq 5$ if and only if $H$ is 3 -colourable.

As a forcing gadget at $v$, we will use the parachute with harness $v$. It is easy to see that in a 2 -backbone 5 -colouring of a parachute, its harness is coloured in $\{1,5\}$.

We consider any embedding of $H$. For each face $\left(x_{1}, x_{2}, \ldots, x_{k}, x_{1}\right)$ of $H$, we put a cycle $\left(z_{1}, z_{2}, \ldots, z_{2 k}, z_{1}\right)$, inside which we put parachutes on every vertex $z_{i}$ for every $1 \leq i \leq 2 k$. We then add the edges $x_{i} z_{2 i} x_{i} z_{2 i+1}$ for all $1 \leq i \leq k$.

Assume that $(G, F)$ has a 2-backbone 5 -colouring $\phi$, then, because of the parachutes, all the vertices in the cycles added inside faces must be coloured in $\{1,5\}$. Moreover consecutive vertices on one such cycles get different colours, so one is coloured 1 and the other is coloured 5 . Hence all the vertices in $H$ are coloured in $\{2,3,4\}$. Hence $\phi$ induces a proper 3 -colouring on $H$ with colours $\{2,3,4\}$.

Reciprocally, assume that $H$ is 3 -colourable. Then there exists a proper 3 -colouring $c$ of $H$ into $\{2,3,4\}$. One can then colour all the cycles inside faces with 1 and 5 . The colouring can then easily be extended into a 2 -backbone 5 -colouring of $(G, F)$.

Theorem 22. The following problem is NP-complete.
Input: A planar graph $G$ and a galaxy $F$ in $G$ with maximum degree 2 .
Question: Is $\mathrm{BBC}_{2}(G, F) \leq 5$ ?

Proof. The proof is identical to the one of Theorem 21. The only difference comes from the forcing gadget, which is more complicated because it cannot contains stars of degree 3 in $F$.

To construct the forcing gadget, we need an auxiliary gadget, called no-3-gadget. It is depicted in Figure 6.

Claim 23. In any 2-backbone 5-colouring of a no-3-gadget, its roof is not coloured in 3 .
Proof. We will denote the vertices of the no-3-gadget by their names in Figure 6. Assume for a contradiction that there is a 2 -backbone 5 -colouring $\phi$ of a no-3-gadget such that $\phi(x)=3$.

Assume first that $\phi(a) \in\{4,5\}$, then $\phi(b) \in\{1,2\}$ and $\{\phi(a), \phi(c)\}=\{4,5\}$. Hence $\phi(d) \in$ $\{1,2\}$ and so $\{\phi(f), \phi(c)\}=\{4,5\}$. Therefore $\phi(e)=3$ and so $\phi(d)=1$. Similarly, if $\phi(a) \in$ $\{1,2\}$, we obtain that $\phi(d)=5$. Hence, $\phi(d) \in\{1,5\}$.

Similarly, $\phi\left(d^{\prime}\right) \in\{1,5\}$. Free to consider $6-\phi$ instead of $\phi$, we may assume that $\phi(d)=1$ and $\phi\left(d^{\prime}\right)=5$. Thus $\phi\left(f^{\prime}\right)=2$.

Now $\phi(g) \in\{3,4\}$. If $\phi(g)=3$, then $\{\phi(i), \phi(h)\}=\{1,5\}$, and if $\phi(g)=4$, then $\{\phi(i), \phi(h)\}=$ $\{1,2\}$. In both cases, one of $h$ and $i$ is coloured 1 , which is impossible because $\phi(d)=1$.


Figure 6: The no-3-gadget with roof $x$ and its symbol.


Figure 7: The forcing gadget with head $v$. (Edges of $E(F)$ are in bold, no-3-gadgets are represented by their symbols.)

The forcing gadget is the one depicted in Figure 7.
Claim 24. In any 2-backbone 5 -colouring of a forcing gadget, its head is coloured in $\{1,5\}$.
Proof. Consider a forcing gadget, whose vertices are named as in Figure 7, and $\phi$ a 2-backbone 5colouring of it.

Let us prove that $\phi(w)=3$ and so that $\phi(v) \in\{1,5\}$. Assume for a contradiction that $\phi(w) \neq 3$. Without loss of generality, we may assume that $\phi(w) \in\{1,2\}$.

Observe that the vertices $x, y, z, x^{\prime}, y^{\prime}, z^{\prime}$ are not assigned 3 because they are roofs of no-3gadgets.

If $\phi(w)=1$, then $(\phi(x), \phi(y), \phi(z))$ and $\left(\phi\left(x^{\prime}\right), \phi\left(y^{\prime}\right), \phi\left(z^{\prime}\right)\right)$ is either $(4,2,5)$ or $(5,2,4)$. Hence the vertices $x, x^{\prime}$ and $z$ are all coloured in $\{4,5\}$, which is impossible, since they form a triangle.

If $\phi(w)=2$, then $(\phi(x), \phi(y), \phi(z))$ and $\left(\phi\left(x^{\prime}\right), \phi\left(y^{\prime}\right), \phi\left(z^{\prime}\right)\right)$ is either $(4,1,5)$ or $(5,1,4)$. Hence the vertices $x, x^{\prime}$ and $z$ are all coloured in $\{4,5\}$, which is impossible, since they form a triangle.

To get the equivalence between the 3-colourability of the original graph $H$ and the existence of a 2 -backbone 5 -colouring of $(G, F)$, it remains to prove that for any $\alpha \in\{1,5\}$, there is a 2 -backbone 5 -colouring of the forcing gadget such that the head is coloured $\alpha$.

We denote the vertices by their names in Figure 7. Set $\phi(w)=3, \phi(x)=\phi\left(y^{\prime}\right)=1, \phi(y)=$ $\phi\left(z^{\prime}\right)=5, \phi(z)=2$ and $\phi\left(x^{\prime}\right)=4$.

Observe that no vertex in $\left\{x, y, z, x^{\prime}, y^{\prime}, z^{\prime}\right\}$ has been coloured 3. Hence, it remains to prove that for any $\beta \in\{1,2,4,5\}$, there is a 2 -backbone 5 -colouring of the forcing gadget such that the head is coloured $\beta$. By the symmetry $\phi \rightarrow 6-\phi$, it suffices to prove that one exists for $\beta \in\{1,2\}$. We denote the vertices by their names in Figure 6. Let us denote by $\bar{\beta}$ the colour of $\{1,2\} \backslash\{\beta\}$.
$\phi(a)=3, \phi(b)=\bar{\beta}, \phi(c)=5, \phi(d)=4, \phi(e)=\beta, \phi(f)=\bar{\beta}, \phi\left(a^{\prime}\right)=3, \phi\left(b^{\prime}\right)=5, \phi\left(c^{\prime}\right)=\bar{\beta}$, $\phi\left(d^{\prime}\right)=5, \phi\left(e^{\prime}\right)=\beta, \phi\left(f^{\prime}\right)=3, \phi(g)=1, \phi(h)=5, \phi(i)=3$.

### 2.3.2 Hamiltonian-path backbone

Theorem 25. The following problem is NP-complete.
Input: A planar graph $G$ with a hamiltonian path $P$.
Question: $\mathrm{BBC}_{2}(G, P) \leq 5$ ?

To prove this theorem, we shall use a reduction similar to the one of Theorem 21. However, we do not reduce directly from Planar 3-Colourability but use an intermediate problem whose NP-completeness is proven by reducing Planar 3-Colourability to it.

This intermediate problem is the following:
Traceable Planar 3-Colourability
Input: A planar graph $G$ with a hamiltonian path $P$.
Question: Is $G 3$-colourable?
Lemma 26. Traceable Planar 3-Colourability is NP-complete.
Proof. Reduction from Planar 3-Colourability. Let $H$ be a connected planar graph. We will construct a planar graph $G$ having a hamiltonian path $P$ such that $\chi(G) \leq 3$ if and only if $\chi(H) \leq 3$.

To do so, we shall construct a sequence of pairs $\left(G_{i}, P_{i}\right)$ for $1 \leq i \leq|V(H)|$ such that $P_{i}$ is a path in the planar connected graph $G_{i},\left|V\left(P_{i}\right)\right|=\left|V\left(G_{i}\right)\right|-|V(H)|+i$, and $\chi\left(G_{i}\right) \leq 3$ if and only if $\chi(H) \leq 3$. Then the path $P:=P_{V(H)}$ will be a hamiltonian path of $G:=G_{V(H)}$ and $\chi(G) \leq 3$ if and only if $\chi(H) \leq 3$.

Let $x$ be a vertex of $H$. We set $G_{1}:=H$ and $P_{1}:=(x)$. Trivially, $\left(G_{1}, P_{1}\right)$ verifies the above property.

Assume now that $i \geq 1$ and let us construct $\left(G_{i+1}, P_{i+1}\right)$ from $\left(G_{i}, P_{i}\right)$. Let $P_{i}=\left(v_{1}, v_{2}, \ldots, v_{\ell}\right)$ be a path. Since $G_{i}$ is connected, there exists $j$ such that $v_{j}$ is adjacent to a vertex $y$ in $V\left(G_{i}\right) \backslash V\left(P_{i}\right)$. If $j=1$, then let $P_{i+1}:=\left(y, v_{1}, v_{2}, \ldots, v_{\ell}\right)$, and $G_{i+1}:=G_{i} ;$ if $j=p$, then let $P_{i+1}:=$ $\left(v_{1}, v_{2}, \ldots, v_{\ell}, y\right)$, and $G_{i+1}:=G_{i}$; if $y$ is also incident to $v_{j+1}$, let $P_{i+1}:=\left(v_{1}, \ldots, v_{j}, y, v_{j+1}, \ldots, v_{\ell}\right)$ ). In those three cases, $\left(G_{i+1}, P_{i+1}\right)$ has trivially the desired property.

So we may assume that $1<j<\ell$ and $y$ is not adjacent to $v_{j+1}$. Let $y_{1}, y_{2}, \ldots y_{r}$ be the neighbours of $v_{j}$ in their order around it such that $v_{j+1}=y_{r}, y_{k}=y$ and $v_{j-1}=y_{q}$ for $q<r$.

Let $G_{i+1}$ be the graph obtained from $G_{i}$ as follows. For all $1 \leq s \leq k-1$, remove the edge $v_{j} y_{s}$, add three vertices $a_{s}, b_{s}, c_{s}$ and the edges $a_{s} b_{s}, b_{s} c_{s}, c_{s} a_{s}, v_{j} a_{s}, v_{j} b_{s}, b_{s} y_{s}$; Add the edges $c_{s} a_{s+1}$ for all $1 \leq s \leq k-2$, and $v_{j+1} a_{1}$. Finally add a vertex $y^{\prime}$ and the edges $y y^{\prime}$ and $y^{\prime} c_{k-1}$. Let $P_{i+1}$ be the path obtained from $P_{i}$ by replacing the edge $v_{j} v_{j+1}$ by the subpath ( $v_{j}, y, c_{k-1}, b_{k-1}, a_{k-1}, \ldots, c_{1}, b_{1}, a_{1}, v_{j+1}$ ). See Figure 8, which illustrates the construction when $k=5$.

$G_{i}$

$G_{i+1}$

Figure 8: Constructing $\left(G_{i+1}, P_{i+1}\right)$ from $\left(G_{i}, P_{i}\right)$ (Egdes of the paths are in bold.)

Clearly, the number of vertices not covered by $P_{i+1}$ in $G_{i+1}$ is one less than the number of vertices not covered by $P_{i}$ in $G_{i}$. So, since $\left|V\left(P_{i}\right)\right|=\left|V\left(G_{i}\right)\right|-|V(H)|+i$, we have $\left|V\left(P_{i+1}\right)\right|=\left|V\left(G_{i+1}\right)\right|-$ $|V(H)|+i+1$.

It remains to prove that $G_{i+1}$ is 3 -colourable if and only if $G_{i}$ is.
Assume first that $G_{i+1}$ admits a proper 3-colouring $\phi$ in $\{1,2,3\}$. We claim that it also induces a proper 3 -colouring of $G_{i}$. Indeed, without loss of generality, we may assume that $\phi\left(v_{j}\right)=1$ and $\phi\left(v_{j+1}\right)=2$. Then for all $1 \leq s \leq k-1, \phi\left(a_{s}\right)=3$ and $\phi\left(c_{s}\right)=2$, so $\phi\left(b_{s}\right)=1$. Hence $\phi\left(y_{s}\right) \neq 1$. Therefore, for all $1 \leq s \leq k-1, \phi\left(y_{s}\right) \neq \phi\left(v_{j}\right)$. Since the $v_{j} y_{s}, 1 \leq s \leq k-1$, are the only edges of $G_{i}$ which are not in $G_{i+1}, \phi$ is a proper 3-colouring of $G_{i}$.

Conversely, assume that $G_{i}$ admits a 3 -colouring $\phi$ in $\{1,2,3\}$. It induces a partial proper 3 colouring of $G$, such that $\phi\left(v_{j}\right) \neq \phi\left(y_{s}\right)$ for all $1 \leq s \leq k-1$. Let us extend it. Without loss of generality, $\phi\left(v_{j}\right)=1$ and $\phi\left(v_{j+1}\right)=2$. For all $1 \leq s \leq k-1$, set $\phi\left(a_{s}\right)=3, \phi\left(b_{s}\right)=1$, and
$\phi\left(c_{s}\right)=2$. Finally, colour $y^{\prime}$ with the colour in $\{1,2,3\} \backslash\left\{\phi(y), \phi\left(c_{k-1}\right)\right\}$. This gives a proper 3 -colouring of $G_{i+1}$.

Proof of Theorem 25. Reduction from Traceable Planar 3-Colourability. Let $(H, Q)$ be an instance of this problem. We shall construct a graph $G$ and a hamiltonian path $P$ of $G$ such that $\mathrm{BBC}_{2}(G, P) \leq 5$ if and only if $\chi(H) \leq 3$. To do so we start from $H$ and for each edge $x y$ of $Q$, we will plug in an edge gadget $E(x y)$ containing a hamiltonian path $P(x y)$ from $x$ to $y$. The union of all the $P(x y), x y \in E(Q)$, will then be a hamiltonian path $P$ of the resulting graph $G$.

To construct the edge gadget, we use an auxiliary forcing gadget depicted in Figure 9. The head of such a gadget is the vertex denoted by $v$ in the figure. Its fringes are the vertices denoted by $a$ and $e$.


Figure 9: The forcing gadget with head $v$ and fringes $a$ and $e$ (left) and its symbol (right)

Claim 27. In any 2 -backbone 5 -colouring of a forcing gadget, the head is coloured in $\{2,4\}$.
Proof. We denote the vertices by their names in Figure 9. Suppose for a contradiction that there is a 2 -backbone 5 -colouring $\phi$ such that $\phi(v) \notin\{2,4\}$. By the symmetry $\phi \rightarrow 6-\phi$, we may assume that $\phi(v) \in\{1,3\}$.

If $\phi(v)=3$, then all the vertices $a, b, c, d, e$ are coloured in $\{1,2,4,5\}$. On the path $(a, b, c, d, e)$, vertices coloured $\{1,2\}$ alternate with vertices coloured $\{4,5\}$. Hence $a, c$, and $e$ are all coloured in $\{1,2\}$, or all coloured in $\{4,5\}$, which is a contradiction as they form a clique.

If $\phi(v)=1$, then all the vertices $a, b, c, d, e$ are coloured in $\{2,3,4,5\}$. Now $\phi(b)$ is at distance 2 from the two distinct colours $\phi(a)$ and $\phi(c)$, hence $\phi(b) \in\{2,5\}$. Similarly, $\phi(d) \in\{2,5\}$. But $\phi(c)$ is at distance 2 from $\phi(b)$ and $\phi(d)$, so $\phi(b)=\phi(d)$. Then the three vertices $a, c$, and $e$ are all coloured in $\{2,3,4,5\} \backslash\{\phi(b)-1, \phi(b), \phi(b)+1\}$, which has cardinality 2 . This is a contradiction as those three vertices form a clique.

Now the edge gadget is the one depicted in Figure 10.
Let us now prove that $\mathrm{BBC}_{2}(G, P) \leq 5$ if and only if $\chi(H) \leq 3$.
Assume first that $(G, P)$ admits a 2-backbone 5 -colouring $\phi$. Since $H$ is a subgraph of $G, \phi$ induces a proper colouring on $H$. We shall prove that every vertex of $H$ is coloured in $\{1,3,5\}$, thus proving that this proper colouring uses (at most) 3 colours.

Every vertex $v$ of $H$ is contained in an edge $x y$ of $Q$, so it is contained in the edge gadget $E(x y)$ in $G$. So it is adjacent to two vertices (namely $v_{1}$ and $v_{2}$ if $v=x$, and $v_{2}$ and $v_{3}$ if $v=y$ ), which are heads of forcing gadgets and adjacent. Hence by Claim 27, one of these vertices is coloured 2 and the other is coloured 4 . Hence $v$ must be coloured in $\{1,3,5\}$.


Figure 10: The edge gadget $E(x y)$ and its hamiltonian path $P(x y)$ in bold (Forcing gadgets are represented by their symbols.)

Let us now assume that $H$ is 3 -colourable. Then there exists a proper colouring $\phi$ of $H$ with $\{1,3,5\}$. Let us now extend into a 2 -backbone 5 -colouring of $(G, P)$. It is sufficient to prove that we can extend it to every edge-gadget.

To extend it to the edge-gadget $E(x y)$ (we use the names of Figure 10), set $\phi\left(v_{1}\right)=\phi\left(v_{3}\right)=2$ and $\phi\left(v_{2}\right)=4$. Now, since for any pair $(\alpha, \beta) \in\{1,2,3,4,5\}^{2}$, there is a 2 -backbone 5 -colouring of the path of length 4 such that the first vertex is coloured $\alpha$ and the last vertex is coloured $\beta$, it suffices to prove that we can extend $\phi$ to the forcing gadget.

Consider such a forcing gadget (with vertex names as in Figure 9). Then $\phi(v) \in\{2,4\}$. By the symmetry $\phi \rightarrow 6-\phi$, we may assume that $\phi(v)=2$. Then setting $\phi(a)=4, \phi(b)=\phi(d)=1$, $\phi(c)=3$ and $\phi(e)=5$, we obtain the desired extension.

Hence, $\mathrm{BBC}_{2}(G, P) \leq 5$.

## $2.4 \quad q$-backbone $(q+3)$-colouring for $q \geq 3$

Theorem 28. For any $q \geq 3$, the following problem is NP-complete.
Input: A planar graph $G$ and a galaxy $F$ in $G$ with maximum degree 3 .
Question: Is $\mathrm{BBC}_{q}(G, F) \leq q+3$ ?

## Proof. Reduction from Planar 3-Colourability.

We shall need the graph, which we call a kite, depicted in Figure 11. The vertex named $t$ in the figure is the tip of the kite, and the one named $u$ is its corner.

Claim 29. If $\phi$ is a $q$-backbone ( $q+3$ )-colouring of a kite such that $\phi(t) \in\{1,2,3, q+1, q+2, q+3\}$, then either $\phi(t) \in\{1,2,3\}$ and $\phi(u)=q+3$, or $\phi(t) \in\{q+1, q+2, q+3\}$ and $\phi(u)=1$.

Proof. Observe that the vertices $v, z_{1}, z_{2}, z_{3}$ are harnesses of parachutes. Thus, by Proposition 14-(i), they must be assigned 1 or $q+3$.

Assume that $\phi(v)=1$, then $\phi\left(z_{1}\right)=\phi\left(z_{2}\right)=\phi\left(z_{3}\right)=q+3$. Thus $\left\{\phi\left(s_{1}\right), \phi\left(s_{2}\right)\right\}=\{q+1, q+2\}$ and so $\phi(u)=q+3$ and $\phi(t) \in\{1,2,3\}$.

Similarly if $\phi(v)=q+3$, we obtain $\phi(u)=1$ and $\phi(t) \in\{q+1, q+2, q+3\}$.
Let $H$ be a planar graph. Let $(G, F)$ be the graph pair obtained from $H$ as follows. Firstly, for each face $f$ of $H$, we create a parachute $P_{f}$ with harness $v_{f}$, and for each vertex $x$ incident to $f$, we


Figure 11: The kite
create a kite $K_{f}(x)$ with tip $x$ and corner $u_{f}(x)$. We then link the vertex $v_{f}$ to all the $u_{f}(x)$. Secondly, for every vertex $x \in V(H)$, we add a vertex $y_{x}$ and the edge $x y_{x}$ in the backbone.

Clearly, the resulting graph $G$ is planar and the resulting backbone $F$ is a galaxy with maximum degree 3.

Let us now prove that $B B C_{q}(G, F) \leq q+3$ if and only if $H$ is 3-colourable.
Assume first that $(G, F)$ admits a $q$-backbone $(q+3)$-colouring $\phi$. Observe that each vertex $x$ in $V(H)$ is coloured in $\{1,2,3, q+1, q+2, q+3\}$, because it is adjacent to $y_{x}$ in $F$.

Let $x$ be a vertex in $V(H)$. Free to consider $q+4-\phi$, we may assume that $\phi(x) \in\{1,2,3\}$. Consider a face $f$ incident to $x$ in $H$. By Claim 29, the kite $K_{f}(x)$ has its corner coloured $q+3$. Together with Proposition 14-(i), this implies that $\phi\left(v_{f}\right)=1$. Thus, the corner of the kites in $f$ cannot be coloured 1, therefore there are coloured $q+3$ and so by Claim 29 , all the vertices incident to $f$ in $H$ are all coloured in $\{1,2,3\}$. Applying this reasoning to each face of $H$, we obtain that all vertices of $H$ are coloured in $\{1,2,3\}$. Hence, $\phi$ induces a proper 3 -colouring on $H$.

Conversely, assume that $H$ admits a proper 3 -colouring $c$. One can extend it into a $q$-backbone $(q+3)$-colouring of $(G, F)$ as follows. For every $x \in V(H)$, we colour $y_{x}$ with $q+3$; for every face $f$, we colour the vertex $v_{f}$ with 1 and the corners of the kites by $q+3$. One can then extend the colouring to each kite (as in the proof of Claim 29) to obtain a $q$-backbone $(q+3)$-colouring of $(G, F)$.

The reduction above can be modified to have a spanning tree $T$ for the backbone in place of the galaxy $F$. It suffices consider a spanning tree $U$ of $H$ and do the following: add a path of length two in the backbone along each edge of the tree $U$; for each kite, add $t z_{3}$ and $v z_{3}$ in the backbone and add paths of length two in the backbone along edges $z_{1} v$ and $z_{2} v$. This will prove the following statement.

Theorem 30. The following problem is NP-complete.
Input: A planar graph $G$ and a spanning tree $T$ of $G$.

Question: Is $\mathrm{BBC}_{q}(G, T) \leq q+3$ ?

## $2.5 \quad q$-backbone $(q+5)$-colouring

Theorem 31. For any $q \geq 4$, the following problem is NP-complete.
Input: A planar graph $G$ and a spanning tree $T$ of $G$.
Question: Is $\mathrm{BBC}_{q}(G, T) \leq q+5$ ?

## Proof. Reduction from Planar 3-Colourability.

Let $H$ be a planar graph. We shall construct a planar graph $G$ together with a spanning tree $T$ such that $H$ is 3-colourable if and only if $B B C_{q}(G, T) \leq q+5$. Take $U$ be a spanning tree of $H$.

We first construct a graph $G^{\prime}$ from $H$ by adding for every edge $e=u v$ of $U$ a vertex $x_{e}$ linked to $u$ and $v$. We let $T^{\prime}$ be the spanning tree of $G^{\prime}$ induced by the new edges. The pair $(G, T)$ is then obtained from $\left(G^{\prime}, T^{\prime}\right)$ by adding a parachute on every vertex. Clearly $G$ is planar as for each edge $e=u v$ the path $u x_{e} v$ can be drawn along the edge $u v$.

Suppose that $(G, T)$ admits a $q$-backbone $(q+5)$-colouring. Then by Proposition 14 -(iii), every vertex in $G^{\prime}$ is coloured in $\{1,2,3, q+3, q+4, q+5\}$. Note that the vertices of $H$ form one of the part of the bipartition of $T^{\prime}$. Hence, the colours of the vertices of $H$ are either all in $\{1,2,3\}$ or all in $\{q+3, q+3, q+5\}$. In both cases, $\phi$ induces a proper 3 -colouring on $H$.

Conversely, it is straightforward to extend a a proper 3-colouring of $H$ into a $q$-backbone $(q+5)$ colouring of $(G, T)$.

## 3 Circular backbone colouring

The following Proposition is an analogue to Proposition 13 and its proof is similar.
Proposition 32. Let $G$ be a graph and let $H$ be a subgraph of $G$ such that $2 \leq \chi(H)<\chi(G)$. Then $\mathrm{CBC}_{q}(G, H) \leq(\chi(G)+q-2) \chi(H)$.

Proof. Let $g$ be a proper $\chi(G)$-colouring of $G$ and $h$ a proper $\chi(H)$-colouring of $H$.
Assume first that $\chi(H)$ is even. Let $f$ be the colouring defined by:

$$
f(v)= \begin{cases}(h(v)-1)(q-2+\chi(G))+g(v), & \text { if } h(v) \text { is odd } \\ (h(v)-1)(q-2+\chi(G))+\chi(G)+1-g(v) & \text { if } h(v) \text { is even }\end{cases}
$$

Let us check that $f$ is a circular $q$-backbone $((\chi(G)+q-2) \chi(H))$-colouring of $(G, H)$. For $1 \leq$ $i \leq \chi(H)$, let $I_{i}=\{(i-1)(q-2+\chi(G))+1, \ldots,(i-1)(q-2+\chi(G))+\chi(G)\}$. Observe that if $h(v)=i$, then $f(v) \in I_{i}$. The $I_{i}$ form intervals of $\mathbb{Z}_{(\chi(G)+q-2) \chi(H)}$. These intervals do not intersect and two consecutives intervals are separated by $q-2$ elements. In particular, if $h(u) \neq h(v)$, then $|f(u)-f(v)| \geq q-1$. Moreover $|f(u)-f(v)|=q-1$ only if $g(u)=g(v)$.

Consider an edge $u v \in E(G)$. By the previous remark, if $h(u) \neq h(v)$, then $f(u) \neq f(v)$. If $h(u)=h(v)$, then $|f(u)-f(v)|=|g(u)-g(v)| \neq 0$, because $g$ is proper.

Consider now an edge $u v \in E(H)$. Then $h(u) \neq h(v)$, so $f(u)$ and $f(v)$ are in different $I_{i}$. If they are in non-consecutive $I_{i}$ (modulo $\chi(H)$ ), then $|f(u)-f(v)| \geq 2 q-2+\chi(G) \geq q$. Assume now that they are in consecutive intervals, then $|f(u)-f(v)| \geq q$ because $g(u) \neq g(v)$.

Assume now that $\chi(H)$ is odd. Let $f$ be the colouring defined by:

$$
f(v)= \begin{cases}1, & \text { if } h(v)=1 \text { and } g(v)=\chi(G), \\ g(v)+1, & \text { if } h(v)=1 \text { and } g(v)<\chi(G), \\ \chi(G)+q-1, & \text { if } h(v)=2 \text { and } g(v)=\chi(G)-1, \\ \chi(G)+q, & \text { if } h(v)=2 \text { and } g(v)=\chi(G), \\ 2 \chi(G)+q-1-g(v), & \text { if } h(v)=2 \text { and } g(v)<\chi(G)-1, \\ (h(v)-1)(q-2+\chi(G))+g(v), & \text { if } h(v) \text { is odd and } h(v)>2, \\ (h(v)-1)(q-2+\chi(G))+\chi(G)+1-g(v) & \text { if } h(v) \text { is even and } h(v)>2 .\end{cases}
$$

Similarly to the even case, one can check that $f$ is a circular $q$-backbone $((\chi(G)+q-2) \chi(H))$ colouring of $(G, H)$.

### 3.1 Planar graphs of girth at least 5

Theorem 33. Let $G$ be a planar graph of girth at least 5 and $M$ a matching in $G$. Then $\operatorname{CBC}_{q}(G, M) \leq$ $2 q+1$.
Proof. Our proof is based on a structural result of Borodin and Glebov [1]. See also [9].
Theorem 34 (Borodin and Glebov [1]). The vertex set of every planar graph of girth at least 5 can be partitioned into an independent set and a set which induces a forest.

Let $(S, F)$ be a partition of $V(G)$ such that $S$ is stable and $F$ induces a forest. Let us colour every vertex of $S$ with 1 . Now since $F$ is a forest, it has an ordering $v_{1}, \ldots, v_{p}$ such that for every $i, v_{i}$ has at most one neighbour in $\left\{v_{1}, \ldots, v_{i-1}\right\}$. We colour the vertices of $F$ according to this ordering as follows. If $v_{i}$ has no neighbour in $\left\{v_{1}, \ldots, v_{i-1}\right\}$, then colour it with $q+1$. If $v_{i}$ has a neighbour $u$ in $\left\{v_{1}, \ldots, v_{i-1}\right\}$ and $u v_{i} \notin E(M)$, then colour it with a colour of $\{q+1, q+2\}$ not assigned to $u$. If $v_{i}$ has a neighbour $u$ in $\left\{v_{1}, \ldots, v_{i-1}\right\}$ and $u v_{i} \in E(M)$, then assign $2 q+1$ (resp. 2 to $v_{i}$ ) if $u$ is coloured $q+1$ (resp. $q+2$ ). It is easy to check that the obtained colouring is a $q$-backbone $\mathbb{Z}_{2 q+1}$-colouring of $(G, M)$.

### 3.2 Circular $q$-backbone $2 q$-colouring

Proposition 35. Let $G$ be a graph and $H$ a spanning connected subgraph of $G$. Then $\operatorname{CBC}_{q}(G, H)=$ $2 q$ if and only if $G$ is bipartite.
Proof. If $G$ is bipartite, then $\chi(G)=\chi(H)=2$. Thus, by Equation (3), $\mathrm{CBC}_{q}(G, H)=2 q$.
Assume now that $(G, H)$ admits a circular $q$-backbone $2 q$-colouring $f$. Let $v$ be a vertex of $G$. Without loss of generality, we may assume that $f(v)=1$. Then all the neighbours of $v$ in $H$ must be coloured $q+1$. And so on, by induction, all the vertices at even distance from $v$ in $H$ are coloured 1 and all the vertices at odd distance from $v$ in $H$ are coloured $q+1$. Since $H$ is connected and spans $G$, it follows that all vertices are coloured 1 or $q+1$, so $G$ is bipartite.

Proposition 35 implies that given a graph $G$ and a spanning connected subgraph $H$, deciding if $\mathrm{CBC}_{q}(G, H)=2 q$ can be done in polynomial time. In contrast, if the condition of $G$ be connected is removed, when $q=2$, the problem becomes NP-complete, as shown by the following theorem.

Theorem 36. The following problem is NP-complete.
Input: A planar graph $G$ and a matching $M$ in $G$.
Question: Is $\mathrm{CBC}_{2}(G, M) \leq 4$ ?

Proof. The problem is trivially in NP since a circular 2-backbone 4-colouring of $(G, F)$ is clearly a certificate.

To prove it is NP-complete, we give a reduction from Not-All-EQUAL 3SAT.
Let $\mathcal{C}=\left\{C_{1}, \ldots, C_{n}\right\}$ be a collection of clauses of size three over a set $U$ of variables. We will construct a graph pair $(G, M)$ such that $M$ is a matching in $G$.

To do so we need some definitions and gadgets.
Colours 1 and 3 are said to be twins and so do the colours 2 and 4 . Trivially two vertices joined by an edge of $M$ receives distinct twin colours. Two colours are siblings if they are equal or twins.

A link with ends $u$ and $v$ and central edge $w_{1} w_{2}$ is a subgraph with vertex set $\left\{u, v, w_{1}, w_{2}\right\}$ and edge set $\left\{u w_{1}, u w_{2}, v w_{1}, v w_{2}, w_{1} w_{2}\right\}$ with $w_{1} w_{2} \in M$. Two ends of a link are said to be linked.
Claim 37. In a circular 2-backbone 4-colouring c, the colours of the ends of a link are siblings.
Proof. The two vertices $w_{1}$ and $w_{2}$ are joined by an edge of $M$, so $\left\{c\left(w_{1}\right), c\left(w_{2}\right)\right\} \in\{\{1,3\} ;\{2,4\}\}$. Hence if $u$ is coloured in $\{1,3\}$ (resp. $\{2,4\}$ ), then $\left\{c\left(w_{1}\right), c\left(w_{2}\right)\right\}$ is $\{2,4\}$ (resp. $\{1,3\}$ ), and so $v$ is coloured in $\{1,3\}$ (resp. $\{2,4\}$ ).

For each variable $u \in U$, we create a variable gadget $G^{u}$ which is obtained from the distinct vertices $a_{1}^{u}, a_{2}^{u}, \ldots, a_{n}^{u}$ by linking, from $1 \leq i \leq n-1$, the vertices $a_{i}^{u}$ and $a_{i+1}^{u}$ by an link with central edge $b_{i}^{u} c_{i}^{u}$.

Claim 37 (and its proof) immediately implies the following.
Claim 38. In a circular 2-backbone 4 -colouring of $G^{u}$, all the $a_{u}^{i}$ are coloured with two sibling colours and all the $b_{u}^{i}$ with the two other colours (which are also siblings).

For each clause $C_{i}=\ell_{1}^{i} \vee \ell_{2}^{i} \vee \ell_{3}^{i}$, we create a triangle $z_{1}^{i} z_{2}^{i} z_{3}^{i}$. Now for $j=1,2,3$, if $\ell_{j}^{i}$ is the nonnegated literal $u$, we join $z_{j}^{i}$ with $a_{i}^{u}$, and if $\ell_{j}^{i}$ is the negated literal $\bar{u}$, we join $z_{j}^{i}$ with $b_{i}^{u}$. Such edges are said to be red. So far, the obtained graph $H$ is not planar, but we can clearly draw it such that only red edges cross. We can now subdivide every red edge into a red path such that every edge is crossed at most once. We then replace the red edges which are not crossed by a link (with the same end) and two red edges $u v$ and $x y$ that cross each other by the crossing gadget depicted in Figure 12. The resulting graph $G$ is planar and it comes with a matching $M$.


Figure 12: The crossing gadget

Claim 39. In a circular 2-backbone 4-colouring of the crossing gadget, the colours of $u$ and $v$ are siblings and the colours of $x$ and $y$ are siblings. In addition, for any 4-tuple $\left\{c_{u}, c_{v}, c_{x}, c_{y}\right\}$ such that $c_{u}$ and $c_{v}$ are siblings and $c_{x}$ and $c_{y}$ are siblings, there is a circular 2-backbone 4-colouring $c$ of the crossing gadget such that $c(u)=c_{u}, c(v)=c_{v}, c(x)=c_{x}$, and $c(y)=c_{y}$.

Proof. Consider first a circular 2-backbone 4-colouring of the crossing gadget. $u$ is linked to $u^{\prime}$, which is linked to $v^{\prime}$, which in turn is linked to $v$. Hence, by Claim 37, the colours of $u$ and $v$ are siblings.

Assume that $x$ is coloured in $\{1,3\}$, then $x^{\prime}$ is also coloured in $\{1,3\}$, say 1 . The vertices $a$ and $b$ are assigned twin colours, so one is coloured 2 and the other 4 . We now distinguish two cases depending on the colour of $u^{\prime}$.

1. Assume $u^{\prime}$ is coloured 3. Then $v^{\prime}$ must also be coloured 3. The vertices $c$ and $d$ are assigned twin colours, so one is coloured 2 and the other 4 . Hence $y^{\prime}$ is coloured 1 .
2. Assume $u^{\prime}$ is coloured in $\{2,4\}$. Without loss of generality, we may assume it is coloured 2. Then $a$ is coloured 4 and $b$ is coloured 2 , so $v^{\prime}$ is coloured 4 . Hence $y^{\prime}$ is coloured in $\{1,3\}$.

In both cases the colour of $x$ and $y^{\prime}$ are siblings, and so, by Claim 37, the colours of $x$ and $y$ are siblings.

For any 4-tuple $\left\{c_{u}, c_{v}, c_{x}, c_{y}\right\}$ such that $c_{u}$ and $c_{v}$ are siblings and $c_{x}$ and $c_{y}$ are siblings, finding the desired circular 2-backbone 4 -colouring is straightforward and left to the reader.

We shall now prove that $\mathcal{C}$ admits a suitable truth assignment if and only if $\mathrm{CBC}_{2}(G, M) \leq 4$.
Assume first that $(G, M)$ admits a circular 2-backbone 4-colouring. Let $\phi$ be the truth assignment defined by $\phi(u)=$ true if all the $a_{u}^{i}$ are coloured in $\{1,3\}$, and $\phi(u)=$ false if all the $a_{u}^{i}$ are coloured in $\{2,4\}$. Note that is well defined by Claim 38. Now by Claims 37 and 39, for each clause $C_{i}=\ell_{1}^{i} \vee \ell_{2}^{i} \vee \ell_{3}^{i}$, the vertex $z_{j}^{i}$ is coloured in $\{1,3\}$ if and only if the literal $\ell_{j}^{i}$ is true. But since $z_{1}^{i} z_{2}^{i} z_{3}^{i}$ is a triangle, at least three colours must appear on these vertices, and so at least one from $\{1,3\}$ and at least one from $\{2,4\}$. Hence, at least one of the literals of $C_{i}$ is true and at least one is false. Thus $\phi$ is suitable.

Reciprocally, assume that $\mathcal{C}$ admits a suitable truth assignment $\phi$. If $\phi(u)=$ true, then colour all the $a_{u}^{i}$ with 1 , all the $b_{u}^{i}$ with 2 and all the $c_{u}^{i}$ with 4 . And if $\phi(u)=f a l s e$, then colour all the $a_{u}^{i}$ with 2 , all the $b_{u}^{i}$ with 1 and all the $c_{u}^{i}$ with 3 . Now, for each clause $C_{i}=\ell_{1}^{i} \vee \ell_{2}^{i} \vee \ell_{3}^{i}$, some literal, say $\ell_{1}^{i}$, is true and some literal, say $\ell_{3}^{i}$, is false. Then assign 1 to $z_{1}^{i}, 2$ to $z_{3}^{i}$, and colour $z_{2}^{i}$ with 3 if $\ell_{2}^{i}$ is true and 4 otherwise. By Claims 37 and 39, this partial colouring may be extended into a circular 2 -backbone 4 -colouring of $(G, M)$.

Theorem 40. The following problem is NP-complete.
Input: A planar graph $G$ and a matching $M$ in $G$.
Question: Is $\mathrm{CBC}_{2}(G, M) \leq 5$ ?

Proof. The reduction is from Planar $C_{5}$-Colouring which is defined as follows:
Input: A planar graph $G$.
Question: Does $G$ have a homomorphism onto $C_{5}$, the cycle of length 5 ?
This was proved to be NP-complete by MacGillivray and Siggers [10].
To make the reduction we need an edge gadget. This gadget is built from the planar graph $H_{1}(u, v)$ together with the matching $M_{1}(u, v)$ depicted in Figure 13.


Figure 13: Graph $H_{1}(u, v)$ with matching $M_{1}(u, v)$ (in bold)

The graph $H_{2}(u, v)$ is obtained from $H_{1}(u, v)$ by replacing the edge $u w$ by $H_{1}(w, u)$. The matching $M_{2}(u, v)$ is then the union of $M_{1}(u, v) \backslash\{u w\}$ and $M_{1}(w, u)$. Observe that $u$ and $v$ are incident to no edges of $M_{2}(u, v)$. The pair $\left(H_{2}(u, v), M_{2}(u, v)\right)$ is the edge gadget.

Broersma et al. [5] proved that in any circular 2-backbone 5-colouring of $\left(H_{1}(u, v), M_{1}(u, v)\right)$, vertices $u$ and $v$ receive colours which are cyclically 2 apart. In addition, it is straightforward to see that any precolouring of $u$ and $v$ with colours that are cyclically 2 apart can be extended into a circular 2-backbone 5 -colouring of $\left(H_{1}(u, v), M_{1}(u, v)\right)$. These two facts imply the following claim.

## Claim 41.

(i) In any circular 2-backbone 5 -colouring of $\left(H_{2}(u, v), M_{2}(u, v)\right)$, vertices $u$ and $v$ receive colours which are cyclically 2 apart.
(ii) Any precolouring of $u$ and $v$ with colours that are cyclically 2 apart can be extended into $a$ circular 2-backbone 5 -colouring of $\left(H_{2}(u, v), M_{2}(u, v)\right)$.

Let $H$ be an instance of Planar $C_{5}$-Colouring. Replace each edge $u v \in E(G)$ by an edge gadget $\left(H_{2}(u, v), M_{2}(u, v)\right)$ to obtain a planar graph $G$ and a matching $M$ (the union of the $M_{2}(u, v)$ ). By Claim 41-(i), every circular 2-backbone 5 -colouring of $(G, M)$ induces a $C_{5}$-colouring of $H$ (the vertices of the $C_{5}$ are the colours ( $1,3,5,2,4$ ). Conversely, by Claim 41-(ii), any $C_{5}$-colouring of $H$ can be extended into a circular 2-backbone 5 -colouring of $(G, M)$. Hence $H$ admits a $C_{5}$-colouring if and only if $(G, M)$ admits a circular 2 -backbone 5 -colouring.

Adding long paths along existing edges to transform the matching into a spanning tree, one derives the following:

Theorem 42. The following problem is NP-complete.
Input: A planar graph $G$ and a spanning tree $T$ of $G$.
Question: Is $\mathrm{CBC}_{2}(G, T) \leq 5$ ?

## 4 Further research

Campos et al. [6] proved that if $G$ is planar and $T$ has diameter at most 3, then $\mathrm{BBC}_{2}(G, T) \leq 5$. Hence one can the find the 2-backbone chromatic number of such a pair in polynomial time. One can ask of the complexity for larger diameter.

Problem 43. For a fixed $d \geq 4$, what is the complexity of finding the 2-backbone chromatic number of $(G, T)$, when $G$ is planar and $T$ a spanning tree of diameter $d$ ?

Since, for any fixed $k \leq 4$, deciding if the 2 -backbone chromatic number of $(G, T)$ is at most $k$ can be done in polynomial time, if Conjecture 1 holds, Problem 43 is equivalent to finding the complexity of deciding if $\mathrm{BBC}_{2}(G, T) \leq 5$.

If $G$ is a triangle-free planar graph, then, by Grötzsch's Theorem [8], it is 3 -colourable, and so $\mathrm{BBC}_{q}(G, H) \leq 2 q+1$ and $\mathrm{CBC}_{q}(G, H) \leq 3 q$ for any subgraph $H$ of $G$. Hence Conjecture 1 and Conjecture 6 for $q=2$, hold when $G$ is triangle-free. A natural next step would be to prove Conjecture 6 for values of $q$ larger than 2 when $G$ is triangle-free.

Steinberg's Conjecture (1976) states that every planar graph without 4- and 5-cycles is 3-colourable. Towards this, Erdős (1991) proposed the following relaxation of Steinberg's Conjecture: Determine the smallest value of $k$, such that every planar graph without cycles of length from 4 to $k$ is 3colourable. The best known bound for such a $k$ is 7 which was proved by Borodin, Glebov, Raspaud and Salavatipour [2]. Hence, an evidence to both Conjecture 6 and Steinberg's Conjecture would be to prove the following:

Conjecture 44. If $G$ is a planar graph without 4- and 5-cycles and $F$ a spanning forest of $G$, then $\mathrm{CBC}_{2}(G, F) \leq 7$.

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