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# Parallel Repetition of entangled games on the uniform distribution

André Chailloux, Scarpa Giannicola

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# Parallel Repetition of entangled games on the uniform distribution

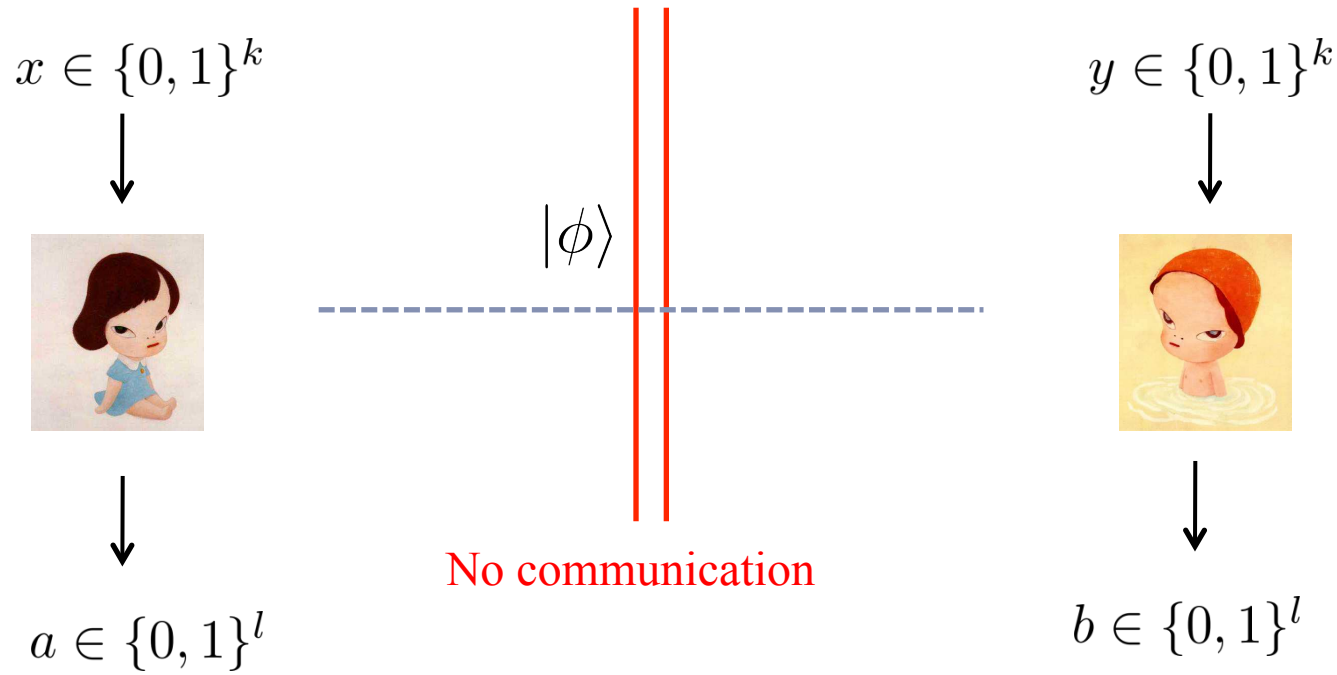
**André Chailloux**

INRIA Rocquencourt, Projet SECRET

Joint work with Giannicola Scarpa (CWI, Amsterdam)

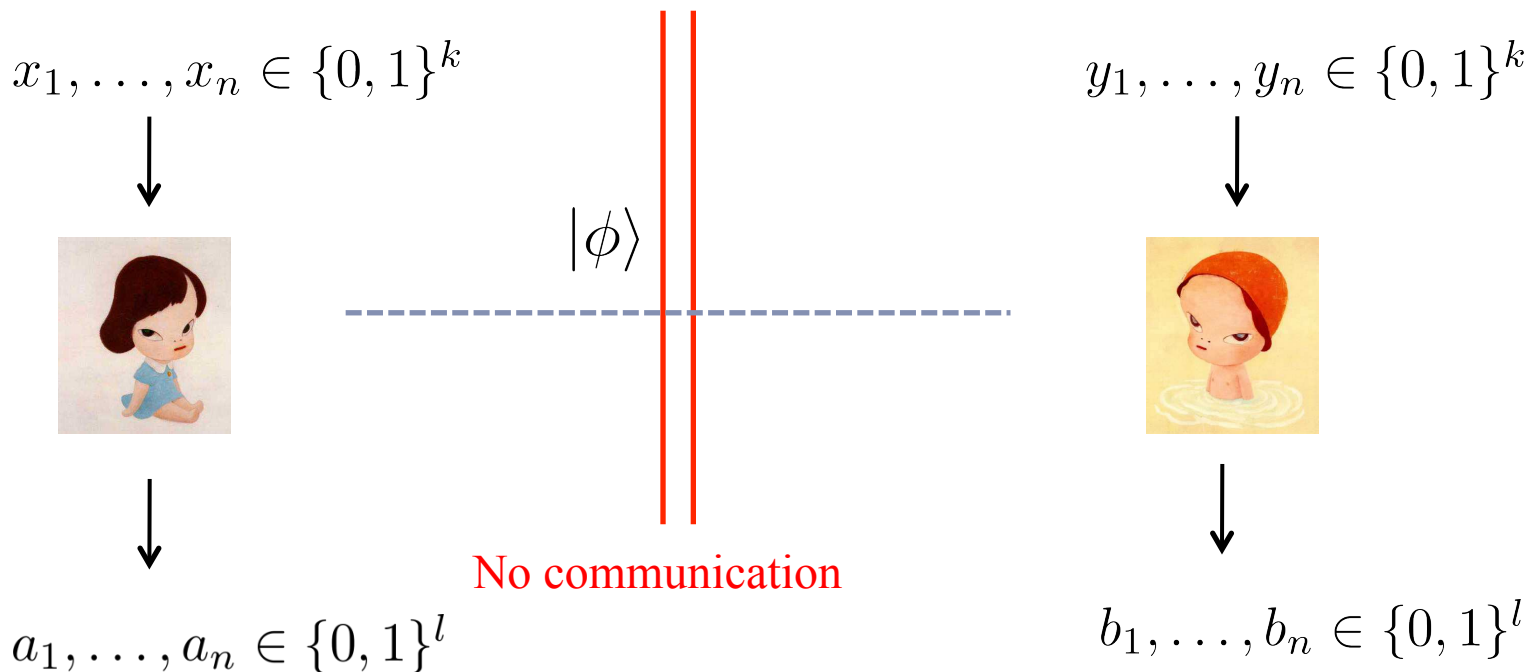
Nancy, JIQ 2013

# Entangled games



- ▶ A game  $G = (V, p)$ .
  - ▶  $(x, y)$  is taken according to distribution  $p$
  - ▶ Alice and Bob win the game if  $V(a, b | x, y) = 1$
  - ▶ Max pr. of winning the game:  $\omega^*(G)$

# Parallel repetition



- ▶ Above game:  $G^n$ . Each  $(x_i, y_i)$  taken according to  $p$ .
- ▶ Alice and Bob win the game if  $\forall i, V(a_i, b_i | x_i, y_i) = 1$
- ▶ Naive strategy: win wp.  $(\omega^*(G))^n$ 
  - ▶ Possible to do better for some games.
  - ▶ What can we say about  $\omega^*(G^n)$  ?

# Why do we care ?

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- ▶ Classically: widely used in complexity (hardness of approximation results)
- ▶ Quantumly
  - ▶ Natural object for studying non locality.
  - ▶ Helps understanding how to use entanglement efficiently
    - ▶ How can we do complicated stuff ?
  - ▶ Also related to hardness of approximation results.

# Previous work

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- ▶ Classically, widely studied.
- ▶ General games [Raz98]  $\omega(G^n) \leq (1 - \varepsilon^3)^{\Omega(n/l)}$  where  $\omega(G) = 1 - \varepsilon$
- ▶ Better results for specific kinds of games.
  
- ▶ Quantumly, also widely studied
  - ▶ XOR games[CleveSlofstraUngerUpadhyay06] (like CHSH): perfect parallel repetition :  $\omega^*(G^n) = (\omega^*(G))^n$
  - ▶ Unique games[KempeRegevToner09]
  - ▶ General games [KempeVidick10]  $\omega^*(G^n) \leq O(\frac{1}{n})$  when  $\omega^*(G) = O(1)$

# Our result

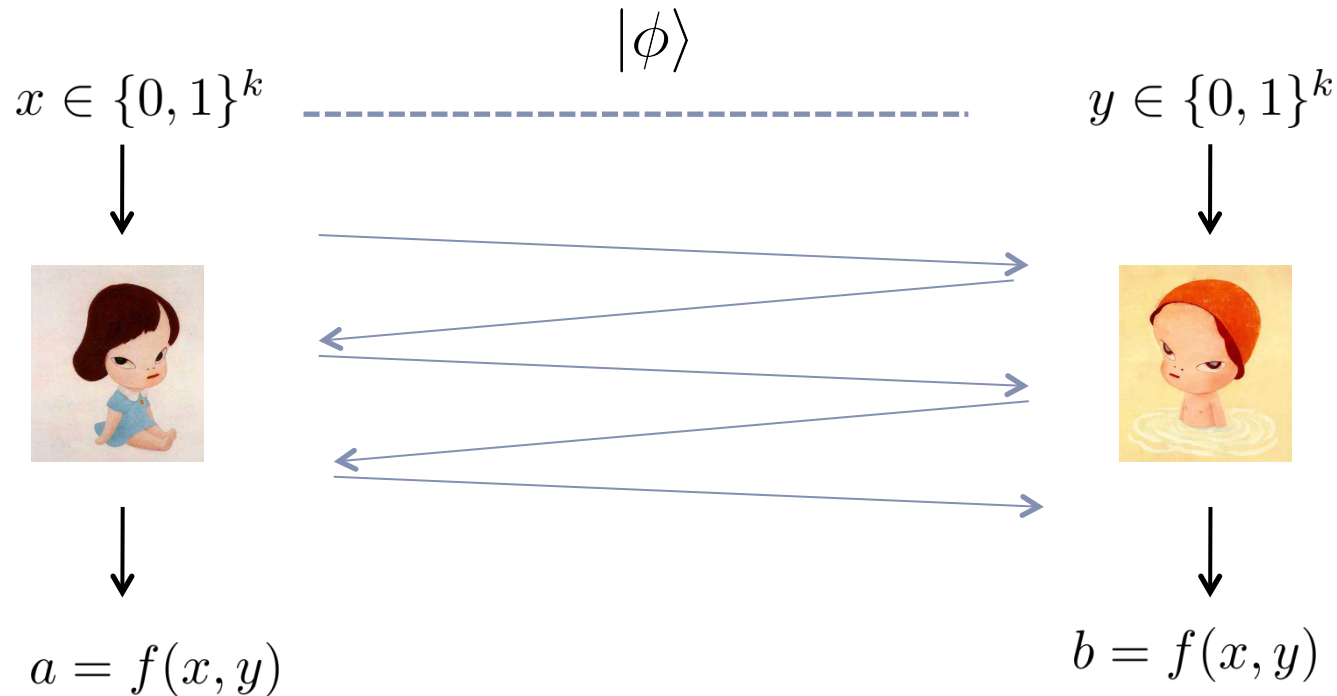
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## THEOREM

For any game  $G$  on the uniform distribution such that  $\omega^*(G) \leq 1 - \varepsilon$ , we have  $\omega^*(G^n) \leq (1 - \varepsilon^2)^{\Omega(\frac{n}{kl})}$

- ▶ Main technique: extend the notion of Interactive Information Cost used in Communication complexity to entangled games.

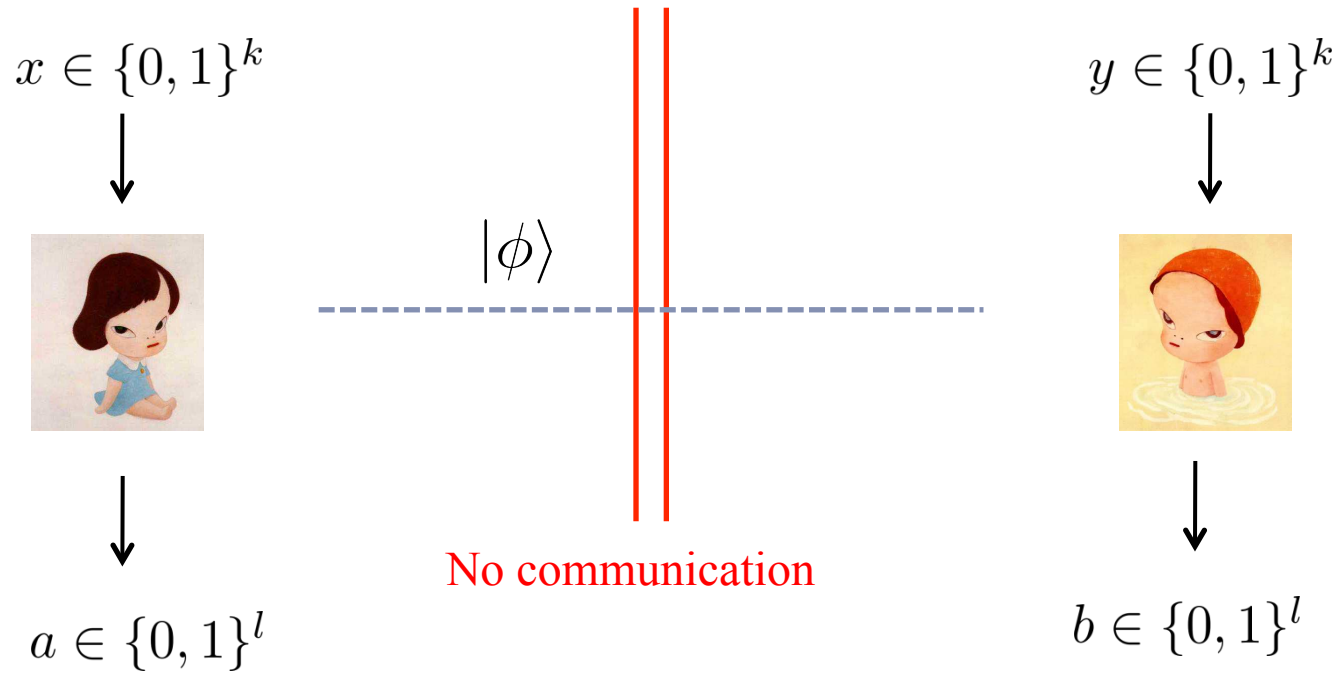
# Information cost in communication complexity



- ▶ CC: minimal amount of communication in order to output  $f(x, y)$
- ▶ IC: amount of information that Alice and Bob need to know about each other's inputs to output  $f(x, y)$
- ▶ IC is very cool when studying CC.
- ▶ Can we do the same for entangled games ?

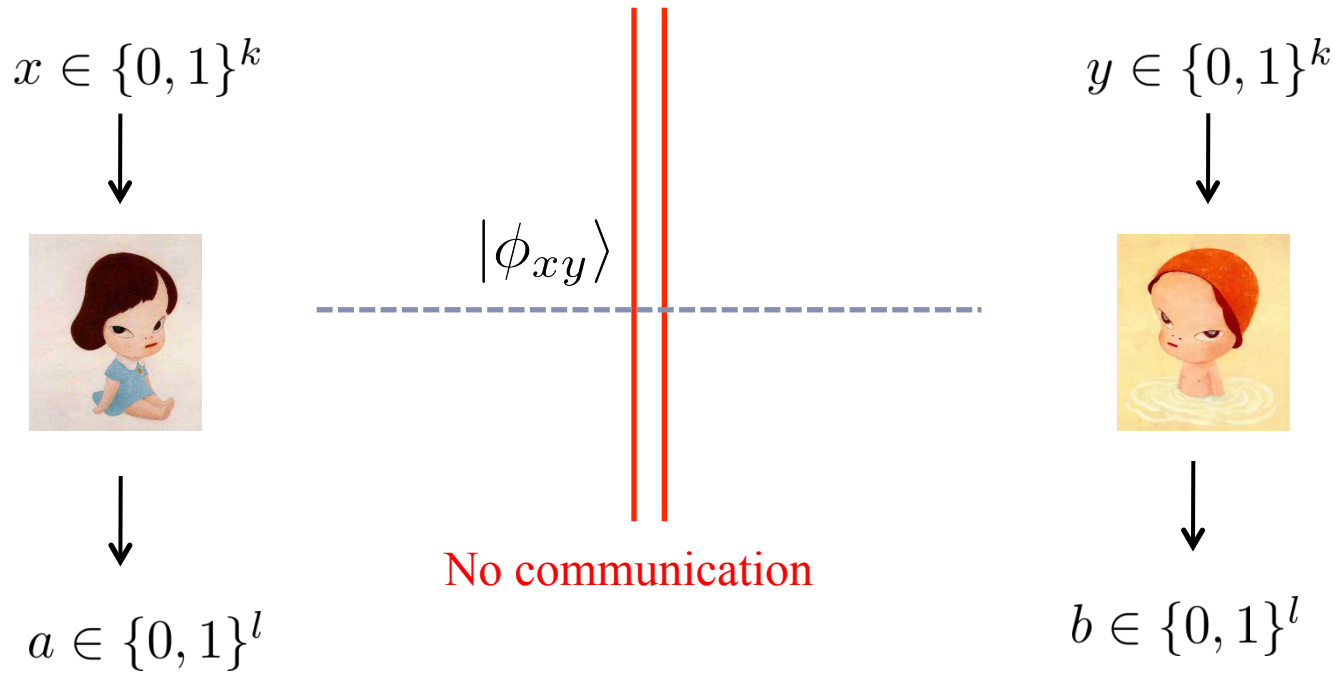


# Information Cost for Entangled games



- ▶ How to extend this to entangled games ?

# Information Cost for Entangled games



- ▶ How to extend this to entangled games ?
- ▶ Advice states: the state Alice and Bob share can depend on  $x, y$
- ▶ IC for games (informal def):

IC(G) = minimal amount of information that these states have to give to Alice & Bob about each other's inputs to win wp. 1 ?

# Is this notion useful ?

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▶ Can we bound  $\omega^*(G)$  using this notion of information cost ?

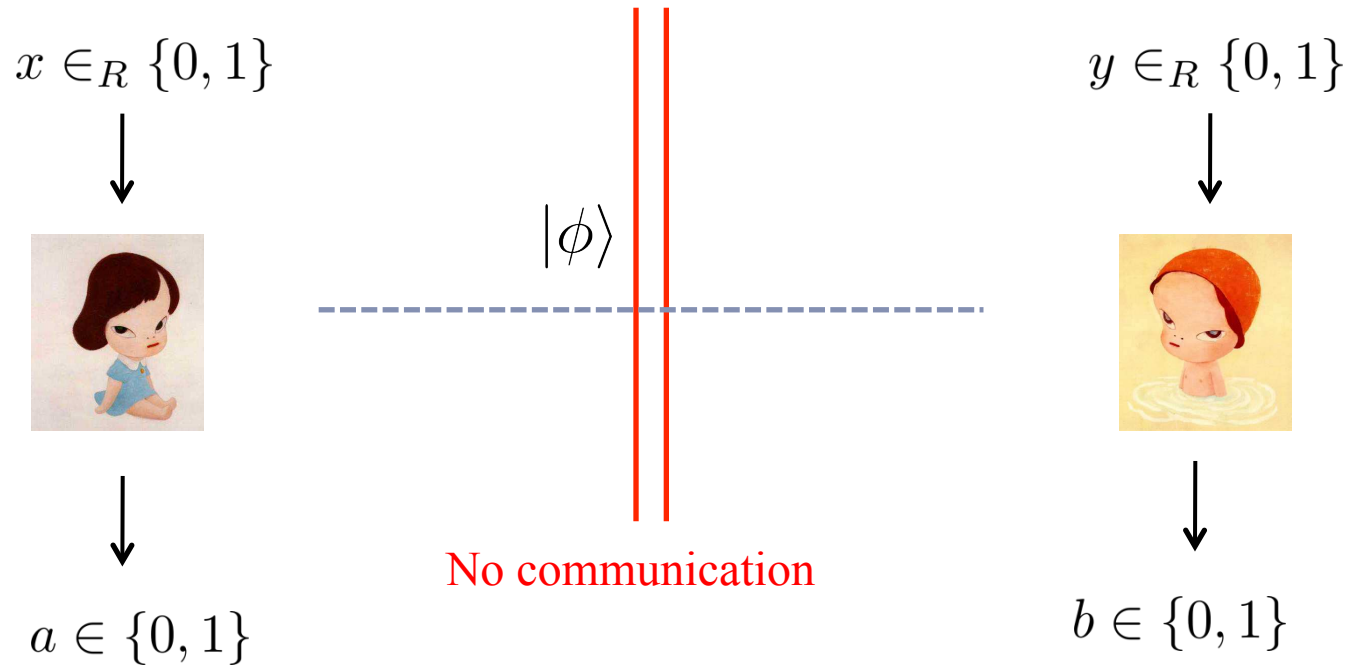
▶ At least, we want:

$$\text{if } \omega^*(G) < 1 \text{ then } \text{IC}(G) > 0$$

▶ Is that the case ? NO

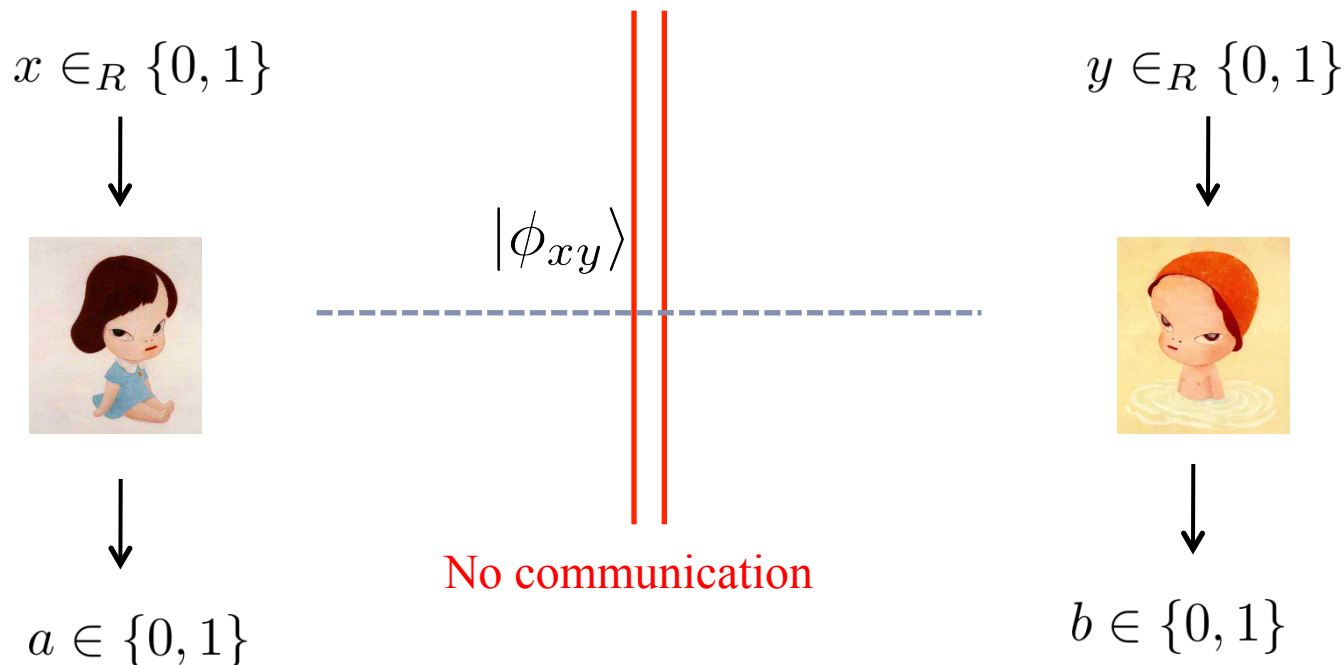
▶ Example: CHSH game

# CHSH Game



- ▶ Alice and Bob win iff.  $a \oplus b = x \cdot y$
- ▶ They can win wp. at most  $\omega^*(CHSH) = \cos^2(\pi/8)$

# CHSH Game with advice states



- ▶ Consider the following advice states

$$|\phi_{00}\rangle = |\phi_{01}\rangle = |\phi_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |\phi_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

- ▶ Measure in the computational basis: win wp. 1
- ▶ Alice has no information about  $y$ , Bob has no information about  $x$ .
- ▶ The information cost useless when studying games.

# The end ?

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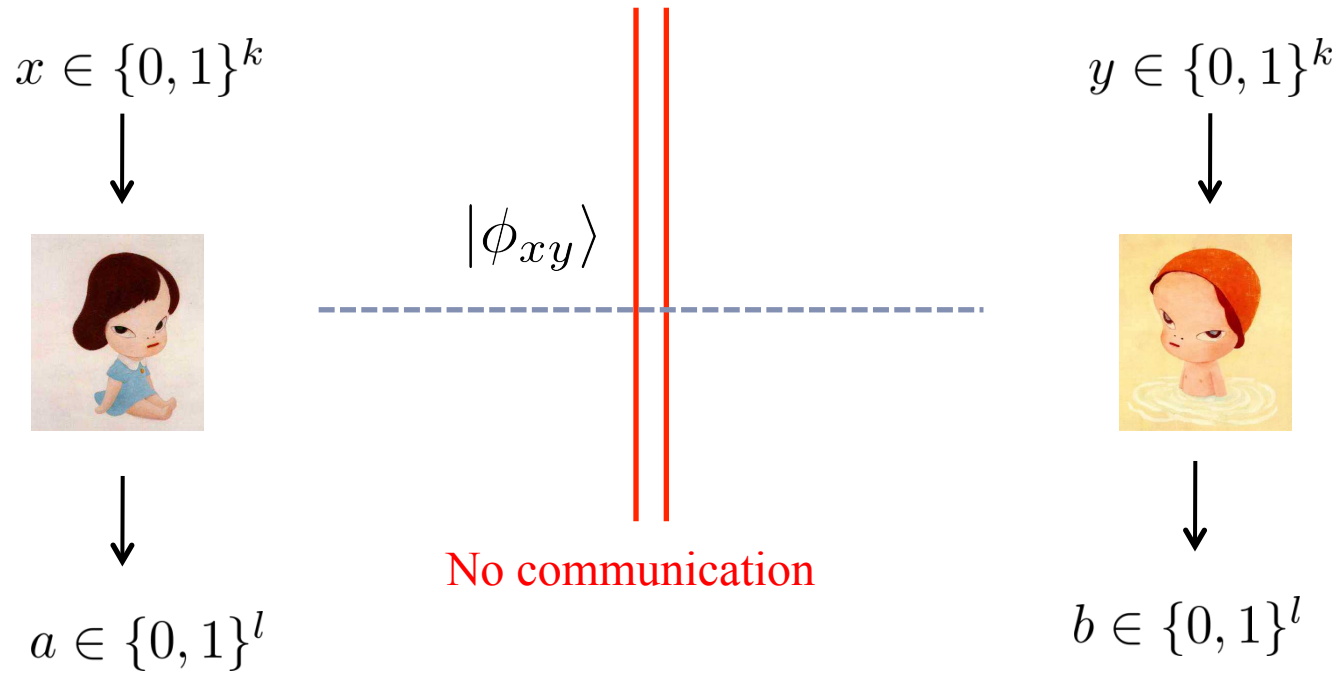
- ▶ We wanted to extend the notion of information cost to entangled game.
- ▶ The notion we defined is pretty useless when studying entangled games.

# Not yet

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- ▶ We wanted to extend the notion of information cost to entangled game.
- ▶ The notion we defined is pretty useless when studying entangled games.
- ▶ We really really want this kind of approach to work.
  - ▶ Appealing to use information theory for entangled games
- ▶ So we cheat a bit.
- ▶ Main idea: allow the players to have a quantum superposition of their inputs.

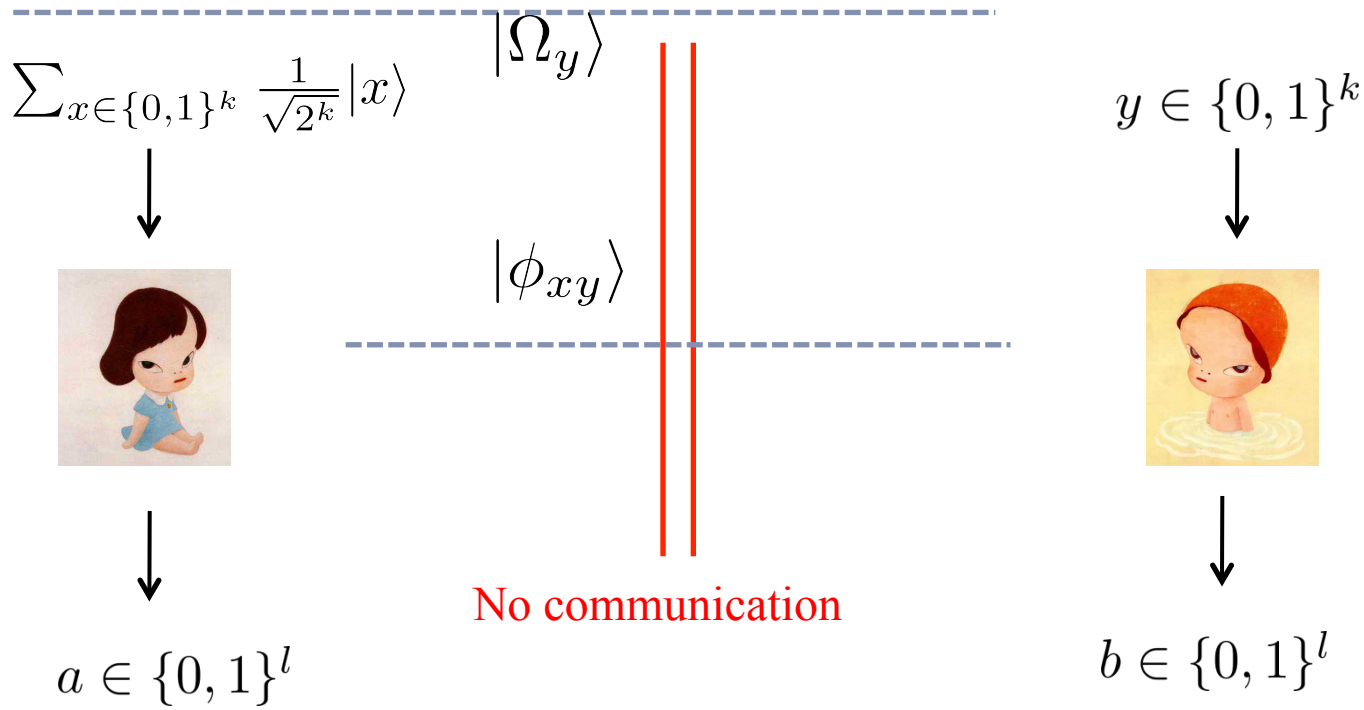
# Normal inputs



- ▶ If Bob has  $y$ , Alice has  $\rho_y^A = \text{Tr}_B \left( \sum_{x \in \{0,1\}^k} \frac{1}{2^k} |x\rangle\langle x| \otimes |\phi_{xy}\rangle\langle\phi_{xy}| \right)$



# Superposed Alice



▶ If Bob has  $y$ , Alice has  $\rho_y^A = \text{Tr}_B \left( \sum_{x \in \{0,1\}^k} \frac{1}{2^k} |x\rangle \langle x| \otimes |\phi_{xy}\rangle \langle \phi_{xy}| \right)$

▶ Alice has a superposition of her inputs: If Bob has  $y$ , Alice has

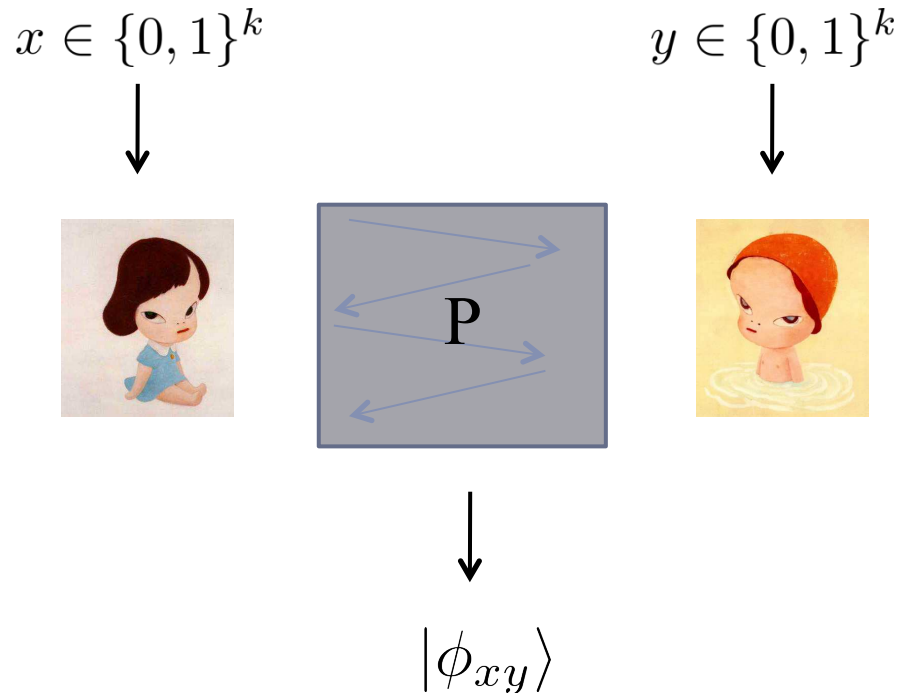
$$\sigma_y^A = \text{Tr}_B (|\Omega_y\rangle) \text{ where } |\Omega_y\rangle = \frac{1}{\sqrt{2^k}} \sum_x |x\rangle |\phi_{xy}\rangle$$

▶ There is entanglement between Alice's superposed input and the advice.

# The Superposed information cost: motivation

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- ▶ Why consider superposed inputs ?



- ▶ How much about  $y$  (or  $x$ ) does this procedure give away ?
- ▶ Better if secure also vs. Alice if she decides to have a superposition of her inputs.
- ▶ Arises in quantum cryptography.

# The Superposed information cost

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$IC(G)$  = minimal amount of information that advice states have to give to Alice & Bob about each other's inputs to win wp. 1 ?

- ▶ Extend this the case where Alice and Bob can have a superposition of their inputs

$SIC(G)$  = minimal amount of information that advice states have to give to superposed Alice & superposed Bob about each other's inputs to win wp. 1 ?

- ▶ Is this notion more useful ?

# CHSH Game with advice states: superposed information.

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- ▶ Advice states

$$|\phi_{00}\rangle = |\phi_{01}\rangle = |\phi_{01}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad |\phi_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

- ▶ Recall  $\rho_y^A = \text{Tr}_B(\sum_{x \in \{0,1\}} \frac{1}{2} |x\rangle\langle x| \otimes |\phi_{xy}\rangle\langle\phi_{xy}|)$  and

$$\sigma_y^A = \text{Tr}_B(|\Omega_{xy}\rangle) \text{ where } |\Omega_{xy}\rangle = \frac{1}{\sqrt{2}} \sum_x |x\rangle |\phi_{xy}\rangle$$

- ▶ Normal Alice: For each  $y$ ,  $\rho_y^A = \frac{I}{4}$  : no info about  $y$

- ▶ Superposed Alice:  $\sigma_0^A = |+\rangle\langle+| \otimes \frac{I}{2}$      $\sigma_1^A = \frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Psi^+\rangle\langle\Psi^+|$

With  $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  and  $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

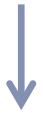
- ▶  $\sigma_0^A \neq \sigma_1^A$  : Superposed Alice has some information about Bob's input  $y$ .

# The Superposed information cost: Definition

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▶ Alice and Bob share  $\rho = \frac{1}{2^{2k}} \sum_{x,y \in \{0,1\}^k} |x\rangle\langle x|_{\mathcal{X}} \otimes |\phi_{xy}\rangle\langle\phi_{xy}|_{\mathcal{AB}} \otimes |y\rangle\langle y|_{\mathcal{Y}}$

▶  $\sigma^A = \frac{1}{2^k} \sum_y |\Omega_y^A\rangle\langle\Omega_y^A| \otimes |y\rangle\langle y|$  where  $|\Omega_y^A\rangle = \frac{1}{\sqrt{2^k}} \sum_x |x\rangle|\phi_{xy}\rangle$



The state that A & B share if Alice has a superposition of her inputs

▶  $\sigma^B = \frac{1}{2^k} \sum_x |x\rangle\langle x| \otimes |\Omega_x^B\rangle\langle\Omega_x^B|$  where  $|\Omega_x^B\rangle = \frac{1}{\sqrt{2^k}} \sum_y |\phi_{xy}\rangle|y\rangle$

▶  $SIC(\rho) = I(Y : XA)_{\sigma^A} + I(X : BY)_{\sigma^B}$



Information that Alice has about Bob's input when they share  $\sigma^A$

# The Superposed information cost: Definition

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- ▶ Superposed information cost of a game

Definition:  $SIC(G) = \inf_{\rho} SIC(\rho)$

Where the inf. is taken over states  $\rho$  st. Alice & Bob win G wp. 1 using  $\rho$

- ▶  $\rho$  is of this form  $\rho = \frac{1}{2^{2k}} \sum_{x,y \in \{0,1\}^k} |x\rangle\langle x|_{\mathcal{X}} \otimes |\phi_{xy}\rangle\langle\phi_{xy}|_{\mathcal{AB}} \otimes |y\rangle\langle y|_{\mathcal{Y}}$

# Upper bounding $SIC(G^n)$

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- ▶ Additivity:  $SIC(G^n) = nSIC(G)$ 
  - ▶ Almost for free because the information cost is a nice IT quantity.
- ▶ Relation to  $\omega^*(G)$  :  $SIC(G) \geq \frac{1-\omega^*(G)}{32 \ln(2)}$ 
  - ▶ One of the main ingredients of the proof.
  - ▶ Shows that the notion of SIC is interesting for games.
- ▶ Putting this together  $SIC(G^n) \geq \frac{n(1-\omega^*(G))}{32 \ln(2)}$ 
  - ▶ It's a hint that it's hard to win  $G^n$  but it's not an actual proof

# How do we show parallel repetition ?

---

- ▶ Let  $t$  that satisfies  $\omega^*(G^n) = 2^{-t}$ . We show that  $t \geq f(n)$  for some function  $f$ .
- ▶ We show that  $SIC^1(G^n) \leq \frac{t \cdot kl}{1 - \omega^*(G)}$

1: we don't use exactly SIC but the spirit is there.



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- ▶ We show that  $SIC^1(G^n) \leq \frac{t \cdot kl}{1 - \omega^*(G)}$
- ▶ We had  $SIC(G^n) \geq \frac{n(1 - \omega^*(G))}{32 \ln(2)}$
- ▶ Putting it all together, we get  $t \geq \frac{n(1 - \omega^*(G))^2}{32 \ln(2) kl}$  which gives

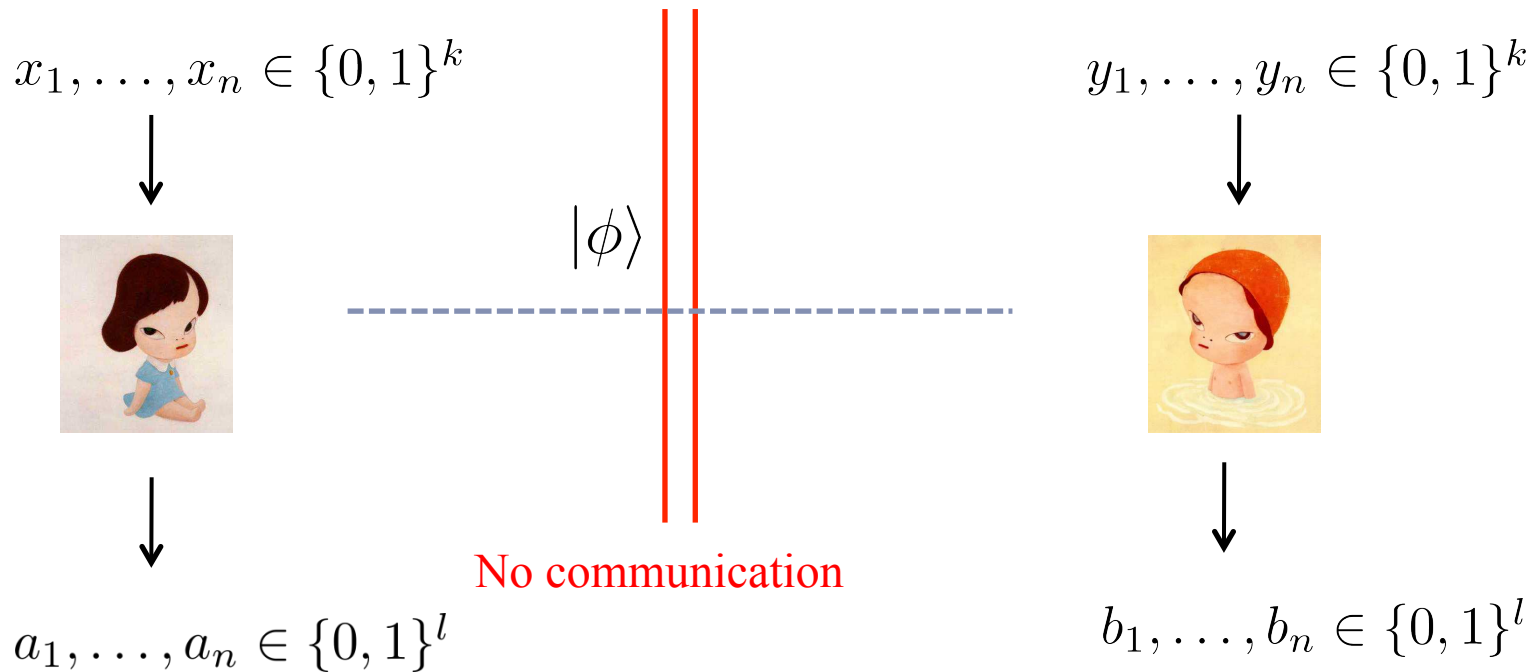
## THEOREM

For any game  $G$  on the uniform distribution such that  $\omega^*(G) \leq 1 - \varepsilon$ , we have  $\omega^*(G^n) \leq (1 - \varepsilon^2)^{\Omega(\frac{n}{kl})}$

1: we don't use exactly SIC but the spirit is there.

# Intuition on why $SIC(G^n) \leq \frac{t \cdot kl}{1 - \omega^*(G)}$

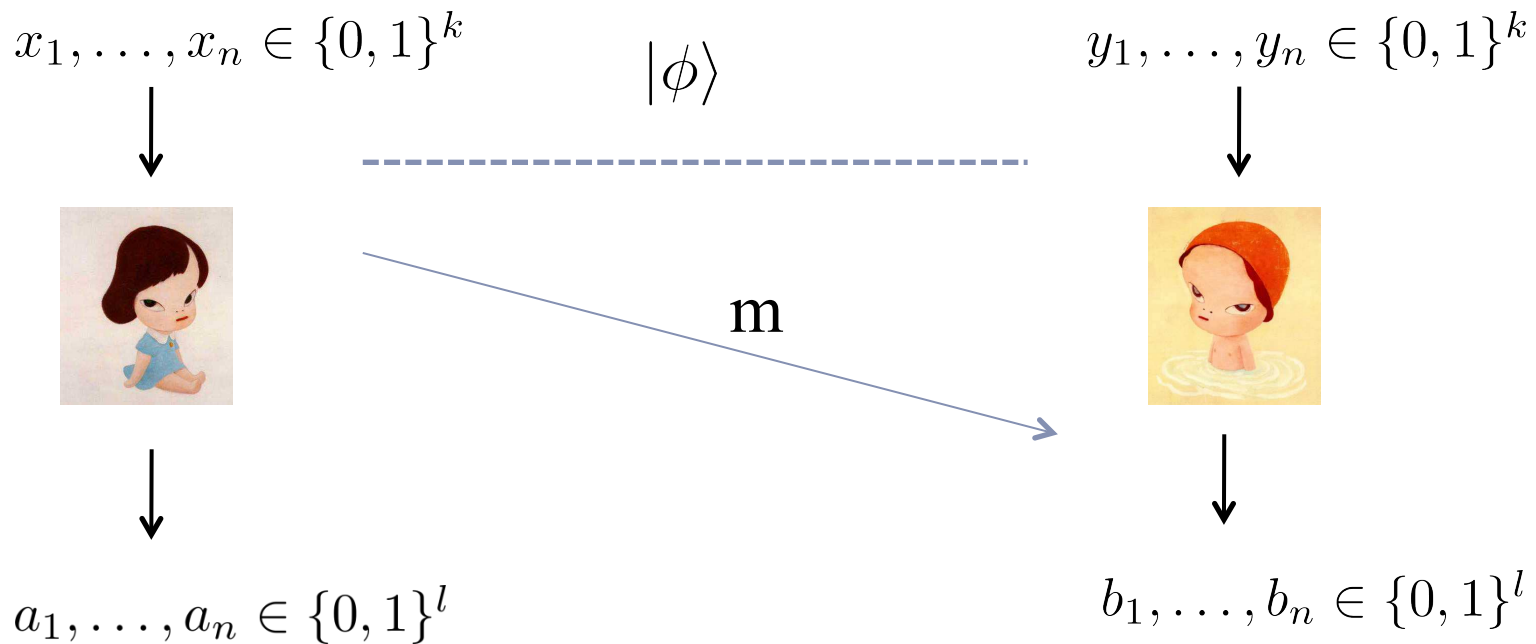
- ▶ Recap: we start from  $G^n$ .



- ▶ We want to show that  $SIC(G^n) \leq \frac{t \cdot kl}{1 - \omega^*(G)}$

# Intuition on why $SIC(G^n) \leq \frac{t \cdot kl}{1 - \omega^*(G)}$

## ► Communication protocol



## ► What amount of bits Alice has to send to Bob st. Alice & Bob win wp 1 ?

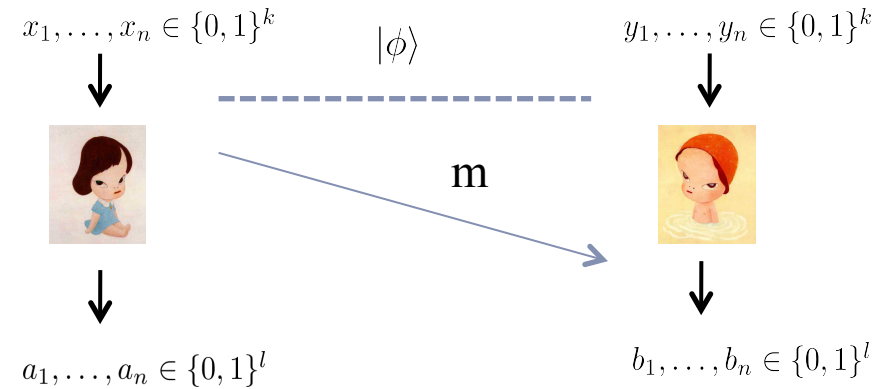
# The communication protocol

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- ▶ This question is a hard question. We show a much weaker statement.
- ▶ Alice can send  $m = O(\frac{t \cdot kl}{\epsilon})$  to Bob.
- ▶ After the message
  - ▶ The probability of winning  $G^n$  doesn't increase
  - ▶ But: Bob knows whether they win the game.
- ▶ That will be enough.

# The communication protocol

- ▶ How do they do ?



- ▶ Alice and Bob play the game optimally  $G^n$ 
  - ▶ They can win wp.  $\omega^*(G^n)$
- ▶ Alice picks  $O(\frac{t}{\epsilon})$  random pairs of input/output  $(x_i, a_i)$  and sends them to Bob
- ▶ Bob checks for each of these pairs that  $V(a_i, b_i | x_i, y_i) = 1$ 
  - ▶ If this holds for each pair, Bob knows that they win  $G^n$  with high pr.

# The communication protocol

---

- ▶ Alice sends  $O(\frac{t}{\varepsilon})$  pairs each of size  $kl$  so  $m = O(\frac{t \cdot kl}{\varepsilon})$
- ▶ Let  $\xi$  the state that Alice and Bob share conditioned on winning.
  - ▶ This state is created by sending  $m = O(\frac{t \cdot kl}{\varepsilon})$  + postselection so it doesn't have high SIC,  $SIC(\xi) = O(\frac{t \cdot kl}{\varepsilon})$
  - ▶ This state allows the players to win  $G^n$  wp. 1
  - ▶ This means that  $SIC(G^n) = O(\frac{t \cdot kl}{\varepsilon})$
  - ▶ The above 3 statements are almost true but that's ok.

# How do we show parallel repetition ?

---

- ▶ Let  $t$  that satisfies  $\omega^*(G^n) = 2^{-t}$ . We show that  $t \geq f(n)$  for some function  $f$ .
- ▶ We show that  $SIC^1(G^n) \leq \frac{t \cdot kl}{1 - \omega^*(G)}$
- ▶ We had  $SIC(G^n) \geq \frac{n(1 - \omega^*(G))}{32 \ln(2)}$
- ▶ Putting it all together, we get  $t \geq \frac{n(1 - \omega^*(G))^2}{32 \ln(2) kl}$  which gives

## THEOREM

For any game  $G$  on the uniform distribution such that  $\omega^*(G) \leq 1 - \varepsilon$ , we have  $\omega^*(G^n) \leq (1 - \varepsilon^2)^{\Omega(\frac{n}{kl})}$

1: we don't use exactly SIC but the spirit is there.

# Conclusion

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- ▶ We introduced the Superposed Information Cost, a powerful tool for the study of entangled games.
- ▶ We managed to use this notion to prove parallel repetition for entangled games.
  
- ▶ Can we remove the assumption on the inputs ?
- ▶ Can we use this tool for other problems related to entangled games or quantum communication complexity ?