

Semi-Supervised Learning for Graph to Signal Mapping: a Graph Signal Wiener Filter Interpretation



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Objectives:

- Use Semi-Supervised Learning to map graphs to smooth time series
- Interpret SSL in terms of graph signal filtering

Context: Duality between Graphs and Time Series as discussed in [2]

Duality between Graphs $G = (V, E)$ and Time Series:

- Time Series to Graph: modelisation by a Markov process on quantiles
- Graph to Time Series

- $|V| = N$ nodes, weight matrix W (w_{ij} weight of $(ij) \in E$, 0 otherwise)
- Question: how to smoothly map nodes to amplitudes intervals?
- Proposition of [2]: Bijective map of i to $p(i) \in 1 \dots N$, minimising:

$$\mathcal{C} = \sum w_{ij} |p(i) - p(j)|$$

- Time series: image of a random walk on the graph under this mapping
- Issue: NP-hard problem, with possibly several optima

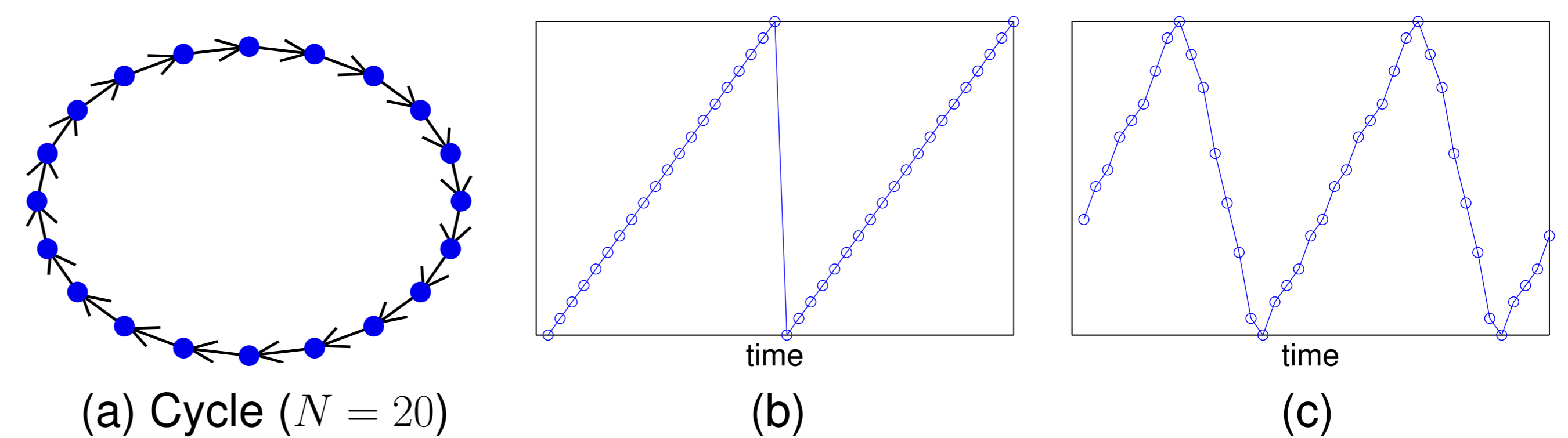


Figure 1: Different graph to signal mappings using method in [2]

SSL and Graphs to Times Series Mapping

Graph-Based SSL for Classification (4)

- Labels $1, \dots, K$ on the nodes of the labelled set $V_L \subset V$
- Label function: matrix Y with $y_{ik} = 1$ if $\text{class}(i) = k$, 0 otherwise
- Diagonal graph degree matrix D , with $d_{ii} = \sum_k w_{ik}$
- Normalised Laplacian matrix $\mathcal{L} = I_N - D^{-1/2} W D^{-1/2}$, with $\|x\|_{\mathcal{L}} = \|\mathcal{L}^{1/2} x\|_2$
- Classifier F minimising:

$$\min_F \left\{ \underbrace{\sum_{k=1}^K \|D^{\sigma-1/2} (F_{\cdot k} - Y_{\cdot k})\|_2^2}_{\text{fitting term}} + \alpha \underbrace{\|D^{\sigma-1/2} F_{\cdot k}\|_{\mathcal{L}}^2}_{\text{penalty}} \right\}$$

- Regularising parameter α , canonical parameter σ
→ Methods: Laplacian $\sigma = 0$, Normalised Laplacian $\sigma = 1/2$, PageRank $\sigma = 1$
- Closed-form solution [1]:

$$F_{\cdot k} = \underbrace{\left[(1 + \alpha) I_N - \alpha D^{-\sigma} W D^{\sigma-1} \right]^{-1}}_{C_{\sigma, \alpha}} Y_{\cdot k}$$

- Classification: $\text{class}(i) = \text{argmax}_k F_{ik}$

SSL for a Graph to Time Series Mapping (Contribution 1)

- Classifier used to quantify the relative distance to centroids
- $K = 2 = |V_L|$: two nodes diametrically opposed with different labels
- Mapping from nodes to amplitudes:

$$X(i) = \frac{F_{i,1} - F_{i,2}}{F_{i,1} + F_{i,2}}$$

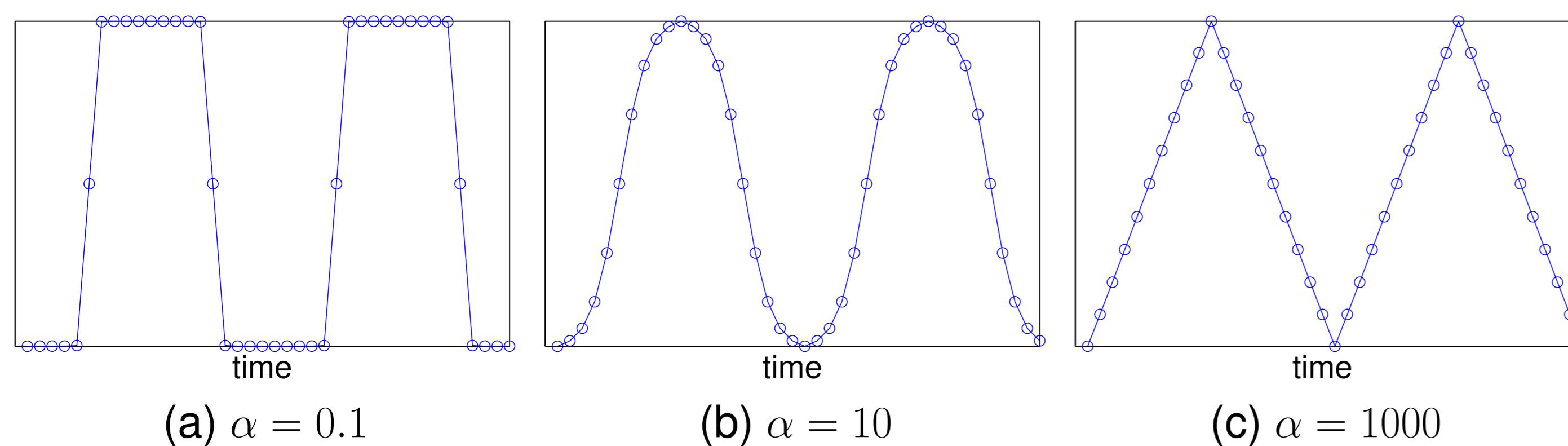


Figure 2 : SSL on the cycle of Figure 1(a)

Graph Signal Wiener Filter Interpretation

Graph Signal Filtering (3)

- Graph Signal: $y : V \rightarrow \mathbb{C}$
- Linear filter: $x = Hy$
- Graph shift operator: $y \mapsto Wy$
- Shift-Invariant filter $H: \forall y, H(Wy) = W(Hy)$
- H Linear Shift-Invariant (LSI) iff $H = h(W)$ (with h a polynomial)

SSL as Graph Filter (Contribution 2)

- Assumption: W symmetric doubly stochastic: $W = U \Lambda U^T$
- $C_{\sigma, \alpha} = U g_{\alpha}(\Lambda) U^T$ with $g_{\alpha}(\lambda) = [(1 + \alpha) - \alpha \lambda]^{-1}$
- Interpolation polynomial h_{α} of g_{α} on $\text{diag}(\Lambda)$: $C_{\sigma, \alpha} = U h_{\alpha}(\Lambda) U^T = h_{\alpha}(W)$
→ Graph-Based SSL is a LSI graph filter of kernel $C_{\sigma, \alpha}$
- Eigenvalues ω_i^2 of the Normalised Graph Laplacian
- Eigenvalue λ_i of W : $\lambda_i = 1 - \omega_i^2$
- Eigenfrequency response $\tilde{g}_{\alpha}(\omega_i) = g_{\alpha}(1 - \omega_i^2) = [1 + \alpha \omega_i^2]^{-1}$
→ Wiener filter for a graph signal + Gaussian Noise, and SNR of $1/\alpha$

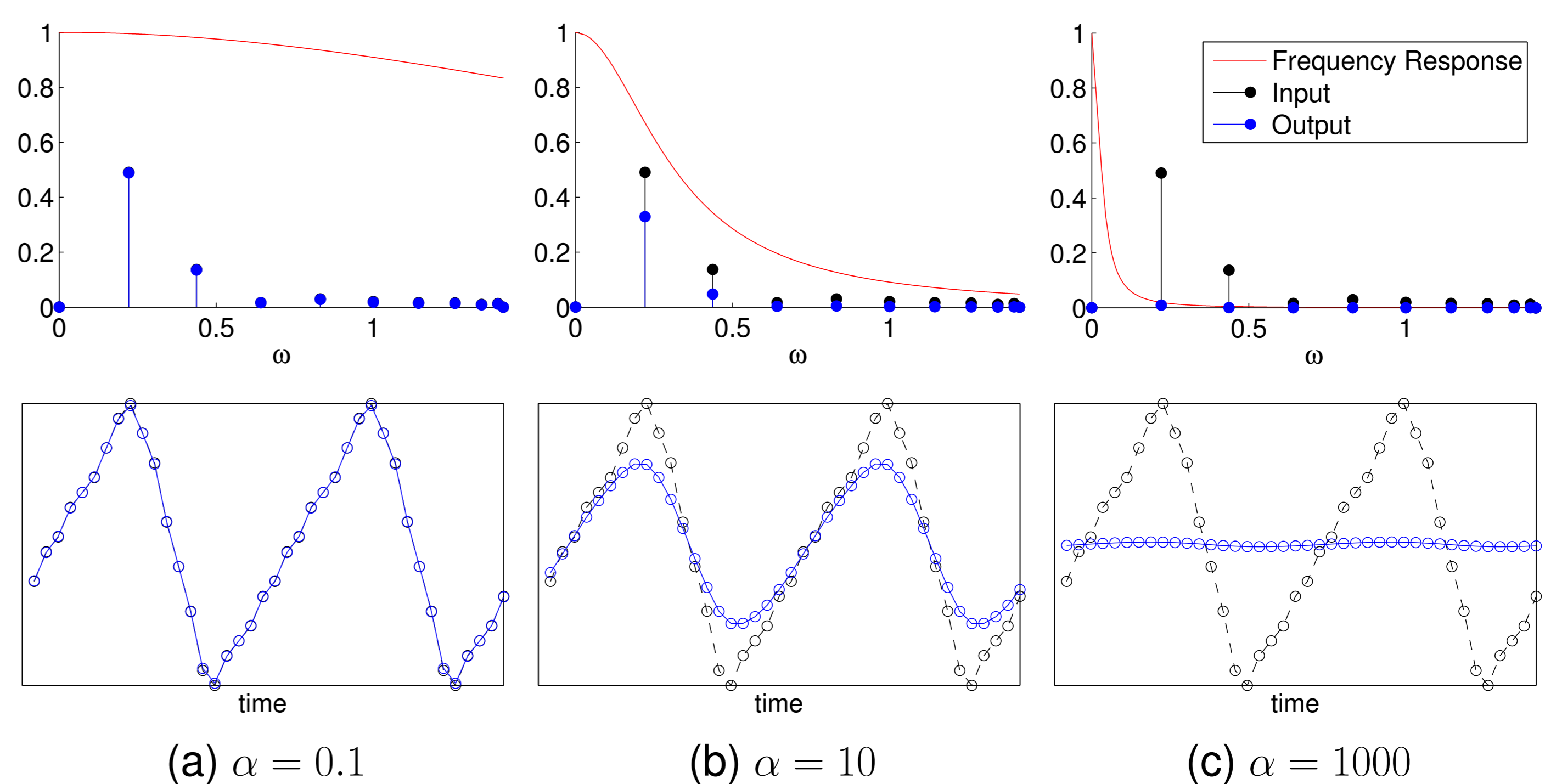


Figure 3: SSL Filtering of the signal of Figure 1(c)

Conclusion:

- Contributions: (i) efficient graph to smooth time series mapping, (ii) Graph-Based SSL interpretation as a graph signal filter
- Ongoing work: (i) investigation of non-stochastic weight matrices, (ii) study the influence of σ , (iii) more than 2 classes

References

- [1] K. Avrachenkov, P. Gonçalves, A. Mishenin, and M. Sokol. Generalized Optimization Framework for Graph-based Semi-supervised Learning. In *SDM*, 2012.
- [2] A.S.L.O. Campanharo, M.I. Siring, R.D. Malmgren, F.M. Ramos, and L.A.N. Amaral. Duality between Time Series and Networks. *PLoS one*, 6(8):e23378, 2011.
- [3] A. Sandryhaila and J.M.F. Moura. Discrete Signal Processing on Graphs. *Signal Processing, IEEE Transactions on*, 61(7):1644–1656, 2013.
- [4] X. Zhu and A.B. Goldberg. *Introduction to Semi-Supervised Learning*, volume 3 of *Synthesis Lectures on Artificial Intelligence and Machine Learning*. M. & C., 2009.