



HAL
open science

Maximization Coloring Problems on graphs with few P4s

Victor Campos, Claudia Linhares Sales, Ana Karolinnna Maia de Oliviera,
Rudini Sampaio

► **To cite this version:**

Victor Campos, Claudia Linhares Sales, Ana Karolinnna Maia de Oliviera, Rudini Sampaio. Maximization Coloring Problems on graphs with few P4s. *Discrete Applied Mathematics*, 2014, 164 (2), pp.539-546. 10.1016/j.dam.2013.10.031 . hal-00951135

HAL Id: hal-00951135

<https://inria.hal.science/hal-00951135>

Submitted on 24 Feb 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Maximization Coloring Problems on graphs with few P_4 s

V. Campos* C. Linhares Sales* A. K. Maia† R. Sampaio*

February 24, 2014

Abstract

Given a graph $G = (V, E)$, a *greedy coloring* of G is a proper coloring such that, for each two colors $i < j$, every vertex of $V(G)$ colored j has a neighbor with color i . The greatest k such that G has a greedy coloring with k colors is the *Grundy number* of G . A *b-coloring* of G is a proper coloring such that every color class contains a vertex which is adjacent to at least one vertex in every other color class. The greatest integer k for which there exists a *b-coloring* of G with k colors is its *b-chromatic number*. Determining the Grundy number and the *b-chromatic number* of a graph are NP-hard problems in general.

For a fixed q , the $(q, q - 4)$ -graphs are the graphs for which no set of at most q vertices induces more than $q - 4$ distinct induced P_4 s. In this paper, we obtain polynomial-time algorithms to determine the Grundy number and the *b-chromatic number* of $(q, q - 4)$ -graphs, for a fixed q . They generalize previous results obtained for cographs and P_4 -sparse graphs, classes strictly contained in the $(q, q - 4)$ -graphs.

1 Introduction

Let $G = (V, E)$ be a finite undirected graph, without loops or multiple edges. A *k-coloring* of G is a surjective mapping $c : V(G) \rightarrow \{1, 2, \dots, k\}$ such that $c(u) \neq c(v)$ for any edge $uv \in E$. The sets of vertices S_1, \dots, S_k with colors $1, 2, \dots, k$, respectively, that form a partition of $V(G)$ in stable sets, are called *color classes*. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G admits a k -coloring. It is well known that determining $\chi(G)$ is a NP-hard problem.

Hence lots of heuristics have been developed to color a graph. One of the most basic and used is the greedy algorithm. Given an order v_1, v_2, \dots, v_n of the vertices of G , the greedy algorithm colors the vertices of G assigning to v_i the minimum positive integer that was not already assigned to its neighbors in the set $\{v_1, \dots, v_{i-1}\}$. Such a coloring is called a *greedy coloring*. The maximum number of colors of a greedy coloring of a graph G , over all possible orderings of the vertices of $V(G)$, is the *Grundy number* of G and it is denoted by $\Gamma(G)$.

Zaker [1] showed that, for any fixed k , one can decide in polynomial time if a given graph has Grundy number at least k (that is, deciding if $\Gamma(G) \geq k$ is fixed parameter tractable on k). However determining the Grundy number of a graph is NP-hard [1]. Moreover, in 2010, Havet and Sampaio [2] proved that it is NP-complete to decide if $\Gamma(G) = \Delta(G) + 1$. In addition, Asté et al. [3] showed that, for any constant $c \geq 1$, it is NP-complete to decide if $\Gamma(G) \leq c \cdot \chi(G)$.

Another alternative way of dealing with the coloring problem is to try to improve any coloring c of the graph by applying some strategy, obtaining from c a coloring with a smaller number of colors.

*ParGO Research Group, Federal University of Ceará, Campus do Pici, Bloco 910, 60455-760 - Fortaleza - Brazil. Partially supported by CNPq/Brazil and FUNCAP/Brazil. (email: [campos, linhares, rudini]@lia.ufc.br)

†COATI Project, I3S (CNRS/UNSA) & INRIA, 2004 route des Lucioles BP 93, 06902 Sophia-Antipolis Cedex, France. Partially supported by ANR Blanc STINT and CAPES/Brazil. (email: karol.maia@inria.fr)

Observe that, if c has a color class S_i such that for every vertex $v \in S_i$, there is at least one other color class S_j such that v does not have neighbors in S_j , we could eliminate S_i by recoloring every vertex v from S_i with the color j that does not appear in its neighborhood. A vertex v from S_i is said to be *dominant* if v is adjacent to at least one vertex in S_j for all $j \neq i$. It is easy to see that if every color class $S_i \in c$ has a dominant vertex, then it is not possible to improve c by applying the above strategy.

A *b-coloring* of G is a coloring such that every color class contains a dominant vertex. The *b-chromatic number* $\chi_b(G)$ of a graph G is the maximum number k such that there exists a *b-coloring* of G with k colors. Observe that the *b-chromatic number* of G measures the worst performance of the improvement strategy of a coloring described previously. This parameter has been introduced by R. W. Irving and D. F. Manlove [4]. They proved that determining the *b-chromatic number* is polynomial-time solvable for trees, but it is NP-hard for general graphs. In [5], Kratochvíl, Tuza and Voigt proved that computing the *b-chromatic number* is NP-hard even if G is a connected bipartite graph.

Let $G = (V, E)$ be a graph. We say that G is a P_4 if $V(G) = \{w, x, y, z\}$ and $E(G) = \{wx, xy, yz\}$, that is, an induced path on four vertices. We say that w and z are the *endpoints* and x and y the *midpoints* of the P_4 .

A *cograph* is a P_4 -free graph and a *P_4 -sparse graph* is a graph G such that each subset of G with five vertices induces at most one P_4 . The P_4 -sparse graphs, introduced in [6], generalize cographs and can be recognized in linear time [7].

Many NP-hard problems were proved to be polynomial-time solvable on cographs and P_4 -sparse graphs. In particular, polynomial-time algorithms were presented to solve the problem of determining the Grundy number and the *b-chromatic number* for these graphs [8, 9, 10].

Babel and Olariu [11] defined a graph as *$(q, q - 4)$ -graph* if no set of at most q vertices induces more than $q - 4$ distinct P_4 s. For example, cographs and P_4 -sparse graphs are precisely $(4, 0)$ -graphs and $(5, 1)$ -graphs, respectively.

Our main result (Theorem 1) says that, for every fixed integer $q > 0$, there is a polynomial algorithm to obtain the Grundy number and the *b-chromatic number* of a $(q, q - 4)$ -graph.

Theorem 1 (Main result). *Let $q > 0$ be a fixed integer. The Grundy number and the b-chromatic number of a $(q, q - 4)$ -graph G can be computed in polynomial time.*

This paper is organized as follows. Section 2 contains structural results for $(q, q - 4)$ -graphs. Section 3 presents the results used to calculate the Grundy number of these graphs and in Section 4 we show how to determine their *b-chromatic number*.

2 Decomposing $(q, q - 4)$ -graphs

A graph H is *p -connected* if, for every partition of $V(G)$ into nonempty disjoint sets V_1 and V_2 , there exists an (V_1, V_2) -crossing P_4 , that is, an induced P_4 containing vertices from both V_1 and V_2 . A *p -connected graph H is separable* if there exists a partition of $V(G)$ into nonempty disjoint subsets V_1 and V_2 such that each (V_1, V_2) -crossing P_4 has its *midpoints* in V_1 and its *endpoints* in V_2 . We say that (V_1, V_2) is the *separation* of H and H_1 and H_2 are the graphs $H[V_1]$ and $H[V_2]$, respectively. A maximal *p -connected induced subgraph* is called a *p -component*. Vertices which are not contained in a nontrivial *p -component* are called *weak*.

A *decomposition tree* of a graph G is a tree T_G , where the leaves are subsets of vertices of G and each non-leaf node v in T_G , with children v_1, \dots, v_l , represents the subgraph of G , denoted by $G(v)$, induced by the leaves of the subtree of T_G rooted by v . Moreover, v is labelled according to its relation with the graphs $G(v_1), \dots, G(v_l)$. Clearly, the intersection of the leaves must be empty and their union must be the set of vertices of G . The root node of T_G represents the original graph G .

In [12], Jamison and Olariu suggest a decomposition tree for general graphs, called *primeval decomposition tree*, which can be computed in linear time [12]. The leaves of its decomposition tree are p -connected graphs and its weak vertices, and its internal nodes are labelled *union*, *join* or *p -component*.

If the label of a node v is *union*, $G(v)$ is the disjoint union of $G(v_1), \dots, G(v_l)$, that is, the set of vertices of $G(v)$ is the union of the set of vertices of $G(v_1), \dots, G(v_l)$ and the set of edges of $G(v)$ is the union of the set of edges of $G(v_1), \dots, G(v_l)$.

If the label of a node v is *join*, $G(v)$ is the join of $G(v_1), \dots, G(v_l)$, that is, the set of vertices of $G(v)$ is the union of the set of vertices of $G(v_1), \dots, G(v_l)$ and the set of edges of $G(v)$ is the union of the set of edges of $G(v_1), \dots, G(v_l)$, in addition to all the possible edges between the vertices of $G(v_1), \dots, G(v_l)$.

If v is labelled *p -component*, it has two children on the tree: a separable p -component H , which is a leaf on the primeval decomposition tree and an internal node that represents the graph $G(v) - H$. Moreover, every vertex from $G(v) - H$ is adjacent to every vertex in H_1 and to no vertex in H_2 .

A graph is a *spider* if its vertex set can be partitioned into three sets S , K and R in such a way that S is a stable set, K is a clique, all the vertices of R are adjacent to all the vertices of K and to none of the vertices of S and there exists a bijection $f : S \rightarrow K$ such that, for all $s \in S$, either the neighborhood of s $N(s) = \{f(s)\}$ (and it is a *thin spider*) or $N(s) = K - \{f(s)\}$ (and it is a *thick spider*). We say that the spider is without head if $R = \emptyset$.

In [11], Babel and Olariu also proved that the primeval decomposition of a $(q, q - 4)$ -graph has a special property: every node v on the tree labelled as p -component is such that its separable p -component H is a headless spider or it has less than q vertices. If H is the headless spider, it is easy to see that H_1 is the clique and H_2 is the stable set. Since every vertex from $V(G(v) - H)$ is adjacent to every vertex in H_1 and non-adjacent to every vertex in H_2 , we have that $G(v)$ is itself a spider with head, where $G(v) - H$ is the head.

In this paper, we calculate the Grundy number and the b -chromatic number of $(q, q - 4)$ -graphs through bottom-up traversal on their primeval decomposition tree. More specifically we solve the case in which a node v on is labelled as p -component and the p -connected component H of $G(v)$ is a graph with less than q vertices. In the remaining non-trivial cases, we use some results in [9] and [10] to calculate $\Gamma(G(v))$ and $\chi_b(G(v))$, respectively.

3 Greedy Coloring of $(q, q - 4)$ -graphs

As first shown in [8], if G is the disjoint union of two graphs G_1 and G_2 , then $\Gamma(G) = \max\{\Gamma(G_1), \Gamma(G_2)\}$. On the other hand, if G is the join of two graphs G_1 and G_2 , then $\Gamma(G) = \Gamma(G_1) + \Gamma(G_2)$. In [9] is shown how to determine the Grundy number for spiders.

Lemma 2 ([9]). *Let G be a spider with partition (S, K, R) and n vertices. If G is a spider and $\Gamma(R)$ is given, then $\Gamma(G)$ can be determined in linear time.*

Let $G = (V, E)$ be a graph. A subset M of V with $1 \leq |M| \leq |V|$ is called a *module* if each vertex in $V - M$ is either adjacent to all vertices of M or to none of them. A module M is called a *homogeneous set* if $1 < |M| < |V|$. The graph obtained from G by shrinking every maximal homogeneous set to one single vertex is called the *characteristic graph* of G .

A graph is called *split graph* if its vertex set has a partition (K, S) such that K induces a clique and S induces a stable set.

Lemma 3 ([13]). *A p -connected graph H is separable if and only if its characteristic graph is a split graph.*

Note that, if M_1 and M_2 are two modules of a graph G such that $M_1 \cap M_2 = \emptyset$, then either the edges from $\{\{v, w\} : v \in M_1, w \in M_2\}$ belong to G or G has none of such edges.

Recall Lemma 3. Clearly, if the characteristic graph of a separable p -component H with separation (V_1, V_2) is the split graph (K, S) , then every maximal homogeneous set $M_i^1 \subseteq V_1$ shrinks to a vertex v_i^1 in the clique K , and every maximal homogeneous set $M_j^2 \subseteq V_2$ shrinks to a vertex v_j^2 in the stable set S . We say that $H[M_j^i] = H_j^i$.

Let H be a separable p -component with separation (V_1, V_2) . Observe that $H_1 = H[V_1]$ is the join of H_1^1, \dots, H_l^1 , since, between the graphs induced by two modules in the same graph, or there exist all the edges or none between them, and H_1^1, \dots, H_l^1 are the graphs induced by the strong maximal modules of H_1 . So, $\Gamma(H_1)$ is the Grundy number of the join of the graphs H_1^1, \dots, H_l^1 , which is $\sum_{i=1}^l \Gamma(H_i^1)$. Similarly, the Grundy number of some H_j^2 in $H_2 = H[V_2]$ with its neighborhood in H_1 is the Grundy number of the join of these graphs.

In [9], a relation between the Grundy number of a graph and the Grundy number of its modules is shown.

Proposition 4. *Let G, H_1, \dots, H_n be disjoint graphs such that $n = |V(G)|$ and let $V(G) = \{v_1, \dots, v_n\}$. Let G' be the graph obtained by replacing $v_i \in V(G)$ by H_i , in such a way that there exist all the edges between the vertices of H_i and H_j , $i \neq j$, if and only if $v_i v_j \in E(G)$. Then for every greedy coloring of G' at most $\Gamma(H_i)$ colors contain vertices of the induced subgraph $G'[V(H_i)] \subseteq G'$, for all $i \in \{1, \dots, n\}$.*

According to Proposition 4, a greedy coloring of a graph G restricted to its modules is a greedy coloring to them. The following result is a simple generalization of a result in [9]:

Lemma 5. *Let G be a graph and let M be a module of G such that $G[M] = H$ and in a greedy coloring that generates $\Gamma(G)$ there are k colors in H . Let G' be the graph obtained from G by replacing H by a complete graph K_k . Then, $\Gamma(G) = \Gamma(G')$.*

Proof. Let c be the coloring that generates $\Gamma(G)$. Let $A = \{\alpha_1, \dots, \alpha_k\}$ be the set of colors of c appearing on H . Let the vertices of the complete graph that replaces H on G' be w_1, \dots, w_k and c' be the coloring of G' defined by $c'(w_i) = \alpha_i$ for $i \in \{1, \dots, k\}$ and $c'(v) = c(v)$ for each vertex $v \in V(G) - M$. It is a simple matter to check that c' is a greedy coloring of G' . Hence $\Gamma(G') \geq \Gamma(G)$. Now let $\{S_1, \dots, S_k\}$ be a greedy k -coloring of H and c' be a greedy $\Gamma(G')$ -coloring of G' . It is important to see that there is a greedy k -coloring of H , by Proposition 4. Let $B = \{\beta_1, \dots, \beta_k\}$ be the set of colors appearing on K_k with $\beta_1 < \dots < \beta_k$. Let c be the coloring of G which, for every $1 \leq i \leq k$, assigns the color β_i to the vertices from S_i . Clearly, c is a greedy coloring of G . So $\Gamma(G) \geq \Gamma(G')$. \square

We denote by θ_H an order that produces a coloring with $\Gamma(H)$ colors for H . In particular, we denote by θ_j^i an order that produces a coloring with $\Gamma(H_j^i)$ colors for H_j^i . Theorem 6 is the main result of this section.

Theorem 6. *Let G be a $(q, q-4)$ -graph containing a separable p -component H with separation (V_1, V_2) and at most q vertices, such that every vertex in $R = G - H$ is adjacent to all vertices in $H_1 = H[V_1]$ and to no vertex in $H_2 = H[V_2]$. Let H_1^1, \dots, H_l^1 be the graphs induced by the maximal homogeneous sets of H_1 and H_1^2, \dots, H_m^2 the graphs induced by the maximal homogeneous sets of H_2 . Given $\chi(R)$ and $\Gamma(R)$, let G' be the graph obtained from G by replacing R by a complete graph $K_{\Gamma(R)}$. Then:*

- (a) *If $\Gamma(R) \geq \max_{1 \leq i \leq m} \Gamma(H_i^2)$, then $\Gamma(G) = \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$;*
- (b) *If $\Gamma(R) < \max_{1 \leq i \leq m} \Gamma(H_i^2)$, then $\Gamma(G) = \Gamma(G')$.*

Proof. (a) If we give an order to the greedy algorithm starting by $\theta_R, \theta_1^1, \dots, \theta_l^1$, we have a greedy coloring of G with at least $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ colors, since $R \cup H_1$ is the join of R, H_1^1, \dots, H_l^1 . So, we have to prove that $\Gamma(G) \leq \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$. Suppose by contradiction that there is a greedy coloring c of G with more than $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ colors and let c_{max} be the highest color in c . Consider the following cases:

(i) There is a vertex $v \in R$ colored c_{max} :

Let $c' = c(R \cup H_1)$. All colors in c should appear in c' , since v , to be colored c_{max} , has to be adjacent to vertices colored with all colors different from c_{max} , and a vertex in R has neighbors only in $R \cup H_1$. So, c' has more than $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ colors. Note that c' is not a greedy coloring to $R \cup H_1$, because a greedy coloring to $R \cup H_1$ has at most $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ colors, since $R \cup H_1$ is the join of R, H_1^1, \dots, H_l^1 . Thus, there is a vertex $u \in R \cup H_1$ colored t that has no neighbor colored f in $R \cup H_1$, for some $f < t$. Such vertex should be in H_1 , since all neighbors of vertices in R are in $R \cup H_1$. Then, $u \in H_i^1$ has a neighbor $w \in H_j^2$ colored f . Note that there exist all edges between H_i^1 and H_j^2 . Some vertex $z \in R \cup H_1$ is also colored f . It is easy to see that $z \notin R$, otherwise u would have a neighbor in $R \cup H_1$ colored f , since every vertex from R is adjacent all vertex in H_1 . Lemma 3 shows that there is all possible edges between two modules of H_1 . So, $z \notin H_s^1$, for $s \neq i$, because in this case also u would have a neighbor in $R \cup H_1$ already colored f . Therefore $z \in H_i^1$ and consequently z is adjacent to w , since there must exist all possible edges between H_i^1 and H_j^2 . But both are colored f , and this coloring would be improper.

(ii) There is a vertex $v \in H_2$ colored c_{max} :

For some $s \in \{1, \dots, m\}$, let $v \in H_s^2$ and $c' = c(H_s^2 \cup N(H_s^2)), N(H_s^2)$ beeing the graphs induced by the maximal homogeneous sets of H_1 such that the vertices are adjacent to the vertices of H_s^2 . All colors in c should appear in c' , since v has to be adjacent to vertices colored with all colors different from c_{max} and a vertex in H_s^2 has neighbors only in $(H_s^2) \cup N(H_s^2)$. So, c' has more than $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ colors. Note that $\Gamma(R) \geq \max_{1 \leq i \leq m} \Gamma(H_i^2)$ implies $\Gamma(R) \geq \Gamma(H_s^2)$. Therefore, $\Gamma(H_s^2) + \sum_{i \in N(H_s^2)} \Gamma(H_i^1) \leq \Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$. Then c' is not a greedy coloring to $(H_s^2) \cup N(H_s^2)$, because a greedy coloring to it has at most $\Gamma(H_s^2) + \sum_{i \in N(H_s^2)} \Gamma(H_i^1)$ colors, since $H_s^2 \cup N(H_s^2)$ is the join of $H_s^2, H_i^1, \forall i \in N(H_s^2)$. Thus, there is a vertex $u \in H_s^2 \cup N(H_s^2)$, colored t , that has no neighbor colored f in $H_s^2 \cup N(H_s^2)$, for some $f < t$. Such vertex should be in H_1 , because all neighbors of vertices in H_s^2 are in $H_s^2 \cup N(H_s^2)$. So, $u \in H_i^1$, where $H_i^1 \in N(H_s^2)$, has a neighbor $w \in R \cup H_1 - N(H_s^2)$ colored f . Observe that some vertex $z \in H_s^2 \cup N(H_s^2)$ is also colored f . It is easy to see that $z \notin H_s^2$. Otherwise, u would have a neighbor in $H_s^2 \cup N(H_s^2)$ colored f since every vertex in H_s^2 is adjacent to every vertex in $N(H_s^2)$. For the same reason, $z \notin H_j^1$, for $j \neq i$ and $j \in N(H_s^2)$. Therefore $z \in H_i^1$, but there is all possible edges between H_i^1 and $R \cup H_1 - N(H_s^2)$, what makes w and z neighbors. But both w and z are colored f , and this coloring would be improper.

(iii) There is a vertex $v \in H_1$ colored c_{max} :

To receive a color bigger than $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$, v must have at least $\Gamma(R) + \sum_{i=1}^l \Gamma(H_i^1)$ neighbors of different colors. From its neighborhood in R , v has at most $\Gamma(R)$ neighbors with different colors, by Proposition 4. From the neighborhood of v in H_i^1 , for $i \in \{1, \dots, l\}$, v has at most $\sum_{i=1}^l \Gamma(H_i^1) - 1$ (its own color), also by Proposition 4. So, it must appear another color c_n in a vertex $w \in H_j^2$, where $V(H_j^2) \in N(v)$. Since the vertices in R have no neighborhood with H_2 , c_n must be bigger than all colors in R and w must be neighbor of vertices colored with all colors in R . All these colors must appear in H_j^2 , because the neighbors from w outside H_j^2 are

vertices in H_1 , all neighbors from all vertices in R and, therefore, with different colors of R . We know that in H_j^2 appears at most $\Gamma(H_j^2)$ colors, what makes w to have at most $\Gamma(H_j^2) - 1$ neighbors colored differently in H_j^2 . But we know $\Gamma(H_j^2) \leq \Gamma(R)$ implies $\Gamma(H_j^2) - 1 < \Gamma(R)$. So, all colors of R cannot appear on the neighborhood of w , and such vertex cannot receive a different color.

(b) Since $\Gamma(R) < \max_{1 \leq i \leq m} \Gamma(H_i^2)$, in a greedy $\Gamma(G)$ -coloring of G , by Proposition 4, there are $p < q$ colors on R . We do not know the exact value of p , but we know that p goes from $\chi(R)$ to $\Gamma(R)$. By Lemma 5, we can replace R by a complete graph on p vertices and we can obtain all possible ordinations of $V(G)$, which are $(q+p)!$ in total. So, we can calculate all greedy colorings for G in $\sum_{p=\chi(R)}^{\Gamma(R)} (p+q)! \leq q(2q)! = O(1)$ steps, for a fixed q . \square

4 b -coloring of $(q, q-4)$ -graphs

In [10], Bonomo et al. presented a dynamic programming polynomial-time algorithm to compute the b -chromatic number of a P_4 -sparse graph. For this, they introduced the *dominance vector* of a graph.

Definition 7. Let G be a graph. Given a coloring of G , a vertex v is said to be dominant if v is adjacent to at least one vertex colored within each of the colors not assigned to v . The *dominance vector* dom_G of G is such that $dom_G[t]$ is the maximum number of distinct color classes admitting dominant vertices in any coloring of G with t colors, where $\chi(G) \leq t \leq |V(G)|$.

Note that a graph G admits a b -coloring with t colors if and only if $dom_G[t] = t$. So, the b -chromatic number $\chi_b(G)$ is the maximum number t such that $dom_G[t] = t$. Thus, once calculated the dominance vector of a graph, we have its b -chromatic number. Bonomo et al. [10] proved that calculating the dominance vector is polynomial-time solvable for cographs and P_4 -sparse graphs.

Lemmas 8 and 9 below from [10] show how to obtain the dominance vector for disjoint unions, joins and spiders. The calculation of $\chi(G)$ is from [14] and [15].

Lemma 8 (Dominance vector for union and join operations [10]). *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $V_1 \cap V_2 = \emptyset$ and let $t \geq \chi(G)$. If $G = G_1 \cup G_2$, then $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$ and*

$$dom_G[t] = \min\{t, dom_{G_1}[t] + dom_{G_2}[t]\}.$$

If $G = G_1 \vee G_2$, let $a = \max\{\chi(G_1), t - |V(G_2)|\}$ and $b = \min\{|V(G_1)|, t - \chi(G_2)\}$. Then, $\chi(G) = \chi(G_1) + \chi(G_2)$ and

$$dom_G[t] = \max_{a \leq j \leq b} \{dom_{G_1}[t] + dom_{G_2}[t - j]\}.$$

Lemma 9 (Dominance vector for spiders [10]). *Let G be a spider with partition (S, K, R) , where $k = |S| = |K| \geq 2$. If R is empty, consider $\chi(G[R]) = 0$ and $dom_{G[R]}[0] = 0$. Thus, $\chi(G) = k + \chi(G[R])$ and*

(a) *If G is a thin spider, then*

$$dom_G[i] = \begin{cases} k + dom_{G[R]}[i - k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\ k, & \text{if } i = k + |R| + 1, \\ 0, & \text{if } i > k + |R| + 1 \end{cases}$$

(b) If G is a thick spider, then

$$dom_G[i] = \begin{cases} k + dom_{G[R]}[i - k], & \text{if } k + \chi(G[R]) \leq i \leq k + |R|, \\ \min\{k, 4k - 2i + 2|R|\}, & \text{if } k + |R| + 1 \leq i \leq 2k + |R|, \\ 0, & \text{if } i > 2k + |R| \end{cases}$$

Using these lemmas, Bonomo et al. proved the theorem below.

Theorem 10 (Bonomo et al. [10]). *The dominance vector and the b -chromatic number of a cograph or P_4 -sparse graph can be computed in $O(n^3)$ time.*

Let $G = (V, E)$ be a graph and M be a module of G . Let $G_M = G[M]$ and let $N(M)$ be the neighborhood of a vertex in M . Let H, H_1 and H_2 be the subgraphs of G induced by $V \setminus M, N(M)$ and $V(H) \setminus N(M)$, respectively. If H has less than q vertices, G is obtained by applying p -component(q) operation over $(G_M, H = (H_1, H_2))$.

To calculate $dom_G[t]$, auxiliary lemma below shows us that there exists a good coloring such that: (a) all colors appears in M or H_1 or (b) vertices of M have distinct colors. Given a coloring c of G and a subgraph G' of G , let $n(C)$ be the number of colors used in C and let (C, G') be the restriction of the coloring C to G' .

Lemma 11. *If $\chi(G) \leq t \leq |V(G)|$, then there is a proper coloring C of G with t colors that maximizes the number of color classes with dominant vertices such that $n(C) = n(C, H_1) + n(C, G_M)$ or $n(C, G_M) = |V(G_M)|$.*

Proof. Let C be a coloring of G with t colors that maximizes the number of color classes with dominant vertices and then maximizes $n(C, G_M)$. Since M is a module, each vertex in G_M is adjacent to all vertices in H_1 . Thus, $n(C) \geq n(C, H_1) + n(C, G_M)$. Suppose that C does not satisfy the lemma. Since $n(C) > n(C, H_1) + n(C, G_M)$, then there is a color c that appears only in vertices of H_2 and thus no vertex of G_M is dominant in C . Since $n(C, G_M) < |V(G_M)|$, then there are two vertices v and v' of G_M that have the same color in C . Consider the coloring C' obtained from C by coloring v with color c . Note that any dominant vertex in C is also a dominant vertex in C' and thus C' also has a maximum number of color classes with dominant vertices among colorings with t colors. Note that $n(C', G_M) > n(C, G_M)$. Suppose again that C' does not satisfy the lemma. So, we can repeat this argument until we obtain a coloring C^* such that all vertices of G_M are colored with distinct colors. Thus, $n(C^*, G_M) = |V(G_M)|$ as desired. \square

Applying this lemma, we have four possible cases:

- (a) all colors appears in M or H_1
 - (a.1) There is no dominant vertex in H_2
 - (a.2) There is a dominant vertex in H_2
- (b) Vertices of M have distinct colors
 - (b.1) There are colors in M that are not in H
 - (b.2) Every color in M appears in H

Case (b.2) is easy to handle because it implies that $|M| \leq |V(H)|$. Since we will force that $|V(H)| \leq q$, we can obtain all colorings of G with t colors in constant time. To deal with cases (a.1), (a.2) and (b.1), we have to define some parameters.

Let $\mathcal{C}(t)$ be the set of all colorings of H with t colors and let $\mathcal{C}(t, t')$ be subset of $\mathcal{C}(t)$ with colorings of H such that H_1 uses t' colors. Let $C \in \mathcal{C}(t, t')$. For $H' \subseteq H$, let $c(C, H')$ denote the set of colors used in H' . We say that a vertex v in H_1 is partially dominant if v is adjacent to at least one vertex receiving each color in $c(C, H_1)$. Let $d_1(C)$ be the number of color classes of C with partially dominant vertices in H_1 . Let $d_2(C)$ be the number of color classes of $c(C, H_2) \setminus c(C, H_1)$ with a dominant vertex. Let $d_3(C)$ be the number of color classes in $c(C, H_1)$ with either a dominant vertex in H_2 or a partially dominant vertex in H_1 . Let $J \subseteq c(C, H_2) \setminus c(C, H_1)$. We say that a vertex v in H_1 is \bar{J} -dominant if v is adjacent to at least one vertex receiving each color in $c(C, H) \setminus J$. Let $d_4(C, J)$ be the number of color classes of C with either a dominant vertex in H_2 or a \bar{J} -dominant vertex in H_1 and $d_5(C, j) = \sup\{d_4(C, J) \mid J \subseteq c(C, H_2) \setminus c(C, H_1), |J| = j\}$.

Let $\chi(G) \leq t \leq |V|$, let $t_1 = \max\{t - |V(G_M)|, 0\}$, let $t_2 = \min\{|V(H_1)|, t - \chi(G_M)\}$, let $t_3 = \min\{|V(H)|, t\}$, let $t_4 = \min\{t - |V(G_M)|, |V(H_1)|\}$ and let

$$\begin{aligned} \tau_1(t) &= \sup_{\substack{t_1 \leq t' \leq t_2 \\ t' \leq t \leq t_3}} \{dom_{G_M}[t - t'] + d_1(C) \mid C \in \mathcal{C}(\hat{t}, t')\} \\ \tau_2(t) &= \sup_{t_1 \leq t' \leq t_2} \{\min\{t - t', d_2(C) + dom_{G_M}[t - t']\} + d_3(C) \mid C \in \mathcal{C}(t, t')\} \\ \tau_3(t) &= \sup_{\substack{t_1 \leq \hat{t} \leq t_3 \\ 0 \leq t' \leq t_4}} \{d_5(C, \hat{t} + |V(G_M)| - t) \mid C \in \mathcal{C}(\hat{t}, t')\} \end{aligned}$$

Excluding case (b.2) by forcing that $|V(G)| > 2|V(H)|$, we have the important lemma below.

Lemma 12. *If $\chi(G) \leq t \leq |V(G)|$ and $|V(G)| > 2|V(H)|$, then*

$$dom_G[t] = \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}.$$

Proof. Let C be a coloring of G with t colors that maximizes the number of color classes with dominant vertices. According to Lemma 11, suppose that either $n(C, H_1) + n(C, G_M) = t$ or $n(C, H_1) + n(C, G_M) < t$ and $n(C, G_M) = |V(G_M)|$. Let $\hat{t} = n(C, H)$.

The first case considered is (a) when $n(C, H_1) + n(C, G_M) = t$. Note that if v is a dominant vertex in C , then v is dominant in (C, G_M) if $v \in V(G_M)$ and v is partially dominant in (C, H) if $v \in V(H_1)$. Let $t' = n(C, H_1)$. Since $\chi(G_M) \leq n(C, G_M) = t - n(C, H_1) \leq |V(G_M)|$, then $t - |V(G_M)| \leq t' \leq t - \chi(G_M)$. We also get that $t' \leq |V(H_1)|$ and, thus, $t_1 \leq t' \leq t_2$.

Now, consider (a.1) that there is no dominant vertex in H_2 . In this case, $t' \leq \hat{t} \leq \min\{|V(H)|, t\}$. We also have that the number of color classes of C with dominant vertices of colors that appear in H_1 is precisely $d_1(C, H)$ and the with dominant vertices of colors that appear in G_M is at most $dom_{G_M}[t - t']$. Thus, if $n(C, H_1) + n(C, G_M) = t$ and there is no dominant vertex of C in H_2 , then $dom_G[t] \leq \tau_1(t)$.

Now, consider (a.2) that there is at least one dominant vertex u in H_2 . Since u is adjacent to every other color of C and every neighbour of u is in H , then $\hat{t} = t$. Note that the number of color classes of C with dominant vertices of colors that appear in H_1 is precisely $d_3(C, H)$. The number of color classes of C with dominant vertices of colors that appear in G_M is at most $\min\{t - t', d_2(C) + dom_G[t - t']\}$. Thus, if $n(C, H_1) + n(C, G_M) = t$ and there is at least one dominant vertex of C in H_2 , then $dom_G[t] \leq \tau_2(t)$.

The second case considered is (b) when $n(C, H_1) + n(C, G_M) < t$ and $n(C, G_M) = |V(G_M)|$. If (b.2) $c(C, G_M) \subseteq c(C, H)$, then $n(C, G_M) = |V(G_M)|$ implies that $|M| \leq |V(H)|$ and $|V(G)| \leq |V(H)|$, a contradiction. Thus, (b.1) there is a color unique to vertices in G_M . Note also that $n(C, H_1) + n(C, G_M) < t$ implies that there is a color unique to vertices in H_2 . Thus, all dominant vertices of C are in H_1 . Note that $\hat{t} \leq |V(H)|$ and $\hat{t} \leq t$ and, thus, $\hat{t} \leq t_3$. We also have that $t \leq \hat{t} + |V(G_M)|$ which implies that $\hat{t} \geq t_1$. Let $J = c(C, G_M) \cap c(C, H)$. Note that $t = \hat{t} + |V(G_M)| - |J|$ which implies that

$|J| = \hat{t} + |V(G_M)| - t$. Since J is a subset of $c(C, H) \setminus c(C, H_1)$, then $|J| = \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$ which implies that $t' \leq t - |V(G_M)|$. Since H_1 has at most $V(H_1)$ colors, then $t' \leq t_4$. Now, note that every dominant vertex of C is a J -dominant vertex of H_1 in (C, H) . Thus, the number of color classes with dominant vertices in C is $d_4((C, H), J)$, which is at most $d_5((C, H), \hat{t} + |V(G_M)| - t)$. Thus, if $n(C, H_1) + n(C, G_M) < t$ and $|V| > 2|V(H)|$, then $dom_G[t] \leq \tau_3(t)$.

We can conclude from the previous paragraphs that if $|V| > 2|V(H)|$, then $dom_G[t] \leq \max\{\tau_1(t), \tau_2(t), \tau_3(t)\}$. To conclude this proof, it remains to prove that $dom_G[t] \geq \tau_i(t)$, for $i \in \{1, 2, 3\}$. Let C_H be a coloring of H with \hat{t} colors and $t' = n(C_H, H_1)$. We break into cases depending on C_H being related to each of the parameters $\tau_i(t)$. To do so, let C_M be a coloring of G_M with $t - t'$ colors and $dom_{G_M}[t - t']$ color classes with dominant vertices and C'_M be a coloring of G_M with $|V(G_M)|$ colors.

Suppose that $t_1 \leq t' \leq t_2$. If $\hat{t} \leq t$, then rename the colors in $c(C_H, H_2) \setminus c(C_H, H_1)$ to colors in the set $c(C_M)$ and let C be the coloring of G obtained by piecing together this coloring with C_M . Note that C has precisely t colors and there are $dom_{G_M}[t - t']$ color classes with dominant vertices in colors of $c(C, G_M)$ and $d_1(C_H)$ color classes with dominant vertices in colors of $c(C, H_1)$. Since $c(C, G_M) \cap c(C, H_1) = \emptyset$, then C has at least $dom_{G_M}[t - t'] + d_1(C)$ color classes with dominant vertices. This implies that $dom_G[t] \geq \tau_1(t)$.

Now, suppose that $\hat{t} = t$. Let $c(C_M) = \{c_1, \dots, c_{t-t'}\}$ and suppose that the color classes with indices in $\{1, \dots, dom_{G_M}[t - t']\}$ have dominant vertices in C_M . Now let C'_H be obtained from C_H by renaming the colors in $c(C_H, H_2) \setminus c(C_H, H_1)$ to colors in $c(C_M)$ in such a way that the color classes with the highest indices have dominant vertices. Note that this is possible as $c(C_H, H_2) \setminus c(C_H, H_1)$ has size precisely $t - t'$. Let C be obtained by piecing together the colorings C_M and C'_H . Note that C has precisely t colors, $d_3(C_H)$ color classes in $c(C, H_1)$ with dominant vertices and $\min\{t - t', d_2(C) + dom_{G_M}[t - t']\}$ color classes in $c(C, G_M)$ with dominant vertices. This implies that $dom_G[t] \geq \tau_2(t)$.

Now, suppose that $0 \leq t' \leq t_4$ and $t_1 \leq \hat{t} \leq t_3$. Note that $0 \leq \hat{t} + |V(G_M)| - t \leq \hat{t} - t'$, as $\hat{t} \geq t_1 = t - |V(G_M)|$ and $t' \leq t_4 \leq t - |V(G_M)|$. Thus, let J be a subset of $c(C_H, H_2) \setminus c(C_H, H_1)$ such that $d_4(C_H, J) = d_5(C_H, \hat{t} + |V(G_M)| - t)$. Let C'_H be obtained by renaming the colors of C_H in the set J to colors in C'_M so that $|c(C'_H) \cap c(C'_M)| = |J| = \hat{t} + |V(G_M)| - t$. Let C be obtained by piecing together the colorings C'_H and C'_M . Note that $n(C) = n(C, H) + n(C, G_M) - |J| = \hat{t} + |V(G_M)| - |J| = t$. This implies that $dom_G[t] \geq \tau_3(t)$. \square

Lemma 13. *Let $q > 0$ be a fixed integer, let H be a graph with less than q vertices and let H_1 and H_2 be induced subgraphs of H such that $V(H_1)$ and $V(H_2)$ are a vertex partition of H . Given a graph G_M with n vertices, let G be the graph obtained by applying p -component operation over $(G_M, H = (H_1, H_2))$ (just join all edges between G_M and H_1). Then, given the chromatic number $\chi(G_M)$ and the dominance vector dom_M of G_M , we can calculate the chromatic number $\chi(G)$ in time $\Theta(n)$ and the dominance vector dom_G of G in time $\Theta(n^2)$.*

Proof. Since $|V(H)| \leq q$, where q is an integer fixed, we have that parameters $\tau_1(t)$, $\tau_2(t)$ and $\tau_3(t)$ can be obtained in linear time (once fixed t' and \hat{t} , the value in sup can be obtained in constant time that depends only on q). If $|V(G)| \leq 2|V(H)| \leq 2q$, then we can calculate $dom_G[t]$ in constant time. If $|V(G)| > 2|V(H)|$, then, applying Lemma 12, we have $dom_G[t]$ in linear time. So we can obtain the dominance vector dom_G of G in time $\Theta(n^2)$ for all possible values of t . \square

References

- [1] M. Zaker, Results on the Grundy chromatic number of graphs, Discrete Math. 306 (23) (2006) 3166–3173.

- [2] F. Havet, L. Sampaio, On the Grundy number of a graph, in: Proceedings of the International Symposium on Parameterized and Exact Computation(IPEC), Lecture Notes on Computer science 6478, December 2010.
- [3] M. Asté, F. Havet, C. Linhares-Sales, Grundy number and products of graphs, *Discrete Math.* 310 (9) (2010) 1482–1490.
- [4] R. W. Irving, D. F. Manlove, The b -chromatic number of a graph, *Discrete Appl. Math.* 91 (1-3) (1999) 127–141.
- [5] J. Kratochvíl, Z. Tuza, M. Voigt, On the b -chromatic number of a graph, *Lecture Notes in Computer Science*.
- [6] C. T. Hoang, Perfect graphs, ProQuest LLC, Ann Arbor, MI, 1985, thesis (Ph.D.)—McGill University (Canada).
- [7] B. Jamison, S. Olariu, Recognizing P_4 -sparse graphs in linear time, *SIAM J. Comput.* 21 (2) (1992) 381–406.
- [8] A. Gyárfás, J. Lehel, On-line and first fit colorings of graphs, *J. Graph Theory* 12 (2) (1988) 217–227. doi:10.1002/jgt.3190120212.
URL <http://dx.doi.org/10.1002/jgt.3190120212>
- [9] J. C. S. Araújo, C. Linhares-Sales, Grundy number of P_4 -classes, in: LAGOS'09—V Latin American Algorithms, Graphs and Optimization Symposium, Vol. 35 of *Electron. Notes Discrete Math.*, Elsevier Sci. B. V., Amsterdam, 2009, pp. 21–27.
- [10] F. Bonomo, G. Durán, F. Maffray, J. Marengo, M. Valencia-Pabon, On the b -coloring of cographs and P_4 -sparse graphs, *Graphs Combin.* 25 (2) (2009) 153–167.
- [11] L. Babel, S. Olariu, On the structure of graphs with few P_4 s, *Discrete Appl. Math.* 84 (1-3) (1998) 1–13.
- [12] B. Jamison, S. Olariu, p -components and the homogeneous decomposition of graphs, *SIAM J. Discrete Math.* 8 (3) (1995) 448–463.
- [13] L. Babel, T. Kloks, J. Kratochvíl, D. Kratsch, H. Müller, S. Olariu, Efficient algorithms for graphs with few P_4 's, *Discrete Math.* 235 (1-3) (2001) 29–51, *combinatorics (Prague, 1998)*.
- [14] D. G. Corneil, Y. Perl, L. Stewart, Cographs: recognition, applications and algorithms, in: Proceedings of the fifteenth Southeastern conference on combinatorics, graph theory and computing (Baton Rouge, La., 1984), Vol. 43, 1984, pp. 249–258.
- [15] B. Jamison, S. Olariu, Linear time optimization algorithms for P_4 -sparse graphs, *Discrete Appl. Math.* 61 (2) (1995) 155–175.