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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*Splitting a tournament into two subtournaments with  
given minimum outdegree*

Frédéric Havet — Bernard Lidický

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## Splitting a tournament into two subtournaments with given minimum outdegree

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**Abstract:** A  $(k_1, k_2)$ -outdegree-splitting of a digraph  $D$  is a partition  $(V_1, V_2)$  of its vertex set such that  $D[V_1]$  and  $D[V_2]$  have minimum outdegree at least  $k_1$  and  $k_2$ , respectively. We show that there exists a minimum function  $f_T$  such that every tournament of minimum outdegree at least  $f_T(k_1, k_2)$  has a  $(k_1, k_2)$ -outdegree-splitting, and  $f_T(k_1, k_2) \leq k_1^2/2 + 3k_1/2 + k_2 + 1$ . We also show a polynomial-time algorithm that finds a  $(k_1, k_2)$ -outdegree-splitting of a tournament if one exists, and returns ‘no’ otherwise. We give better bound on  $f_T$  and faster algorithms when  $k_1 = 1$ .

**Key-words:** tournament, splitting, outdegree

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## Partager un tournoi en deux sous-tournois de degré sortant minimum prescrits

**Résumé :** Un  $(k_1, k_2)$ -partage d'un digraphe  $D$  est une partition  $(V_1, V_2)$  de son ensemble de sommets telle que  $D[V_1]$  et  $D[V_2]$  soient de degré sortant minimum au moins  $k_1$  et  $k_2$ , respectivement. Nous établissons l'existence d'une fonction (minimum)  $f_T$  telle que tout tournoi de degré sortant minimum au moins  $f_T(k_1, k_2)$  a un  $(k_1, k_2)$ -partage, et que  $f_T(k_1, k_2) \leq k_1^2/2 + 3k_1/2 + k_2 + 1$ . Nous donnons également un algorithme en temps polynomial qui trouve un  $(k_1, k_2)$ -partage d'un tournoi s'il en existe un et renvoie 'non' sinon. Nous donnons de meilleures bornes sur  $f_T$  et des algorithmes plus rapides pour  $k_1 = 1$ .

**Mots-clés :** tournoi, partage, degré sortant

## 1 Introduction

Let  $D$  be a digraph. For a vertex  $v \in V(D)$  the *outdegree* of  $v$ , denoted by  $d_D^+(v)$ , is the number of arcs directed away from  $v$ . The minimum outdegree over all vertices of  $D$  is denoted by  $\delta^+(D)$ . We drop  $D$  in  $d_D^+(v)$  and  $\delta^+(D)$  if it is clear from the context.

A  $(k_1, k_2)$ -*outdegree-splitting* of a digraph  $D$  is a partition  $(V_1, V_2)$  of its vertex set such that  $D[V_1]$  and  $D[V_2]$  have minimum outdegree at least  $k_1$  and  $k_2$ , respectively. A digraph admitting a  $(k_1, k_2)$ -outdegree-splitting is said to be  $(k_1, k_2)$ -*outdegree-splittable*.

**Problem 1** (Alon [1]). Is there a function  $f$  such that every digraph with minimum outdegree  $f(k_1, k_2)$  has a  $(k_1, k_2)$ -outdegree-splitting?

The existence of the corresponding function  $f$  for the undirected analogue is easy and has been observed by many authors. Stiebitz [12] even proved the following tight result: if the minimum degree of an undirected graph  $G$  is  $d_1 + d_2 + \dots + d_k$ , where each  $d_i$  is a non-negative integer, then the vertex set of  $G$  can be partitioned into  $k$  pairwise disjoint sets  $V_1, \dots, V_k$ , so that for all  $i$ , the induced subgraph on  $V_i$  has minimum degree at least  $d_i$ . This is clearly tight, as shown by an appropriate complete graph.

Problem 1 is equivalent to the following:

**Problem 2.** Is there a function  $f'(k_1, k_2)$  such that every digraph with minimum outdegree  $f'(k_1, k_2)$  has two disjoint (induced) subdigraphs, one of them with minimum outdegree  $k_1$  and the other with minimum outdegree  $k_2$ ?

This follows from the following proposition.

**Proposition 3.** *Let  $D$  be a digraph with minimum outdegree at least  $k_1 + k_2 - 1$ . If  $D$  contains two disjoint subdigraphs  $D_1$  and  $D_2$  such that  $\delta^+(D_1) = k_1$  and  $\delta^+(D_2) = k_2$ , then  $D$  has a  $(k_1, k_2)$ -outdegree-splitting.*

*Proof.* Consider two disjoint digraphs  $D_1$  and  $D_2$  with  $\delta^+(D_1) = k_1$  and  $\delta^+(D_2) = k_2$  such that  $V(D_1) \cup V(D_2)$  is maximum. Suppose for a contradiction that  $S = V(D) \setminus (V(D_1) \cup V(D_2))$  is not empty. Then every vertex  $s \in S$  has at most  $k_1 - 1$  outneighbours in  $D_1$  otherwise  $D_1 + s$  and  $D_2$  contradict the maximality of  $D_1$  and  $D_2$ . Hence every vertex of  $S$  has at least  $k_1 + k_2 - 1 - (k_1 - 1) = k_2$  outneighbours in  $D - D_1$ . It follows that  $D - D_1$  has minimum degree  $k_2$ . So  $D_1$  and  $D - D_1$  contradicts the maximality of  $D_1$  and  $D_2$ .  $\square$

**Corollary 4.**  $f(k_1, k_2) \leq \max\{f'(k_1, k_2), k_1 + k_2 - 1\}$ .

This implies in particular that  $f(1, 1) = f'(1, 1) = 3$ . Indeed Thomassen [13] showed that every digraph of minimum outdegree at least 3 has two disjoint cycles. (In this paper, paths and cycles are always directed.)

This is a special case of Bermond-Thomassen Conjecture [3]:

**Conjecture 5** (Bermond and Thomassen [3]). *Every digraph with  $\delta^+ \geq 2k - 1$  contains  $k$  disjoint cycles.*

Note that Alon [1] proved that if  $\delta^+ \geq 64k$  there are  $k$  disjoint cycles.

A *tournament* is a digraph such that for every two distinct vertices  $u, v$  there is exactly one arc with ends  $\{u, v\}$  (so, either the arc  $uv$  or the arc  $vu$  but not both).

In this paper, we settle Problem 1 for tournaments.

**Theorem 6.** *Every tournament of minimum outdegree at least  $k_1^2/2 + 3k_1/2 + k_2 + 1$  has a  $(k_1, k_2)$ -outdegree-splitting.*

To prove Theorem 6, we shall prove the following theorem.

**Theorem 7.** *Every tournament with minimum outdegree at least  $k$  has a subtournament with minimum outdegree  $k$  and order at most  $k^2/2 + 3k/2 + 1$ .*

We can then easily derive Theorem 6.

*Proof of Theorem 6.* Let  $T$  be a tournament of minimum outdegree at least  $k_1^2/2 + 3k_1/2 + k_2 + 1$ . By Theorem 7, there exists a subtournament  $T_1$  with minimum outdegree at least  $k_1$  and order at most  $k_1^2/2 + 3k_1/2 + 1$ . Let  $T_2 = T - T_1$ . Then  $\delta^+(T_2) \geq \delta^+(T) - |V(T_1)| \geq k_2$ . Hence  $(V(T_1), V(T_2))$  is a  $(k_1, k_2)$ -outdegree-splitting.  $\square$

In fact, we prove a more general statement than Theorem 7 (Theorem 17). This enables us to prove the following generalization of Theorem 6.

**Theorem 8.** *Let  $m = \max\{k_1^2/2 + 3k_1/2 + k_2 + u_1 + 1, k_1 + u_2\}$ , let  $T$  be a tournament of minimum outdegree at least  $m$  and let  $U_1$  and  $U_2$  be two disjoint subsets of  $V(T)$  of cardinality  $u_1$  and  $u_2$  respectively. Then there is a  $(k_1, k_2)$ -outdegree-splitting  $(V_1, V_2)$  of  $T$  such that  $U_1 \subseteq V_1$  and  $U_2 \subseteq V_2$ .*

The bound of Theorem 6 is certainly not tight. Theorem 6 asserts that every tournament with minimum outdegree 4 has a  $(1, 1)$ -outdegree-splitting, but we know that having outdegree 3 is sufficient.

**Problem 9.** What is the minimum integer  $f_T(k_1, k_2)$  such that every tournament with minimum outdegree at least  $f_T(k_1, k_2)$  has a  $(k_1, k_2)$ -outdegree-splitting?

Theorem 6 implies that  $f_T(1, k) \leq k + 3$ . We describe examples implying  $f_T(1, k) \geq k + 2$ , and we conjecture that this lower bound is the exact value.

**Conjecture 10.** *For any positive integer  $k$ ,  $f_T(1, k) = k + 2$ .*

In Section 4, we establish this conjecture for  $k \in \{2, 3, 4\}$ , that is, we prove  $f_T(1, 2) = 4$ ,  $f_T(1, 3) = 5$ , and  $f_T(1, 4) = 6$ .

Next we consider problems of deciding whether a digraph admits a  $(k_1, k_2)$ -outdegree-splitting.

**OUTDEGREE SPLITTING**

Input: A digraph  $D$  and two positive integers  $k_1$  and  $k_2$ .

Question: Does  $D$  admit a  $(k_1, k_2)$ -outdegree-splitting?

Particular cases of this problem are when  $k_1$  and  $k_2$  are fixed integers and not part of the input. Hence for every fixed  $k_1, k_2$ , we have the following problem.

**$(k_1, k_2)$ -OUTDEGREE-SPLITTING**

Input: A digraph  $D$ .

Question: Does  $D$  admit a  $(k_1, k_2)$ -outdegree-splitting?

**Theorem 11.**  $(1, 1)$ -OUTDEGREE-SPLITTING is polynomial-time solvable.

*Proof.* Let us describe a polynomial-time algorithm solving  $(1, 1)$ -OUTDEGREE SPLITTING.

If the input digraph  $D$  has a vertex with outdegree 0, then the answer is ‘no’ because this vertex has outdegree 0 in any subdigraph of  $D$  containing it. Henceforth we may assume that  $\delta^+(D) \geq 1$ .

It is well-known that a digraph with outdegree at least 1 contains a cycle. Therefore, Proposition 3 implies that a digraph with minimum outdegree at least 1 admits a  $(1, 1)$ -outdegree-splitting if and only if it contains two disjoint cycles. Thus it is enough to decide whether  $D$  contains two disjoint cycles.

But deciding whether a digraph contains two disjoint cycles can be done in polynomial time as shown by McCuaig [7]. (See also [9].)  $\square$

In Section 5, we consider the restriction of these problems to tournaments.

**TOURNAMENT OUTDEGREE-SPLITTING**

Input: A tournament  $T$  and two positive integers  $k_1$  and  $k_2$ .

Question: Does  $T$  admit a  $(k_1, k_2)$ -outdegree-splitting?

**TOURNAMENT  $(k_1, k_2)$ -OUTDEGREE-SPLITTING**

Input: A tournament  $T$ .

Question: Does  $T$  admit a  $(k_1, k_2)$ -outdegree-splitting?

**TOURNAMENT  $(1, 1)$ -OUTDEGREE-SPLITTING** is a particular case of  $(1, 1)$ -OUTDEGREE-SPLITTING, and thus is polynomial-time solvable. In Theorem 31, we show that, more generally, for any  $k_1, k_2$ , **TOURNAMENT  $(k_1, k_2)$ -OUTDEGREE-SPLITTING** can be solved in  $O(n^{k^2/2+3k/2+3})$  time. We then describe a faster algorithm solving **TOURNAMENT  $(1, k_2)$ -OUTDEGREE-SPLITTING**. It runs in  $O(n^3)$  time for  $k_2 \geq 2$  and in  $O(n^2)$  time for  $k_2 = 1$ . In view of these results, it is natural to ask the following.

**Problem 12.** Is **TOURNAMENT OUTDEGREE-SPLITTING** fixed-parameter tractable with  $(k_1, k_2)$  as a parameter? In other words, can we solve **TOURNAMENT OUTDEGREE SPLITTING** in  $F(k_1, k_2)P(n)$  time, where  $F$  is an arbitrary computable function and  $P$  is a polynomial in the order  $n$  of the input tournament?

Finally, in Section 6, we present some possible directions for further research.

## 2 Definitions and folklore on tournaments

The *score sequence* of a tournament  $T$ , denoted by  $s(T)$ , is the non-decreasing sequence of outdegrees of its vertices. Landau [6] characterized the non-decreasing sequences of integers that are score sequences.

**Theorem 13** (Landau [6]). *A non-decreasing sequence of non-negative integers  $(s_1, s_2, \dots, s_n)$  is a score sequence if and only if:*

$$(i) \quad s_1 + s_2 + \dots + s_i \geq \binom{i}{2}, \text{ for } i = 1, 2, \dots, n-1, \text{ and}$$

$$(ii) \quad s_1 + s_2 + \dots + s_n = \binom{n}{2}.$$

Condition (ii) in the above theorem implies directly the following proposition.

**Proposition 14.** *Every tournament of order  $2k$  has minimum degree less than  $k$ .*

**Corollary 15.**  $f_T(k_1, k_2) \geq k_1 + k_2 + 1$ .

*Proof.* Let  $T$  be a  $(k_1 + k_2)$ -regular tournament of order  $2k_1 + 2k_2 + 1$ . In every bipartition  $(V_1, V_2)$  of  $V(T)$ , either  $|V_1| \leq 2k_1$  or  $|V_2| \leq 2k_2$ . Thus, by Proposition 14, either  $\delta^+(T[V_1]) < k_1$  or  $\delta^+(T[V_2]) < k_2$ .  $\square$

An  $\ell$ -*cycle* is a cycle of length  $\ell$ . A tournament  $T$  is *transitive* if it contains no cycles. The score sequence of a transitive tournament of order  $n$  is  $(0, 1, \dots, n-1)$ .

We denote by  $tt_3(T)$  the number of transitive subtournaments of order 3 in  $T$  and by  $c_3(T)$  number of 3-cycles in  $T$ . Since a tournament of order 3 is either a transitive tournament or a 3-cycle, we have

$$tt_3(T) + c_3(T) = \binom{|T|}{3}.$$



Now if  $v$  is a vertex, the number of transitive subtournaments of order 3 with source  $v$  is  $\binom{d^+(v)}{2}$ . Hence

$$tt_3(T) = \sum_{v \in V(T)} \binom{d^+(v)}{2}.$$

A digraph  $D$  is *strongly connected* or *strong* if there is a path from  $u$  to  $v$  for every  $u, v \in V(D)$ . A digraph  $D$  is *k-strong* if  $D - X$  is strong for every  $X \subset V(D)$  where  $|X| \leq k - 1$ . A (strong) *component* of  $D$  is a strong subdigraph of  $D$  which is maximal by inclusion.

Let  $T$  be a tournament. Let  $T_1, T_2, \dots, T_m$  be the components of  $T$ . Then  $(V(T_1), V(T_2), \dots, V(T_m))$  is a partition of  $V(T)$  and without loss of generality, we may suppose that  $T_i \rightarrow T_j$  whenever  $i < j$ . In this case we say that  $T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_m$  is the *decomposition* of  $T$ . Component  $T_1$  is said to be the *initial* component of  $T$  and  $T_m$  its *terminal* component.

A vertex is *pancyclic* in a digraph  $D$  if, for every  $3 \leq \ell \leq |D|$ , it is contained in an  $\ell$ -cycle. To contain a pancyclic vertex, a tournament must contain a hamiltonian cycle. Therefore, it must be strong according to Camion's theorem [4]. Moon [8] showed that this condition is sufficient.

**Theorem 16** (Moon [8]). *Every vertex of a strong tournament is pancyclic.*

We sometimes use the results of this section without referring to them.

### 3 Small subtournament of minimum outdegree $k$

We now prove Theorem 7. In fact, we prove a more general theorem whose particular case with  $U = \emptyset$  is Theorem 7.

**Theorem 17.** *Let  $T$  be a tournament with minimum outdegree at least  $k$  and  $U \subseteq V(T)$  be a subset of vertices. There is a subtournament  $T'$  of  $T$  with minimum outdegree  $k$  such that  $U \subseteq V(T')$  and  $|V(T')| \leq |U| + k^2/2 + 3k/2 + 1$ .*

*Proof.* For every  $p$ , we prove the result for all sets  $U$  of size  $p$  by induction on  $|V(T)|$ , the result holding trivially if  $|V(T)| \leq p + k^2/2 + 3k/2 + 1$ .

Let  $T$  be a tournament of order at least  $p + k^2/2 + 3k/2 + 2$  with minimum outdegree at least  $k$  and  $U$  a set of  $p$  vertices of  $T$ . Let  $S$  be the set of vertices of degree  $k$  in  $T$ . There are  $k|S|$  arcs with their tail in  $S$ . Among them  $|S|(|S| - 1)/2$  are in  $S$  and the remaining ones have their heads out of  $S$ . Hence  $|N^+(S)| \leq |S| + k|S| - |S|(|S| - 1)/2$ . Now the polynomial  $P(x) = (k+3/2)x - x^2/2$  increases on  $[0, k+3/2]$  and decreases on  $[k+3/2, +\infty[$ . Moreover  $P(k+1) = P(k+2) = k^2/2 + 3k/2 + 1$ . Consequently,  $|N^+(S)| \leq k^2/2 + 3k/2 + 1$ .

Since  $|V(T)| \geq p + k^2/2 + 3k/2 + 2$ , there is a vertex  $v$  which is not in  $N^+(S) \cup U$ . Thus  $T - v$  has minimum outdegree at least  $k$  and by induction  $T - v$  (and thus also  $T$ ) has a subtournament  $T'$  with minimum outdegree  $k$  such that  $U \subseteq V(T')$  and  $|V(T')| \leq |U| + k^2/2 + 3k/2 + 1$ .  $\square$

The bound  $k^2/2 + 3k/2 + 1$  is Theorem 7 is tight in the following sense.

**Proposition 18.** *For every non-negative integer  $k$ , and for every  $n \geq k^2/2 + 3k/2 + 1$ , there is a tournament  $T(n, k)$  of order  $n$  and a set  $W \subset V(T)$  of order  $n - k^2/2 + 3k/2 + 1$  such that for every  $U \subset W$ , every subtournament  $T'$  with minimum outdegree  $k$  such that  $U \subseteq V(T')$  has order at least  $|U| + k^2/2 + 3k/2 + 1$ .*

*Proof.* Consider the disjoint union of a strong tournament  $S$  of order  $k + 1$  and a transitive tournament  $TT$  of order  $k(k + 1)/2$ . Set  $V(S) = \{s_1, \dots, s_{k+1}\}$ . Partition  $V(TT)$  into  $k + 1$  sets  $A_1, \dots, A_{k+1}$  such that  $|A_i| = k - d_S^+(s_i)$ . This is possible since  $\sum_{i=1}^{k+1} d_S^+(s_i) = k(k + 1)/2$ , so  $\sum_{i=1}^{k+1} (k - d_S^+(s_i)) = |V(TT)|$ . Now for each

$i$ , add the arc  $s_i a$  for all  $a \in A_i$ , and all the arcs  $bs_i$  for all  $b \in V(TT) \setminus A_i$ . The resulting tournament  $R$  has order  $k^2/2 + 3k/2 + 1$  and minimum outdegree  $k$ .

Let  $R'$  be a subtournament of  $R$  with outdegree at least  $k$ .

It must contain a vertex of  $S$ , because all subtournaments of  $TT$  are transitive. But each element  $s$  of  $S$  has outdegree exactly  $k$  in  $R$ , so if  $s \in V(R')$ , then  $N_R^+(s) \subset V(R')$ . Since  $S$  is strong, it has a hamiltonian cycle by Camion's Theorem, and so  $V(S) \subset V(R')$ . But by construction, every vertex in  $R$  is dominated by a vertex in  $S$ , and thus must be in  $R'$ . Hence  $R = R'$ .

Set  $p = n - k^2/2 + 3k/2 + 1$ . Let  $T(n, k)$  be a tournament obtained from the disjoint union of  $R$  and the transitive tournament  $TT_p$  of order  $p$  by adding all arcs from  $TT_p$  towards  $R$ . Then, for any set  $U \subset V(TT_p)$ , every subtournament of  $T(n, k)$  with minimum outdegree  $k$  containing  $U$  must also contain  $V(R)$  and thus has order at least  $|U| + k^2/2 + 3k/2 + 1$ .  $\square$

We can then easily derive Theorem 8 from Theorem 17.

*Proof of Theorem 8.* Let  $T$  be a tournament of minimum outdegree at least  $m$ . The tournament  $T - U_2$  has minimum outdegree at least  $k_1$  because  $|U_2| = u_2$ . Thus, by Theorem 17, there exists a subtournament  $T_1$  of  $T - U_2$  with minimum degree at least  $k_1$  and order at most  $k_1^2/2 + 3k_1/2 + u_1 + 1$  such that  $U_1 \subseteq V(T_1)$ . Set  $V_1 = V(T_1)$ ,  $T_2 = T - T_1$ , and  $V_2 = V(T_2)$ . By definition,  $U_1 \subseteq V_1$  and  $U_2 \subseteq V_2$ . Now  $\delta^+(T_2) \geq \delta^+(T) - |V_1| \geq k_2$ . Hence  $(V(T_1), V(T_2))$  is a  $(k_1, k_2)$ -outdegree-splitting.  $\square$

### 3.1 Outdegree-critical tournaments

Theorem 7 can be rephrased in terms of  $k$ -outdegree-critical tournament. A tournament  $T$  is said to be  $k$ -outdegree-critical if it has minimum outdegree  $k$  and all its proper subtournaments have outdegree less than  $k$ . Theorem 7 implies that all  $k$ -outdegree-critical tournaments have bounded size. Hence a natural problem is the following.

**Problem 19.** Describe the  $k$ -outdegree-critical tournaments.

The unique 1-outdegree-critical tournament is the 3-cycle.

We now show that the 2-outdegree-critical tournaments are those depicted in Figure 1.

**Theorem 20.** Every tournament with minimum outdegree 2 has a subtournament isomorphic to one of those depicted in Figure 1.

*Proof.* By induction on  $|V(T)|$ , the result holding trivially when  $|V(T)| < 5$ .

Let  $T$  be a tournament of order at least 5 with minimum outdegree 2. Every vertex  $v$  has an inneighbour  $u$  such that  $d^+(u) = 2$ , for otherwise  $T - v$  has minimum outdegree at least 2 and by induction  $T - u$  (and thus also  $T$ ) has a subtournament with minimum outdegree 2 and with order 5 or 6.

Let  $S$  be the set of vertices of outdegree 2 in  $T$ . By the previous remark,  $S$  is not empty and  $T[S]$  has minimum indegree at least 1. Hence  $T[S]$  contains a 3-cycle  $C = (x_1, x_2, x_3)$ . For  $i = 1, 2, 3$ , let  $y_i$  be the outneighbour of  $x_i$  in  $V(T) \setminus \{x_1, x_2, x_3\}$ . If all  $y_i$  are distinct, then each  $y_i$  dominates  $\{x_1, x_2, x_3\} \setminus \{x_i\}$ , and so  $T[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$  is one of the tournaments  $A_6, B_6, C_6$  and  $D_6$ . If  $y_1 = y_2 = y_3$ , let  $z_1$  and  $z_2$  be the two outneighbours of  $y_1$ . These two vertices dominate  $\{x_1, x_2, x_3\}$ , so  $T[\{x_1, x_2, x_3, y_1, z_1, z_2\}]$  is isomorphic to  $E_6$ . If  $y_1 = y_2 \neq y_3$ , then  $y_3$  dominates  $x_1$  and  $x_2$ , and  $y_1$  dominates  $x_3$ . If  $y_1$  dominates  $y_3$ , then  $T[\{x_1, x_2, x_3, y_1, y_3\}]$  is isomorphic to  $R_5$ . If  $y_1$  is dominated by  $y_3$ , let  $z$  be an outneighbour of  $y_1$  distinct from  $x_3$ . The vertex  $z$  dominates  $\{x_1, x_2\}$ , so  $T[\{x_1, x_2, x_3, y_1, y_3, z\}]$  is isomorphic to  $C_6$  or  $D_6$ .  $\square$

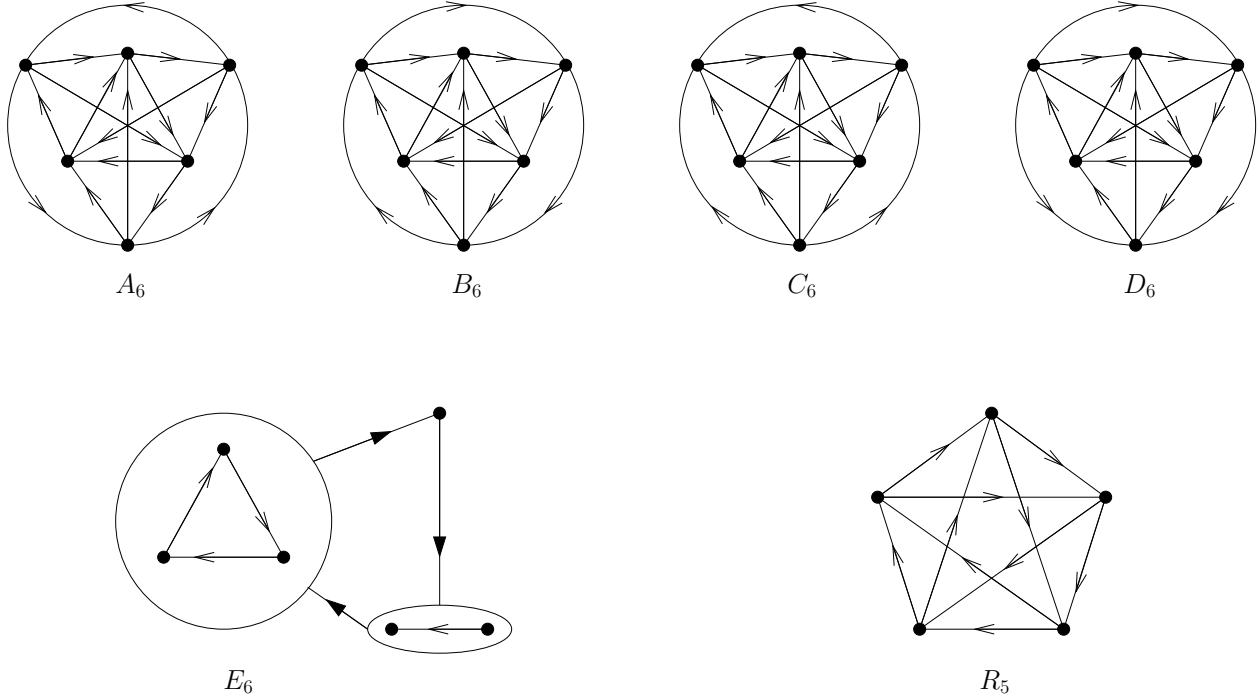


Figure 1: The 2-outdegree-critical tournaments.

## 4 $(1, k)$ -outdegree-splitting of tournaments

### 4.1 Improved upper bound for $f_T(1, k)$

A 3-cycle  $C$  in a tournament  $T$  is said to be  $k$ -good if  $\delta^+(T - C) \geq k$ . Clearly, if  $C$  is a  $k$ -good 3-cycle, then  $(V(C), V(T - C))$  is a  $(1, k)$ -splitting of  $T$ .

**Lemma 21.** *Let  $k$  be an integer and let  $T$  be a strong tournament with minimum outdegree at least  $k + 2$ . Let  $S$  be the set of vertices with outdegree  $k + 2$  in  $T$ . If  $T$  has no  $k$ -good 3-cycle, then the following hold.*

- (i) *Every arc is dominated by a vertex in  $S$ .*
- (ii) *For every vertex  $v$ , the subtournament  $T[N^-(v) \cap S]$  has minimum indegree 1 and at least five vertices.*
- (iii)  $|V(T)| \leq \frac{1}{10}(k + 7)(k + 8)$ .

*Proof.* Suppose that  $T$  contains no  $k$ -good 3-cycle. A 3-cycle  $C$  in  $T$  is  $S$ -dominated if there is a vertex  $x \in S$  dominating  $C$ . Clearly, a 3-cycle in  $T$  is  $k$ -good if and only if it is not  $S$ -dominated. Hence all 3-cycles are  $S$ -dominated.

(i) Let  $uv$  be an arc. Since  $T$  is strong, there is a 3-cycle  $C$  containing  $u$  by Theorem 16. This cycle is dominated by a vertex  $s \in S$ . If  $s$  dominates  $v$ , then  $s$  dominates the arc  $uv$ . If not, then  $uvs$  is a 3-cycle. This cycle is dominated by a vertex  $s'$  in  $S$ , which thus dominates  $uv$ .

(ii) Let  $v$  be a vertex of  $T$ . By (i),  $v$  is dominated by a vertex in  $S$ , so  $N^-(v) \cap S$  is not empty. For any vertex  $s \in N^-(v) \cap S$ , the arc  $sv$  is dominated by a vertex  $s' \in S$ , which is distinct from  $s$ . Hence  $T[N^-(v) \cap S]$  has indegree at least 1 and thus contains a 3-cycle  $s_1s_2s_3$ . This 3-cycle is dominated by a vertex  $s \in S$ .

Assume first  $s \rightarrow v$ . By (i) the arc  $sv$  is dominated by a vertex  $s'$  of  $S$ . Clearly  $s' \notin \{s_1, s_2, s_3\}$ , because  $s'$  dominates  $s_1s_2s_3$ . Hence  $s_1, s_2, s_3, s, s'$  are five vertices in  $N^-(v) \cap S$ .

Assume now that  $v \rightarrow s$ . Then  $ss_1v$  is a 3-cycle which is dominated by a vertex  $s'$ . This vertex is in  $N^-(v) \cap S$  and is distinct from  $s_2, s_3$  because it dominates  $s$ . Furthermore, by (i) there is a vertex  $t$  of  $S$  dominating  $s'v$ . If  $t \notin \{s_1, s_2, s_3\}$ , then  $s_1, s_2, s_3, s', t$  are five vertices in  $N^-(v) \cap S$ . So we may assume that  $t \in \{s_1, s_2, s_3\}$  and, without loss of generality,  $t = s_2$ . Now, there is a vertex  $s''$  dominating the 3-cycle  $ss_2v$ . This vertex is distinct from  $s_1, s_3$  because it dominates  $s$ , and is distinct from  $s'$  because it dominates  $s_2$ . Hence,  $s_1, s_2, s_3, s', s''$  are five vertices in  $N^-(v) \cap S$ .

(iii) By (ii), every vertex has at least four inneighbours in  $S$ . Thus

$$|V(T)| = |N^+(S)| \leq |S| + \frac{1}{5} \left( (k+2)|S| - \binom{|S|}{2} \right).$$

But the polynomial  $Q(x) = x + \frac{1}{5} \left( (k+2)x - \binom{x}{2} \right) = \frac{1}{10}x(2k+15-x)$  increases on  $[0, k+15/2]$  and decreases on  $[k+15/2, +\infty[$  and  $Q(k+7) = Q(k+8) = \frac{1}{10}(k+7)(k+8)$ . Consequently,  $|V(T)| \leq \frac{1}{10}(k+7)(k+8)$ .  $\square$

**Theorem 22.** *Let  $k$  be an integer in  $\{1, 2, 3, 4\}$ . If  $T$  is a tournament with minimum outdegree at least  $k+2$ , then  $T$  contains a  $k$ -good 3-cycle.*

*Proof.* It is sufficient to prove the result for strong tournaments. Indeed if  $T$  is not strong, then its terminal component  $T'$  has also outdegree at least  $k+2$ . Moreover, every 3-cycle that is  $k$ -good in  $T'$  is also  $k$ -good in  $T$ .

Henceforth, we may assume that  $T$  is strong. Let  $S$  be the set of vertices with outdegree  $k+2$  in  $T$ .

- Assume  $k \in \{1, 2\}$ . Then every vertex of  $S$  has outdegree at most 4 in  $T[S]$ , so  $T[S]$  has a vertex with indegree at most 4. Thus, by Lemma 21-(ii),  $T$  has a 1-good 3-cycle.
- Assume  $k = 3$ . Since  $\delta^+(T) \geq 5$ , then  $|V(T)| \geq 11$ . By Lemma 21-(iii), we have the result if  $|V(T)| > 11$ . Henceforth we may assume  $|V(T)| = 11$ , so  $T$  is 5-regular. Hence  $tt_3(T) = \sum_{v \in V(T)} \binom{5}{2} = 110$ . Thus  $c_3(T) = \binom{11}{3} - 110 = 55$ . Now a tournament of order 5 contains at most five 3-cycles, and it contains exactly five if and only if it is  $R_5$  the 2-regular tournament on 5-vertices. If all the 3-cycles are dominated, the outneighbourhood of every vertex induces an  $R_5$ . But then a vertex  $u$  dominates at most two inneighbours of any other vertex  $v$ . Now if  $T$  had no  $k$ -good 3-cycles, then by Lemma 21-(ii), for every vertex  $v$  the subtournament  $T[N^-(v)]$  would have a 3-cycle, which cannot be dominated and thus is  $k$ -good, a contradiction.
- Assume  $k = 4$ . Since  $\delta^+(T) \geq 6$ , then  $|V(T)| \geq 13$ . By Lemma 21-(iii), we have the result if  $|V(T)| > 13$ . Henceforth we may assume  $|V(T)| = 13$ , so  $T$  is 6-regular. It is possible to test all 6-regular graphs on 13 vertices using a simple computer program and verify that each of them has at least one good 3-cycle. The source code of the computer program is available at <http://kam.mff.cuni.cz/~bernard/pub/6-regular.cpp>\*

$\square$

**Corollary 23.** *For  $k \in \{1, 2, 3, 4\}$ ,  $f_T(1, 2) = k+2$ .*

*Proof.* Let  $k \in \{1, 2, 3, 4\}$ . Theorem 22 implies  $f_T(1, k) \leq k+2$  and Corollary 15 yields  $f_T(1, k) \geq k+2$ .  $\square$

We believe that Theorem 22 can be extended to all values of  $k$ .

**Conjecture 24.** *Let  $k$  be a positive integer. If  $T$  is a tournament with minimum outdegree at least  $k+2$ , then  $T$  contains a  $k$ -good 3-cycle.*

\*We plan to post the code on arXiv for a more permanent storage.

A first step to prove this conjecture is the following.

**Conjecture 25.** *Let  $k$  be a positive integer. If  $T$  is a  $(k+2)$ -regular tournament, then  $T$  contains a  $k$ -good 3-cycle.*

If true Conjecture 24 would be best possible.

**Proposition 26.** *Let  $k$  be a positive integer. For any  $n \geq 3k + 3$ , there is a tournament of order  $n$  with minimum outdegree  $k + 1$  that does not admit any  $(1, k)$ -outdegree-splitting.*

*Proof.* Let  $n \geq 3k + 3$ . Let  $T$  be a tournament of order  $n$  whose vertex set can be partitioned into  $(X_1, X_2, \{x\})$  such that  $X_1 \rightarrow X_2$ ,  $X_2 \rightarrow x$ ,  $x \rightarrow X_1$ ,  $T[X_1]$  is a transitive tournament of order  $n - 2k - 2$ , and  $T[X_2]$  is a  $k$ -regular tournament.

Clearly,  $\delta^+(T) = k + 1$ . Let us now prove that  $T$  has no  $(1, k)$ -outdegree-splitting.

Suppose for a contradiction that  $T$  admits a  $(1, k)$ -outdegree-splitting  $(V_1, V_2)$ . The set  $V_2$  must contain a vertex in  $X_2$  because  $T[X_1 \cup \{x\}]$  is transitive. The subtournament  $T[V_1]$  contains a 3-cycle  $C$ . This cycle either contains  $x$  or is contained in  $C_1$ .

- If  $C$  contains  $x$ , then  $C = xx_1x_2$  with  $x_1 \in X_1$  and  $x_2 \in X_2$ . But  $T[X_2]$  is  $k$ -regular, so it is strong. Thus there is a vertex  $u$  of  $V_2 \cap X_2$  dominating a vertex in  $V_1 \cap X_2$ . Thus  $u$  has outdegree at most  $k - 1$  in  $T[V_2]$ , a contradiction.
- If  $C$  is contained in  $C_1$ , then  $|V_2 \cap X_2| \leq 2k - 2$ . Therefore  $T[V_2 \cap X_2]$  has a vertex  $u$  with outdegree less than  $k - 1$ . This vertex  $u$  has outdegree less than  $k$  in  $T[V_2]$ , a contradiction.

□

## 4.2 Existence of $k$ -good 3-cycles

A result of Song [11] states that every 2-strong tournament of order at least 6 can be split into a 3-cycle and a strong subtournament unless it is  $P_7$ , the Paley tournament of order 7. Since  $P_7$  is 3-regular, it has a 1-good 3-cycle by Theorem 22. Therefore we obtain the following.

**Theorem 27.** *Every 2-strong tournament of order at least 6 has a 1-good 3-cycle and thus admits a  $(1, 1)$ -outdegree-splitting.*

In fact, having a 1-good 3-cycle is equivalent to having a  $(1, 1)$ -outdegree-splitting.

**Proposition 28.** *Let  $T$  be a tournament. Then  $T$  has a  $(1, 1)$ -outdegree-splitting if and only if it has a 1-good 3-cycle  $C$ .*

*Proof.* As we already observed, if  $C$  is a 1-good 3-cycle, then  $T$  has a  $(1, 1)$ -outdegree-splitting.

Conversely, suppose that  $T$  admits a  $(1, 1)$ -outdegree-splitting  $(V_1, V_2)$ . Then for  $i = 1, 2$ ,  $T[V_i]$  contains a 3-cycle  $C_i$ . Let  $S_2$  be the largest set such that  $V_2 \subseteq S_2 \subseteq V(T - C_1)$  and  $\delta^+(T[S_2]) \geq 1$ . If  $S_2 = V(T - C_1)$ , then  $C_1$  is  $k$ -good. If not, then let  $R = V(T - C_1) \setminus S_2$ . By definition,  $S_2 \rightarrow R$ . Thus  $\delta^+(T - C_2) \geq 1$ , and  $C_2$  is  $k$ -good. □

Unfortunately, Proposition 28 cannot be generalized for larger value of  $k$  in the sense that there are tournaments with a  $(1, k)$ -splitting and no  $k$ -good 3-cycles. Furthermore, there are such tournaments with minimum outdegree  $k + 1$ ; this shows that the condition of having minimum outdegree  $k + 2$  in Conjecture 24 is best possible.

**Proposition 29.** *Let  $k$  be an integer greater than 1. There exists a tournament of order at  $3k + 3$  with minimum outdegree  $k + 1$  such that  $T$  has a  $(1, k)$ -splitting but no  $k$ -good 3-cycles.*

*Proof.* Let  $T$  be a tournament whose vertex set can be partitioned into  $(X_1, X_2, X_3, \{x\})$  such that  $X_1 \rightarrow X_2$ ,  $X_1 \cup X_2 \rightarrow X_3$ ,  $X_3 \rightarrow x$ ,  $x \rightarrow X_1 \cup X_2$ ,  $T[X_1]$  is a transitive tournament of order  $k - 2$ , and  $T[X_2]$  is a 3-cycle and  $T[X_3]$  is a  $k$ -regular tournament.

Clearly,  $(X_1 \cup X_2 \cup \{x\}, X_3)$  is a  $(1, k)$ -splitting of  $T$ .

Let us now prove that no 3-cycle is  $k$ -good. There are three kinds of 3-cycles:  $T[X_2]$ , 3-cycles contained in  $T[X_3]$ , and 3-cycles of the form  $xyz$  with  $y \in X_1 \cup X_2$  and  $z \in X_3$ .

- $T[X_2]$  is not  $k$ -good, because  $x$  has outdegree less than  $k$  in  $T - X_2$ .
- If  $C$  is a 3-cycle in  $T[X_3]$ , then  $T[X_3] - C$  has at most  $2k - 2$  vertices and thus contains a vertex  $v$  of outdegree less than  $k - 1$ . Therefore  $v$  has outdegree less than  $k$  in  $T - C$ . So  $C$  is not  $k$ -good.
- If  $C$  is a 3-cycle of the form  $xyz$  with  $y \in X_1 \cup X_2$  and  $z \in X_3$ , then every inneighbour  $v$  of  $z$  in  $T[X_3]$  has outdegree less than  $k$  in  $T - C$ . So  $C$  is not  $k$ -good.

□

**Problem 30.** For any fixed  $k \geq 2$ , are there infinitely many strong tournaments with minimum outdegree  $k + 1$  that have a  $(1, k)$ -splitting but no  $k$ -good 3-cycles?

## 5 Finding outdegree splittings in tournaments

**Theorem 31.** For every positive integers  $k_1$  and  $k_2$ , TOURNAMENT  $(k_1, k_2)$ -OUTDEGREE-SPLITTING is polynomial-time solvable.

*Proof.* Let  $g(k) = k^2/2 + 3k/2 + 1$ . Let  $T$  be a tournament of order  $n$ . If  $T$  has a  $(k_1, k_2)$ -outdegree-splitting  $(V_1, V_2)$ , then  $V_1$  contains a subset  $S_1$  of size at most  $g(k_1)$  such that  $\delta^+(T[S_1]) \geq k_1$ .

The algorithm considers all subsets  $S_1$  of order at most  $g(k_1)$ . For each of them, we first check if  $\delta^+(T[S_1]) \geq k_1$ . If no, we proceed to the next subtournament. If yes, we check if there is a  $(k_1, k_2)$ -outdegree-splitting  $(V_1, V_2)$  such that  $S_1 \subseteq V_1$  using a procedure `extend`( $S_1$ ). If this procedure, returns ‘yes’, then we also return ‘yes’. If not we proceed to the next subtournament.

The procedure `extend`( $S_1$ ) proceeds as follows. If  $S_1 = V(T)$ , return ‘no’. If  $T - S_1$  has minimum outdegree at least  $k_2$ , we return  $(S_1, V(T) \setminus S_1)$ . Otherwise, pick a vertex  $x$  of  $V(T) \setminus S_1$  having outdegree less than  $k_2$  in  $T - S_1$  and return `extend`( $S_1 \cup \{x\}$ ).

The procedure `extend` runs in  $O(n^2)$ -time. (We only need to make  $O(n)$  updates on the score sequence). At worse, we run it for each subset  $S_1$  of size at most  $g(k_1)$ . There are  $O(n^{g(k_1)})$  such subsets. Hence the algorithm runs in  $O(n^{g(k_1)+2})$  time. □

The running time of the algorithm given in the proof of Theorem 31 is certainly not optimal. When  $k_1 = 1$ , running time is  $O(n^5)$ . We now give a faster algorithm, that runs in  $O(n^3)$  time for  $k_1 = 1$  and  $k_2 \geq 2$  and in  $O(n^2)$  time for  $k_1 = k_2 = 1$ . This algorithm is also faster than the one described in Theorem 11.

The key ingredients are the following three statements. The first one is an immediate extension of Proposition 3 with an identical proof, which translates into a  $O(n^2)$ -time algorithm.

**Proposition 32.** Let  $D$  be a digraph of order  $n$ . If  $D$  contains two disjoint digraphs  $D_1, D_2$  such that  $\delta^+(D_i) = k_i$  for  $i = 1, 2$  and  $d_D^+(v) \geq k_1 + k_2 - 1$  for all  $v \in V(D - (D_1 \cup D_2))$ , then  $D$  admits a  $(k_1, k_2)$ -outdegree-splitting. Moreover such a  $(k_1, k_2)$ -outdegree-splitting can be found in  $O(n^2)$  time.

The second one is an algorithmic version of Theorem 17.

**Proposition 33.** *Let  $T$  be a tournament with minimum outdegree at least  $k$ . One can find in  $O(n^3)$  time a subtournament  $T'$  of  $T$  with minimum outdegree  $k$  such that  $|V(T')| \leq k^2/2 + 3k/2 + 1$ .*

*Proof.* By the proof of Theorem 17, if  $|V(T)| > k^2/2 + 3k/2 + 1$ , then it contains a vertex  $x$  such that  $T - x$  has minimum outdegree at least  $k$ . Such a vertex can be found in  $O(n^2)$  time, by finding the set  $S$  of vertices with outdegree  $k$ , and taking  $x$  not in  $S \cup N^+(S)$ . We then recursively apply the procedure to  $T - x$ . As we reduce the order of the tournament at most  $n$  times, we find the desired subtournament  $T'$  in  $O(n^3)$  time.  $\square$

**Lemma 34.** *Let  $T$  be a tournament and  $v$  a vertex of  $T$ . If  $T$  has a  $(1, k)$ -outdegree-splitting  $(V_1, V_2)$  with  $v_1 \in V_1$ , then there is a 3-cycle  $C_1$  in  $T[V_1]$  such that  $v \in V(C_1)$  or  $V(C_1) \subseteq N^+(v)$ .*

*Proof.* Let  $N_1 = N^+(v) \cap V_1$ . If  $T[N_1]$  has a cycle, then it is the desired 3-cycle. Otherwise,  $T[N_1]$  is a transitive tournament. Now the sink  $w$  of  $T[N_1]$  has an outneighbour  $u$  in  $T[V_1]$ , which is necessarily an inneighbour of  $v$ , by definition of  $N_1$ . Therefore  $uvw$  is the desired 3-cycle.  $\square$

**Theorem 35.** (i) TOURNAMENT  $(1, 1)$ -OUTDEGREE-SPLITTING can be solved in  $O(n^2)$  time;  
(ii) for all  $k \geq 2$ , TOURNAMENT  $(1, k)$ -OUTDEGREE-SPLITTING can be solved in  $O(n^3)$  time.

*Proof.* (i) Let us describe a procedure  $(1, 1)$ -split( $T$ ) that given a tournament  $T$  returns ‘yes’ if it admits a  $(1, 1)$ -outdegree-splitting, and returns ‘no’, otherwise.

0. We first compute the outdegree of every vertex and we determine  $\delta^+(T)$ . This can be done in  $O(n^2)$  time.
1. If  $\delta^+(T) = 0$ , then the tournament  $T$  has no  $(1, 1)$ -outdegree-splitting, and we return ‘no’.
2. If  $\delta^+(T) \geq 3$ , the answer is ‘yes’, by Corollary 23.
3. If  $\delta^+(T) \in \{1, 2\}$ , let  $v$  be a vertex of degree 1 or 2 in  $T$ . Without loss of generality, one may look for a  $(1, 1)$ -outdegree-splitting  $(V_1, V_2)$  of  $T$  such that  $v \in V_1$ . For every  $w \in N^+(v)$  and  $u \in N^+(w) \setminus N^+(v)$ , we check whether  $T - \{u, v, w\}$  contains a 3-cycle. If yes for at least one choice of  $\{u, v, w\}$ , the answer is ‘yes’ by Proposition 3 since  $\delta^+(T) \geq k$ . If not, then return ‘no’. This is valid by Lemma 34.

Given its score sequence, checking if a tournament of order  $n$  contains a 3-cycle can be done in  $O(n)$  by checking whether the score sequence is distinct from  $(0, 1, 2, \dots, n - 1)$ , the score sequence of the transitive tournament. Since the score sequence of  $T - \{u, v, w\}$  can be obtained in linear time from the list of outdegrees of  $T$ , checking if  $T - \{u, v, w\}$  contains a cycle can be done in  $O(n)$  time.

Now since  $v$  has degree at most 2, the procedure considers at most  $2(n - 1)$  subtournaments  $T - \{u, v, w\}$ . Therefore  $(1, 1)$ -split runs in  $O(n^2)$  time.

(ii) Let us describe a procedure  $(1, k)$ -split( $T$ ) that given a tournament  $T$  returns ‘yes’ if  $T$  admits a  $(1, k)$ -outdegree-splitting, and return ‘no’, otherwise.

0. We first compute the outdegree of every vertex and we determine  $\delta^+(T)$ . This can be done in  $O(n^2)$  time.
1. If  $\delta^+(T) = 0$ , then the tournament  $T$  has no  $(1, k)$ -outdegree-splitting, and we return ‘no’.

2. If  $1 \leq \delta^+(T) \leq k - 1$ , let  $U_1$  be the set of vertices of degree less than  $k$  in  $T$ . Clearly, for any  $(1, 2)$ -outdegree-splitting  $(V_1, V_2)$  of  $T$ ,  $U_1 \subseteq V_1$ . Let  $v$  be a vertex of  $U_1$ . For every  $w \in N^+(v)$  and  $u \in N^+(w) \setminus N^+(v)$ , we check whether  $T - (U_1 \cup \{u, v, w\})$  contains a subtournament of minimum outdegree  $k$  using the procedure `Outdegree-k-Subtournament` described below. If yes for at least one choice of  $\{u, v, w\}$ , the answer is ‘yes’ by Proposition 32 since all vertices of  $V(T) \setminus U_1$  have outdegree at least  $k$  in  $T$ . If not, then return ‘no’. This is valid by Lemma 34.
3. If  $\delta^+(T) \geq k$ , then we first find a subtournament  $T'$  of  $T$  with  $\delta(T') \geq k$  and  $|V(T')| \leq k^2/2 + 3k/2 + 1$ . If  $T - T'$  contains a 3-cycle, then  $T$  admits a  $(1, k)$ -outdegree-splitting by Proposition 3, and so we return ‘yes’. If not then  $T - T'$  is a transitive tournament and all 3-cycles of  $T$  intersect  $T'$  and therefore there are at most  $(k^2/2 + 3k/2 + 1)n^2$  of them. For each 3-cycle  $C$ , we check with `Outdegree-k-Subtournament` whether  $T - C$  contains a subtournament of minimum outdegree  $k$ . If yes, for one of them, then we return ‘yes’ because there is a  $(1, k)$ -outdegree-splitting by Proposition 3. If not, then we return ‘no’.

**Remark 36.** In the above procedure, one can shorten Step 3 if  $\delta^+(T) \geq k + 2$ . In this case, by Corollary 23, we can directly return ‘yes’.

The procedure `Outdegree-k-Subtournament`( $T$ ) takes as an input the tournament  $T$  as well as its list of outdegrees and a list  $L$  of vertices in the transitive tournament  $T - T'$  ordered in increasing order of their outdegrees. Observe that the list of outdegrees is already computed when `degree-(1, k)-split` call this procedure and the order of  $T - T'$  can be computed just once after computing  $T'$ . First, we alter the list of outdegrees by keeping the outdegrees for vertices in  $T'$  but for vertices in  $T - T'$  we count only outneighbours in  $T'$ . At each step, `Outdegree-k-Subtournament` first checks  $V(T)$  and returns ‘no’ if  $V(T) = \emptyset$ , otherwise it tries to find a vertex  $v$  with  $d^+(v) < k$ . Notice that possible candidates for  $v$  are only vertices in  $T'$  and the first  $k$  vertices in  $L$ . If there is no such vertex  $v$ , it returns ‘yes’. Otherwise it removes  $v$  and tries again. If  $v \in V(T')$ , then it decreases the outdegree of all inneighbours of  $v$  and if  $v \notin V(T')$ , then it decreases outdegrees only for inneighbours from  $V(T')$ . The total time spent on a vertex  $v \in V(T')$  is  $O(n)$ , which gives  $O(V(T')n) = O(n)$  in total. The total time spent on a vertex  $v \notin V(T')$  is  $O(1)$ , which gives  $O(n)$  in total. Therefore, `Outdegree-k-Subtournament` runs in  $O(n)$  time.

Now Step 1 runs in constant time. In Step 2, there are at most  $k+1$  candidates for  $w$ , and thus `Outdegree-k-Subtournament` is called less than  $(k + 1)n$  times. Therefore Step 2 runs in  $O(n^3)$  time. Step 3 first finds a small subtournament  $T'$  with outdegree  $k$ , which can be done in  $O(n^3)$  time by Proposition 33. Then it runs  $O(n^2)$  times `Outdegree-k-Subtournament`. Therefore Step 3 runs in  $O(n^3)$  time.

Overall `(1, k)-split` runs in  $O(n^3)$  time. □

The procedure `(1, k)-split`( $T$ ) can be modified to find a  $(1, k)$ -outdegree-splitting if it exists, using Proposition 32 instead of Proposition 3.

In contrast, the procedure `(1, 1)-split`( $T$ ) cannot be instantly modified into a procedure that finds a  $(1, 1)$ -outdegree-splitting if it exists. However, using a similar approach, we now describe such a procedure.

**Theorem 37.** *One can find a  $(1, 1)$ -outdegree-splitting of a tournament in  $O(n^2)$  time.*

*Proof.* Let us describe a procedure `(1, 1)-findsplit`( $T$ ) that returns a  $(1, 1)$ -outdegree-splitting of the tournament  $T$  if it admits one, and return ‘no’, otherwise.

We first compute the outdegree of every vertex and we determine  $\delta^+(T)$ .

If  $T$  contains a vertex of outdegree 0, then we return ‘no’. If  $\delta^+(T) \geq 4$ , then we pick a vertex  $x$  and find a 3-cycle  $C$  containing  $x$ . Such a cycle can be found in  $O(n^2)$  by testing if there is an arc from  $N^+(x)$  to  $N^-(x)$ . We return  $(V(C), V(T - C))$ . This is valid since  $\delta^+(T - C) \geq \delta^+(T) - |V(C)| \geq 1$ .



If  $\delta^+(T) \leq 3$ , we choose a vertex  $v$  such that  $d^+(v) \in \{1, 2, 3\}$ . If  $T[N^+(v)]$  induces a 3-cycle, then we check whether  $T - N^+(v)$  contains a cycle  $C$ . If yes, we extend  $(T[N^+(v)], C)$  into a  $(1, 1)$ -outdegree-splitting by Proposition 32. If not, for every  $w \in N^+(v)$  and  $u \in N^+(w) \setminus N^+(v)$ , we check if  $T - \{u, v, w\}$  contains a cycle  $C(uvw)$ . If yes for at least one choice of  $\{u, w\}$ , then we extend  $(uvw, C(uvw))$  into a  $(1, 1)$ -outdegree-splitting by Proposition 32 and we return ‘no’ otherwise. This is valid by Lemma 34.

Since there are at most three candidates for  $w$ , there are  $O(n)$  cases to check. Therefore  $(1, 1)$ -findsplit runs in  $O(n^2)$  time.  $\square$

**Remark 38.** The proof of Proposition 28 yields a  $O(n^2)$ -time procedure to find a 1-good 3-cycle given a  $(1, 1)$ -outdegree-splitting. Combining this procedure with  $(1, 1)$ -findsplit, we obtain a  $O(n^2)$ -time algorithm that finds a 1-good 3-cycle in a tournament if it exists, and returns ‘no’ otherwise.

## 6 Further research

### 6.1 Splittable score sequences

Being  $(1, 1)$ -outdegree-splittable is not determined by the score sequence. For example, the two tournaments depicted Figure 2 have score sequences  $(2, 2, 2, 2, 3, 4)$  but the one to the left has no  $(1, 1)$ -outdegree-splitting (See Proposition 26) while the one to the right admits the  $(1, 1)$ -outdegree-splitting  $(\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\})$ .

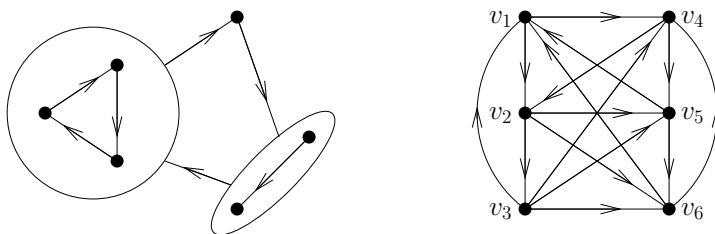


Figure 2: Non- $(1, 1)$ -outdegree-splittable and  $(1, 1)$ -outdegree-splittable tournaments with the same score sequence

However there are score sequences  $s$  such that all tournaments with score sequence  $s$  are  $(1, 1)$ -outdegree-splittable. Such score sequences are said to be  $(1, 1)$ -outdegree-splittable. For example, Theorem 22 implies that  $(s_1, \dots, s_n)$  is  $(1, 1)$ -outdegree-splittable.

**Problem 39.** Which score sequences are  $(1, 1)$ -outdegree-splittable?

### 6.2 Erdős-Posa property for digraphs with minimum outdegree $k$

McCuaig’s algorithm [7] relies on the theorem stating that a digraph  $D$  has either two disjoint cycles or a set  $S$  of at most three vertices such that  $D - S$  is acyclic. More generally, Reed et al. [9] showed that cycles in digraphs have the Erdős-Posa property.

**Theorem 40** (Reed et al. [9]). *For every positive integer  $n$ , there exists an integer  $t(n)$  such that for every digraph  $D$ , either  $D$  has a  $n$  pairwise-disjoint cycles, or there exists a set  $T$  of at most  $t(n)$  vertices such that  $D - T$  is acyclic.*

It is then natural to ask whether digraphs with maximum outdegree  $k$  have the the Erdős-Posa property.

**Problem 41.** Let  $k$  be a fixed integer. For every positive integer  $n$ , does there exist an integer  $t_k(n)$  such that for every digraph  $D$ , either  $D$  has a  $n$  pairwise-disjoint subdigraphs with minimum outdegree  $k$ , or there exists a set  $T$  of at most  $t_k(n)$  vertices such that  $\delta^+(D - T) < k$  ?

### 6.3 Strong connectivity and outdegree-splitting with prescribed vertices

Any  $f_T(k_1, k_2)$ -strong tournament has minimum outdegree at least  $f_T(k_1, k_2)$  and thus admits a  $(k_1, k_2)$ -outdegree-splitting. Therefore, it is natural to ask the following.

**Problem 42.** What is the minimum integer  $h_T(k_1, k_2)$  such that every  $h_T(k_1, k_2)$ -strong tournament  $T$  of order at least  $2k_1 + 2k_2 + 2$  contains a  $(k_1, k_2)$ -outdegree-splitting?

The condition  $|V(T)| \geq 2k_1 + 2k_2 + 2$  in the above problem is just to avoid the small tournaments that cannot have any  $(k_1, k_2)$ -outdegree-splitting for cardinality reasons. Clearly,  $h_T(k_1, k_2) \leq f_T(k_1, k_2)$ . But it is very likely that  $h_T(k_1, k_2)$  is smaller than  $f_T(k_1, k_2)$ . As mentioned in the beginning of Subsection 4.2, a result of Song [11] implies that  $h_T(1, 1) \leq 2$  (In fact  $h_T(1, 1) = 2$  because a 1-strong tournament  $T$  with a vertex  $v$  such that  $T - v$  is a transitive tournament has clearly no  $(1, 1)$ -outdegree-splitting.) whereas  $f_T(1, 1) = 3$ .

One might also ask similar questions for outdegree-splitting with prescribed vertices (as in Theorem 8). Bang-Jensen et al. [2] proved that if  $T$  is a tournament of order 8 and  $xy$  an arc in  $T$  such that  $T \setminus xy$  is 2-strong, then  $T$  contains an outdegree-1-splitting  $(V_x, V_y)$  with  $x \in V_x$  and  $y \in V_y$ .

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