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REDUCTION OF THE RESONANCE ERROR IN NUMERICAL HOMOGENIZATION II: CORRECTORS, EXTRAPOLATION, AND SPECTRAL THEORY

ANTOINE GLORIA AND ZAKARIA HABIBI

Abstract. This paper is the companion article of [17]. One common drawback among numerical homogenization methods is the presence of the so-called resonance error, which roughly speaking is a function of the ratio ε/ρ , where ρ is a typical macroscopic lengthscale and ε is the typical size of the heterogeneities. In the present work, we make a systematic use of regularization and extrapolation to reduce this resonance error at the level of the approximation of homogenized coefficients and correctors. We quantify this reduction in the periodic case and in the case of Poisson random inclusions. We also report on a systematic numerical study in dimension 2, which demonstrates the efficiency of the method and the sharpness of the analysis. Last, we combine this approach to numerical homogenization methods, prove the asymptotic consistency in the case of locally stationary ergodic symmetric coefficients using spectral theory, and give quantitative estimates in the case of periodic coefficients.

Keywords: numerical homogenization, resonance error, effective coefficients, correctors, periodic, random.

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1. INTRODUCTION

Numerical homogenization methods are designed to solve partial differential equations for which the operator is heterogeneous spatially at a small scale. Such problems arise in many applications such as diffusion in porous media or composite materials. By numerical homogenization, we mean that we compute not only the mean field solution of the heterogeneous problem, but also the local fluctuations, which may be important in many applications. In the recent years, two main types of methods were introduced: methods which rely on homogenization to reduce the complexity of the problem (see [4], [25, 26], [9, 8], and [15] e.g.), and methods which rely on the structure of the solution space independently of homogenization structures (see [29, 30, 31], [24], [5]). Both approaches are nonlinear in the sense that the vector space in which the solution is approximated depends on the operator itself.

In this article we focus on methods based on the homogenization theory. More precisely, we shall address the following prototypical scalar linear elliptic equation: For some $1 \gg \varepsilon > 0$, find $u_\varepsilon \in H_0^1(D)$ such that

$$-\nabla \cdot A_\varepsilon \nabla u_\varepsilon = f_\varepsilon \quad \text{in } D, \quad (1.1)$$

on a domain D with $f_\varepsilon \in H^{-1}(D)$. Here, the spatial dependence of the operator is encoded in the function A_ε , whose frequencies are of order ε^{-1} . Academic examples are of the form: $A_\varepsilon(x) = A(x/\varepsilon)$, with A periodic, quasi-periodic or stationary in an ergodic stochastic setting. More realistic models can be of the form: $A_\varepsilon(x) = A(x, x/\varepsilon)$, where $A(x, \cdot)$ may be periodic, quasi-periodic or stationary for all $x \in D$, provided some suitable cross-regularity holds (see [2, 7] e.g.). In all these examples, ε refers to the *actual* lengthscale of the heterogeneities. Numerical homogenization methods suffer from the so-called resonance error, which is roughly speaking related to scale separation. Since this error often dominates the error due to discretization (see for instance [1, Section 4] for a related discussion), its reduction is one of the biggest challenges in numerical homogenization of linear elliptic PDEs. For general coefficients this was successfully achieved in [31] and [24]. Yet the price to pay is the ‘‘oversampling’’ size (see below for details). In the present contribution we take advantage of homogenization structures (such as stationary ergodic coefficients) to reduce the resonance error without increasing significantly the oversampling size. The aim of this paper is to introduce a general method which is consistent with homogenization and successfully reduces the resonance error in three main academic examples of heterogeneous media: periodic, quasi-periodic, and random with Poisson inclusions.

To this end we shall improve the approach proposed in [17] (and further developed in [20]) to reduce the resonance error, and extend it to the approximation of correctors, which are the central objects of the homogenization theory. This will allow us to combine this method of reduction of the resonance error with standard numerical homogenization methods. Some of the results were announced in [19]. The starting point in [17] is the effective computation of

$$\xi \cdot A_{\text{hom}} \xi = \mathcal{M}\{(\xi + \nabla \phi) \cdot A(\xi + \nabla \phi)\}, \quad (1.2)$$

where ϕ is the solution (in a suitable sense) of

$$-\nabla \cdot A(\xi + \nabla \phi) = 0 \quad \text{in } \mathbb{R}^d, \quad (1.3)$$

and $\mathcal{M}\{\cdot\}$ is the average operator on \mathbb{R}^d :

$$\mathcal{M}\{\mathcal{E}\} := \lim_{R \uparrow \infty} \int_{Q_R} \mathcal{E}(x) dx, \quad (1.4)$$

with $Q_R := (-R, R)^d$. Such quantities are well-defined in periodic, quasi-periodic and stochastic homogenization (see [28] e.g.). The naive approach to approximate (1.2) consists in replacing ϕ by ϕ_R , the weak solution in $H_0^1(Q_R)$ of

$$-\nabla \cdot A(\xi + \nabla \phi_R) = 0 \quad \text{in } Q_R,$$

and the average on \mathbb{R}^d by an average on Q_R , which yields the approximation A_R of A_{hom}

$$\xi \cdot A_R \xi := \int_{Q_R} (\xi + \nabla \phi_R(x)) \cdot A(x) (\xi + \nabla \phi_R(x)) dx. \quad (1.5)$$

In the case of periodic coefficients A , it is elementary to show that

$$|A_R - A_{\text{hom}}| \lesssim R^{-1}. \quad (1.6)$$

This is the so-called resonance error. To reduce this error, we have introduced the following alternative to approximate A_{hom} . Let $\phi_{T,R}$ be the unique weak solution in $H_0^1(Q_R)$ of

$$T^{-1} \phi_{T,R} - \nabla \cdot A(\xi + \nabla \phi_{T,R}) = 0 \quad \text{in } Q_R,$$

where $T > 0$ controls the importance of the zero-order term and $R > 0$ is the size of the finite domain Q_R . The homogenized coefficients are approximated by taking the filtered average

$$\xi \cdot A_{T,R,L} \xi := \int_{Q_R} (\xi + \nabla \phi_{T,R}(x)) \cdot A(x) (\xi + \nabla \phi_{T,R}(x)) \mu_L(x) dx, \quad (1.7)$$

where μ_L is a smooth averaging function (whose properties will be detailed later on, see Definition 4) compactly supported in $Q_L = (-L, L)^d$, $L \leq R$. Then, as showed in [17, Theorem 1], for

- $T = L^2(\log L)^{-4}$,
- $R = 3L/2$,

and an averaging function of order 3, we have in the periodic case

$$|A_{T,R,L} - A_{\text{hom}}| \lesssim R^{-4} \log^8 R, \quad (1.8)$$

which is much better than (1.6). This approach also yields improvements over the naive approach in the case of random independent and identically distributed coefficients for discrete elliptic equations, see [18]. Yet, it is not clear in general whether the approximation (1.7) remains uniformly elliptic for all values of the parameters, which is an issue of importance in practice. In this contribution we shall show that a variant of (1.7) ensures the uniform ellipticity of the approximation (without changing in a quantitative way the convergence properties) and that Richardson extrapolations of $\nabla \phi_{T,R}$ with respect to T further reduce the resonance error.

Let us now go back to the original problem (1.1), and show how the method introduced in [17] can be used to reduce the resonance error in the numerical solution of (1.1). We first assume that A_ε satisfies a homogenization property, that is there exist uniformly elliptic (not necessarily constant) coefficients A_{hom} such that the solution u_ε of (1.1) converges weakly in $H^1(D)$ to the unique weak solution u_{hom} in $H_0^1(D)$ of

$$-\nabla \cdot A_{\text{hom}} \nabla u_{\text{hom}} = f \quad \text{in } D \quad (1.9)$$

for all suitable f . Such a homogenization property typically takes place when A_ε is the combination of a smooth function and an oscillating part at scale $\varepsilon > 0$. Unfortunately, A_{hom} is not explicit in general. Its approximation is one part of numerical homogenization.

A very general approach consists in averaging out the equation at a “mesoscale” $\rho \gg \varepsilon$. More precisely, for all $\rho \geq \varepsilon > 0$, let $A_{\rho,\varepsilon}$ be defined by

$$\xi \cdot A_{\rho,\varepsilon}(x)\xi := \inf \left\{ \int_{D \cap B(x,\rho)} (\xi + \nabla \phi) \cdot A_\varepsilon(y)(\xi + \nabla \phi) dy, \phi \in H_0^1(D \cap B(x,\rho)) \right\} \quad (1.10)$$

for all $\xi \in \mathbb{R}^d$, where $B(x,\rho)$ is the ball centered at x and of radius ρ . An approximation of u_{hom} is then given by the weak solution $u_{\rho,\varepsilon}$ in $H_0^1(D)$ of

$$-\nabla \cdot A_{\rho,\varepsilon}(x)\nabla u_{\rho,\varepsilon}(x) = f(x) \quad \text{in } D. \quad (1.11)$$

In particular (see [19] for the linear case, and [15] for more general results), we have

$$\lim_{\rho \downarrow 0} \lim_{\varepsilon \downarrow 0} \|u_{\rho,\varepsilon} - u_{\text{hom}}\|_{H^1(D)} = 0, \quad (1.12)$$

thus ensuring the consistency of the approach in the framework of the homogenization theory (the only assumption is that A_ε H-converges towards A_{hom}). The main difference between (1.11) and (1.1) is that $A_{\rho,\varepsilon}$ does not oscillate at scale ε , which is a big advantage for the numerical practice. In order to make use of the strategy of [17] recalled above for the computation of homogenized coefficients, we change variables $x \rightsquigarrow \varepsilon^{-1}x$ to make the oscillations of A_ε be of order 1. Equation (1.10) now turns into

$$\xi \cdot A_{\rho,\varepsilon}(x)\xi := \inf \left\{ \int_{\frac{D-x}{\varepsilon} \cap B(0,\frac{\rho}{\varepsilon})} (\xi + \nabla \phi) \cdot A(y)(\xi + \nabla \phi) dy, \phi \in H_0^1\left(\frac{D-x}{\varepsilon} \cap B\left(0,\frac{\rho}{\varepsilon}\right)\right) \right\}, \quad (1.13)$$

where $A(y) := A_\varepsilon(x + \varepsilon y)$. It is then clear that (1.13) can be seen as the approximation (1.5) of some homogenized coefficients A_{hom} . In particular, if A is periodic, the error $|A_{\rho,\varepsilon} - A_{\text{hom}}|$ scales as ε/ρ . To reduce this resonance error, techniques such as oversampling have been introduced (see e.g. [12], [9]). In particular, if we denote by $\phi_{\rho,\varepsilon} \in H_0^1\left(\frac{D-x}{\varepsilon} \cap B\left(0,\frac{\rho}{\varepsilon}\right)\right)$ the minimizer in (1.13), a possibility is to take the following variant of (1.13):

$$\xi \cdot A_{\rho,\varepsilon}(x)\xi := \int_{\frac{D-x}{\varepsilon} \cap B(0,\frac{\rho}{\varepsilon})} (\xi + \nabla \phi_{\rho,\varepsilon}) \cdot A(y)(\xi + \nabla \phi_{\rho,\varepsilon}) \mu_{\rho,\varepsilon}(y) dy, \quad (1.14)$$

where $\mu_{\rho,\varepsilon}$ is a suitable averaging function with support compactly included in $B\left(0,\frac{\rho}{\varepsilon}\right)$. Although this method does not change the scaling ε/ρ of the error for the approximation of homogenized coefficients in the periodic setting, it may reduce the prefactor of the error (the multiplicative constant in front of ε/ρ). It also improves the approximation of the corrector itself, see [16, Tables 3.1 & 3.2] for elementary numerical examples. In general however, it is not clear under which conditions on A_ε and $\mu_{\rho,\varepsilon}$ the coefficients $A_{\rho,\varepsilon}$ defined in (1.14) are uniformly elliptic. In order to circumvent this difficulty, reduce the resonance error, and obtain convergence rates valid for any coefficients A_ε , Henning and Peterseim introduced in [24] an original oversampling approach which amounts to solving a problem of type (1.10) with an additional orthogonality constraint on a domain of size $\rho |\ln \rho|$ instead of ρ . One expects this method to be optimal in general in the sense that for some coefficients A_ε , the size $\rho |\ln \rho|$ of the “oversampling region” cannot

be reduced, and it is also not clear in the analysis of [24] whether the size can be reduced for specific coefficients which enjoy homogenization properties. A similar achievement is obtained in [31] by Owhadi, Zhang and Berlyand, and the same remark applies on the oversampling size. Progress in that direction is related as much to homogenization as to numerical analysis of linear elliptic PDEs. The aim of this article is to introduce a numerical homogenization method for the *restricted class of locally stationary ergodic coefficients* with the following properties

- (1) the approximation of the homogenized coefficients is uniformly elliptic;
- (2) the resonance error can be quantified and is reduced;
- (3) the oversampling size remains of order ρ .

We conclude this introduction by a quick presentation of our method. We replace (1.14) by

$$\xi \cdot A_{T,k,\rho,\varepsilon}(x)\xi := \int_{\frac{D-x}{\varepsilon} \cap B(0, \frac{\rho}{\varepsilon})} (\xi + \nabla \phi_{T,k,\rho,\varepsilon} - \langle \nabla \phi_{T,k,\rho,\varepsilon} \rangle_{x,\mu_{\rho,\varepsilon}}) \cdot A(y) (\xi + \nabla \phi_{T,k,\rho,\varepsilon} - \langle \nabla \phi_{T,k,\rho,\varepsilon} \rangle_{x,\mu_{\rho,\varepsilon}}) \mu_{\rho,\varepsilon}(y) dy, \quad (1.15)$$

where $\mu_{\rho,\varepsilon}$ is a suitable averaging function, $T > 0$ is a parameter, $k \in \mathbb{N}$, $\phi_{T,k,\rho,\varepsilon}$ are Richardson extrapolations (with respect to T) of the unique weak solution in $\phi_{T,1,\rho,\varepsilon} \in H_0^1\left(\frac{D-x}{\varepsilon} \cap B(0, \frac{\rho}{\varepsilon})\right)$ of the regularized cell problem

$$T^{-1} \phi_{T,1,\rho,\varepsilon} - \nabla \cdot A(\xi + \nabla \phi_{T,1,\rho,\varepsilon}) = 0 \quad \text{in } \frac{D-x}{\varepsilon} \cap B\left(0, \frac{\rho}{\varepsilon}\right),$$

and for all $\psi \in L^2\left(\frac{D-x}{\varepsilon} \cap B\left(0, \frac{\rho}{\varepsilon}\right)\right)$,

$$\langle \psi \rangle_{x,\mu_{\rho,\varepsilon}} := \frac{1}{\int_{\frac{D-x}{\varepsilon} \cap B(0, \frac{\rho}{\varepsilon})} \mu_{\rho,\varepsilon}(y) dy} \int_{\frac{D-x}{\varepsilon} \cap B(0, \frac{\rho}{\varepsilon})} \psi(y) \mu_{\rho,\varepsilon}(y) dy.$$

Then, $A_{T,k,\rho,\varepsilon}$ is uniformly elliptic, and we may approximate u_{hom} by the weak solution $u_{T,k,\rho,\varepsilon}$ in $H_0^1(D)$ of

$$-\nabla \cdot A_{T,k,\rho,\varepsilon} \nabla u_{T,k,\rho,\varepsilon} = f. \quad (1.16)$$

From the analysis point of view, the first natural question concerns consistency: Under which assumptions on A_ε and in which sense does the convergence

$$\lim_{T \uparrow \infty, \rho \downarrow 0} \lim_{\varepsilon \downarrow 0} \|u_{T,k,\rho,\varepsilon} - u_{\text{hom}}\| = 0 \quad (1.17)$$

hold? Can we obtain convergence rates in function of T , k , ρ , ε ? A second issue is the approximation of ∇u_ε knowing $u_{T,k,\rho,\varepsilon}$ and the quantification of the error, that is a numerical corrector (in the spirit of [15, Theorem 2], see also [18] in the linear case) to approximate the fluctuations of the original solution u_ε around the mean field approximation $u_{T,k,\rho,\varepsilon}$. And the last issue is the discretization of this approach and its relation with standard numerical homogenization methods.

The article is organized as follows. We recall in Section 2 standard results on stochastic homogenization of self-adjoint operators and spectral theory, and introduce a method based on regularization and extrapolation to approximate homogenized coefficients and correctors. We then use spectral theory to prove the consistency of the approximations in the stationary ergodic setting, and obtain in addition optimal quantitative estimates in the case of periodic coefficients and in the case of random Poisson inclusions. In Section 3

we devise computable proxies for these approximations, prove sharp estimates on the approximation error, and display the results of numerical tests in dimension 2 on both symmetric and nonsymmetric periodic and quasiperiodic coefficients. The numerical tests confirm the interest of the method and the sharpness of the analysis. In Section 4, we finally show how the regularization and extrapolation method can be combined with standard numerical homogenization methods, and prove the asymptotic consistency in the locally stationary ergodic symmetric case. In the case of periodic coefficients we provide with a quantitative error analysis based on Section 2 and Section 3, which shows that the resonance error is drastically reduced.

We make use of the following notation:

- $d \geq 2$ is the dimension;
- For all D open subset of \mathbb{R}^d , $|D|$ denotes its d -dimensional Lebesgue measure, \int_D the integral on D , and \int_D is a notation for $\frac{1}{|D|} \int_D$;
- $Q := (-1, 1)^d$, $Q_R := (-R, R)^d$, $B_R(x) := \{y \in \mathbb{R}^d : |x - y| < R\}$, $B_R := B_R(0)$ for all $x \in \mathbb{R}^d$ and $R \in \mathbb{R}^+$;
- $H_{\#}^1(Q)$ denotes the closure in $H^1(Q)$ of smooth Q -periodic functions with zero mean;
- \lesssim and \gtrsim stand for \leq and \geq up to a multiplicative constant which only depends on the dimension d and the constants α, β (see Definition 1 below) if not otherwise stated;
- When both \lesssim and \gtrsim hold, we simply write \sim ;
- we use \gg instead of \gtrsim when the multiplicative constant is (much) larger than 1.

2. HOMOGENIZATION, SPECTRAL THEORY, AND RICHARDSON EXTRAPOLATION

2.1. Stochastic homogenization. We first recall some standard qualitative results, which will be used in the core of the article and whose proofs can be found in [32].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space (we denote by $\langle \cdot \rangle$ the associated expectation). We shall say that the family of mappings $(\theta_z)_{z \in \mathbb{R}^d}$ from Ω to Ω is a strongly continuous measure-preserving ergodic translation group if:

- $(\theta_z)_{z \in \mathbb{R}^d}$ has the group property: $\theta_0 = \text{Id}$ (the identity mapping), and for all $x, y \in \mathbb{R}^d$, $\theta_{x+y} = \theta_x \circ \theta_y$;
- $(\theta_z)_{z \in \mathbb{R}^d}$ preserves the measure: for all $x \in \mathbb{R}^d$, and every measurable set $F \in \mathcal{F}$, $\theta_x F$ is measurable and $\mathbb{P}(\theta_x F) = \mathbb{P}(F)$;
- $(\theta_z)_{z \in \mathbb{R}^d}$ is strongly continuous: for any measurable function f on Ω , the function $(\omega, x) \mapsto f(\theta_x \omega)$ defined on $\Omega \times \mathbb{R}^d$ is measurable (with the Lebesgue measure on \mathbb{R}^d);
- $(\theta_z)_{z \in \mathbb{R}^d}$ is ergodic: for all $F \in \mathcal{F}$, if for all $x \in \mathbb{R}^d$, $\theta_x F \subset F$, then $\mathbb{P}(F) \in \{0, 1\}$.

Definition 1. For all $d \geq 2$, $\beta \geq \alpha > 0$, we define $\mathcal{M}_{\alpha\beta}$ as the set of $d \times d$ real matrices such that for all $\xi \in \mathbb{R}^d$,

$$\xi \cdot A\xi \geq \alpha|\xi|^2, \quad |A\xi| \leq \beta|\xi|,$$

and $\mathcal{M}_{\alpha\beta}^{\text{sym}}$ the subset of symmetric matrices of $\mathcal{M}_{\alpha\beta}$.

Let $0 < \alpha \leq \beta < \infty$, and let $A \in L^\infty(\Omega, \mathcal{M}_{\alpha\beta})$. We define the stationary extension of A (still denoted by A) on $\mathbb{R}^d \times \Omega$ as follows:

$$A : (x, \omega) \mapsto A(x, \omega) := A(\theta_x \omega).$$

Homogenization theory ensures that the solution operator associated with $-\nabla \cdot A(\cdot/\varepsilon, \omega) \nabla$ converges as $\varepsilon > 0$ vanishes to the solution operator of $-\nabla \cdot A_{\text{hom}} \nabla$ for \mathbb{P} -almost every ω , where A_{hom} is a deterministic elliptic matrix characterized as follows. For all $\xi, \zeta \in \mathbb{R}^d$, and \mathbb{P} -almost every ω ,

$$\begin{aligned} \xi \cdot A_{\text{hom}} \zeta &= \lim_{R \uparrow \infty} \int_{Q_R} (\xi + \nabla \phi^\xi(x, \omega)) \cdot A(x, \omega) (\zeta + \nabla \phi^\zeta(x, \omega)) dx \\ &= \left\langle (\xi + \nabla \phi^\xi) \cdot A(\zeta + \nabla \phi^\zeta) \right\rangle, \end{aligned} \quad (2.1)$$

where $\phi^\xi : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}$ is Borel measurable, is such that $\phi^\xi(0, \cdot) = 0$ almost surely, $\nabla \phi^\xi$ is stationary, $\langle \nabla \phi^\xi \rangle = 0$, and $\phi^\xi(\cdot, \omega) \in H_{\text{loc}}^1(\mathbb{R}^d)$ is almost surely a distributional solution of the corrector equation

$$-\nabla \cdot A(x, \omega) (\xi + \nabla \phi^\xi(x, \omega)) = 0 \text{ in } \mathbb{R}^d, \quad (2.2)$$

and likewise for ϕ^ζ .

The proof of existence and uniqueness of these correctors is obtained by regularization, and we consider for all $T > 0$ the stationary solution ϕ_T^ξ with zero expectation of the equation

$$T^{-1} \phi_T^\xi(x, \omega) - \nabla \cdot A(x, \omega) (\xi + \nabla \phi_T^\xi(x, \omega)) = 0 \text{ in } \mathbb{R}^d. \quad (2.3)$$

Indeed, as proved in [23, Lemma 2.4] in the case of symmetric coefficients (the proof holds in the case of nonsymmetric coefficients as well), for all $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$, the regularized corrector equation admits a unique solution ϕ_T^ξ in the class of functions $v \in H_{\text{loc}}^1(\mathbb{R}^d)$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{B_1(x)} (v^2(x) + |\nabla v(x)|^2) dx < \infty. \quad (2.4)$$

Furthermore, the solution ϕ_T^ξ satisfies the uniform a priori estimate

$$\sup_{x \in \mathbb{R}^d} \int_{B_{\sqrt{T}}(x)} (T^{-1} (\phi_T^\xi(x))^2 + |\nabla \phi_T^\xi(x)|^2) dx \lesssim \sqrt{T}^d. \quad (2.5)$$

Note that the existence of regularized correctors does not require any assumption on the coefficients A besides uniform boundedness from below and above. In the case when A is a stationary field, the associated random field ϕ_T^ξ of solutions is stationary, and in addition the equation has an equivalent form in the probability space, to which we can apply the Lax-Milgram theorem. This formulation requires the definition of a differential calculus in the probability space: the differential operator ∇_i (for $i \in \{1, \dots, d\}$) has a counterpart in probability, denoted by D_i and defined by

$$D_i f(\omega) = \lim_{h \downarrow 0} \frac{f(\theta_{h e_i} \omega) - f(\omega)}{h}.$$

These are the infinitesimal generators of the d one-parameter strongly continuous unitary groups on $L^2(\Omega)$ defined by the translations in each of the d directions. These operators

commute and are closed and densely defined on $L^2(\Omega)$. We denote by $\mathcal{H}(\Omega)$ the domain of $D = (D_1, \dots, D_d)$. This subset of $L^2(\Omega)$ is a Hilbert space for the norm

$$\|f\|_{\mathcal{H}}^2 = \langle |Df|^2 \rangle + \langle f^2 \rangle.$$

Since the groups are unitary, the operators are skew-adjoint so that we have an integration by parts formula: for all $f, g \in \mathcal{H}(\Omega)$

$$\langle f D_i g \rangle = - \langle g D_i f \rangle.$$

The equivalent form of the regularized corrector equation for stationary coefficients is as follows:

$$T^{-1} \phi_T^\xi(0, \cdot) - D \cdot A(0)(\xi + D \phi_T^\xi(0, \cdot)) = 0, \quad (2.6)$$

which admits a unique weak solution $\phi_T^\xi(0, \cdot) \in \mathcal{H}(\Omega)$, that is such that for all $\psi \in \mathcal{H}(\Omega)$,

$$\left\langle T^{-1} \phi_T^\xi(0, \cdot) \psi + D \psi \cdot A(0)(\xi + D \phi_T^\xi(0, \cdot)) \right\rangle = 0. \quad (2.7)$$

One may prove using the integration by parts formula that $D \phi_T^\xi(0, \cdot)$ is bounded in $L^2(\Omega)$ and converges weakly in $L^2(\Omega)$ to some potential field $\Phi^\xi \in L^2(\Omega)$. Using then the spectral representation of the translation group we may prove uniqueness of the corrector ϕ^ξ (which is such that $\nabla \phi^\xi(x, \omega) = \Phi^\xi(\theta_x \omega)$ under the ergodicity assumption), see [32]. Note that whereas we have $\nabla \phi_T^\xi(x, \omega) = D \phi_T^\xi(0, \theta_x \omega)$ for all $T > 0$, ϕ^ξ is not a stationary field and $D \phi^\xi(0, \cdot)$ is not a well-defined quantity a priori (except in the periodic setting), only $\nabla \phi^\xi$ is.

We recall that the case of periodic and quasi-periodic coefficients can be recast in this stochastic framework up to randomizing the origin of the periodic cell. In particular, for Q -periodic coefficients, we then have $\mathcal{H}(\Omega) = H_{\#}^1(Q)$, and the expectation is replaced by the spatial average on Q . We conclude by an example of random coefficients we shall more specifically consider in this article: Random Poisson inclusions. By the ‘‘Poisson ensemble’’ we understand the following probability measure: Let the configuration of points $\mathcal{P} := \{x_n\}_{n \in \mathbb{N}}$ on \mathbb{R}^d be distributed according to the Poisson point process with density one. This means the following

- For any two disjoint (Lebesgue measurable) subsets D and D' of \mathbb{R}^d we have that the configuration of points in D and the configuration of points in D' are independent. In other words, if X is a function of \mathcal{P} that depends on \mathcal{P} only through $\mathcal{P}|_D$ and X' is a function of \mathcal{P} that depends on \mathcal{P} only through $\mathcal{P}|_{D'}$ we have

$$\langle X X' \rangle_0 = \langle X \rangle_0 \langle X' \rangle_0, \quad (2.8)$$

where $\langle \cdot \rangle_0$ denotes the expectation w. r. t. the Poisson point process.

- For any (Lebesgue measurable) bounded subset D of \mathbb{R}^d , the number of points in D is Poisson distributed; the expected number is given by the Lebesgue measure of D .

With any realization $\mathcal{P} = \{x_n\}_{n \in \mathbb{N}}$ of the Poisson point process, we associate the coefficient field A via

$$A(x) = \left\{ \begin{array}{ll} \alpha & \text{if } x \in \bigcup_{n=1}^{\infty} B_{\frac{1}{2}}(x_n) \\ \beta & \text{else} \end{array} \right\} \text{Id.}$$

This defines a probability measure $\langle \cdot \rangle$ on Ω by ‘‘push-forward’’ of $\langle \cdot \rangle_0$.

2.2. Approximation of homogenized coefficients and correctors. The starting point in [17] is the observation that the solution $\phi_T^\xi \in H_{\text{loc}}^1(\mathbb{R}^d)$ of (2.3) can be approximated on a bounded domain Q_L by the weak solution $\phi_{T,R}^\xi \in H_0^1(Q_R)$ of

$$T^{-1}\phi_{T,R}^\xi - \nabla \cdot A(\xi + \nabla\phi_{T,R}^\xi) = 0 \text{ in } Q_R \quad (2.9)$$

up to an error which is of infinite order measured in units of $\frac{R-L}{\sqrt{T}}$. This is in contrast to the corrector ϕ^ξ for which the approximation using homogeneous Dirichlet boundary conditions yields a surface term of order $\frac{1}{R}$. Since we have easy access to a proxy for ϕ_T^ξ , a natural question is how to approximate the gradient of the corrector $\nabla\phi^\xi$ and the homogenized coefficients A_{hom} using ϕ_T^ξ instead of ϕ^ξ . In [17], the following approximations are considered in the case of symmetric coefficients A :

$$\nabla\phi^\xi \rightsquigarrow \nabla\phi_T^\xi, \quad \xi \cdot A_{\text{hom}}\xi \rightsquigarrow \xi \cdot A_T\xi := \left\langle (\xi + \nabla\phi_T^\xi) \cdot A(\xi + \nabla\phi_T^\xi) \right\rangle.$$

In the case of symmetric periodic coefficients, it is shown that $\left\langle |\nabla\phi^\xi - \nabla\phi_T^\xi|^2 \right\rangle \sim T^{-2}$ and $|A_T - A_{\text{hom}}| \sim T^{-2}$. An explicit family of higher-order approximations $\{\tilde{A}_{T,k}\}_{k \in \mathbb{N}}$ has then been introduced in [20] based on spectral theory which satisfies for symmetric periodic coefficients $|\tilde{A}_{T,k} - A_{\text{hom}}| \sim T^{-2k}$. More precisely, the first three approximations of A_{hom} are given by:

$$\begin{aligned} \xi \cdot \tilde{A}_{T,1}\xi &= \left\langle (\xi + \nabla\phi_T^\xi) \cdot A(\xi + \nabla\phi_T^\xi) \right\rangle, \\ \xi \cdot \tilde{A}_{T,2}\xi &= \left\langle (\xi + \nabla\phi_T^\xi) \cdot A(\xi + \nabla\phi_T^\xi) \right\rangle - 3T^{-1} \left\langle (\phi_T^\xi)^2 \right\rangle - 2T^{-1} \left\langle (\phi_{T/2}^\xi)^2 \right\rangle \\ &\quad + 5T^{-1} \left\langle \phi_T^\xi \phi_{T/2}^\xi \right\rangle, \\ \xi \cdot \tilde{A}_{T,3}\xi &= \left\langle (\xi + \nabla\phi_T^\xi) \cdot A(\xi + \nabla\phi_T^\xi) \right\rangle - \frac{55}{9}T^{-1} \left\langle (\phi_T^\xi)^2 \right\rangle - 8T^{-1} \left\langle (\phi_{T/2}^\xi)^2 \right\rangle \\ &\quad - \frac{4}{9}T^{-1} \left\langle (\phi_{T/4}^\xi)^2 \right\rangle + \frac{41}{3}T^{-1} \left\langle \phi_T^\xi \phi_{T/2}^\xi \right\rangle - \frac{22}{9}T^{-1} \left\langle \phi_T^\xi \phi_{T/4}^\xi \right\rangle \\ &\quad + \frac{10}{3}T^{-1} \left\langle \phi_{T/2}^\xi \phi_{T/4}^\xi \right\rangle. \end{aligned}$$

These results have two drawbacks:

- Approximation of homogenized coefficients: It is not clear how to extend the definition of $\tilde{A}_{T,k}$ to nonsymmetric coefficients;
- Approximation of correctors: The approximation of $\nabla\phi^\xi$ by $\nabla\phi_T^\xi$ yields an $L^2(\Omega)$ error which saturates at T^{-1} .

The aim of this section is to propose approximations of $\nabla\phi^\xi$ and A_{hom} that reduce these drawbacks. As noticed in [22], the family $\{\tilde{A}_{T,k}\}_{k \in \mathbb{N}}$ can be seen as extrapolations in spectral variables of A_T . The main idea of this section consists in approximating the corrector ϕ^ξ by Richardson extrapolations of ϕ_T^ξ . In order to treat nonsymmetric coefficients, we also consider the corrector associated with the adjoint operator (that is with A replaced by ${}^T A$), that we abusively call the adjoint corrector.

Definition 2 (Richardson extrapolation of regularized correctors). Let $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$ be some coefficients, and ${}^T A$ be the associated adjoint coefficients. For all $T > 0$, $\xi \in \mathbb{R}^d$ and $\zeta \in \mathbb{R}^d$, we consider the regularized corrector $\phi_T^\xi \in H_{\text{loc}}^1(\mathbb{R}^d)$ associated with A and

direction ξ and the regularized corrector $\overline{\phi}_T^\xi$ associated with ${}^T A$ and direction ζ (that is the unique distributional solutions in $H_{\text{loc}}^1(\mathbb{R}^d)$ of (2.3) satisfying (2.4) with A and ξ , and ${}^T A$ and ζ , respectively). The families $\{\phi_{T,k}^\xi\}_{k \in \mathbb{N}}$ and $\{\overline{\phi}_{T,k}^\zeta\}_{k \in \mathbb{N}}$ of Richardson extrapolations are defined by the following induction: $\phi_{T,1}^\xi := \phi_T^\xi$, $\overline{\phi}_{T,1}^\zeta := \overline{\phi}_T^\zeta$, and for all $k \in \mathbb{N}$,

$$\phi_{T,k+1}^\xi := \frac{1}{2^k - 1} (2^k \phi_{2T,k}^\xi - \phi_{T,k}^\xi), \quad \overline{\phi}_{T,k+1}^\zeta := \frac{1}{2^k - 1} (2^k \overline{\phi}_{2T,k}^\zeta - \overline{\phi}_{T,k}^\zeta). \quad (2.10)$$

In the case when A has the ‘‘averaging property’’ (this holds for instance in the stationary stochastic case), we may define approximations of the homogenized coefficients associated with the Richardson extrapolations of the regularized correctors:

Definition 3 (Approximation of homogenized coefficients). Let $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$ be some coefficients, and ${}^T A$ be the associated adjoint coefficients. For all $T > 0$, $\xi \in \mathbb{R}^d$ and $\zeta \in \mathbb{R}^d$, we let $\{\phi_{T,k}^\xi\}_{k \in \mathbb{N}}$ and $\{\overline{\phi}_{T,k}^\zeta\}_{k \in \mathbb{N}}$ be the families of Richardson extrapolations of Definition 2, and we define the family $\{A_{T,k}\}_{k \in \mathbb{N}}$ of associated approximations of the homogenized coefficients by

$$\zeta \cdot A_{T,k} \xi := \mathcal{M}\{(\zeta + \nabla \overline{\phi}_{T,k}^\zeta) \cdot A(\xi + \nabla \phi_{T,k}^\xi)\},$$

whenever the limit in (1.4) exists.

In the case of stationary ergodic coefficients A , Definition 3 makes sense and we have

$$\zeta \cdot A_{T,k} \xi = \left\langle (\zeta + \nabla \overline{\phi}_{T,k}^\zeta) \cdot A(\xi + \nabla \phi_{T,k}^\xi) \right\rangle.$$

In the following paragraph we show, using the spectral theory introduced in [32], that the approximations $\nabla \phi_{T,k}^\xi$ and $A_{T,k}$ are consistent with $\nabla \phi^\xi$ and A_{hom} in the case of stationary ergodic symmetric coefficients, in the limit $T \uparrow +\infty$.

2.3. Asymptotic consistency by spectral theory. Spectral theory has played an important role in the theory of stochastic homogenization of self-adjoint operators since the seminal contribution by Papanicolaou and Varadhan [32]. In this paragraph, we assume that A is a stationary ergodic field of symmetric coefficients, and shall prove:

Theorem 1. *Let $A \in L^\infty(\Omega, \mathcal{M}_{\alpha\beta}^{\text{sym}})$ be stationary ergodic symmetric coefficients, ϕ^ξ be the associated corrector in some direction $\xi \in \mathbb{R}^d$, $|\xi| = 1$, and A_{hom} be the associated homogenized coefficients. For all $k \in \mathbb{N}$ and $T > 0$, let $\phi_{T,k}^\xi$ and $A_{T,k}$ be as in Definitions 2 and 3. Then,*

$$\lim_{T \uparrow \infty} \left\langle |\nabla \phi_{T,k}^\xi - \nabla \phi^\xi|^2 \right\rangle = 0, \quad \lim_{T \uparrow \infty} |A_{T,k} - A_{\text{hom}}| = 0.$$

To establish these results, we shall indeed prove a more precise statement in the form of a spectral representation of the errors. This will allow us in the following paragraph to turn this theorem quantitative in some specific situations. To this end, let us recall the spectral theory introduced in [32].

Let $\mathcal{L} = -D \cdot AD$ be the operator defined on $\mathcal{H}(\Omega)$ as a quadratic form. We denote by \mathcal{L} its Friedrichs extension on $L^2(\Omega)$. This operator is a nonnegative self-adjoint operator, so that by the spectral theorem it admits a spectral resolution

$$\mathcal{L} = \int_{\mathbb{R}^+} \lambda G(d\lambda).$$

Proposition 1. *In the context of Theorem 1, we have*

$$\left\langle |\nabla\phi_{T,k}^\xi - \nabla\phi|^2 \right\rangle \sim \int_{\mathbb{R}^+} \frac{T^{-2k}}{\lambda(T^{-1} + \lambda)^{2k}} de_{\mathfrak{d}}(\lambda), \quad (2.11)$$

$$|A_{T,k} - A_{\text{hom}}| \sim \int_{\mathbb{R}^+} \frac{T^{-2k}}{\lambda(T^{-1} + \lambda)^{2k}} de_{\mathfrak{d}}(\lambda), \quad (2.12)$$

where $e_{\mathfrak{d}}$ denotes the projection of the spectral resolution G of $\mathcal{L} = -D \cdot AD$ onto the local drift defined by $\mathfrak{d} := D \cdot A\xi \in \mathcal{H}(\Omega)'$.

Combined with the integrability property $\int_{\mathbb{R}^+} \frac{1}{\lambda} de_{\mathfrak{d}}(\lambda) < \infty$ proved in [32], Proposition 1 yields Theorem 1 by the Lebesgue dominated convergence theorem.

Proof of Proposition 1. We split the proof into three steps.

Step 1. Reduction to an estimate on the spectral density.

Let $\tilde{T} \geq T > 0$. By ellipticity and the spectral theorem,

$$\begin{aligned} \left\langle |\nabla\phi_T^\xi - \nabla\phi_{\tilde{T}}^\xi|^2 \right\rangle &\lesssim \left\langle (\nabla\phi_T^\xi - \nabla\phi_{\tilde{T}}^\xi) \cdot A(\nabla\phi_T^\xi - \nabla\phi_{\tilde{T}}^\xi) \right\rangle \\ &= \int_{\mathbb{R}^+} \lambda \left(\frac{1}{T^{-1} + \lambda} - \frac{1}{\tilde{T}^{-1} + \lambda} \right)^2 de_{\mathfrak{d}}(\lambda) \\ &= \int_{\mathbb{R}^+} \frac{(T^{-1} - \tilde{T}^{-1})^2 \lambda}{(T^{-1} + \lambda)^2 (\tilde{T}^{-1} + \lambda)^2} de_{\mathfrak{d}}(\lambda). \end{aligned}$$

Since, as proved in [32],

$$\int_{\mathbb{R}^+} \frac{1}{\lambda} de_{\mathfrak{d}}(\lambda) < \infty, \quad (2.13)$$

this implies by the Lebesgue dominated convergence theorem that $\nabla\phi_T^\xi$ is a Cauchy sequence in $L^2(\Omega)$, which therefore converges to $\nabla\phi^\xi$ strongly in $L^2(\Omega)$ (and not only weakly). Let now $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function such that $\psi(\lambda) \lesssim \frac{1}{\lambda}$ and set $\Psi := \psi(\mathcal{L})\mathfrak{d}$. Then, by the spectral theorem, on the one hand,

$$\left\langle |\nabla\Psi|^2 \right\rangle \lesssim \left\langle \nabla\Psi \cdot A\nabla\Psi \right\rangle = \int_{\mathbb{R}^+} \lambda \psi(\lambda)^2 de_{\mathfrak{d}}(\lambda) < \infty,$$

and on the other hand,

$$\left\langle (\nabla\Psi - \nabla\phi_{\tilde{T}}^\xi) \cdot A(\nabla\Psi - \nabla\phi_{\tilde{T}}^\xi) \right\rangle = \int_{\mathbb{R}^+} \lambda \left(\psi(\lambda) - \frac{1}{\tilde{T}^{-1} + \lambda} \right)^2 de_{\mathfrak{d}}(\lambda).$$

Combined with the convergence of $\nabla\phi_{\tilde{T}}^\xi$ to $\nabla\phi^\xi$ in $L^2(\Omega)$ as $\tilde{T} \uparrow \infty$ and the Lebesgue dominated convergence theorem, this turns into:

$$\left\langle (\nabla\Psi - \nabla\phi^\xi) \cdot A(\nabla\Psi - \nabla\phi^\xi) \right\rangle = \int_{\mathbb{R}^+} \lambda \left(\psi(\lambda) - \frac{1}{\lambda} \right)^2 de_{\mathfrak{d}}(\lambda). \quad (2.14)$$

For all $T > 0$, set

$$\psi_{T,1} : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \lambda \mapsto \psi_{T,1}(\lambda) = \frac{1}{T^{-1} + \lambda},$$

and define by induction for all $k \in \mathbb{N}$,

$$\psi_{T,k+1} : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \lambda \mapsto \psi_{T,k+1}(\lambda) = \frac{1}{2^k - 1} (2^k \psi_{2T,k}(\lambda) - \psi_{T,k}(\lambda)).$$

These functions are continuous and satisfy $\psi_{T,k}(\lambda) \lesssim \frac{1}{\lambda}$ (where the multiplicative constant depends on T and k). By definition of the Richardson extrapolation of the regularized corrector (cf. Definition 2), we also have $\phi_{T,k}^\xi = \psi_{T,k}(\mathcal{L})\mathfrak{d}$ for all $T > 0$ and $k \in \mathbb{N}$. Finally, by ellipticity,

$$\left\langle |\nabla \phi_{T,k}^\xi - \nabla \phi^\xi|^2 \right\rangle \lesssim \left\langle (\nabla \phi_{T,k}^\xi - \nabla \phi^\xi) \cdot A(\nabla \phi_{T,k}^\xi - \nabla \phi^\xi) \right\rangle \lesssim \left\langle |\nabla \phi_{T,k}^\xi - \nabla \phi^\xi|^2 \right\rangle,$$

so that it enough to estimate the Dirichlet form to prove (2.11). We shall prove that for all $T > 0$, $k \in \mathbb{N}$, and $\lambda > 0$,

$$\frac{1}{\lambda} - \psi_{T,k}(\lambda) = 2^{-\frac{1}{2}k(k-1)}T^{-k} \frac{1}{\lambda \prod_{i=0}^{k-1} ((2^i T)^{-1} + \lambda)}. \quad (2.15)$$

The desired estimate (2.11) will then follow from the combination of (2.14), (2.15), and (2.13) — the latter to prove the equivalence and not only the upper bound.

Step 2. Proof of (2.15).

We proceed by induction. For $k = 1$, (2.15) reduces to

$$\frac{1}{\lambda} - \psi_{T,1}(\lambda) = \frac{1}{T^{-1} + \lambda} - \frac{1}{\lambda} = T^{-1} \frac{1}{\lambda(T^{-1} + \lambda)}.$$

Assume that (2.15) holds at step $k \in \mathbb{N}$. Then,

$$\begin{aligned} \frac{1}{\lambda} - \psi_{2T,k}(\lambda) &= 2^{-\frac{1}{2}k(k-1)}(2T)^{-k} \frac{1}{\lambda \prod_{i=0}^{k-1} ((2^{i+1}T)^{-1} + \lambda)} \\ &= 2^{-k} \frac{T^{-1} + \lambda}{(2^k T)^{-1} + \lambda} 2^{-\frac{1}{2}k(k-1)}T^{-k} \frac{1}{\lambda \prod_{i=0}^{k-1} ((2^i T)^{-1} + \lambda)} \\ &= 2^{-k} \frac{T^{-1} + \lambda}{(2^k T)^{-1} + \lambda} \left(\frac{1}{\lambda} - \psi_{T,k}(\lambda) \right) \end{aligned} \quad (2.16)$$

Hence the induction rule for $\psi_{T,k+1}$ and (2.15) at step $k \in \mathbb{N}$ yield

$$\begin{aligned} \frac{1}{\lambda} - \psi_{T,k+1}(\lambda) &= \frac{2^k}{2^k - 1} \left(\frac{1}{\lambda} - \psi_{2T,k+1}(\lambda) \right) - \frac{1}{2^k - 1} \left(\frac{1}{\lambda} - \psi_{T,k+1}(\lambda) \right) \\ &\stackrel{(2.16)}{=} \frac{1}{2^k - 1} \left(\frac{1}{\lambda} - \psi_{T,k}(\lambda) \right) \left(\frac{T^{-1} + \lambda}{(2^k T)^{-1} + \lambda} - 1 \right) \\ &= 2^{-k} T^{-1} \left(\frac{1}{\lambda} - \psi_{T,k}(\lambda) \right) \frac{1}{(2^k T)^{-1} + \lambda} \\ &\stackrel{(2.15) \text{ at step } k}{=} 2^{-\frac{1}{2}k(k+1)}T^{-(k+1)} \frac{1}{\lambda \prod_{i=0}^k ((2^i T)^{-1} + \lambda)}, \end{aligned}$$

as desired.

Step 3. Proof of (2.12).

Recall the weak form of the corrector equation for ϕ^ξ : For all $\Psi \in \mathcal{H}(\Omega)$,

$$\left\langle \mathbf{D}\Psi \cdot A(\xi + \nabla \phi^\xi) \right\rangle = 0, \quad (2.17)$$

and since $\nabla \phi_T^\xi(0, \cdot) \rightarrow \nabla \phi^\xi(0, \cdot)$ strongly in $L^2(\Omega, \mathbb{R}^d)$, we also have

$$\left\langle \nabla \phi^\xi \cdot A(\xi + \nabla \phi^\xi) \right\rangle = 0. \quad (2.18)$$

By symmetry of A and (2.17) & (2.18) we then have for all $\Psi \in \mathcal{H}(\Omega)$,

$$\begin{aligned}
& \langle (\xi + D\Psi) \cdot A(\xi + D\Psi) \rangle - \langle (\xi + \nabla\phi^\xi) \cdot A(\xi + \nabla\phi^\xi) \rangle \\
&= \langle (D\Psi - \nabla\phi^\xi) \cdot A(\xi + D\Psi) \rangle + \langle (\xi + \nabla\phi^\xi) \cdot A(D\Psi - \nabla\phi^\xi) \rangle \\
&\stackrel{\text{symmetry}}{=} \langle (D\Psi - \nabla\phi^\xi) \cdot A(\xi + D\Psi) \rangle + \langle (D\Psi - \nabla\phi^\xi) \cdot A(\xi + \nabla\phi^\xi) \rangle \\
&\stackrel{(2.17)\&(2.18)}{=} \langle (\xi + \nabla\phi^\xi) \cdot A(D\Psi - \nabla\phi^\xi) \rangle - \langle (D\Psi - \nabla\phi^\xi) \cdot A(\xi + \nabla\phi^\xi) \rangle \\
&= \langle (D\Psi - \nabla\phi^\xi) \cdot A(D\Psi - \nabla\phi^\xi) \rangle.
\end{aligned}$$

Estimate (2.12) then follows from the definition of $A_{T,k}$ and (2.11) by taking $\Psi = \phi_{T,k}^\xi(0, \cdot)$ in this identity. \square

2.4. Quantitative analysis. Whereas Theorem 1 proves the consistency of the approach by regularization and extrapolation in the case of symmetric stationary ergodic coefficients, Proposition 1 allows one to turn the consistency result quantitative for more specific statistics. We shall consider two representative examples of the ergodic setting: non-decaying correlations in the form of periodic coefficients, and finite correlation-length in the form of coefficients given by random Poisson inclusions.

In the case of periodic coefficients, the Poincaré inequality on the torus shows that \mathcal{L} has a spectral gap, so that there exists $\mu > 0$ such that the non-negative measure $e_\mathfrak{d}$ satisfies $e_\mathfrak{d}((0, \mu)) = 0$, and for all $k \in \mathbb{N}$

$$\int_{\mathbb{R}^+} \frac{1}{\lambda^k} d_\mathfrak{d}(\lambda) = \int_\mu^\infty \frac{1}{\lambda^k} d_\mathfrak{d}(\lambda) \stackrel{(2.13)}{\lesssim} \mu^{-k} + 1 < \infty,$$

which makes the estimates of Proposition 1 explicit in T . In the case of periodic coefficients, one can even by-pass the use of spectral theory, and prove by elementary elliptic PDE theory that the quantitative estimates hold for non-symmetric coefficients as well.

Theorem 2. *Let $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$ be Q -periodic (non-necessarily symmetric) and A_{hom} be the associated homogenized coefficients. For all unit vectors $\xi, \zeta \in \mathbb{R}^d$ we denote by $\phi^\xi \in H_{\#}^1(Q)$ and $\overline{\phi}^\zeta \in H_{\#}^1(Q)$ the associated corrector and adjoint corrector, respectively (recall that these correctors exist as a consequence of the Fredholm alternative on the torus). For all $T > 0$ and $k \in \mathbb{N}$, let $\phi_{T,k}^\xi$ and $\overline{\phi}_{T,k}^\zeta$ be the associated regularized correctors of Definition 2, and $A_{T,k}$ the approximations of the homogenized coefficients of Definition 3. Then, we have*

$$\|\nabla(\phi_{T,k}^\xi - \phi^\xi)\|_{L^2(Q)}, \|\nabla(\overline{\phi}_{T,k}^\zeta - \overline{\phi}^\zeta)\|_{L^2(Q)} \lesssim T^{-k}, \quad (2.19)$$

and

$$|A_{\text{hom}} - A_{T,k}| \lesssim T^{-2k}, \quad (2.20)$$

where the multiplicative constants depend on k , next to α, β , and d .

Proof. We split the proof into two steps.

Step 1. Proof of (2.19).

We only prove the estimate for $\phi_{T,k}^\xi$. We first claim that $\phi_{T,k}^\xi$ satisfies for all $k \in \mathbb{N}$ the equation

$$T^{-1}\phi_{T,k+1}^\xi - \nabla \cdot A(\xi + \nabla\phi_{T,k+1}^\xi) = T^{-1}\phi_{2T,k}^\xi. \quad (2.21)$$

We proceed by induction. For $k = 1$, this follows from a direct calculation. Assume (2.21) holds at step k . Then, by the induction assumption for $\phi_{T,k+1}^\xi$ and $2^{k+1}\phi_{2T,k+1}^\xi$, we have

$$\begin{aligned} T^{-1}\phi_{T,k+1}^\xi - \nabla \cdot A(\xi + \nabla\phi_{T,k+1}^\xi) &= T^{-1}\phi_{2T,k}^\xi, \\ T^{-1}(2^k\phi_{2T,k+1}^\xi) - \nabla \cdot A(2^{k+1}\xi + \nabla 2^{k+1}\phi_{2T,k+1}^\xi) &= 2^k T^{-1}\phi_{4T,k}^\xi, \end{aligned}$$

so that the definition (2.10) of the Richardson extrapolation at step $k + 1$ yields

$$\begin{aligned} T^{-1}\left(\frac{2^k}{2^{k+1}-1}\phi_{2T,k+1}^\xi - \frac{1}{2^{k+1}-1}\phi_{T,k+1}^\xi\right) - \nabla \cdot A(\xi + \nabla\phi_{T,k+2}^\xi) \\ = T^{-1}\left(\frac{2^k}{2^{k+1}-1}\phi_{4T,k}^\xi - \frac{1}{2^{k+1}-1}\phi_{2T,k}^\xi\right), \end{aligned}$$

that is, using the identity $\frac{2^k}{2^{k+1}-1} = \frac{2^{k+1}}{2^{k+1}-1} - \frac{2^{k+1}-2^k}{2^{k+1}-1}$,

$$\begin{aligned} T^{-1}\phi_{T,k+2}^\xi - \nabla \cdot A(\xi + \nabla\phi_{T,k+2}^\xi) \\ = \frac{T^{-1}}{2^{k+1}-1}\left((2^{k+1}-2^k)\phi_{2T,k+1}^\xi + 2^k\phi_{4T,k}^\xi - \phi_{2T,k}^\xi\right) \\ \stackrel{(2.10) \text{ at step } k}{=} \frac{T^{-1}}{2^{k+1}-1}\left((2^{k+1}-2^k)\phi_{2T,k+1}^\xi + (2^k-1)\phi_{2T,k+1}^\xi\right) \\ = T^{-1}\phi_{2T,k+1}^\xi, \end{aligned}$$

as claimed.

From equation (2.21) we deduce that

$$T^{-1}(\phi_{T,k+1}^\xi - \phi^\xi) - \nabla \cdot A\nabla(\phi_{T,k+1}^\xi - \phi^\xi) = T^{-1}(\phi_{2T,k}^\xi - \phi^\xi).$$

Testing this equation with test-function $\phi_{T,k+1}^\xi - \phi^\xi$, integrating by parts and using Poincaré's inequality on $H_{\#}^1(Q)$ then yields

$$\|\nabla(\phi_{T,k+1}^\xi - \phi^\xi)\|_{L^2(Q)} \lesssim T^{-1}\|\nabla(\phi_{2T,k}^\xi - \phi^\xi)\|_{L^2(Q)}.$$

This proves (2.19) by induction, starting with the elementary energy estimate

$$\|\nabla(\phi_{T,1}^\xi - \phi^\xi)\|_{L^2(Q)} \lesssim T^{-1}\|\phi^\xi\|_{L^2(Q)}\|\phi_{T,1}^\xi - \phi^\xi\|_{L^2(Q)},$$

combined itself with Poincaré's inequality.

Step 2. Proof of (2.20).

The proof is similar to Step 3 in the proof of Proposition 1. We recall the weak form of the corrector and adjoint corrector equations: For all $\Psi \in H_{\#}^1(Q)$

$$\int_Q \nabla\Psi \cdot {}^T A(\zeta + \nabla\bar{\phi}^\zeta) = 0, \quad \int_Q \nabla\Psi \cdot A(\xi + \nabla\phi^\xi) = 0. \quad (2.22)$$

For all $\Psi_1, \Psi_2 \in H_{\#}^1(Q)$, we then have using (2.22) once with $\Psi = \Psi_1 - \phi^\xi$ and ${}^T A$ and once with $\Psi = \Psi_2 - \bar{\phi}^\zeta$ and A :

$$\begin{aligned}
& \int_Q (\zeta + \nabla \Psi_2) \cdot A(\xi + \nabla \Psi_1) - \int_Q (\zeta + \nabla \bar{\phi}^\zeta) \cdot A(\xi + \nabla \phi^\xi) \\
&= \int_Q (\nabla \Psi_2 - \nabla \bar{\phi}^\zeta) \cdot A(\xi + \nabla \Psi_1) + \int_Q (\zeta + \nabla \bar{\phi}^\zeta) \cdot A(\nabla \Psi_1 - \nabla \phi^\xi) \\
&= \int_Q (\nabla \Psi_2 - \nabla \bar{\phi}^\zeta) \cdot A(\xi + \nabla \Psi_1) + \int_Q (\nabla \Psi_1 - \nabla \phi^\xi) \cdot {}^T A(\zeta + \nabla \bar{\phi}^\zeta) \\
&\stackrel{(2.22)}{=} \int_Q (\nabla \Psi_2 - \nabla \bar{\phi}^\zeta) \cdot A(\xi + \nabla \Psi_1) - \int_Q (\nabla \Psi_2 - \nabla \bar{\phi}^\zeta) \cdot A(\xi + \nabla \phi^\xi) \\
&= \int_Q (\nabla \Psi_2 - \nabla \bar{\phi}^\zeta) \cdot A(\nabla \Psi_1 - \nabla \phi^\xi).
\end{aligned}$$

Hence (2.20) follows from (2.19) by taking $\Psi_1 = \phi_{T,k}^\xi$ and $\Psi_2 = \bar{\phi}_{T,k}^\zeta$ in the identity above. \square

Let us turn now to the second example and consider symmetric coefficients given by random Poisson inclusions. In [23], Otto and the first author obtained the following bounds on the associated spectral measure. With the notation of Theorem 1, we say that the spectral exponents are *at least* $(\gamma_1, \gamma_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ if for all $0 < \lambda \leq 1$,

$$e_{\partial}([0, \lambda]) \lesssim \lambda^{\gamma_1} (\log^{\gamma_2}(\lambda^{-1}) + 1). \quad (2.23)$$

As a by-product of [23, Theorem 2] (see in particular [23, Step 2, proof of Theorem 2]), we have in the case of Poisson random inclusions:

$$\begin{aligned}
d = 2 : & \quad \gamma_1 = 2, & \quad \gamma_2 > 0, \\
2 < d < 6 : & \quad \gamma_1 = d/2 + 1, & \quad \gamma_2 = 0, \\
d = 6 : & \quad \gamma_1 = 4, & \quad \gamma_2 = 1, \\
d > 6 : & \quad \gamma_1 = 4, & \quad \gamma_2 = 0.
\end{aligned} \quad (2.24)$$

From these estimates of the spectral exponents, one may deduce the following error estimates for the regularization and extrapolation methods in the case of Poisson random inclusions.

Theorem 3. *Let the symmetric coefficients $A \in L^\infty(\Omega, \mathcal{M}_{\alpha\beta}^{\text{sym}})$ be given by the random Poisson inclusions, and let A_{hom} be the associated homogenized coefficients. For all unit vectors $\xi \in \mathbb{R}^d$, let ϕ^ξ be the corrector in direction ξ , and for all $T > 0$ and $k \in \mathbb{N}$, let $\phi_{T,k}^\xi$ be the extrapolation of the regularized corrector of Definition 2, and $A_{T,k}$ be the approximation of the homogenized coefficients of Definition 3. Let $k(d) := \lceil \frac{d}{4} \rceil$ (where $\lceil \cdot \rceil$ denotes here the smallest integer larger or equal to). Then there exists some $q > 0$ such that, for all $d \geq 2$ and for all $k' \geq \min\{k(d), 2\}$, we have*

$$\left\langle |\nabla \phi_{T,k'}^\xi - \nabla \phi^\xi|^2 \right\rangle \lesssim \begin{cases} d = 2 & : T^{-1} \log^q T \\ 2 < d < 6 & : T^{-d/2}, \\ d = 6 & : T^{-3} \log T, \\ d > 6 & : T^{-3}. \end{cases} \quad (2.25)$$

Likewise, for all $k' \geq \min\{k(d), 2\}$

$$|A_{T,k'} - A_{\text{hom}}| \lesssim \begin{cases} d = 2 & : T^{-1} \log^q T \\ 2 < d < 6 & : T^{-d/2}, \\ d = 6 & : T^{-3} \log T, \\ d > 6 & : T^{-3}. \end{cases} \quad (2.26)$$

In the estimates above, the multiplicative constant depends on k' , next to α, β , and d .

Remark 1. In the case of discrete linear elliptic equations on \mathbb{Z}^d with independent and identically distributed coefficients, Neukamm, Otto, and the first author proved in [22] the following *optimal* values of the spectral exponents: $(\frac{d}{2} + 1, 0)$ for all $d \geq 2$. In particular, this can be used to prove the following *optimal* estimates for the regularization and extrapolation method in this discrete setting: For all $k' \geq k(d)$,

$$\langle |\nabla \phi_{T,k'}^\xi - \nabla \phi^\xi|^2 \rangle, |A_{T,k'} - A_{\text{hom}}| \lesssim T^{-d/2}.$$

We believe that these estimates hold as well in the case of continuum equations for coefficients given by the random Poisson inclusions.

Remark 2. In view of Remark 1, Theorem 3 is optimal in terms of scaling for $d = 2, 3, 4, 5, 6$ up to logarithms in dimensions $d = 2$ and $d = 6$. In particular, this implies that for dimensions of interest in practice (say, $d = 2$ and $d = 3$), $k' = 1$ is enough to reach the optimal scaling and Richardson extrapolation does not reduce the error further for random coefficients with finite correlation-length (as opposed to the case of periodic coefficients).

Proof of Theorem 3. Theorem 3 is a direct consequence of Proposition 1 and of [23]. The fundamental theorem of calculus and Fubini's theorem imply that for all $f \in C^1((0, 1])$,

$$\begin{aligned} \int_0^1 f(\lambda) de_\partial(\lambda) &= - \int_{\lambda=0}^1 \int_{\hat{\lambda}=\lambda}^1 f'(\hat{\lambda}) d\hat{\lambda} de_\partial(\lambda) + f(1) \int_{\lambda=0}^1 de_\partial(\lambda) \\ &= - \int_{\hat{\lambda}=0}^1 f'(\hat{\lambda}) e_\partial([0, \hat{\lambda}]) d\hat{\lambda} + f(1) e_\partial([0, 1]). \end{aligned} \quad (2.27)$$

Used with $f(\lambda) = \frac{1}{\lambda(T^{-1} + \lambda)^{2k'}}$ and combined with Proposition 1, this yields for all $k' \in \mathbb{N}$

$$\begin{aligned} \langle |\nabla \phi_{T,k'}^\xi - \nabla \phi|^2 \rangle &\lesssim \int_{\mathbb{R}^+} \frac{T^{-2k'}}{\lambda(T^{-1} + \lambda)^{2k'}} de_\partial(\lambda) \\ &\leq \int_0^1 \frac{T^{-2k'}}{\lambda(T^{-1} + \lambda)^{2k'}} de_\partial(\lambda) + \int_{\mathbb{R}^+} \frac{T^{-2k'}}{\lambda} de_\partial(\lambda) \\ &\stackrel{(2.27) \& (2.13)}{\lesssim} \int_0^1 \frac{T^{-2k'}}{\lambda^2(T^{-1} + \lambda)^{2k'}} e_\partial([0, \lambda]) d\lambda + T^{-2k'}. \end{aligned}$$

We then appeal to [23, Theorem 2] in the form of (2.24), and assume that $2k' + 1 \geq \gamma_1$ (which yields the definition of $k(d)$). According to (2.24), $\gamma_1 \geq 2$, so that we have in that

case

$$\begin{aligned}
\left\langle |\nabla \phi_{T,k'}^\xi - \nabla \phi|^2 \right\rangle &\lesssim T^{-\gamma_1+1} \int_0^1 \frac{1}{\lambda^{\gamma_1}(T^{-1} + \lambda)} e_{\mathfrak{d}}([0, \lambda]) d\lambda + T^{-2k'} \\
&\stackrel{(2.23)\&(2.24)}{\lesssim} T^{-\gamma_1+1} \int_0^1 \frac{\log^{\gamma_2}(\lambda^{-1}) + 1}{T^{-1} + \lambda} d\lambda + T^{-2k'} \\
&\stackrel{\lambda \sim T\lambda}{=} T^{-\gamma_1+1} \int_0^T \frac{(\log(\lambda^{-1}) + \log T)^{\gamma_2} + 1}{1 + \lambda} d\lambda + T^{-2k'} \\
&\lesssim T^{-\gamma_1+1} \left[\log^{\gamma_2} T \int_0^T \frac{1}{1 + \lambda} + \int_0^1 \log^{\gamma_2}(\lambda^{-1}) \right] + T^{-2k'} \\
&\stackrel{(2.13)}{\lesssim} T^{-\gamma_1+1} \log^{\gamma_2} T,
\end{aligned}$$

which yields the claim. \square

3. NUMERICAL APPROXIMATION OF HOMOGENIZED COEFFICIENTS AND CORRECTORS

The aim of this section is to introduce numerical approximations of correctors and homogenized coefficients based on the regularization and extrapolation method introduced in Section 2. In order to make this method of any practical use one has to make the different quantities at stake computable and control the approximation errors: this is the objective of the first subsection. The last two subsections are dedicated to a systematic numerical study of the method, which illustrates both the interest of the approach and the sharpness of the analysis. More precisely, Subsection 3.2 displays results of numerical tests for symmetric periodic and quasi-periodic coefficients, whereas Subsection 3.3 treats non-symmetric periodic and quasi-periodic coefficients. For some numerical tests in a discrete random setting, we refer the reader to [18] and [13].

3.1. From abstract to computable approximations. As recalled in the introduction, the motivation to use the regularization approach is the observation that the solution $\phi_T^\xi \in H_{\text{loc}}^1(\mathbb{R}^d)$ (which exists for any $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$ and is unique in the class of functions that satisfy (2.4)) of

$$T^{-1} \phi_T^\xi - \nabla \cdot A(\xi + \nabla \phi_T^\xi) = 0 \text{ in } \mathbb{R}^d$$

is much easier to approximate on bounded domains than the solution $\phi^\xi \in H_{\text{loc}}^1(\mathbb{R}^d)$ (whose existence is only known to hold under structure assumptions, e. g. almost-sure existence in the stationary ergodic case) of

$$-\nabla \cdot A(\xi + \nabla \phi^\xi) = 0 \text{ in } \mathbb{R}^d.$$

This fact relies on the exponential decay of the Green's function associated with the operator $T^{-1} - \nabla \cdot A \nabla$.

In particular, this observation takes the following general form which holds for *any* coefficient field A .

Theorem 4. *Let $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$, $\xi \in \mathbb{R}^d$ be a unit vector, and ϕ_T^ξ be the associated regularized corrector of Definition 2. For all R , we let $\phi_{T,R}^\xi \in H_0^1(Q_R)$ be the unique weak*

solution of: For all $\psi \in H_0^1(Q_R)$,

$$\int_{Q_R} T^{-1} \psi \phi_{T,R}^\xi + \nabla \psi \cdot A(\xi + \nabla \phi_{T,R}^\xi) = 0. \quad (3.1)$$

Then there exists $c > 0$ depending only on α, β and d such that for all $0 < L \leq R$ with $R \sim R - L \gtrsim \sqrt{T}$, we have

$$\int_{Q_L} |\nabla \phi_{T,R}^\xi - \nabla \phi_T^\xi|^2 \lesssim R^d T \exp\left(-c \frac{R-L}{\sqrt{T}}\right). \quad (3.2)$$

Remark 3. In the case of periodic coefficients, we proved in [17, Theorem 3.1] that (3.2) can be upgraded to

$$\int_{Q_L} |\nabla \phi_{T,R}^\xi - \nabla \phi_T^\xi|^2 \lesssim R^d \sqrt{T} \exp\left(-c \frac{R-L}{\sqrt{T}}\right). \quad (3.3)$$

Remark 4. By definition of Richardson extrapolation, in the context of Theorem 4 and Definition 2, we have for all $k \in \mathbb{N}$:

$$\int_{Q_L} |\nabla \phi_{T,k,R}^\xi - \nabla \phi_{T,k}^\xi|^2 \lesssim R^d 2^{k-1} T \exp\left(-c \frac{R-L}{\sqrt{2^{k-1}T}}\right). \quad (3.4)$$

In the case of periodic coefficients, this estimate can be upgraded, as in Remark 3, to

$$\int_{Q_L} |\nabla \phi_{T,k,R}^\xi - \nabla \phi_{T,k}^\xi|^2 \lesssim R^d \sqrt{2^{k-1}T} \exp\left(-c \frac{R-L}{\sqrt{2^{k-1}T}}\right). \quad (3.5)$$

Combined with Theorem 2, this yields in particular:

$$\int_{Q_L} |\nabla \phi_{T,k,R}^\xi - \nabla \phi^\xi|^2 \lesssim L^d T^{-2k} + R^d \sqrt{2^{k-1}T} \exp\left(-c \frac{R-L}{\sqrt{2^{k-1}T}}\right). \quad (3.6)$$

Before we turn to proof of Theorem 4, let us make some comments on the approximation of homogenized coefficients. Unlike correctors, homogenized coefficients are averaged quantities. In particular, in order to make the approximations $A_{T,k}$ of Definition 3 computable, one needs to approximate both the extrapolation $\phi_{T,k}^\xi$ of the regularized corrector and the averaging operator \mathcal{M} itself. Since we have already addressed the approximation of the corrector in Theorem 4, it only remains to approximate the averaging operator on domains Q_L . As recalled in the introduction, a first possibility is to replace \mathcal{M} by the average on Q_L . This yields however a very slow convergence for correlated fields, as the periodic example shows: If \mathcal{E} is a periodic (non constant) integrable function, then

$$|\mathcal{M}(\mathcal{E}) - \frac{1}{|Q_L|} \int_{Q_L} \mathcal{E}| \sim \frac{1}{L}. \quad (3.7)$$

In order to enhance the convergence rate, we have introduced in [17] a filtered average, inspired by the work by Blanc and Le Bris in [6].

Definition 4. A function $\mu : [-1, 1] \rightarrow \mathbb{R}^+$ is said to be a filter of order $p \geq 0$ if μ is continuous, even, non-increasing on $[0, 1]$, and satisfies

- (i) $\mu \in C^{p-1}([-1, 1]) \cap W^{p,\infty}((-1, 1))$,
- (ii) $\int_{-1}^1 \mu(x) dx = 1$,
- (iii) $\mu^{(k)}(-1) = \mu^{(k)}(1) = 0$ for all $k \in \{0, \dots, p-1\}$.

Remark 5. For $p = 0$, the conditions (i) and (iii) are void. The average on $[-1, 1]$ (as used in (3.7) in the multidimensional version) is a filter of order 0.

Then, we may replace the average on Q_L by the filtered average with filter $\mu_L : Q_L \rightarrow \mathbb{R}^+$ given by

$$\mu_L(x) := L^{-d} \prod_{i=1}^d \mu(L^{-1}x_i),$$

where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. This yields the following formula for the computable approximation of $A_{T,k}$ in directions $\zeta, \xi \in \mathbb{R}^d$ and on a box Q_R with average on Q_L :

$$\zeta \cdot A'_{T,k,R,L,p} \xi := \int_{Q_L} (\zeta + \nabla \bar{\phi}_{T,k,R}^\zeta(x)) \cdot A(x) (\xi + \nabla \phi_{T,k,R}^\xi(x)) \mu_L(x) dx. \quad (3.8)$$

We shall also use a variant of this definition, for which we can show the uniform ellipticity (see in particular Step 1 of the proof of Theorem 8 below):

$$\begin{aligned} \zeta \cdot A_{T,k,R,L,p} \xi := \int_{Q_L} & \left(\zeta + \nabla \bar{\phi}_{T,k,R}^\zeta(x) - \left\langle \nabla \bar{\phi}_{T,k,R}^\zeta \right\rangle_{\mu_L} \right) \\ & \cdot A(x) \left(\xi + \nabla \phi_{T,k,R}^\xi(x) - \left\langle \nabla \phi_{T,k,R}^\xi \right\rangle_{\mu_L} \right) \mu_L(x) dx, \end{aligned} \quad (3.9)$$

where for all $\psi \in L^1(Q_L)$,

$$\langle \psi \rangle_{\mu_L} := \int_{Q_L} \psi(x) \mu_L(x) dx.$$

The following general convergence result holds:

Theorem 5. Let $d \geq 2$, $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$, $\zeta, \xi \in \mathbb{R}^d$ be unit vectors, and $\bar{\phi}_{T,k}^\zeta, \phi_{T,k}^\xi$ be the associated Richardson extrapolations of the regularized corrector and adjoint corrector for $T > 0$ and $k \in \mathbb{N}$, cf. Definition 2. If A , $\nabla \bar{\phi}_{T,k}^\zeta$, and $\nabla \phi_{T,k}^\xi$ jointly have the averaging property, then for all $p \in \mathbb{N}_0$,

$$\lim_{L,R \rightarrow \infty} \zeta \cdot A_{T,k,R,L,p} \xi = \lim_{L,R \rightarrow \infty} \zeta \cdot A'_{T,k,R,L,p} \xi = \zeta \cdot A_{T,k} \xi,$$

where $A_{T,k,R,L,p}$ and $A'_{T,k,R,L,p}$ are as in (3.9) and (3.8), and $A_{T,k}$ as in Definition 3.

The joint averaging property required in this theorem holds for instance in the stationary ergodic setting almost surely (but it could also hold for general ergodic algebras).

Proof. We split the proof into two steps.

Step 1. Proof that $A'_{T,k,R,L,p} \rightarrow A_{T,k}$.

By Remark 4, it is enough to show that

$$\lim_{L \rightarrow \infty} \int_{Q_L} (\zeta + \nabla \bar{\phi}_{T,k}^\zeta(x)) \cdot A(x) (\xi + \nabla \phi_{T,k}^\xi(x)) \mu_L(x) dx = \zeta \cdot A_{T,k} \xi.$$

We shall prove the claim for any function $\psi \in L^1_{\text{loc}}(\mathbb{R}^d)$ which has the averaging property and satisfies $\sup_{x \in \mathbb{R}^d} \int_{Q_\rho(x)} |\psi(x)| dx < \infty$ for some $\rho > 0$ (recall that $\phi_{T,k}$ satisfies (2.4)).

By the co-area formula, we rewrite the μ_L -average as

$$\langle \psi \rangle_{\mu_L} = \int_0^{\mu(0)} L^{-d} \int_{Q_{L \max\{\mu^{-1}(t)\}}} \psi(x) dx dt.$$

so that the claim follows from the Lebesgue dominating convergence since for all $0 < t < \max_{[-1,1]} \mu$, by the averaging property,

$$\lim_{L \uparrow \infty} \int_{Q_{L \max\{\mu^{-1}(t)\}}} \psi(x) dx = \mathcal{M}(\psi).$$

Step 2. Proof that $A_{T,k,R,L,p} \rightarrow A_{T,k}$

In view of Step 1, it suffices to show that

$$\lim_{L, R-L \uparrow \infty} |A_{T,k,R,L,p} - A'_{T,k,R,L,p}| = 0.$$

By definition of $A_{T,k,R,L,p}$ and $A'_{T,k,R,L,p}$, the claim follows if

$$\lim_{L, R-L \uparrow \infty} \left\langle \nabla \phi_{T,k,R}^\xi \right\rangle_{\mu_L} = 0,$$

which, by Remark 4

$$\left| \left\langle \nabla \phi_{T,k,R}^\xi \right\rangle_{\mu_L} - \left\langle \nabla \phi_{T,k}^\xi \right\rangle_{\mu_L} \right| \lesssim \sqrt{T} \left(\frac{R}{L} \right)^d \exp \left(-c \frac{R-L}{\sqrt{T}} \right),$$

we may prove in the equivalent form of

$$\lim_{L \uparrow \infty} \left\langle \nabla \phi_{T,k}^\xi \right\rangle_{\mu_L} = 0. \quad (3.10)$$

Since $\phi_{T,1}^\xi$ satisfies (2.3), we learn from the De Giorgi-Nash-Moser theory that $\phi_{T,1}^\xi$ is locally Hölder-continuous, and from (2.5) we infer that the Hölder exponent is indeed uniform on \mathbb{R}^d . Hence, (2.5) implies by an elementary interpolation argument that

$$\sup_{\mathbb{R}^d} |\phi_{T,1}| \lesssim \sqrt{T}. \quad (3.11)$$

Likewise this bound holds for $\phi_{T,k}$ by definition of Richardson extrapolation. We are now in position to prove (3.10). On the one hand, since $\nabla \phi_{T,k}$ has the averaging property and satisfies (2.4), Step 1 shows that

$$\lim_{L \uparrow \infty} \left\langle \nabla \phi_{T,k}^\xi \right\rangle_{\mu_L} = \mathcal{M}(\nabla \phi_{T,k}^\xi).$$

On the other hand, by the divergence theorem and the uniform bound (3.11),

$$|\mathcal{M}(\nabla \phi_{T,k}^\xi)| = \lim_{L \uparrow \infty} \left| \int_{B_L} \nabla \phi_{T,k}^\xi \right| = \lim_{L \uparrow \infty} \frac{1}{|B_L|} \left| \int_{\partial B_L} \phi_{T,k}^\xi \cdot n \right| \lesssim \lim_{L \uparrow \infty} \frac{\sqrt{T}}{L} = 0. \quad \square$$

In the case of periodic coefficients we may turn this qualitative convergence result quantitative:

Theorem 6. *Let $d \geq 2$, $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$ be Q -periodic (non-necessarily symmetric), μ be a filter of order $p \geq 0$, $k \in \mathbb{N}$, and A_{hom} , $A_{T,k,R,L,p}$ and $A'_{T,k,R,L,p}$ be the homogenized coefficients and their approximations (3.9) and (3.8), respectively. Then in the regime $R^2 \gtrsim T \gtrsim R$, $R \geq L \sim R \sim R-L$, there exists $c > 0$ depending only on α, β and d such that we have*

$$|A_{T,k,R,L,p} - A_{\text{hom}}|, |A'_{T,k,R,L,p} - A_{\text{hom}}| \lesssim L^{-(p+1)} + T^{-2k} + T^{1/4} \exp \left(-c \frac{R-L}{\sqrt{2^{k-1}T}} \right), \quad (3.12)$$

where the multiplicative constant depends on k and p next to α, β , and d .

Proof. The estimate for $|A'_{T,k,R,L,p} - A_{\text{hom}}|$ is a direct consequence of the combination of Theorem 2, Remark 4, and [17, Theorem 3.1]. It only remains to prove that the estimate for $|A_{T,k,R,L,p} - A_{\text{hom}}|$ is a consequence of the estimate for $|A'_{T,k,R,L,p} - A_{\text{hom}}|$. By definition of $A'_{T,k,R,L,p}$ and $A_{T,k,R,L,p}$ and an elementary energy estimate, it is enough to prove that in the desired regime of R, L, T ,

$$\left| \left\langle \nabla \phi_{T,k,R}^\xi \right\rangle_{\mu_L} \right| \lesssim L^{-(p+1)} + T^{1/4} \exp\left(-c \frac{R-L}{\sqrt{2^{k-1}T}}\right).$$

By Remark 4, this follows from

$$\left| \left\langle \nabla \phi_{T,k}^\xi \right\rangle_{\mu_L} \right| \lesssim L^{-(p+1)},$$

which is itself a consequence of [17, Lemma 3.1] since $\mathcal{M}(\nabla \phi_{T,k}^\xi) = 0$. \square

In the random setting one cannot expect to have an almost sure *quantitative* convergence result. The combination of Remark 4 and [23, Theorem 1 and Remark 2.3] yields for symmetric coefficients given by the random Poisson inclusion process:

Theorem 7. *Let $d \geq 2$, $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$ be random Poisson inclusions coefficients, μ be a filter of order $p \geq 1$, and A_{hom} and $A'_{T,k',R,L,p}$ be the homogenized matrix and its approximation (1.7) respectively, where $R^2 \gtrsim T \gtrsim R$, $R \geq L \sim R \sim R - L$, and $k' \in \mathbb{N}$. Then, there exists $c > 0$ and $q > 0$ depending only on α, β and d such that we have for all $k' \geq \min\{k(d) = \lfloor \frac{d+1}{4} \rfloor, 2\}$,*

$$\begin{aligned} \left\langle |A'_{T,k',R,L,p} - A_{\text{hom}}|^2 \right\rangle^{1/2} &\lesssim \sqrt{T} \exp\left(-c \frac{R-L}{\sqrt{2^{k-1}T}}\right) + L^{-d/2} \begin{cases} d=2 & : \log^q L \\ d>2 & : 1 \end{cases} \\ &+ \begin{cases} d=2 & : T^{-1} \log^q T, \\ 2 < d < 6 & : T^{-d/2}, \\ d=6 & : T^{-3} \log T, \\ d>6 & : T^{-3}, \end{cases} \end{aligned} \quad (3.13)$$

where the multiplicative constant depends on k' and p , next to α, β , and d .

Remark 6. The triangle inequality combined with energy estimates yields for all $1 \leq i \leq d$

$$\begin{aligned} &\left\langle (e_i \cdot (A_{T,k',R,L,p} - A_{\text{hom}})e_i)^2 \right\rangle^{1/2} \\ &\lesssim \left\langle (e_i \cdot (A'_{T,k',R,L,p} - A_{\text{hom}})e_i)^2 \right\rangle^{1/2} + \left\langle \left\langle \nabla \phi_{T,k,R}^{e_i} \right\rangle_{\mu_L}^2 + \left\langle \nabla \phi_{T,k,R}^{e_i} \right\rangle_{\mu_L}^4 \right\rangle^{1/2}. \end{aligned}$$

On the one hand, by the energy estimate $\int_{Q_R} |\nabla \phi_{T,k,R}^{e_i}| \lesssim R^d$, one has

$$\left\langle \nabla \phi_{T,k,R}^{e_i} \right\rangle_{\mu_L}^4 \lesssim \left\langle \nabla \phi_{T,k,R}^{e_i} \right\rangle_{\mu_L}^2 \left(\frac{R}{L}\right)^2.$$

On the other hand, by Remark 4,

$$\left| \left\langle \nabla \phi_{T,k,R}^{e_i} - \nabla \phi_{T,k}^{e_i} \right\rangle_{\mu_L} \right| \lesssim \sqrt{T} \exp\left(-c \frac{R-L}{\sqrt{2^{k-1}T}}\right),$$

and by a simplified version of the string of arguments that leads to [23, Theorem 1], we can prove that for all $1 \leq j \leq d$

$$\text{var} \left[e_j \cdot \int_{Q_L} \nabla \phi_{T,k} \mu_L \right] = \text{var} \left[e_j \cdot \int_{Q_L} \phi_{T,k} \nabla \mu_L \right] \lesssim L^{-d} \begin{cases} d=2 & : \log^{2q} L \\ d>2 & : 1 \end{cases}$$

in the desired regime of L and T . Hence, the same estimate as (3.13) holds for $A_{T,k',R,L,p}$.

Remark 7. If the spectral exponents were known to be $(d/2 - 1, 0)$ (as we expect they are), then the estimate would not saturate at $d = 6$ and we would have for all $k' \geq k(d)$,

$$\langle |A_{T,k',R,L,p} - A_{\text{hom}}|^2 \rangle^{1/2} \lesssim \sqrt{T} \exp \left(-c \frac{R-L}{\sqrt{2^{k-1}T}} \right) + L^{-d/2} \begin{cases} d=2 & : \log L, \\ d>2 & : 1. \end{cases}$$

Remark 8. Note that the convergence rate does not depend on the order of the filter provided it is at least of order 1 (the only fact used in the proof is that μ is smooth and vanishes at -1 and 1). This is due to the continuum stationarity and to the fact the result is not almost-sure.

We conclude this subsection with the proof of Theorem 4, which is a generalization of the corresponding result for symmetric periodic coefficients, see [19], to any coefficients $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$.

Proof of Theorem 4. The argument relies on the exponential decay of the Green's function, Caccioppoli's inequality, and the uniform a priori estimate (2.5).

Recall that for all $A \in L^\infty(\mathbb{R}^d, \mathcal{M}_{\alpha\beta})$ and all $T > 0$, the Green's function $G_T(\cdot, y)$ is for all $y \in \mathbb{R}^d$ the unique distributional solution in $W^{1,1}(\mathbb{R}^d)$ of the equation

$$T^{-1}G_T(x, y) - \nabla_x \cdot A(x) \nabla_x G_T(x, y) = \delta(x - y).$$

Combined with the Nash-Aronson estimate, the integration in time of the associated parabolic Green's function times $\exp(-T^{-1}t)$ yields the following pointwise bound uniform wrt A : For all $|x - y| \gtrsim \sqrt{T}$,

$$0 \leq G_T(x, y) \lesssim \frac{1}{|x - y|^{d-2}} \exp \left(-c \frac{|x - y|}{\sqrt{T}} \right). \quad (3.14)$$

(For symmetric coefficients, an alternative proof based on Agmon's positivity method can be found in [17, Appendix B].)

In the rest of this proof, we drop the superscript ξ . Arguing by density in (3.2), one may assume by a standard regularization argument that A is smooth, so that ϕ_T and $\phi_{T,R}$ are smooth on Q_R by elliptic regularity. By assumption there exists $\gamma \sim 1$ such that $R - L \geq 2\gamma\sqrt{T}$. By definition of ϕ_T and $\phi_{T,R}$, we have

$$\begin{cases} T^{-1}(\phi_T - \phi_{T,R}) - \nabla \cdot A(\nabla \phi_T - \nabla \phi_{T,R}) & = 0 & \text{in } Q_R, \\ \phi_T - \phi_{T,R} & = \phi_T & \text{on } \partial Q_R. \end{cases}$$

Set $\phi_1 = \chi \phi_T$ where $\chi \in C^\infty(\overline{Q}_R, \mathbb{R}^+)$ is such that $\chi|_{\partial Q_R} = 1$, $\chi|_{Q_{R-\gamma\sqrt{T}}} = 0$, and $|\nabla \chi| \lesssim \sqrt{T}^{-1}$. Since ϕ_T satisfies (2.5) and ϕ_1 vanishes on $Q_{R-\gamma\sqrt{T}}$, we have

$$\|\phi_1\|_{L^2(Q_R)}^2 \lesssim (R^{d-1}\sqrt{T})T, \quad \|\nabla \phi_1\|_{L^2(Q_R)}^2 = \|\chi \nabla \phi_T + \phi_T \nabla \chi\|_{L^2(Q_R)}^2 \lesssim R^{d-1}\sqrt{T}. \quad (3.15)$$

The function

$$\phi_2 := \phi_T - \phi_{T,R} - \phi_1 \quad (3.16)$$

is smooth and satisfies the equation

$$\begin{cases} T^{-1}\phi_2 - \nabla \cdot A\nabla\phi_2 = -T^{-1}\phi_1 + \nabla \cdot A\nabla\phi_1 & \text{in } Q_R, \\ \phi_2 = 0 & \text{on } \partial Q_R. \end{cases} \quad (3.17)$$

Let $G_{T,R} : Q_R \times Q_R \rightarrow \mathbb{R}^+$ be the Green function associated with the operator $(T^{-1} - \nabla \cdot A\nabla)$ on Q_R with homogeneous Dirichlet boundary conditions. Since the RHS of (3.17) is smooth, the Green representation formula yields

$$\phi_2(x) = - \int_{Q_R} (T^{-1}\phi_1(y)G_{T,R}(x,y) + \nabla G_{T,R}(x,y) \cdot A(y)\nabla\phi_1(y))dy.$$

By Cauchy-Schwarz' inequality, this turns for all $x \in Q_L \subset Q_{R-2\gamma\sqrt{T}}$ into

$$\begin{aligned} |\phi_2(x)| &\lesssim T^{-1}\|\phi_1\|_{L^2(Q_R)} \left(\int_{Q_R \setminus Q_{R-\gamma\sqrt{T}}} G_{T,R}(x,y)^2 dy \right)^{1/2} \\ &\quad + \|\nabla\phi_1\|_{L^2(Q_R)} \left(\int_{Q_R \setminus Q_{R-\gamma\sqrt{T}}} |\nabla G_{T,R}(x,y)|^2 dy \right)^{1/2}. \end{aligned} \quad (3.18)$$

To control the first RHS term of (3.18), we appeal to (3.14) and to the maximum principle in the form of $0 \leq G_{T,R} \leq G_T$. For the second RHS term of (3.18), we use in addition Caccioppoli's inequality. Let $\eta : Q_R \rightarrow \mathbb{R}^+$ be such that $\eta|_{Q_R \setminus Q_{R-\gamma\sqrt{T}}} = 1$, $\eta|_{Q_{R-\frac{3}{2}\gamma\sqrt{T}}} = 0$, and $|\nabla\eta| \lesssim \sqrt{T}^{-1}$, and test the equation for $G_{T,R}(x, \cdot)$ with $\eta^2 G_{T,R}(x, \cdot) \in H_0^1(Q_R)$. This yields after integration by parts

$$\begin{aligned} 0 &= T^{-1} \int_{Q_R} \eta^2(y) G_{T,R}(x,y)^2 dy + \int_{Q_R} \nabla G_{T,R}(x,y) \cdot A(y) \nabla(\eta(y)^2 G_{T,R}(x,y)) dy \\ &= T^{-1} \int_{Q_R} \eta^2(y) G_{T,R}(x,y)^2 dy + \int_{Q_R} \nabla(\eta(y) G_{T,R}(x,y)) \cdot A(y) \nabla(\eta(y) G_{T,R}(x,y)) dy \\ &\quad - \int_{Q_R} G_{T,R}(x,y)^2 \nabla\eta(y) \cdot A(y) \nabla\eta(y) dy. \end{aligned}$$

Hence, by uniform ellipticity of A ,

$$\int_{Q_R} |\nabla(\eta(y) G_{T,R}(x,y))|^2 dy \lesssim \int_{Q_R} G_{T,R}(x,y)^2 |\nabla\eta(y)|^2 dy,$$

that is,

$$\int_{Q_R \setminus Q_{R-\gamma\sqrt{T}}} |\nabla G_{T,R}(x,y)|^2 dy \lesssim T^{-1} \int_{Q_R \setminus Q_{R-\frac{3}{2}\gamma\sqrt{T}}} G_{T,R}(x,y)^2 dy. \quad (3.19)$$

We insert (3.19) into (3.18), appeal to (3.14) and (3.15), and use that for all $x \in Q_{L+1}$ and $y \in Q_R \setminus Q_{R-\frac{3}{2}\gamma\sqrt{T}}$, we have $|x - y| \gtrsim R - L$, so that

$$\begin{aligned}
|\phi_2(x)| &\lesssim (T^{-1}\|\phi_1\|_{L^2(Q_R)} + \sqrt{T}^{-1}\|\nabla\phi_1\|_{L^2(Q_R)}) \left(\int_{Q_R \setminus Q_{R-\frac{3}{2}\gamma\sqrt{T}}} G_{T,R}(x,y)^2 dy \right)^{1/2} \\
&\stackrel{(3.15)}{\lesssim} (R^{d-1}\sqrt{T})\sqrt{T}^{-1} \frac{1}{(R-L)^{d-2}} \exp\left(-c\frac{R-L}{\sqrt{T}}\right) \\
&= \sqrt{T} \frac{R^{d-1}}{(R-L)^{d-1}} \frac{R-L}{\sqrt{T}} \exp\left(-c\frac{R-L}{\sqrt{T}}\right) \\
&\stackrel{R-L \sim R}{\lesssim} \sqrt{T} \exp\left(-c\frac{R-L}{\sqrt{T}}\right), \tag{3.20}
\end{aligned}$$

where c is a positive constant which may change from line to line but does not depend on T , L , and R . Hence,

$$\int_{Q_{L+1}} \phi_2(x)^2 dx \lesssim R^d T \exp\left(-c\frac{R-L}{\sqrt{T}}\right).$$

Another use of Caccioppoli's inequality, this time for ϕ_2 (recall that the RHS of (3.17) vanishes identically in $Q_{R-\gamma\sqrt{T}}$), yields by definition (3.16) of ϕ_2 and since ϕ_1 vanishes on Q_L :

$$\begin{aligned}
\int_{Q_L} |\nabla(\phi_T - \phi_{T,R})(x)|^2 dx &= \int_{Q_L} |\nabla\phi_2(x)|^2 dx \lesssim \int_{Q_{L+1}} \phi_2(x)^2 dx \\
&\lesssim R^d T \exp\left(-c\frac{R-L}{\sqrt{T}}\right)
\end{aligned}$$

as desired. \square

3.2. Numerical tests for symmetric coefficients. In the first paragraph we treat the case of a discrete elliptic equation on \mathbb{Z}^d with periodic coefficients. The advantage of this example is that there is no additional approximation error besides the machine precision since everything is already discrete. This allows us to explore the behavior of the method for large R , which is not possible for a continuum equation due to necessary discretization of the periodic cell (the total number of degrees of freedom becomes rapidly very large). We then address two examples of a continuum equation: periodic and quasi-periodic coefficients. The conclusion is as follows (at least in the periodic case): Whereas the first error term T^{-2k} of the RHS of (3.6) in Remark 4 seems to be dominant in the regime of moderate R , the RHS of (3.12) in Theorem 6 is dominated for moderate R by the error term $L^{-(p+1)}$ due to the averaging process, regardless of the error-term in T^{-2k} . This tends to show that numerical homogenization methods which only rely on approximations of the corrector (and not on the approximation of homogenized coefficients) may be likely to perform better in terms of the resonance error (at least when the ratio $R = \rho/\varepsilon$ is only moderately large).

3.2.1. Warm-up example: Discrete periodic coefficients. Consider the discrete elliptic equation

$$-\nabla^* \cdot A(\xi + \nabla\phi) = 0 \quad \text{in } \mathbb{Z}^2, \tag{3.21}$$

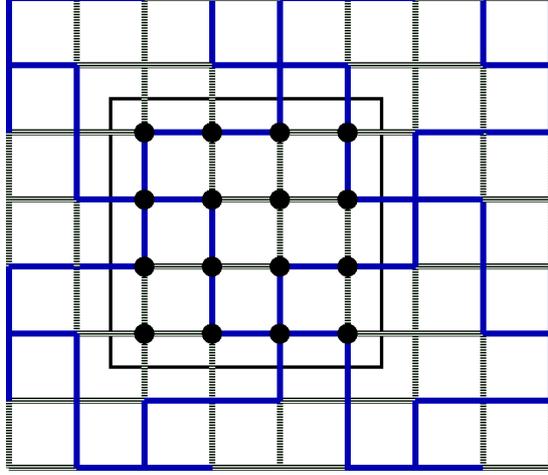


FIGURE 1. Periodic cell in the discrete case

where for all $u : \mathbb{Z}^2 \rightarrow \mathbb{R}$,

$$\nabla u(x) := \begin{bmatrix} u(x + \mathbf{e}_1) - u(x) \\ u(x + \mathbf{e}_2) - u(x) \end{bmatrix}, \quad \nabla^* u(x) := \begin{bmatrix} u(x) - u(x - \mathbf{e}_1) \\ u(x) - u(x - \mathbf{e}_2) \end{bmatrix},$$

and

$$A(x) := \text{diag}[a(x, x + \mathbf{e}_1), a(x, x + \mathbf{e}_2)].$$

The matrix A is $[0, 4]^2$ -periodic, and sketched on a periodic cell on Figure 1. In the example considered, $a(x, x + \mathbf{e}_1)$ and $a(x, x + \mathbf{e}_2)$ represent the conductivities 1 or 100 of the horizontal edge $[x, x + \mathbf{e}_1]$ and the vertical edge $[x, x + \mathbf{e}_2]$ respectively, according to the colors on Figure 1. The homogenization theory for such discrete elliptic operators is similar to the one of the continuum setting (see for instance [33] in this two-dimensional case). Likewise, the analogue of Theorem 6 holds, and our aim is to investigate the behavior of the error in the regime of R large. By rotation invariance the homogenized matrix A_{hom} is a multiple of the identity. It can be evaluated numerically (note that we do not make any other error than the machine precision). Its numerical value is $A_{\text{hom}} = 26.240099009901 \dots$

We have considered the first two approximations formulas $A'_{T,1,R,L,\infty}$ and $A'_{T,2,R,L,\infty}$ of A_{hom} , cf. (3.8), as well as the naive approximation $A'_{\infty,1,R,L,\infty}$ (where $\phi_{\infty,R} \in H_0^1(Q_R)$ is the solution of (3.1) for $T = \infty$, that is, without zero-order term). In all the tests, for $A'_{T,1,R,L,p}$ and $A'_{T,2,R,L,p}$ we have taken the following parameters:

- zero-order term $T = R/10$,
- filter of order $p = \infty$ with $L = R/3$.

This yields the following theoretical predictions according to (the discrete counterpart of) Theorem 6:

$$|A_{\text{hom}} - A'_{\infty,1,R,L,\infty}| \lesssim R^{-1}, \quad |A_{\text{hom}} - A'_{T,1,R,L,\infty}| \lesssim R^{-2}, \quad |A_{\text{hom}} - A'_{T,2,R,L,\infty}| \lesssim R^{-4}.$$

The numerical tests have been performed up to $R = 100$ periodic cells per dimension (that is 400 points per dimension, $1.6 \cdot 10^5$ degrees of freedom). They confirm the theoretical predictions, as can be seen of Figure 2. Yet, for $k = 1$ and even more for $k = 2$, the

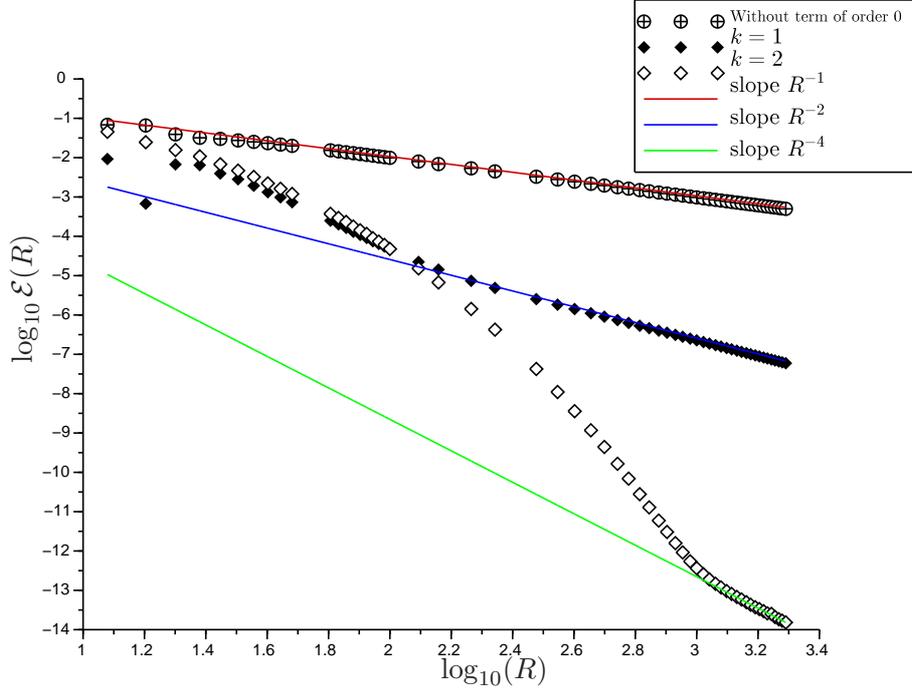


FIGURE 2. Symmetric discrete periodic example. Bound on $R \mapsto \mathcal{E}(R) := |A_{T,k,R,L,\infty} - A_{\text{hom}}|$ for $(T, k) = (\infty, 1), (R/10, 1), (R/10, 2)$, log-log scale (slopes -1 , -2 , and -4).

theoretical predictions are only attained for very large R . Indeed, recall that Theorem 6 provides more details on the error: For all $p \in \mathbb{N}_0$,

$$|A_{\text{hom}} - A'_{T,k,R,L,\infty}| \leq C_p R^{-(p+1)} + C_k R^{-2k} + C_k R^{1/4} \exp(-c\sqrt{2^{-k+1}R}),$$

where $C_p \uparrow \infty$ as $p \uparrow \infty$. In the tests, the last error-term is negligible and only the first two error-terms are observed. In particular, the error due to averaging is super-algebraic for large R but may be dominant for moderate R . This is indeed what we observe on Figure 2. For $A_{\infty,1,R,L,\infty}$ the error due to the approximation of the correctors dominate for all R so that the error curve is a straight line of slope -1 . For $A_{T,1,R,L,\infty}$ and $A_{T,2,R,L,\infty}$ however, for small R , both errors coincide and seem to decay super-algebraically — this is the averaging error — until they cross a straight line of slopes -2 and -4 , respectively, when the systematic error “ R^{-2k} ” starts to dominate. This illustrates that, due to averaging, the computational approximation of the homogenized coefficients can be effectively much slower than the computational approximation of the correctors for moderate R (which is the regime of interest), although the asymptotic error (for large R) is the same.

3.2.2. Symmetric periodic example. We come back to the continuum setting addressed in this article, and consider the following coefficients $A : \mathbb{R}^2 \rightarrow \mathcal{M}_{\alpha\beta}^{\text{sym}}$,

$$A(x) = \left(\frac{2 + 1.8 \sin(2\pi x_1)}{2 + 1.8 \cos(2\pi x_2)} + \frac{2 + \sin(2\pi x_2)}{2 + 1.8 \cos(2\pi x_1)} \right) \text{Id}, \quad (3.22)$$

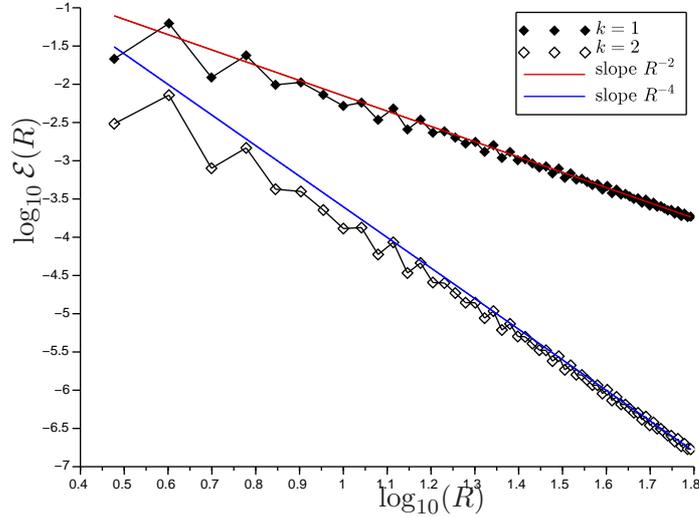


FIGURE 3. Symmetric periodic example. Bound on $R \mapsto \mathcal{E}(R) := \int_{Q_{R/6}} |\nabla \phi_{T,k,R}^\xi - \nabla \phi^\xi|^2$ for $k = 1, 2$, log-log scale (slopes -2 and -4).

used as benchmark tests in [27]. In this case, $\alpha \simeq 0.35$, $\beta \simeq 20.5$, and $A_{\text{hom}} \simeq 2.75 \text{Id}$. The value of A_{hom} can be approximated on a single periodic cell with periodic boundary conditions. We have considered the first two approximation formulas $A_{T,1,R,L,p}$ and $A_{T,2,R,L,p}$ of A_{hom} , cf. (3.9). In all the tests, we have taken the following parameters:

- zero-order term $T = R/100$,
- filter of order $p = 3$ with $L = R/3$.

Theorem 6 yields

$$|A_{T,1,R,L,3} - A_{\text{hom}}| \lesssim R^{-2}, \quad |A_{T,2,R,L,4} - A_{\text{hom}}| \lesssim R^{-4}. \quad (3.23)$$

Likewise, (3.6) in Remark 4 yields:

$$\frac{1}{L^d} \int_{Q_{L/2}} |\nabla \phi_{T,k,R}^\xi - \nabla \phi^\xi|^2 \lesssim R^{-2k}. \quad (3.24)$$

The numerical tests have been performed with FreeFEM++ [14] and R ranges from 3 to 60. The results for the approximation of the correctors are in perfect agreement with (3.24), which describes correctly the decay, even for moderate R , see Figure 3. For the homogenized coefficients however, the theoretical predictions of (3.23) are only met for larger R . Indeed, for smaller R , the averaging error dominates, see Figure 4. As in the discrete case, the errors for $k = 1$ and $k = 2$ are “super-algebraic” and coincide until they cross the straight lines of slopes -2 and -4 , respectively, when the systematic error R^{-2k} dominates. With respect to the tests in the discrete case, there is an additional feature: the overall error displays periodic oscillations, especially for $k = 2$. Whereas the systematic error T^{-2k} decays monotonically (as illustrated on Figure 3), the averaging error does not: When L is a multiple of the period, the error gets smaller, whence the periodic modulation in L . For $k = 1$, the order of magnitude of these oscillations vanish as R

gets larger (and the averaging error becomes negligible wrt the systematic error). Indeed, the periodic modulation is still there but the systematic error $T^{-2k} \sim R^{-2}$ dominates the averaging error $L^{-(p+1)} \sim R^{-4}$, so that this effect is not relevant asymptotically. For $k = 2$ and $p = 4$, although the averaging error is of order L^{-5} whereas the systematic error is of order T^{-4} in Theorem 6, the averaging error for $R \leq 60$ is still a non-negligible part of the overall error (of the same order as the systematic error). To further illustrate this fact we have plotted on Figure 5 the approximation errors $|A_{T,2,R,L,3} - A_{\text{hom}}|$ and $|A_{T,2,R,L,4} - A_{\text{hom}}|$, the difference of which is only due to the averaging function: for $p = 3$, the averaging error is of order $L^{-4} \sim T^{-4}$ and the oscillations due to averaging are of the same order as the systematic error for all R whereas for $p = 4$, $L^{-5} \ll T^{-4}$ and the oscillations are smaller wrt to the systematic error as R gets larger. Again, these tests illustrate the fact that the computational approximation of the homogenized coefficients can be much slower than the computational approximation of the correctors for moderate R , due to averaging.

3.2.3. Symmetric quasi-periodic example. We turn now to a quasi-periodic example and consider the following coefficients:

$$A(x) = \begin{pmatrix} 4 + \cos(2\pi(x_1 + x_2)) + \cos(2\pi\sqrt{2}(x_1 + x_2)) & 0 \\ 0 & 6 + \sin^2(2\pi x_1) + \sin^2(2\pi\sqrt{2}x_1) \end{pmatrix}. \quad (3.25)$$

In this case, $\alpha \geq 2$, $\beta \leq 8$, so that the ellipticity ratio $4 = \frac{8}{2}$ is much smaller than in the previous two examples (100 and 60, respectively). On the one hand, this makes the homogenization procedure easier, and we expect approximation errors to be much smaller. On the other hand, in the quasi-periodic setting, we have no easy access to the homogenized coefficients (as opposed to the periodic setting where A_{hom} is given by a cell formula), so that one needs a reliable error estimator. In this elementary example of quasi-periodic coefficients, one expects the elliptic operator \mathcal{L} to have a spectral gap. We thus expect from Proposition 1 that for all $k \in \mathbb{N}$,

$$\begin{aligned} |A_{T,k} - A_{\text{hom}}| &\lesssim T^{-2k}, \\ \langle |\nabla \phi_{T,k}^\xi - \nabla \phi^\xi|^2 \rangle &\lesssim T^{-2k}. \end{aligned}$$

In particular, for all $k' > k$, the triangle inequality then yields:

$$\begin{aligned} |A_{T,k} - A_{\text{hom}}| &\lesssim T^{-2k'} + |A_{T,k} - A_{T,k'}|, \\ \langle |\nabla \phi_{T,k}^\xi - \nabla \phi^\xi|^2 \rangle &\lesssim T^{-2k'} + \langle |\nabla \phi_{T,k}^\xi - \nabla \phi_{T,k'}^\xi|^2 \rangle. \end{aligned}$$

Hence, provided $|A_{T,k} - A_{T,k'}| \gtrsim T^{-2k'}$, the quantity $|A_{T,k} - A_{T,k'}|$ is an estimator for $|A_{T,k} - A_{\text{hom}}|$. This allows us not to bias the test with some arbitrary guess of A_{hom} . We have considered the first two approximations formulas $A_{T,1,R,L,p}$ and $A_{T,2,R,L,p}$ of A_{hom} and taken $k' = 3$ as a reference for the homogenized coefficients. In all the tests, we have taken the following parameters:

- zero-order term $T = R/100$,
- filter of order $p = 4$ with $L = R/3$.

We then expect

$$\begin{aligned} |A_{T,1,R,L,4} - A_{\text{hom}}| &\lesssim R^{-5} + R^{-6} + |A_{T,1,R,L,4} - A_{T,3,R,L,4}|, \\ |A_{T,2,R,L,4} - A_{\text{hom}}| &\lesssim R^{-5} + R^{-6} + |A_{T,2,R,L,4} - A_{T,3,R,L,4}|, \end{aligned}$$

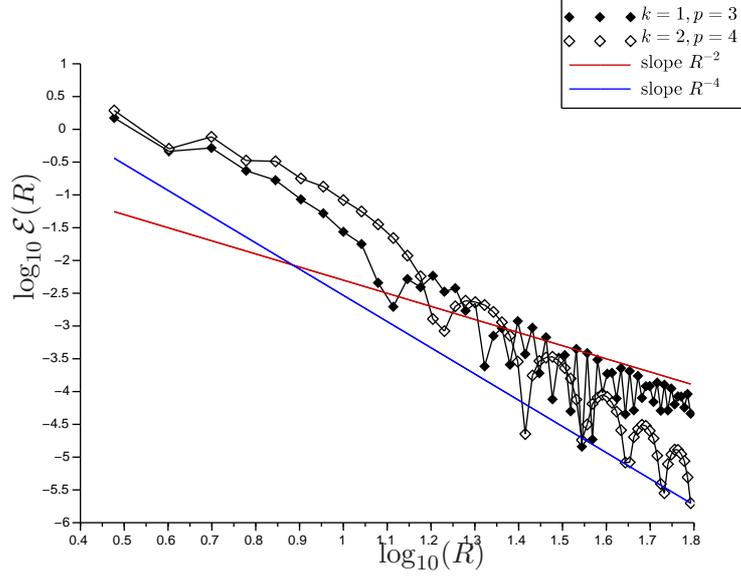


FIGURE 4. Symmetric periodic example. Bound on $R \mapsto \mathcal{E}(R) := |A_{T,k,R,L,p} - A_{\text{hom}}|$ for $(k, p) = (1, 3), (2, 4)$, log-log scale (slopes -2 and -4).

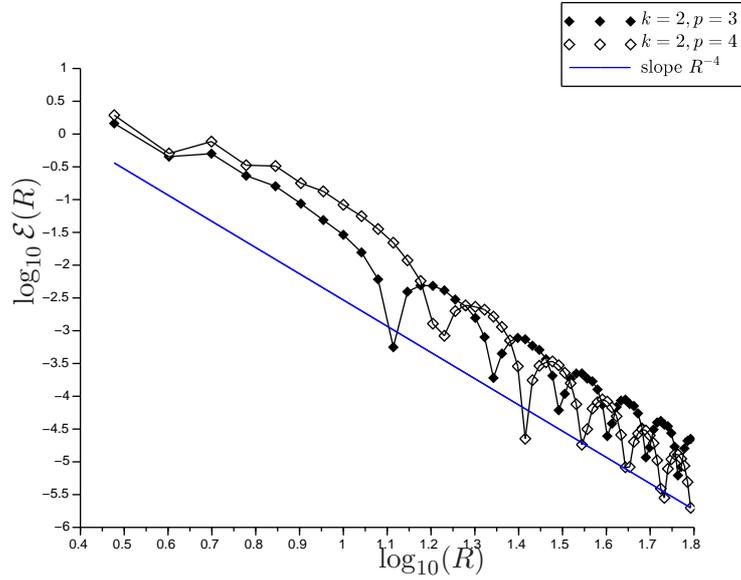


FIGURE 5. Symmetric periodic example. Bound on $R \mapsto \mathcal{E}(R) := |A_{T,2,R,L,p} - A_{\text{hom}}|$ for $p = 3, 4$, log-log scale (slopes -4).

where the error-term R^{-5} is due to averaging (cf. $p = 4$) and the second error-term due to the systematic error (for $k' = 3$). The choice of the filter $p = 4$ minimizes the error due to averaging (which is expected to be of higher order than the systematic error for $k = 1, 2$).

Note however that the chosen estimator for $|A_{T,k,R,L,4} - A_{\text{hom}}|$ is not very sensitive to the averaging function since the same averaging function is taken both in $A_{T,k,R,L,4}$ for $k = 1, 2$ and in $A_{T,3,R,L,4}$. This explains why there are not many oscillations on Figure 6 (besides for small R), where the errors for $k = 1$ and $k = 2$ decay at the predicted rates R^{-2} and R^{-4} , respectively.

For the approximation of correctors, we recall that

$$\langle |\nabla\phi_{T,k}^\xi - \nabla\phi^\xi|^2 \rangle = \lim_{R \uparrow \infty} \int_{Q_{R/2}} |\nabla\phi_{T,k}^\xi - \nabla\phi^\xi|^2.$$

Assuming that the error with respect to T is distributed uniformly over \mathbb{R}^d , one may expect that for $L \gg 1$,

$$\int_{Q_{L/2}} |\nabla\phi_{T,k}^\xi - \nabla\phi^\xi|^2 \sim \lim_{R \uparrow \infty} \int_{Q_{R/2}} |\nabla\phi_{T,k}^\xi - \nabla\phi^\xi|^2,$$

so that our guess for the approximation of the correctors with $T = R/100$ and $L = R/3$,

$$\int_{Q_{L/2}} |\nabla\phi_{T,k,R}^\xi - \nabla\phi^\xi|^2 \lesssim R^{-2k},$$

will be confronted to the results of numerical experiments in the form of

$$\int_{Q_{L/2}} |\nabla\phi_{T,k,R}^\xi - \nabla\phi_{T,k',R}^\xi|^2 \lesssim R^{-2k} \quad (3.26)$$

for $3 = k' > k = 1, 2$. On Figure 7, we obtain a monotonic decay with the expected convergence rates -2 and -4 , which suggests that the correctors are again better-approximated than the homogenized coefficients.

3.3. Numerical tests for non-symmetric coefficients. The tests, expected errors, and results are similar to those of the symmetric setting (although for the proofs one cannot appeal to spectral theory).

3.3.1. Non-symmetric periodic example. We consider the following coefficients:

$$A(x) = \begin{pmatrix} \frac{2+1.8\sin(2\pi x_1)}{2+1.8\cos(2\pi x_2)} + \frac{2+\sin(2\pi x_2)}{2+1.8\cos(2\pi x_1)} & 2 + \sin(2\pi x_1) \cos(2\pi x_2) \\ -2 + \sin(2\pi x_1) \cos(2\pi x_2) & \frac{2+1.8\sin(2\pi x_1)}{2+1.8\cos(2\pi x_2)} + \frac{2+\sin(2\pi x_2)}{2+1.8\cos(2\pi x_1)} \end{pmatrix}. \quad (3.27)$$

We take $L = R/3$, $T = R/100$, and filters of orders $p = 3, 4$. As expected, Figures 8 and 9 are in agreement with the theoretical predictions for $(k, p) = (1, 3), (2, 4)$:

$$\int_{Q_{L/2}} |\nabla\phi_{T,k,R}^\xi - \nabla\phi^\xi|^2 \lesssim R^{-2k}, \quad |A_{T,k,R,L,p} - A_{\text{hom}}| \lesssim R^{-2k}.$$

3.3.2. Non-symmetric quasi-periodic example. We consider the following coefficients:

$$A(x) = \begin{pmatrix} 4 + \cos(2\pi(x_1 + x_2)) + \cos(2\pi\sqrt{2}(x_1 + x_2)) & 2 + \sin(2\pi x_1) \cos(2\pi x_2) \\ -2 + \sin(2\pi x_1) \cos(2\pi x_2) & 6 + \sin^2(2\pi x_1) + \sin^2(2\pi\sqrt{2}x_1) \end{pmatrix}, \quad (3.28)$$

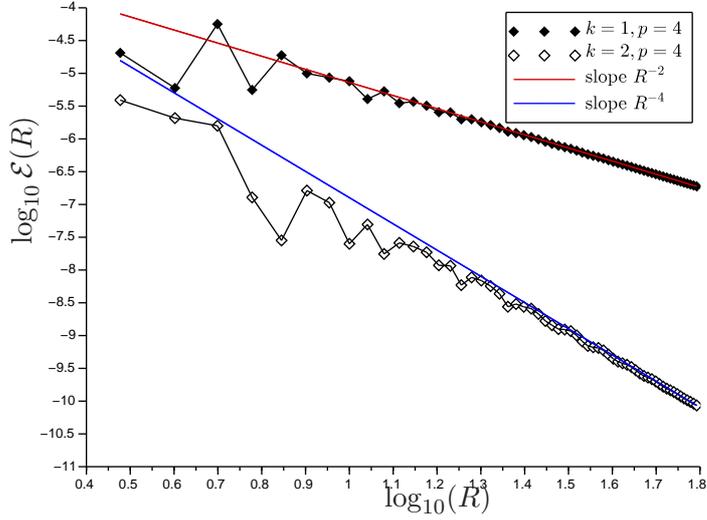


FIGURE 6. Symmetric quasi-periodic example. Bound on $R \mapsto \mathcal{E}(R) := |A_{T,k,R,L,4} - A_{\text{hom}}|$ for $k = 1, 2$, log-log scale (slopes -2 and -4).

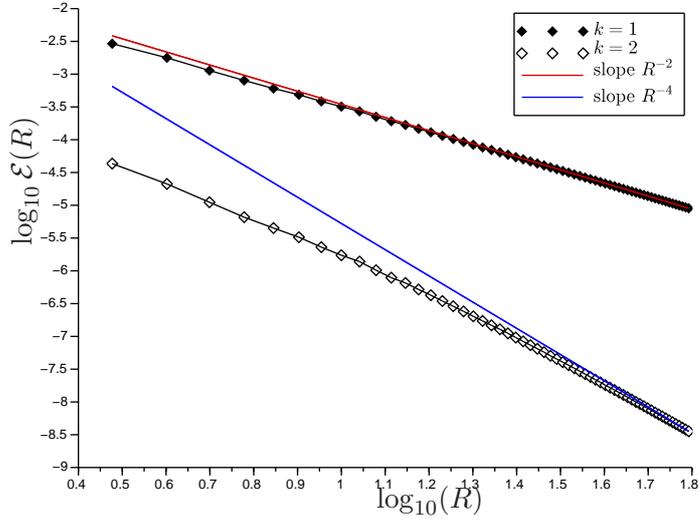


FIGURE 7. Symmetric quasi-periodic example. Bound on $R \mapsto \mathcal{E}(R) := \int_{Q_{R/6}} |\nabla \phi_{T,k,R}^\xi - \nabla \phi_{T,3,R}^\xi|^2$ for $k = 1, 2$, log-log scale (slopes -2 and -4).

We take $L = R/3$, $T = R/100$, and a filter of order $p = 4$. The results are similar as for the symmetric case and Figures 10 and 11 show that for $k = 1, 2$,

$$\int_{Q_{L/2}} |\nabla \phi_{T,k,R}^\xi - \nabla \phi_{T,3,R}^\xi|^2 \lesssim R^{-2k}, \quad |A_{T,k,R,L,3} - A_{T,3,R,L,3}| \lesssim R^{-2k}.$$

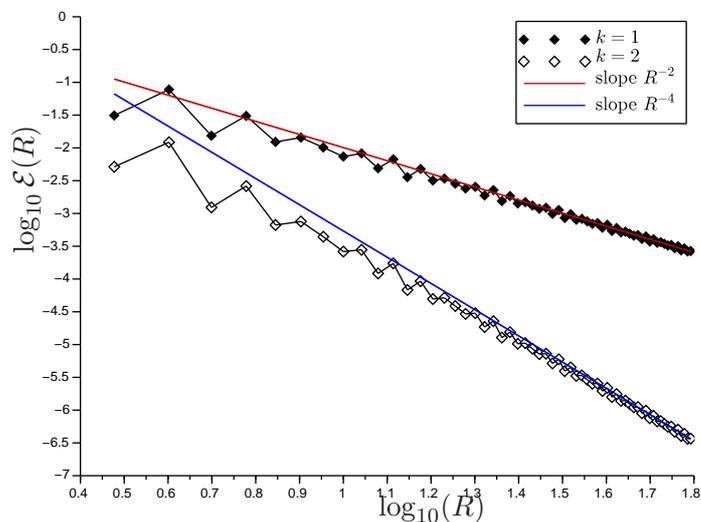


FIGURE 8. Non-symmetric periodic example. Bound on $R \mapsto \mathcal{E}(R) := f_{Q_{R/6}} |\nabla \phi_{T,k,R}^\xi - \nabla \phi^\xi|^2$ for $k = 1, 2$, log-log scale (slopes -2 and -4).

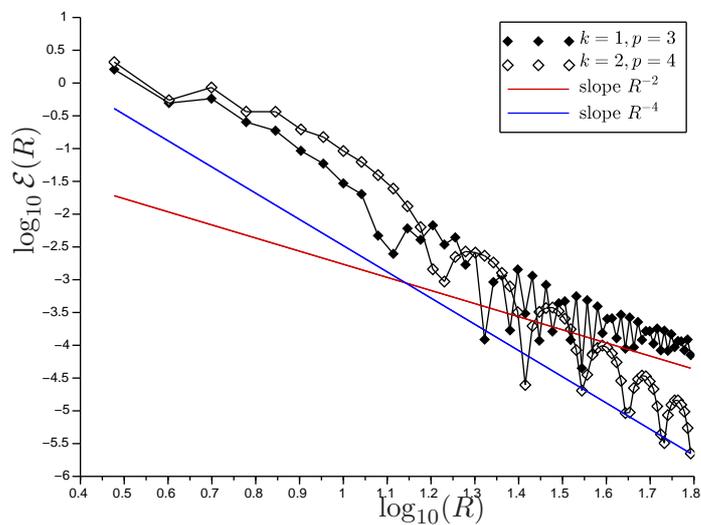


FIGURE 9. Non-symmetric periodic example. Bound on $R \mapsto \mathcal{E}(R) := |A_{T,k,R,L,p} - A_{\text{hom}}|$ for $(k, p) = (1, 3), (2, 4)$, log-log scale (slopes -2 and -4).

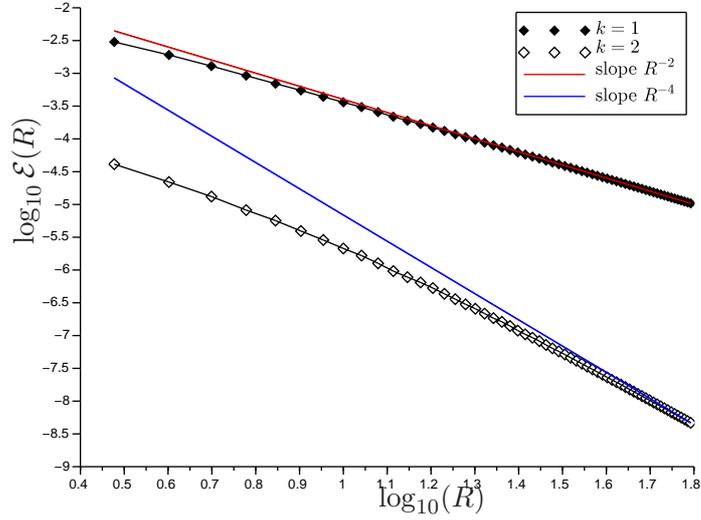


FIGURE 10. Non-symmetric quasi-periodic example. Bound on $R \mapsto \mathcal{E}(R) := f_{Q_{R/6}} |\nabla \phi_{T,k,R}^\xi - \nabla \phi_{T,3,R}^\xi|^2$ for $k = 1, 2$, log-log scale (slopes -2 and -4).

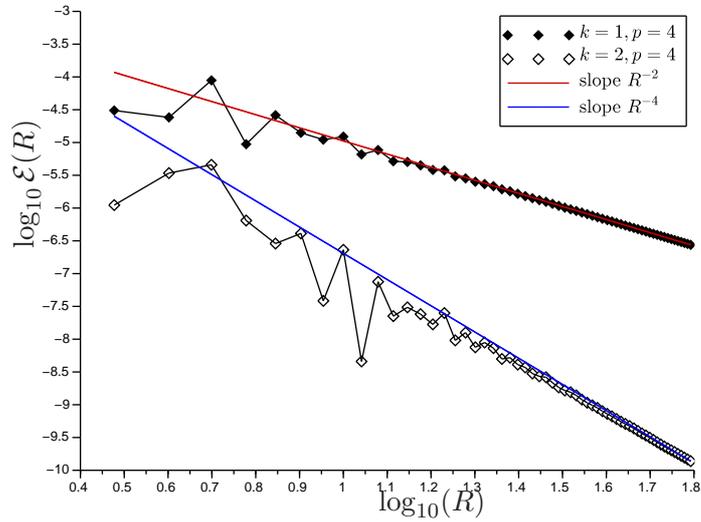


FIGURE 11. Non-symmetric quasi-periodic example. Bound on $R \mapsto \mathcal{E}(R) := |A_{T,k,R,L,4} - A_{T,3,R,L,4}|$ for $k = 1, 2$, log-log scale (slopes -2 and -4).

4. COMBINATION WITH NUMERICAL HOMOGENIZATION METHODS

The aim of this last section is to show that the regularization and extrapolation method to approximate homogenized coefficients and correctors can be used to reduce the resonance error in standard numerical homogenization methods. Following [15, 16] we first introduce a general analytical framework and prove the consistency in the case of symmetric locally stationary ergodic coefficients. We then recover variants of standard numerical homogenization methods after discretization. We conclude the presentation by a quantitative convergence analysis in the case of periodic coefficients.

4.1. Analytical framework. Let $A_\varepsilon : D \times \Omega \rightarrow \mathcal{M}_{\alpha\beta}^{\text{sym}}$ be a family of random symmetric diffusion coefficients parametrized by $\varepsilon > 0$. We make the assumption that A_ε is locally stationary and ergodic (and assume some cross-regularity).

Hypothesis 1. There exists a random Carathéodory function (that is continuous in the first variable and measurable in the second variable) $\tilde{A} : D \times \mathbb{R}^d \times \Omega \rightarrow \mathcal{M}_{\alpha\beta}^{\text{sym}}$, and a constant $\kappa > 0$ such that

- For all $x \in D$, the random field $\tilde{A}(x, \cdot, \cdot)$ is stationary on $\mathbb{R}^d \times \Omega$, and ergodic;
- (*cross-regularity*) \tilde{A} is κ -Lipschitz in the first variable: for all $x, y \in D$, for almost every $z \in \mathbb{R}^d$, and for almost every realization $\omega \in \Omega$,

$$|\tilde{A}(x, z, \omega) - \tilde{A}(y, z, \omega)| \leq \kappa|x - y|;$$

- For all $x \in D$, for and all $\varepsilon > 0$, and for almost every realization $\omega \in \Omega$,

$$A_\varepsilon(x, \omega) := \tilde{A}\left(x, \frac{x}{\varepsilon}, \omega\right)$$

almost surely.

Under Hypothesis 1 A_ε H-converges on D to some deterministic diffusion function $A_{\text{hom}} : D \rightarrow \mathcal{M}_{\alpha\beta}^{\text{sym}}$ almost surely, which is κ -Lipschitz and characterized for all $x \in D$ and $\xi \in \mathbb{R}^d$ by

$$\xi \cdot A_{\text{hom}}(x)\xi = \left\langle (\xi + \nabla\phi(x, 0, \cdot)) \cdot \tilde{A}(x, 0, \cdot)(\xi + \nabla\phi(x, 0, \cdot)) \right\rangle,$$

where $\phi(x, \cdot, \cdot)$ is the corrector in direction ξ associated with the stationary coefficients $\tilde{A}(x, \cdot, \cdot)$ (x is treated as a parameter). The proof is standard: by H-compactness, for almost every realization, A_ε H-converges up to extraction to some limit A^* . Using the locality of H-convergence and the fact that \tilde{A} is uniformly Lipschitz in the first variable, one concludes that $A^*(x) = A_{\text{hom}}(x)$ almost surely. Note that if we weaken the cross-regularity assumption from Lipschitz continuity on D to continuity on \overline{D} , the result holds true as well — although the proof of the homogenization result is much more subtle, see [7, Theorem 4.1.1].

The combination of the regularization and extrapolation method with the analytical framework of [15, 16] yields the following local approximation of A_{hom} in the symmetric case.

Definition 5. Let $\mu : \mathbb{R}^+ \rightarrow [0, 1]$ have support in $[-1, 1]$ and be such that $\int_{-1}^1 \mu = 1$. For all $\rho > 0$, let $\mu_\rho : \mathbb{R}^d \rightarrow \mathbb{R}^+$ denote the ρ -rescaled multidimensional version $y \mapsto \rho^{-d} \prod_{i=1}^d \mu(y_i/\rho)$ of μ (the support of which is in Q_ρ). For all $\delta > 1$, $\rho > 0$, $T > 0$, $k > 0$,

and $\varepsilon > 0$, we denote by $A_{T,k,\rho,\varepsilon}^\delta : D \rightarrow \mathcal{M}_d(\mathbb{R})$ the function defined by: For all $\xi \in \mathbb{R}^d$ and for $x \in D$,

$$\xi \cdot A_{T,k,\rho,\varepsilon}^\delta(x)\xi := \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \cdot A_\varepsilon(x+y)(\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y))\mu_\rho(y)dy, \quad (4.1)$$

where $T_{-x}D := \{y \in \mathbb{R}^d \mid y+x \in D\}$, $v_{T,k}^{\delta\rho,\varepsilon}$ is defined by Richardson extrapolation of the unique weak solution $v_{T,1}^{\delta\rho,\varepsilon}$ in $H_0^1(Q_{\delta\rho} \cap T_{-x}D)$ of

$$(T\varepsilon^2)^{-1}v_{T,1}^{\delta\rho,\varepsilon}(x,y) - \nabla \cdot A_\varepsilon(x+y)(\xi + \nabla_y v_{T,1}^{\delta\rho,\varepsilon}(x,y)) = 0, \quad (4.2)$$

and $P_{\rho,x}$ is the projection on μ_ρ -mean free fields of $L^2(Q_\rho \cap T_{-x}D)$: For all $\psi \in L^2(Q_\rho \cap T_{-x}D)$,

$$P_{\rho,x}(\psi) := \psi - \frac{1}{\int_{Q_\rho \cap T_{-x}D} \mu_\rho(y)dy} \int_{Q_\rho \cap T_{-x}D} \psi(y)\mu_\rho(y)dy. \quad (4.3)$$

This definition can be adapted to non-symmetric coefficients as in Definition 3. Yet the convergence analysis is still not clear to us in that case (except for locally periodic coefficients).

Remark 9. In (4.1) we use $P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x, \cdot)$ instead of simply $\nabla_y v_{T,k}^{\delta\rho,\varepsilon}(x, \cdot)$ (as presented in [19]). As we shall see, this has the advantage to ensure that $A_{T,k,\rho,\varepsilon}^\delta$ be uniformly elliptic, while it is still consistent with the homogenization theory of locally stationary ergodic coefficients, and still reduces the resonance error.

The scaling of the zero-order term in (4.2) may be surprising since it makes the zero-order term diverge as $\varepsilon \downarrow 0$. It is however natural, as can be seen on the case of periodic coefficients: If $A_\varepsilon(y) = A(y/\varepsilon)$ with A periodic, the change of variables $y = \varepsilon z$ turns (4.2) into the equation

$$T^{-1}w(z) - \nabla \cdot A(x+z)(\xi + \nabla w(z)) = 0 \quad \text{in } Q_{\varepsilon^{-1}\delta\rho} \cap (T_{-x}D)/\varepsilon,$$

which is of the form of (2.9). This shows in particular that it is crucial that ε be the ‘‘actual lengthscale’’ of the problem, and not an abstract parameter, which, in this locally random case, is encoded in the assumption $A_\varepsilon(x) = \tilde{A}(x, \frac{x}{\varepsilon})$.

Under Hypothesis 1, the following convergence result holds:

Theorem 8. *Let D be a domain with a Lipschitz boundary. For all $\varepsilon > 0$, let A_ε satisfy Hypothesis 1, and for all $\rho > 0$, $\delta > 1$, $T > 0$, and $k \in \mathbb{N}$, let $A_{T,k,\rho,\varepsilon}^\delta$ be as in Definition 5. Then there exists $\gamma > 0$ depending only on D such that $A_{T,k,\rho,\varepsilon}^\delta \in L^\infty(D, \mathcal{M}_{\alpha'\beta'}^{\text{sym}})$ with $\alpha' = \gamma\alpha$ and $\beta' = 2^d(1 + \delta^d)\beta$. In addition, for all $x \in D$,*

$$\limsup_{T \uparrow \infty} \limsup_{\rho \downarrow 0} \limsup_{\varepsilon \downarrow 0} |A_{T,k,\rho,\varepsilon}^\delta(x) - A_{\text{hom}}(x)| = 0, \quad (4.4)$$

almost surely. The limit in T in (4.4) is uniform in ρ for all x at positive distance from ∂D . In particular, we also have for all $1 \leq p < \infty$

$$\limsup_{\rho \downarrow 0, T \uparrow \infty} \limsup_{\varepsilon \downarrow 0} \|A_{T,k,\rho,\varepsilon}^\delta - A_{\text{hom}}\|_{L^p(D)} = 0, \quad (4.5)$$

where the order of the limits between ρ and T is irrelevant.

From this theorem, one may directly deduce the convergence of the numerical homogenization method:

Corollary 1. *Under the assumptions of Theorem 8, for all $f \in H^{-1}(D)$ there exists a unique weak solution $u_{T,\rho,\varepsilon}^\delta \in H_0^1(D)$ of*

$$-\nabla \cdot A_{T,k,\rho,\varepsilon}^\delta \nabla u_{T,k,\rho,\varepsilon}^\delta = f. \quad (4.6)$$

In addition it satisfies

$$\limsup_{\rho \downarrow 0, T \uparrow \infty} \limsup_{\varepsilon \downarrow 0} \|u_{T,k,\rho,\varepsilon}^\delta - u_{\text{hom}}\|_{H^1(D)} = 0 \quad (4.7)$$

almost surely, where $u_{\text{hom}} \in H_0^1(D)$ is the unique weak solution of

$$-\nabla \cdot A_{\text{hom}} \nabla u_{\text{hom}} = f.$$

As a consequence of H -convergence we also have that

$$\limsup_{\rho \downarrow 0, T \uparrow \infty} \limsup_{\varepsilon \downarrow 0} \|u_{T,k,\rho,\varepsilon}^\delta - u_\varepsilon\|_{L^2(D)} = 0 \quad (4.8)$$

almost surely, where u_ε is the unique weak solution in $H_0^1(D)$ of

$$-\nabla \cdot A_\varepsilon \nabla u_\varepsilon = f.$$

The order of the limits in ρ and T are irrelevant in (4.7) and (4.8).

As mentioned in the introduction, by numerical homogenization we mean not only the approximation of the mean-field solution but also the approximation of the local fluctuations. These are the object of the following corrector result (in the spirit of [4]).

Definition 6. Let $H > 0$, $I_H \in \mathbb{N}$, and let $\{Q_{H,i}\}_{i \in \llbracket 1, I_H \rrbracket}$ be a covering of D in disjoint subdomains of diameter of order H . We define a family (M_H) of approximations of the identity on $L^2(D)$ associated with $Q_{H,i}$: for every $w \in L^2(D)$ and $H > 0$,

$$M_{H,i}(w) = \int_{Q_{H,i} \cap D} w, \quad M_H(w) = \sum_{i=1}^{I_H} M_{H,i}(w) 1_{Q_{H,i} \cap D}.$$

Let $\delta > 1$ be as in Theorem 8, and for all $i \in \llbracket 1, I_H \rrbracket$, set

$$Q_{H,i}^\delta := \{x \in \mathbb{R}^d \mid d(x, Q_{H,i}) < (\delta - 1)H\}.$$

With the notation of Corollary 1, we define the numerical correctors $\gamma_{T,k,1,\rho,\varepsilon}^{\delta,H,i}$ associated with $u_{T,k,\rho,\varepsilon}^\delta$ as the unique weak solution in $H_0^1(Q_{H,i}^\delta \cap D)$ of

$$(T\varepsilon^2)^{-1} \gamma_{T,k,1,\rho,\varepsilon}^{\delta,H,i} - \nabla \cdot A_\varepsilon \left(M_{H,i}(\nabla u_{T,k,\rho,\varepsilon}^\delta) + \nabla \gamma_{T,k,1,\rho,\varepsilon}^{\delta,H,i} \right) = 0, \quad (4.9)$$

which generates $\gamma_{T,k,k',\rho,\varepsilon}^{\delta,H,i}$ for all $k' > 1$ by Richardson extrapolation, cf. Definition 2. We then set for all $k' \in \mathbb{N}$ and for all $1 \leq i \leq I_H$,

$$\nabla u_{T,k,k',\rho,\varepsilon}^{\delta,H,i} := M_H(\nabla u_{T,k,\rho,\varepsilon}^\delta)|_{Q_{H,i} \cap D} + (\nabla \gamma_{T,k,k',\rho,\varepsilon}^{\delta,H,i})|_{Q_{H,i} \cap D} \in L^2(Q_{H,i} \cap D),$$

and finally define the numerical corrector

$$C_{T,k,k',\rho,\varepsilon}^{\delta,H} = \sum_{i=1}^{I_H} \nabla u_{T,k,k',\rho,\varepsilon}^{\delta,H,i} 1_{Q_{H,i} \cap D}.$$

Remark 10. For the definition of the numerical corrector, the extrapolation parameter k' needs not be the same as the extrapolation parameter k used for $A_{T,k,\rho,\varepsilon}^\delta$. Indeed, as shown in Paragraph 4.3 in the case of periodic coefficients, it makes sense to take $k' = 2k$ in terms of scaling.

The following numerical corrector result holds:

Theorem 9. *Under the assumptions of Corollary 1, the corrector of Definition 6 satisfies for all $k' \in \mathbb{N}$,*

$$\limsup_{\rho \downarrow 0, T \uparrow \infty, H \downarrow 0} \limsup_{\varepsilon \downarrow 0} \left\| \nabla u_\varepsilon - C_{T,k,k',\rho,\varepsilon}^{\delta,H} \right\|_{L^2(D)} = 0 \quad (4.10)$$

almost surely, where the order of the limits in ρ, T , and H is irrelevant.

We first prove Theorem 8, and then turn to the proof of Theorem 9.

Proof of Theorem 8. We split the proof into three steps.

Step 1. Uniform ellipticity of $A_{T,k,\rho,\varepsilon}^\delta$.

By regularity of the domain D , there exists $\gamma > 0$ such that for all $x \in D$,

$$\int_{Q_\rho \cap T_{-x}D} \mu_\rho \geq \gamma. \quad (4.11)$$

By uniform ellipticity of A_ε , and then expanding the square, we obtain

$$\begin{aligned} \xi \cdot A_{T,k,\rho,\varepsilon}^\delta(x)\xi &\geq \alpha \int_{Q_\rho \cap T_{-x}D} |\xi + P_{\rho,x}(\nabla v_{T,k}^{\delta\rho,\varepsilon})(x,y)|^2 \mu_\rho(y) dy \\ &= \alpha \int_{Q_\rho \cap T_{-x}D} |\xi|^2 \mu_\rho(y) dy + \alpha \int_{Q_{\delta\rho} \cap T_{-x}D} |P_{\rho,x}(\nabla v_{T,k}^{\delta\rho,\varepsilon})(x,y)|^2 \mu_\rho(y) \\ &\quad + 2\alpha \xi \cdot \underbrace{\int_{Q_\rho \cap T_{-x}D} P_{\rho,x}(\nabla v_{T,k}^{\delta\rho,\varepsilon})(x,y) \mu_\rho(y) dy}_{\stackrel{(4.3)}{=} 0} \\ &\stackrel{(4.11)}{\geq} \gamma \alpha |\xi|^2. \end{aligned}$$

We turn now to the upper bound. Since $P_{\rho,x}$ is the $L^2(Q_\rho \cap T_{-x}D)$ -projection on μ_ρ -mean free functions, for all $\psi \in L^2(Q_\rho \cap T_{-x}D)$,

$$\int_{Q_\rho \cap T_{-x}D} (P_{x,\rho}\psi)^2(y) \mu_\rho(y) dy \leq \int_{Q_\rho \cap T_{-x}D} \psi^2(y) \mu_\rho(y) dy. \quad (4.12)$$

By uniform ellipticity of A_ε , definition of μ_ρ , expanding the square as above, and using (4.3) and (4.12), we have

$$\begin{aligned}
\xi \cdot A_{T,k,\rho,\varepsilon}^\delta(x)\xi &\leq \beta \int_{Q_\rho \cap T_{-x}D} |\xi + P_{\rho,x}(\nabla v_{T,k}^{\delta\rho,\varepsilon})(x,y)|^2 \mu_\rho(y) dy & (4.13) \\
&\stackrel{(4.3)}{=} \beta |\xi|^2 \int_{Q_\rho \cap T_{-x}D} \mu_\rho(y) dy + \beta \int_{Q_\rho \cap T_{-x}D} |P_{\rho,x}(\xi + \nabla v_{T,k}^{\delta\rho,\varepsilon})(x,y)|^2 \mu_\rho(y) dy \\
&\stackrel{(4.12)}{\leq} \beta 2^d |\xi|^2 + \beta \int_{Q_\rho \cap T_{-x}D} |\xi + \nabla v_{T,k}^{\delta\rho,\varepsilon}(x,y)|^2 \mu_\rho(y) dy \\
&\leq \beta 2^d |\xi|^2 + \beta \rho^{-d} \int_{Q_{\delta\rho} \cap T_{-x}D} |\xi + \nabla v_{T,k}^{\delta\rho,\varepsilon}(x,y)|^2 dy \\
&\leq \beta 2^d (1 + \delta^d) |\xi|^2,
\end{aligned}$$

where in the last line we used an energy estimate for $v_{T,k}^{\delta\rho,\varepsilon}(x, \cdot)$. In the sequel we shall refer to (4.13) as an estimate of the $L^2(Q_\rho \cap T_{-x}D)$ -norm of $\xi + P_{\rho,x}(\nabla v_{T,k}^{\delta\rho,\varepsilon})(x, \cdot)$.

Step 2. From local ergodicity to ergodicity.

In this step we introduce a proxy $\tilde{A}_{T,k,\rho,\varepsilon}^\delta(x)$ for $A_{T,k,\rho,\varepsilon}^\delta(x)$ which is uniformly close to $A_{T,k,\rho,\varepsilon}^\delta(x)$ in ρ , and for which we can use the results of Section 2. For all $x \in D$, we define $\tilde{A}_{T,k,\rho,\varepsilon}^\delta(x)$ by: For all unit vectors $\xi \in \mathbb{R}^d$,

$$\begin{aligned}
\xi \cdot \tilde{A}_{T,k,\rho,\varepsilon}^\delta(x)\xi &:= \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \\
&\quad \cdot \tilde{A}(x, \frac{x+y}{\varepsilon})(\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho dy,
\end{aligned}$$

where $\tilde{v}_{T,k}^{\delta\rho,\varepsilon}(x, \cdot)$ is (for $k > 1$) the Richardson extrapolation of the unique weak solution $\tilde{v}_{T,1}^{\delta\rho,\varepsilon}(x, \cdot) \in H_0^1(Q_{\delta\rho} \cap T_{-x}D)$ of

$$(T\varepsilon^2)^{-1} \tilde{v}_{T,1}^{\delta\rho,\varepsilon}(x,y) - \nabla \cdot \tilde{A}(x, \frac{x+y}{\varepsilon})(\xi + \nabla_y \tilde{v}_{T,1}^{\delta\rho,\varepsilon}(x,y)) = 0.$$

We shall prove that

$$|A_{T,k,\rho,\varepsilon}^\delta(x) - \tilde{A}_{T,k,\rho,\varepsilon}^\delta(x)| \lesssim \rho \tag{4.14}$$

uniformly in x, T, ε , the randomness, and where the multiplicative constant depends on k next to α, β and d . We rewrite the difference as: For all $\xi \in \mathbb{R}^d$ with $|\xi| = 1$,

$$\begin{aligned}
& \xi \cdot (A_{T,k,\rho,\varepsilon}^\delta(x) - \tilde{A}_{T,k,\rho,\varepsilon}^\delta(x))\xi \\
&= \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \cdot A_\varepsilon(x+y) (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy \\
&\quad - \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \cdot \tilde{A}(x, \frac{x+y}{\varepsilon}) (\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy \\
&= \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \cdot \tilde{A}(x, \frac{x+y}{\varepsilon}) (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy \\
&\quad - \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \cdot \tilde{A}(x, \frac{x+y}{\varepsilon}) (\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy \\
&\quad + \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \\
&\quad \quad \cdot (A_\varepsilon(x+y) - \tilde{A}(x, \frac{x+y}{\varepsilon})) (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy.
\end{aligned}$$

Using that

$$|A_\varepsilon(x+y) - \tilde{A}(x, \frac{x+y}{\varepsilon})| = |\tilde{A}(x+y, \frac{x+y}{\varepsilon}) - \tilde{A}(x, \frac{x+y}{\varepsilon})| \leq \kappa|y|, \quad (4.15)$$

that $|\xi| = 1$, and concluding as for (4.13), we may bound the third term of the RHS:

$$\begin{aligned}
& \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \\
&\quad \cdot (A_\varepsilon(x+y) - \tilde{A}(x, \frac{x+y}{\varepsilon})) (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy \\
&\quad \leq \kappa\rho \int_{Q_\rho \cap T_{-x}D} |\xi + \nabla_y v_{T,k}^{\delta\rho,\varepsilon}(x,y)|^2 \mu_\rho(y) dy \lesssim \rho. \quad (4.16)
\end{aligned}$$

We now turn to the first two terms, which we write in the form

$$\begin{aligned}
& \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \cdot \tilde{A}(x, \frac{x+y}{\varepsilon}) (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy \\
&\quad - \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \cdot \tilde{A}(x, \frac{x+y}{\varepsilon}) (\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy \\
&= \int_{Q_\rho \cap T_{-x}D} P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon} - \nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y) \\
&\quad \quad \cdot \tilde{A}(x, \frac{x+y}{\varepsilon}) (\xi + P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon})(x,y)) \mu_\rho(y) dy \\
&\quad + \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y)) \\
&\quad \quad \cdot \tilde{A}(x, \frac{x+y}{\varepsilon}) P_{\rho,x}(\nabla_y v_{T,k}^{\delta\rho,\varepsilon} - \nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon})(x,y) \mu_\rho(y) dy,
\end{aligned}$$

so that by the continuity of $P_{\rho,x}$ on $L^2(Q_\rho \cap T_{-x}D)$ and using the same a priori estimate as for (4.16) (following the argument for (4.13)), it enough to prove that

$$\int_{Q_{\delta\rho} \cap T_{-x}D} |\nabla_y v_{T,k}^{\delta\rho,\varepsilon}(x,y) - \nabla_y \tilde{v}_{T,k}^{\delta\rho,\varepsilon}(x,y)|^2 dy \lesssim \rho^2 \quad (4.17)$$

in order to deduce (4.14) from (4.16). By definition of the Richardson extrapolation, this estimate directly follows from the Lipschitz bound (4.15) and an energy estimate on the equation satisfied by $v_{T,1}^{\delta\rho,\varepsilon}(x,\cdot) - \tilde{v}_{T,1}^{\delta\rho,\varepsilon}(x,\cdot)$:

$$\begin{aligned} (T\varepsilon^2)^{-1} (v_{T,1}^{\delta\rho,\varepsilon}(x,y) - \tilde{v}_{T,1}^{\delta\rho,\varepsilon}(x,y)) - \nabla_y \tilde{A}(x, \frac{x+y}{\varepsilon}) \nabla_y (v_{T,1}^{\delta\rho,\varepsilon}(x,y) - \nabla_y \tilde{v}_{T,1}^{\delta\rho,\varepsilon}(x,y)) \\ = \nabla_y \cdot (A_\varepsilon(x+y) - \tilde{A}(x, \frac{x+y}{\varepsilon})) \nabla_y v_{T,1}^{\delta\rho,\varepsilon}(x,y). \end{aligned}$$

Step 3. Limits $\rho \downarrow 0$, $T \uparrow \infty$ and $\varepsilon \downarrow 0$.

For all $x \in D$, there exists $\bar{\rho}(x) > 0$ such that $Q_{\delta\rho} \cap T_{-x}D = Q_{\delta\rho}$ for all $\rho \leq \bar{\rho}(x)$. Assume that $Q_{\delta\rho} \cap T_{-x}D = Q_{\delta\rho}$. In that case, the change of variables $z = y/\varepsilon$ allows one to interpret $\tilde{A}_{T,k,\rho,\varepsilon}^\delta(x)$ as the approximation “ $A_{T,k,R,L,p}(x)$ ” of $A_{\text{hom}}(x)$ defined in (3.9) (with $\tilde{A}(x,\cdot)$ in place of $A(\cdot)$), with the parameters

$$R = \delta \frac{\rho}{\varepsilon}, \quad L = \frac{\rho}{\varepsilon}, \quad \text{and } p \in \mathbb{N}_0. \quad (4.18)$$

By Theorem 5 we then have the almost sure convergence

$$\lim_{\varepsilon \downarrow 0} \tilde{A}_{T,k,\rho,\varepsilon}^\delta(x) = \tilde{A}_{T,k}(x), \quad (4.19)$$

where x is seen as a parameter, and $\tilde{A}_{T,k}(x)$ is associated with the stationary coefficients $y \mapsto \tilde{A}(x,y)$. Note that the limit does not depend on ρ , as expected by stationarity of $\tilde{A}(x,y)$ in y .

Using spectral analysis in the form of Theorem 1 to take the limit $T \uparrow \infty$, this yields for all $x \in D$,

$$\lim_{T \uparrow \infty} \tilde{A}_{T,k}(x) = A_{\text{hom}}(x). \quad (4.20)$$

Since for all $x \in D$, $\bar{\rho}(x) > 0$ (where the function $\bar{\rho}$ is defined at the beginning of this step), the combination of (4.14), (4.19), and (4.20) shows that for all $x \in D$

$$\limsup_{T \uparrow \infty} \limsup_{\rho \downarrow 0} \limsup_{\varepsilon \downarrow 0} |A_{T,k,\rho,\varepsilon}^\delta(x) - A_{\text{hom}}(x)| = 0,$$

and concludes the proof of (4.4).

Note that for any domain \tilde{D} compactly included in D , $\inf_{x \in \tilde{D}} \bar{\rho}(x) > 0$. If $\rho \leq \bar{\rho}(x)$, the limit $\tilde{A}_{T,k}(x)$ of $\tilde{A}_{T,k,\rho,\varepsilon}^\delta(x)$ as $\varepsilon \downarrow 0$ exists almost surely and does not depend on ρ . Hence, for all $x \in \tilde{D}$, if $\rho \leq \inf_{x \in \tilde{D}} \bar{\rho}(x)$ we thus have using (4.19) and (4.14),

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} |A_{T,k,\rho,\varepsilon}^\delta(x) - A_{\text{hom}}(x)| &\leq \lim_{\varepsilon \downarrow 0} |\tilde{A}_{T,k,\rho,\varepsilon}^\delta(x) - A_{\text{hom}}(x)| \\ &\quad + \limsup_{\varepsilon \downarrow 0} |A_{T,k,\rho,\varepsilon}^\delta(x) - \tilde{A}_{T,k,\rho,\varepsilon}^\delta(x)| \\ &\leq |\tilde{A}_{T,k}(x) - A_{\text{hom}}(x)| + \rho, \end{aligned}$$

and the uniformity of the convergence in T w. r. t. $\rho \leq \inf_{x \in \tilde{D}} \bar{\rho}(x)$ follows.

We prove (4.5) now. For all $\rho > 0$, set $D_\rho := \{x \in D \mid \rho < \bar{\rho}(x)\}$. Since D has a Lipschitz boundary,

$$\lim_{\rho \downarrow 0} |D \setminus D_\rho| = 0. \quad (4.21)$$

Let $1 \leq p < \infty$. By the uniform boundedness of A_{hom} and $A_{T,k,\rho,\varepsilon}^\delta$ and the definition of D_ρ , we have

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} \|A_{T,k,\rho,\varepsilon}^\delta - A_{\text{hom}}\|_{L^p(D)} &\leq \limsup_{\varepsilon \downarrow 0} \|A_{T,k,\rho,\varepsilon}^\delta - A_{\text{hom}}\|_{L^p(D_\rho)} \\ &\quad + \limsup_{\varepsilon \downarrow 0} \|A_{T,k,\rho,\varepsilon}^\delta - A_{\text{hom}}\|_{L^p(D \setminus D_\rho)} \\ &\lesssim \|\tilde{A}_{T,k} - A_{\text{hom}}\|_{L^p(D)} + \rho + |D \setminus D_\rho|^{\frac{1}{p}}, \end{aligned} \quad (4.22)$$

and the conclusion follows from (4.21), (4.20), and the Lebesgue dominated convergence theorem. \square

Corollary 1 is a consequence of Theorem 8 (using in particular (4.22)) and of the following lemma:

Lemma 1. *Let $(A_{h_1,h_2})_{h_1,h_2>0}, A \in \mathcal{M}_{\alpha\beta}(D)$ and $(f_{h_1,h_2})_{h_1,h_2>0}, f \in H^{-1}(D)$ be such that $f_{h_1,h_2} \rightarrow f$ in $H^{-1}(D)$ as $h_1, h_2 \downarrow 0$ (in any order), and*

- For all $x \in D$, $\lim_{h_1 \downarrow 0} \lim_{h_2 \downarrow 0} A_{h_1,h_2}(x) = A(x)$;
- There exist two bounded functions $g_1, g_2 : \mathbb{R}^+ \times D \rightarrow \mathbb{R}^+$ such that for all $x \in D$, $\lim_{h \rightarrow 0} g_1(x, h) = \lim_{h \rightarrow 0} g_2(x, h) = 0$, and for all $h_2 > 0$ there exists a measurable subset D_{h_2} of D such that $\lim_{h_2 \rightarrow 0} |D \setminus D_{h_2}| = 0$ and such that for all $x \in D_{h_2}$ and all $h_1 > 0$,

$$|A_{h_1,h_2}(x) - A(x)| \leq g_1(x, h_1) + g_2(x, h_2).$$

Then the unique weak solution $u_{h_1,h_2} \in H_0^1(D)$ of

$$-\nabla \cdot A_{h_1,h_2} \nabla u_{h_1,h_2} = f_{h_1,h_2}$$

converges in $H^1(D)$ as $h_1, h_2 \downarrow 0$ to the unique weak solution u in $H_0^1(D)$ of

$$-\nabla \cdot A \nabla u = f,$$

and the order of convergence of h_1 and h_2 is irrelevant.

Proof of Lemma 1. Subtract the weak forms of the two equations tested with the admissible test-function $u_{h_1,h_2} - u \in H_0^1(D)$. This yields

$$\int_D \nabla(u_{h_1,h_2} - u) \cdot (A_{h_1,h_2} \nabla u_{h_1,h_2} - A \nabla u) = (f_{h_1,h_2} - f, u_{h_1,h_2} - u)_{H^{-1}(D), H_0^1(D)},$$

where $(\cdot, \cdot)_{H^{-1}(D), H_0^1(D)}$ denotes the duality pairing between $H^{-1}(D)$ and $H_0^1(D)$. We rewrite this equation in the form

$$\begin{aligned} &\int_D \nabla(u_{h_1,h_2} - u) \cdot A_{h_1,h_2} \nabla(u_{h_1,h_2} - u) \\ &= - \int_D \nabla(u_{h_1,h_2} - u) \cdot (A_{h_1,h_2} - A) \nabla u + (f_{h_1,h_2} - f, u_{h_1,h_2} - u)_{H^{-1}(D), H_0^1(D)}. \end{aligned} \quad (4.23)$$

Using the ellipticity of $A_{h_1, h_2} \in \mathcal{M}_{\alpha\beta}(D)$, Cauchy-Schwarz', Young's, and Poincaré's inequalities, this turns into

$$\alpha \|\nabla(u_{h_1, h_2} - u)\|_{L^2(D)}^2 \leq \int_D |\nabla u \cdot (A_{h_1, h_2} - A)\nabla u| + \|f_{h_1, h_2} - f\|_{H^{-1}(D)}.$$

By assumption, the second term of the RHS goes to zero as $h_1, h_2 \downarrow 0$. It remains to treat the first term. We split the integral into two terms:

$$\begin{aligned} \int_D |\nabla u \cdot (A_{h_1, h_2} - A)\nabla u| &= \int_{D_{h_2}} |\nabla u \cdot (A_{h_1, h_2} - A)\nabla u| + \int_{D \setminus D_{h_2}} |\nabla u \cdot (A_{h_1, h_2} - A)\nabla u| \\ &\leq \int_D |\nabla u|^2 (g_1(x, h_1) + g_2(x, h_2)) + 2\beta \int_{D \setminus D_{h_2}} |\nabla u|^2. \end{aligned}$$

The first term of the RHS converges to zero as $h_1, h_2 \downarrow 0$ by the Lebesgue dominated convergence theorem, while the second term converges to zero as $h_2 \downarrow 0$ because $|D \setminus D_{h_2}| \downarrow 0$ and $\nabla u \in L^2(D)$. This concludes the proof of the lemma. \square

The proof of Theorem 9, which is rather long and technical, is somewhat counter-intuitive: Although we expect the regularization and extrapolation method to enhance the convergence, our proof rather shows that the method does not worsen the convergence of the standard method based on Dirichlet boundary conditions (without regularization and extrapolation) analyzed in [15]. The proof has essentially two parts: First we prove the numerical corrector result with the standard approximation of the corrector, and we then prove that the difference between that approximation and the one with regularization and extrapolation vanishes in a suitable sense. For the first part, the main idea is to use Tartar's correctors on each element $Q_{H,i} \cap D$ of the partition of D , pass to the limit in ε first, and then in H . The second part of the proof makes crucial use of the ergodic theorem and of Theorem 1.

Proof of Theorem 9. We divide the proof into four steps.

Step 1. Standard corrector result.

For all $1 \leq i \leq I_H$ we define $\gamma_\varepsilon^{H,i}$ as the unique weak solution in $H_0^1(Q_{H,i} \cap D)$ of

$$-\nabla \cdot A_\varepsilon(M_H(\nabla u_{\text{hom}}) + \nabla \gamma_\varepsilon^{H,i}) = 0, \quad (4.24)$$

that we extend by zero on $D \setminus Q_{H,i}$ as a function of $H_0^1(D)$. We then define the following variant of Tartar's corrector:

$$C_\varepsilon^H := \sum_{i=1}^{I_H} \nabla u_\varepsilon^{H,i},$$

where $\nabla u_\varepsilon^{H,i} \in L^2(D)$ has support in $Q_{H,i} \cap D$ and is given by

$$\nabla u_\varepsilon^{H,i} := \left(M_H(\nabla u_{\text{hom}}) + \nabla \gamma_\varepsilon^{H,i} \right) \mathbf{1}_{Q_{H,i} \cap D}. \quad (4.25)$$

The aim of this step is to prove that almost surely

$$\lim_{H \downarrow 0} \lim_{\varepsilon \downarrow 0} \|\nabla u_\varepsilon - C_\varepsilon^H\|_{L^2(D)} = 0. \quad (4.26)$$

As we shall see this result is deterministic and follows from H-convergence.

By ellipticity and symmetry of A_ε ,

$$\begin{aligned} \|\nabla u_\varepsilon - C_\varepsilon^H\|_{L^2(D)}^2 &\lesssim \int_D (\nabla u_\varepsilon - C_\varepsilon^H) \cdot A_\varepsilon (\nabla u_\varepsilon - C_\varepsilon^H) \\ &= \int_D \nabla u_\varepsilon \cdot A_\varepsilon \nabla u_\varepsilon + \int_D C_\varepsilon^H \cdot A_\varepsilon \nabla C_\varepsilon^H \\ &\quad - 2 \sum_{i=1}^{I_H} \int_D \nabla u_\varepsilon^{H,i} \cdot A_\varepsilon \nabla u_\varepsilon. \end{aligned} \quad (4.27)$$

Since $\gamma_\varepsilon^{H,i} \in H_0^1(D)$, the last term of this identity takes the form

$$\sum_{i=1}^{I_H} \int_D \nabla u_\varepsilon^{H,i} \cdot A_\varepsilon \nabla u_\varepsilon = \sum_{i=1}^{I_H} (f, \gamma_\varepsilon^{H,i})_{H^{-1}(D), H_0^1(D)} + \int_D M_H(\nabla u_{\text{hom}}) \cdot A_\varepsilon \nabla u_\varepsilon$$

using the weak form of the defining equation for u_ε . We now argue that

$$\lim_{H \downarrow 0} \lim_{\varepsilon \downarrow 0} \sum_{i=1}^{I_H} \int_D \nabla u_\varepsilon^{H,i} \cdot A_\varepsilon \nabla u_\varepsilon = \int_D \nabla u_{\text{hom}} \cdot A_{\text{hom}} \nabla u_{\text{hom}}. \quad (4.28)$$

Indeed, on the one hand, since $A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_{\text{hom}} \nabla u_{\text{hom}}$ weakly in $L^2(D, \mathbb{R}^d)$ by homogenization and M_H converges to the identity on $L^2(D, \mathbb{R}^d)$,

$$\lim_{H \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_D M_H(\nabla u_{\text{hom}}) \cdot A_\varepsilon \nabla u_\varepsilon = \int_D \nabla u_{\text{hom}} \cdot A_{\text{hom}} \nabla u_{\text{hom}}. \quad (4.29)$$

On the other hand, homogenization ensures that $\gamma_\varepsilon^{H,i} \xrightarrow{\varepsilon \downarrow 0} \gamma_{\text{hom}}^{H,i}$ weakly in $H_0^1(Q_{H,i} \cap D)$ (and therefore weakly in $H_0^1(D)$) almost surely, where $\gamma_{\text{hom}}^{H,i}$ is the unique weak solution in $H_0^1(Q_{H,i} \cap D)$ (extended by zero as a $H_0^1(D)$ function on $D \setminus Q_{H,i}$) of

$$-\nabla \cdot A_{\text{hom}}(M_H(\nabla u_{\text{hom}}) + \nabla \gamma_{\text{hom}}^{H,i}) = 0.$$

Since A_{hom} is κ -Lipschitz, for all $x \in Q_{H,i} \cap D$, $|A_{\text{hom}}(x) - M_{H,i}(A_{\text{hom}})| \lesssim \kappa H$. Hence, an energy estimate yields

$$\begin{aligned} \int_{Q_{H,i} \cap D} \nabla \gamma_{\text{hom}}^{H,i} \cdot A_{\text{hom}} \nabla \gamma_{\text{hom}}^{H,i} &= \int_{Q_{H,i} \cap D} \nabla \gamma_{\text{hom}}^{H,i} \cdot A_{\text{hom}} M_H(\nabla u_{\text{hom}}) \\ &= \left(\int_{Q_{H,i} \cap D} \nabla \gamma_{\text{hom}}^{H,i} \right) \cdot M_{H,i}(A_{\text{hom}}) M_{H,i}(\nabla u_{\text{hom}}) \\ &\quad + \int_{Q_{H,i} \cap D} \nabla \gamma_{\text{hom}}^{H,i} \cdot (A_{\text{hom}} - M_H(A_{\text{hom}})) M_H(\nabla u_{\text{hom}}) \\ &\lesssim \kappa H \|\nabla \gamma_{\text{hom}}^{H,i}\|_{L^2(Q_{H,i} \cap D)} \|\nabla u_{\text{hom}}\|_{L^2(Q_{H,i} \cap D)}, \end{aligned}$$

since $\int_{Q_{H,i} \cap D} \nabla \gamma_{\text{hom}}^{H,i} = 0$ and $M_{H,i}$ is a contraction. Setting $\gamma_{\text{hom}}^H = \sum_{i=1}^{I_H} \gamma_{\text{hom}}^{H,i} 1_{Q_{H,i} \cap D} \in H_0^1(D)$, this implies by Poincaré's inequality

$$\|\gamma_{\text{hom}}^H\|_{H^1(D)} \lesssim H \|\nabla u_{\text{hom}}\|_{L^2(D)} \xrightarrow{H \downarrow 0} 0,$$

so that

$$\lim_{H \downarrow 0} \lim_{\varepsilon \downarrow 0} \sum_{i=1}^{I_H} \left(f, \gamma_\varepsilon^{H,i} \right)_{H^{-1}(D), H_0^1(D)} = \lim_{H \downarrow 0} \left(f, \gamma_{\text{hom}}^H \right)_{H^{-1}(D), H_0^1(D)} = 0. \quad (4.30)$$

Estimate (4.28) then follows from (4.29) and (4.30).

We turn now to the first term of the RHS of (4.27). Since H-convergence implies the convergence of the energy, we have

$$\lim_{\varepsilon \downarrow 0} \int_D \nabla u_\varepsilon \cdot A_\varepsilon \nabla u_\varepsilon = \int_D \nabla u_{\text{hom}} \cdot A_{\text{hom}} \nabla u_{\text{hom}}. \quad (4.31)$$

Likewise, for all $1 \leq i \leq I_H$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{Q_{H,i} \cap D} (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_\varepsilon^{H,i}) \cdot A_\varepsilon (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_\varepsilon^{H,i}) \\ = \int_{Q_{H,i} \cap D} (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_{\text{hom}}^{H,i}) \cdot A_{\text{hom}} (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_{\text{hom}}^{H,i}), \end{aligned}$$

so that by summation over i and defining $\gamma_\varepsilon^H := \sum_{i=1}^{I_H} \gamma_\varepsilon^{H,i} \in H_0^1(D)$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_D (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_\varepsilon^H) \cdot A_\varepsilon (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_\varepsilon^H) \\ = \int_D (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_{\text{hom}}^H) \cdot A_{\text{hom}} (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_{\text{hom}}^H). \end{aligned}$$

Since M_H converges to the identity on $L^2(D)$ and γ_{hom}^H converges to zero in $H^1(D)$, this implies in particular

$$\lim_{H \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_D (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_\varepsilon^H) \cdot A_\varepsilon (M_H(\nabla u_{\text{hom}}) + \nabla \gamma_\varepsilon^H) = \int_D \nabla u_{\text{hom}} \cdot A_{\text{hom}} \nabla u_{\text{hom}}. \quad (4.32)$$

The combination of (4.27), (4.28), (4.31), and (4.32) yields (4.26).

Step 2. First auxiliary numerical corrector and reformulation of (4.10).

In view of (4.26), it enough to prove that almost-surely for all $k' \in \mathbb{N}$,

$$\limsup_{T \uparrow \infty, H, \rho \downarrow 0} \limsup_{\varepsilon \downarrow 0} \|C_{T,k,k',\rho,\varepsilon}^{\delta,H} - C_\varepsilon^H\|_{L^2(D)} = 0. \quad (4.33)$$

We introduce the auxiliary numerical corrector $C_{T,k',\varepsilon}^{\delta,H} \in L^2(D, \mathbb{R}^d)$ defined as follows. The function $\gamma_{T,1,\varepsilon}^{\delta,H,i}$ is associated with u_{hom} as the unique weak solution in $H_0^1(Q_{H,i}^\delta \cap D)$ of

$$(T\varepsilon^2)^{-1} \gamma_{T,1,\varepsilon}^{\delta,H,i} - \nabla \cdot A_\varepsilon \left(M_{H,i}(\nabla u_{\text{hom}}) + \nabla \gamma_{T,1,\varepsilon}^{\delta,H,i} \right) = 0, \quad (4.34)$$

and we define $\gamma_{T,k',\varepsilon}^{\delta,H,i}$ for all $k' > 1$ by Richardson extrapolation, cf. Definition 2. We then set for all $k' \in \mathbb{N}$ and for all $1 \leq i \leq I_H$,

$$\nabla u_{T,k',\varepsilon}^{\delta,H,i} := M_H(\nabla u_{\text{hom}})|_{Q_{H,i} \cap D} + (\nabla \gamma_{T,k',\varepsilon}^{\delta,H,i})|_{Q_{H,i} \cap D} \in L^2(Q_{H,i} \cap D),$$

and finally define the numerical corrector

$$C_{T,k',\varepsilon}^{\delta,H} = \sum_{i=1}^{I_H} \nabla u_{T,k',\varepsilon}^{\delta,H,i} 1_{Q_{H,i}}.$$

Subtracting the equations (4.9) and (4.34) for $\gamma_{T,k,1,\rho,\varepsilon}^{\delta,H,i}$ and $\gamma_{T,1,\varepsilon}^{\delta,H,i}$, respectively, yields the following energy estimate

$$\|\nabla \gamma_{T,k,1,\rho,\varepsilon}^{\delta,H,i} - \nabla \gamma_{T,1,\varepsilon}^{\delta,H,i}\|_{L^2(Q_{H,i}^\delta \cap D)}^2 \lesssim |Q_{H,i}^\delta \cap D| |M_{H,i}(\nabla u_{T,k,\rho,\varepsilon}^\delta - \nabla u_{\text{hom}})|^2, \quad (4.35)$$

so that, using that M_H is a contraction on $L^2(D, \mathbb{R}^d)$,

$$\|C_{T,k,1,\rho,\varepsilon}^{\delta,H} - C_{T,1,\varepsilon}^{\delta,H}\|_{L^2(D)} \lesssim \|\nabla u_{T,k,\rho,\varepsilon}^\delta - \nabla u_{\text{hom}}\|_{L^2(D)}.$$

By definition of the Richardson extrapolation this yields for all $k' \in \mathbb{N}$,

$$\|C_{T,k,k',\rho,\varepsilon}^{\delta,H} - C_{T,k',\varepsilon}^{\delta,H}\|_{L^2(D)} \lesssim \|\nabla u_{T,k,\rho,\varepsilon}^\delta - \nabla u_{\text{hom}}\|_{L^2(D)},$$

and therefore by Corollary 1:

$$\limsup_{T \uparrow \infty, \rho \downarrow 0} \limsup_{\varepsilon \downarrow 0} \|C_{T,k,k',\rho,\varepsilon}^{\delta,H} - C_{T,k',\varepsilon}^{\delta,H}\|_{L^2(D)} = 0, \quad (4.36)$$

uniformly with respect to H , and almost surely. Hence, the claim (4.33) follows from (4.36) provided we prove that almost-surely

$$\lim_{T \uparrow \infty, H \downarrow 0} \lim_{\varepsilon \downarrow 0} \|C_{T,k',\varepsilon}^{\delta,H} - C_\varepsilon^H\|_{L^2(D)} = 0. \quad (4.37)$$

Step 3. Further auxiliary numerical correctors and reformulation of (4.37).

For all $1 \leq i \leq I_H$, fix some arbitrary $x_i \in Q_{H,i}$ and set for all $x \in Q_{H,i}^\delta$, $A_{\varepsilon,i}(x) := \tilde{A}(x_i, \frac{x}{\varepsilon})$. The approximation of A_ε by $A_{\varepsilon,i}$ on $Q_{H,i}$ (and $Q_{H,i}^\delta$) allows us to appeal to the results we proved in Sections 2 and 3 for *stationary* coefficients. The error we make by replacing A_ε by $A_{\varepsilon,i}$ is then controlled using the Lipschitz condition in Hypothesis 1.

We introduce the auxiliary numerical correctors $\tilde{C}_\varepsilon^H, \tilde{C}_{T,k',\varepsilon}^{\delta,H} \in L^2(D, \mathbb{R}^d)$ associated with \tilde{A}_ε as follows. The functions $\tilde{\gamma}_\varepsilon^{H,i}$ and $\tilde{\gamma}_{T,1,\varepsilon}^{\delta,H,i}$ are the unique weak solutions in $H_0^1(Q_{H,i} \cap D)$ and $H_0^1(Q_{H,i}^\delta \cap D)$, respectively, of

$$-\nabla \cdot A_{\varepsilon,i} \left(M_{H,i}(\nabla u_{\text{hom}}) + \nabla \tilde{\gamma}_\varepsilon^{H,i} \right) = 0, \quad (4.38)$$

$$(T\varepsilon^2)^{-1} \tilde{\gamma}_{T,1,\varepsilon}^{\delta,H,i} - \nabla \cdot A_{\varepsilon,i} \left(M_{H,i}(\nabla u_{\text{hom}}) + \nabla \tilde{\gamma}_{T,1,\varepsilon}^{\delta,H,i} \right) = 0, \quad (4.39)$$

and $\tilde{\gamma}_{T,k',\varepsilon}^{\delta,H,i}$ is defined for all $k' > 1$ by Richardson extrapolation of $\tilde{\gamma}_{T,1,\varepsilon}^{\delta,H,i}$, cf. Definition 2. We then set for all $k' \in \mathbb{N}$ and for all $1 \leq i \leq I_H$,

$$\begin{aligned} \nabla \tilde{u}_\varepsilon^{H,i} &:= M_H(\nabla u_{\text{hom}})|_{Q_{H,i} \cap D} + (\nabla \tilde{\gamma}_\varepsilon^{H,i})|_{Q_{H,i} \cap D} \in L^2(Q_{H,i} \cap D), \\ \nabla \tilde{u}_{T,k',\varepsilon}^{\delta,H,i} &:= M_H(\nabla u_{\text{hom}})|_{Q_{H,i} \cap D} + (\nabla \tilde{\gamma}_{T,k',\varepsilon}^{\delta,H,i})|_{Q_{H,i} \cap D} \in L^2(Q_{H,i} \cap D), \end{aligned}$$

and finally define the numerical correctors

$$\tilde{C}_\varepsilon^H = \sum_{i=1}^{I_H} \nabla \tilde{u}_\varepsilon^{H,i} 1_{Q_{H,i} \cap D}, \quad \tilde{C}_{T,k',\varepsilon}^{\delta,H} = \sum_{i=1}^{I_H} \nabla \tilde{u}_{T,k',\varepsilon}^{\delta,H,i} 1_{Q_{H,i} \cap D}.$$

From (4.24) and (4.38) on the one hand, and (4.34) and (4.39) on the other hand, we infer that

$$-\nabla \cdot A_\varepsilon \nabla (\gamma_\varepsilon^{H,i} - \tilde{\gamma}_\varepsilon^{H,i}) = -\nabla \cdot (A_{\varepsilon,i} - A_\varepsilon) (M_H(\nabla u_{\text{hom}}) + \nabla \tilde{\gamma}_\varepsilon^{H,i}) \quad \text{in } Q_{H,i} \cap D,$$

and

$$\begin{aligned} (T\varepsilon^2)^{-1} (\gamma_{T,1,\varepsilon}^{\delta,H,i} - \tilde{\gamma}_{T,1,\varepsilon}^{\delta,H,i}) - \nabla \cdot A_\varepsilon \nabla (\gamma_{T,1,\varepsilon}^{\delta,H,i} - \tilde{\gamma}_{T,1,\varepsilon}^{\delta,H,i}) \\ = -\nabla \cdot (A_{\varepsilon,i} - A_\varepsilon) (M_H(\nabla u_{\text{hom}}) + \nabla \tilde{\gamma}_{T,1,\varepsilon}^{\delta,H,i}) \quad \text{in } Q_{H,i}^\delta \cap D, \end{aligned}$$

so that elementary energy estimates combined with the fact that $\tilde{A}(\cdot, \cdot)$ is κ -Lipschitz with respect to its first variable yields

$$\begin{aligned} \|\nabla \gamma_\varepsilon^{H,i} - \nabla \tilde{\gamma}_\varepsilon^{H,i}\|_{L^2(Q_{H,i} \cap D)}^2 &\lesssim (\kappa H)^2 |Q_{H,i} \cap D| |M_{H,i}(\nabla u_{\text{hom}})|^2, \\ \|\nabla \gamma_{T,1,\varepsilon}^{\delta,H,i} - \nabla \tilde{\gamma}_{T,1,\varepsilon}^{\delta,H,i}\|_{L^2(Q_{H,i} \cap D)}^2 &\lesssim (\kappa H)^2 |Q_{H,i}^\delta \cap D| |M_{H,i}(\nabla u_{\text{hom}})|^2, \end{aligned}$$

from which we deduce, using that M_H is a contraction,

$$\begin{aligned} \|C_\varepsilon^H - \tilde{C}_\varepsilon^H\|_{L^2(D)} &\lesssim H \|\nabla u_{\text{hom}}\|_{L^2(D)}, \\ \|C_{T,k',\varepsilon}^{\delta,H} - \tilde{C}_{T,k',\varepsilon}^{\delta,H}\|_{L^2(D)} &\lesssim H \|\nabla u_{\text{hom}}\|_{L^2(D)}, \end{aligned}$$

uniformly in ε, T, k' , and the realization. To prove (4.37) it is therefore enough to prove that almost surely

$$\lim_{T \uparrow \infty, H \downarrow 0} \lim_{\varepsilon \downarrow 0} \|\tilde{C}_{T,k',\varepsilon}^{\delta,H} - \tilde{C}_\varepsilon^H\|_{L^2(D)} = 0. \quad (4.40)$$

Step 4. Proof of (4.40).

Define $J_1^H := \{1 \leq i \leq I_H \mid Q_{H,i}^\delta \cap D \neq Q_{H,i}^\delta\}$ and $J_2^H := \{1 \leq i \leq I_H, i \notin J_1^H\}$. It suffices to prove that

$$\lim_{T \uparrow \infty, H \downarrow 0} \lim_{\varepsilon \downarrow 0} \|\tilde{C}_{T,k',\varepsilon}^{\delta,H} - \tilde{C}_\varepsilon^H\|_{L^2(D \cap \cup_{i \in J_1^H} Q_{H,i})} = 0, \quad (4.41)$$

$$\lim_{T \uparrow \infty, H \downarrow 0} \lim_{\varepsilon \downarrow 0} \|\tilde{C}_{T,k',\varepsilon}^{\delta,H} - \tilde{C}_\varepsilon^H\|_{L^2(\cup_{i \in J_2^H} Q_{H,i})} = 0. \quad (4.42)$$

We start with the proof of (4.41). Elementary energy estimates based on (4.38) and (4.39) show that

$$\begin{aligned} \|\tilde{C}_{T,k',\varepsilon}^{\delta,H} - \tilde{C}_\varepsilon^H\|_{L^2(D \cap \cup_{i \in J_1^H} Q_{H,i})} &\leq \|\tilde{C}_{T,k',\varepsilon}^{\delta,H}\|_{L^2(D \cap \cup_{i \in J_1^H} Q_{H,i})} + \|\tilde{C}_\varepsilon^H\|_{L^2(D \cap \cup_{i \in J_1^H} Q_{H,i})} \\ &\lesssim \|M_H(\nabla u_{\text{hom}})\|_{L^2(D \cap \cup_{i \in J_1^H} Q_{H,i})} \leq \|\nabla u_{\text{hom}}\|_{L^2(D \cap \cup_{i \in J_1^H} Q_{H,i})} \end{aligned}$$

uniformly in T and ε . Since the measure of the domain of integration $D \cap \cup_{i \in J_1^H} Q_{H,i}$ vanishes as $H \downarrow 0$ and since $\nabla u_{\text{hom}} \in L^2(D)$, estimate (4.41) follows.

We turn now to (4.42), which, by ellipticity of \tilde{A} , is equivalent to

$$\lim_{T \uparrow \infty, H \downarrow 0} \lim_{\varepsilon \downarrow 0} \sum_{i \in J_2^H} \int_{Q_{H,i}} (\tilde{C}_{T,k',\varepsilon}^{\delta,H} - \tilde{C}_\varepsilon^H) \cdot A_{\varepsilon,i} (\tilde{C}_{T,k',\varepsilon}^{\delta,H} - \tilde{C}_\varepsilon^H) = 0.$$

By expanding the square it is enough to prove the following three statements for all $i \in J_2^H$ almost surely:

$$\lim_{\varepsilon \downarrow 0} \int_{Q_{H,i}} \tilde{C}_\varepsilon^H \cdot A_{\varepsilon,i} \tilde{C}_\varepsilon^H = M_{H,i}(\nabla u_{\text{hom}}) \cdot A_{\text{hom}}(x_i) M_{H,i}(\nabla u_{\text{hom}}), \quad (4.43)$$

$$\lim_{\varepsilon \downarrow 0} \int_{Q_{H,i}} \tilde{C}_{T,k',\varepsilon}^{\delta,H} \cdot A_{\varepsilon,i} \tilde{C}_\varepsilon^H = M_{H,i}(\nabla u_{\text{hom}}) \cdot A_{\text{hom}}(x_i) M_{H,i}(\nabla u_{\text{hom}}), \quad (4.44)$$

$$\lim_{T \uparrow \infty} \lim_{\varepsilon \downarrow 0} \int_{Q_{H,i}} \tilde{C}_{T,k',\varepsilon}^{\delta,H} \cdot A_{\varepsilon,i} \tilde{C}_{T,k',\varepsilon}^{\delta,H} = M_{H,i}(\nabla u_{\text{hom}}) \cdot A_{\text{hom}}(x_i) M_{H,i}(\nabla u_{\text{hom}}). \quad (4.45)$$

Identity (4.43) follows from the convergence of the energy associated with the homogenization of (4.38) and the fact that the associated homogenized coefficients are constant (with value $A_{\text{hom}}(x_i)$, since on $Q_{H,i}$, $A_{\varepsilon,i}$ is given by the stationary coefficients $x \mapsto \tilde{A}(x_i, \frac{x}{\varepsilon})$).

From Theorem 5, we learn that almost surely

$$\lim_{\varepsilon \downarrow 0} \int_{Q_{H,i}} \tilde{C}_{T,k',\varepsilon}^{\delta,H} \cdot A_{\varepsilon,i} \tilde{C}_{T,k',\varepsilon}^{\delta,H} = M_{H,i}(\nabla u_{\text{hom}}) \cdot A_{T,k'}(x_i) M_{H,i}(\nabla u_{\text{hom}}),$$

where $A_{T,k'}(x_i)$ is the approximation of the homogenized coefficients associated with the stationary field $x \mapsto \tilde{A}(x_i, x)$ of Definition 3. Estimate (4.45) is now a consequence of Theorem 1.

We conclude with the proof of (4.44). By compensated compactness, since $\tilde{C}_{T,k',\varepsilon}^{\delta,H}$ is curl-free and $A_{\varepsilon,i} \tilde{C}_\varepsilon^H$ is divergence-free on $Q_{H,i}$, it is enough to prove the following almost sure weak convergences in $L^2(Q_{H,i}, \mathbb{R}^d)$:

$$A_{\varepsilon,i} \tilde{C}_\varepsilon^H \rightharpoonup A_{\text{hom}}(x_i) M_{H,i}(\nabla u_{\text{hom}}), \quad \tilde{C}_{T,k',\varepsilon}^{\delta,H} \rightharpoonup M_{H,i}(\nabla u_{\text{hom}}).$$

The first convergence is a consequence of homogenization, which implies convergence of the fluxes. For the second convergence, recall that on $Q_{H,i}$, $\tilde{C}_{T,k',\varepsilon}^{\delta,H} = \nabla \tilde{\gamma}_{T,k',\varepsilon}^{\delta,H,i} + M_{H,i}(\nabla u_{\text{hom}})$, so that we need to prove that $\nabla \tilde{\gamma}_{T,k',\varepsilon}^{\delta,H,i} \rightharpoonup 0$ on $Q_{H,i}$ almost surely. Denote by $\tilde{\gamma}_{T,k',\varepsilon}^i \in H_{\text{loc}}^1(\mathbb{R}^d)$ the Richardson extrapolations of the unique solution $\tilde{\gamma}_{T,1,\varepsilon}^i \in H_{\text{loc}}^1(\mathbb{R}^d)$ of

$$(T\varepsilon^2)^{-1} \tilde{\gamma}_{T,1,\varepsilon}^i - \nabla \cdot \tilde{A}(x_i, \frac{\cdot}{\varepsilon})(M_{H,i}(\nabla u_{\text{hom}}) + \nabla \tilde{\gamma}_{T,1,\varepsilon}^i) = 0 \quad \text{in } \mathbb{R}^d,$$

satisfying (2.4). By (3.4) in Remark 4, we then have (after rescaling by ε):

$$\int_{Q_{H,i}} |\nabla \tilde{\gamma}_{T,k',\varepsilon}^{\delta,H,i} - \nabla \tilde{\gamma}_{T,k',\varepsilon}^i|^2 \lesssim T \exp\left(-c(\delta-1) \frac{H}{\varepsilon} \frac{1}{\sqrt{2^{k'-1}T}}\right).$$

Hence, $\nabla \tilde{\gamma}_{T,k',\varepsilon}^{\delta,H,i}$ weakly converges to zero as $\varepsilon \downarrow 0$ if and only if $\nabla \tilde{\gamma}_{T,k',\varepsilon}^i$ weakly converges to zero. But the latter is a consequence of the ergodic theorem since by construction $\nabla \tilde{\gamma}_{T,k',\varepsilon}^i$ is the ε -rescaled version of the gradient of the corrector, which is stationary and the expectation of which vanishes. □

4.2. Discretization of the analytical framework. In this paragraph we discuss a possible way to discretize the analytical framework introduced in this section. We only present the ‘‘direct approach’’, which allows us to obtain a new variant of the HMM (see [9, 1, 8, 10],[4] for the original method) with regularization and extrapolation. In particular, we shall prove the almost-sure convergence of this numerical approximation

under Hypothesis 1. Note that in [15, 16, 19] we have also introduced a “dual approach”, which allows one to recover variants of the MsFEM (see [25, 26, 12, 27, 3, 11]). We defer the analysis of the dual approach, which is more technical and requires additional ideas, to a subsequent contribution.

In order to solve (4.6), one has first to approximate the coefficients $A_{T,k,\rho,\varepsilon}^\delta$. Let $\xi \in \mathbb{R}^d$. For all $x \in D$, denote by $V_h(x)$ a family of dense subspaces of $H_0^1(Q_{\delta\rho} \cap T_{-x}D)$, and consider the Galerkin approximation $v_{T,k,h}^{\delta\rho,\varepsilon}(x, \cdot)$ of $v_{T,k}^{\delta\rho,\varepsilon}(x, \cdot)$ in $V_h(x)$ (cf. Definition 5), and, following (4.1), set:

$$\begin{aligned} \xi \cdot A_{T,k,\rho,\varepsilon}^{\delta,h}(x)\xi &:= \int_{Q_\rho \cap T_{-x}D} (\xi + P_{\rho,x}(\nabla_y v_{T,k,h}^{\delta\rho,\varepsilon})(x, y)) \\ &\quad \cdot A_\varepsilon(x + y)(\xi + P_{\rho,x}(\nabla_y v_{T,k,h}^{\delta\rho,\varepsilon})(x, y))\mu_\rho(y)dy, \end{aligned}$$

where $P_{\rho,h}$ is as in (4.3).

Remark 11. Since \tilde{A} is Lipschitz-continuous in its first variable, A_{hom} is Lipschitz-continuous as well, and we can replace $A_{T,k,\rho,\varepsilon}^{\delta,h}(x)$ for $x \in D$ such that $Q_{\delta\rho} \cap T_{-x}D \neq Q_{\delta\rho}$ by the value $A_{T,k,\rho,\varepsilon}^{\delta,h}(x')$ at a point x' such that $Q_{\delta\rho} \cap T_{-x'}D = Q_{\delta\rho}$ and such that $|x - x'| \lesssim \rho$ (which is possible since D has a Lipschitz boundary). The convergence analysis is similar and this allows one to avoid possible boundary layers on $Q_\delta \cap T_{-x}D$, at the expense of the regularity of A_{hom} .

The same argument as in Step 1 of the proof of Theorem 8 shows that the approximation $A_{T,k,\rho,\varepsilon}^{\delta,h}$ of $A_{T,k,\rho,\varepsilon}^\delta$ is uniformly elliptic. Let now V_H be a family of dense Galerkin subspaces of $H_0^1(D)$, and denote by $u_{T,k,\rho,\varepsilon}^{\delta,h,H}$ the Galerkin approximation in V_H of the weak solution $u_{T,k,\rho,\varepsilon}^{\delta,h} \in H_0^1(D)$ of

$$-\nabla \cdot A_{T,k,\rho,\varepsilon}^{\delta,h} \nabla u_{T,k,\rho,\varepsilon}^{\delta,h} = f,$$

which exists and is unique by the uniform ellipticity of $A_{T,k,\rho,\varepsilon}^{\delta,h}$. As a first convergence result we have:

$$\lim_{T \uparrow \infty, \rho, H \downarrow 0} \lim_{\varepsilon \downarrow 0} \lim_{h \downarrow 0} \|u_{T,k,\rho,\varepsilon}^{\delta,h,H} - u_{\text{hom}}\|_{H_0^1(D)} = 0, \quad (4.46)$$

where the order of the limits in T, ρ, H is irrelevant. Indeed, by the triangle inequality,

$$\begin{aligned} \|u_{T,k,\rho,\varepsilon}^{\delta,h,H} - u_{\text{hom}}\|_{H_0^1(D)} &\leq \|u_{T,k,\rho,\varepsilon}^{\delta,h,H} - u_{T,k,\rho,\varepsilon}^{\delta,h}\|_{H_0^1(D)} + \|u_{T,k,\rho,\varepsilon}^{\delta,h} - u_{T,k,\rho,\varepsilon}^\delta\|_{H_0^1(D)} \\ &\quad + \|u_{T,k,\rho,\varepsilon}^\delta - u_{\text{hom}}\|_{H_0^1(D)}. \end{aligned}$$

Combined with Céa’s lemma and the triangle inequality, the first RHS term is estimated as follows:

$$\begin{aligned} \|u_{T,k,\rho,\varepsilon}^{\delta,h,H} - u_{T,k,\rho,\varepsilon}^{\delta,h}\|_{H_0^1(D)} &\lesssim \inf_{v_H \in V_H} \|u_{T,k,\rho,\varepsilon}^{\delta,h} - v_H\|_{H_0^1(D)} \\ &\leq \|u_{T,k,\rho,\varepsilon}^{\delta,h} - u_{T,k,\rho,\varepsilon}^\delta\|_{H_0^1(D)} + \|u_{T,k,\rho,\varepsilon}^\delta - u_{\text{hom}}\|_{H_0^1(D)} \\ &\quad + \inf_{v_H \in V_H} \|u_{\text{hom}} - v_H\|_{H_0^1(D)} \end{aligned}$$

so that we have

$$\begin{aligned} \|u_{T,k,\rho,\varepsilon}^{\delta,h,H} - u_{\text{hom}}\|_{H_0^1(D)} &\leq 2\|u_{T,k,\rho,\varepsilon}^{\delta,h} - u_{T,k,\rho,\varepsilon}^{\delta}\|_{H_0^1(D)} + 2\|u_{T,k,\rho,\varepsilon}^{\delta} - u_{\text{hom}}\|_{H_0^1(D)} \\ &\quad + \inf_{v_H \in V_H} \|u_{\text{hom}} - v_H\|_{H_0^1(D)}. \end{aligned}$$

The first RHS term vanishes as $h \downarrow 0$ by Galerkin approximation, the second RHS term converges to zero by Theorem 8, whereas the last RHS term vanishes as $H \downarrow 0$ by Galerkin approximation as well: (4.46) follows.

We turn now to the numerical corrector¹. For clarity we denote by H_1 the discretization parameter of the Galerkin subspaces V_{H_1} of $H_0^1(D)$, and denote by $u_{T,k,\rho,\varepsilon}^{\delta,h,H_1}$ the Galerkin approximation of $u_{T,k,\rho,\varepsilon}^{\delta,h}$ in V_{H_1} . We then follow the strategy of Definition 6 to introduce the desired numerical corrector. Let $H_2 > 0$, $I_{H_2} \in \mathbb{N}$, and let $\{Q_{H_2,i}\}_{i \in \llbracket 1, I_{H_2} \rrbracket}$ be a covering of D in disjoint subdomains of diameter of order H_2 . We denote by (M_{H_2}) the family of approximations of the identity on $L^2(D)$ associated with $Q_{H_2,i}$. Let $\delta > 1$ be as in Theorem 8, and for all $i \in \llbracket 1, I_{H_2} \rrbracket$, set

$$Q_{H_2,i}^\delta := \{x \in \mathbb{R}^d \mid d(x, Q_{H_2,i}) < (\delta - 1)H_2\}.$$

For all $i \in \llbracket 1, I_{H_2} \rrbracket$, let $V_{i,H_2,h}$ be a family of dense subspaces of $H_0^1(Q_{H_2,i}^\delta \cap D)$. With the notation of Corollary 1, we define the functions $\tilde{\gamma}_{T,k,1,\rho,\varepsilon}^{\delta,h,H_1,H_2,i}$ and $\tilde{\gamma}_{T,k,1,\rho,\varepsilon}^{\delta,H_1,H_2,i}$ as the unique weak solutions in $H_0^1(Q_{H_2,i}^\delta \cap D)$ of

$$(T\varepsilon^2)^{-1} \tilde{\gamma}_{T,k,1,\rho,\varepsilon}^{\delta,h,H_1,H_2,i} - \nabla \cdot A_\varepsilon \left(M_{H_2,i} (\nabla u_{T,k,\rho,\varepsilon}^{\delta,h,H_1}) + \nabla \tilde{\gamma}_{T,k,1,\rho,\varepsilon}^{\delta,h,H_1,H_2,i} \right) = 0, \quad (4.47)$$

$$(T\varepsilon^2)^{-1} \tilde{\gamma}_{T,k,1,\rho,\varepsilon}^{\delta,H_1,H_2,i} - \nabla \cdot A_\varepsilon \left(M_{H_2,i} (\nabla u_{T,k,\rho,\varepsilon}^{\delta,H_1}) + \nabla \tilde{\gamma}_{T,k,1,\rho,\varepsilon}^{\delta,H_1,H_2,i} \right) = 0. \quad (4.48)$$

We then define the numerical corrector $\gamma_{T,k,1,\rho,\varepsilon}^{\delta,h,H_1,H_2,i}$ associated with $u_{T,k,\rho,\varepsilon}^{\delta,h,H_1}$ as the Galerkin approximation in $V_{i,H_2,h}$ of $\tilde{\gamma}_{T,k,1,\rho,\varepsilon}^{\delta,h,H_1,H_2,i}$ in $H_0^1(Q_{H_2,i}^\delta \cap D)$. The functions $\gamma_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2,i}$ and $\tilde{\gamma}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2,i}$ are obtained for all $k' \in \mathbb{N}$ by Richardson extrapolation, cf. Definition 2. We also set for all $k' \in \mathbb{N}$ and for all $1 \leq i \leq I_{H_2}$,

$$\begin{aligned} \nabla \tilde{u}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2,i} &:= M_{H_2} (\nabla u_{T,k,\rho,\varepsilon}^{\delta,H_1})|_{Q_{H_2,i} \cap D} + (\nabla \tilde{\gamma}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2,i})|_{Q_{H_2,i} \cap D} \in L^2(Q_{H_2,i} \cap D), \\ \nabla u_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2,i} &:= M_{H_2} (\nabla u_{T,k,\rho,\varepsilon}^{\delta,h,H_1})|_{Q_{H_2,i} \cap D} + (\nabla \tilde{\gamma}_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2,i})|_{Q_{H_2,i} \cap D} \in L^2(Q_{H_2,i} \cap D), \end{aligned}$$

and finally define numerical correctors

$$\tilde{C}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2} := \sum_{i=1}^{I_{H_2}} \nabla \tilde{u}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2,i} 1_{Q_{H_2,i} \cap D}, \quad C_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2} := \sum_{i=1}^{I_{H_2}} \nabla u_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2,i} 1_{Q_{H_2,i} \cap D}. \quad (4.49)$$

Our second result is the following almost sure convergence:

$$\lim_{T \uparrow \infty, \rho, H_1, H_2 \downarrow 0} \lim_{\varepsilon \downarrow 0} \lim_{h \downarrow 0} \|C_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2} - \nabla u_\varepsilon\|_{L^2(D)} = 0, \quad (4.50)$$

where the order of the limits in T, ρ, H_1, H_2 is irrelevant. In particular, one may take $\rho = H_1 = H_2$ in practice.

¹The number of sub/super-scripts may not ease the reading, and we encourage the reader to simply consider $H_1 = H_2 = \rho$ everywhere — although we shall need all the parameters in the proof.

By the triangle inequality,

$$\begin{aligned} \|C_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2} - \nabla u_\varepsilon\|_{L^2(D)} &\leq \|C_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2} - \tilde{C}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2}\|_{L^2(D)} \\ &\quad + \|\tilde{C}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2} - C_{T,k,k',\rho,\varepsilon}^{\delta,H_2}\|_{L^2(D)} + \|C_{T,k,k',\rho,\varepsilon}^{\delta,H_2} - \nabla u_\varepsilon\|_{L^2(D)}, \end{aligned}$$

where $C_{T,k,k',\rho,\varepsilon}^{\delta,H_2}$ is the corrector of Definition 6 (with H_2 in place of H). We start the analysis with the first RHS term. Note that $C_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2}$ depends doubly on h : once through the Galerkin approximation in $V_{i,H_2,h}$ and once through the approximation $M_{H_2,i}(\nabla u_{T,k,\rho,\varepsilon}^{\delta,h,H_1})$ of $M_{H_2,i}(\nabla u_{T,k,\rho,\varepsilon}^{\delta,H_1})$. Since $\lim_{h \downarrow 0} M_{H_2,i}(\nabla u_{T,k,\rho,\varepsilon}^{\delta,h,H_1}) = M_{H_2,i}(\nabla u_{T,k,\rho,\varepsilon}^{\delta,H_1})$ and since the solution of (4.47) depends continuously on $M_{H_2,i}(\nabla u_{T,k,\rho,\varepsilon}^{\delta,h,H_1})$ in $H_0^1(Q_{H_2,i}^\delta \cap D)$, the Galerkin approximation $\gamma_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2,i}$ converges to $\tilde{\gamma}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2,i}$ in $H_0^1(Q_{H_2,i}^\delta \cap D)$ as $h \downarrow 0$. Hence,

$$\lim_{h \downarrow 0} \|C_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2} - \tilde{C}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2}\|_{L^2(D)} = 0.$$

We continue with the second term, which measures the error between the corrector associated with $u_{T,k,\rho,\varepsilon}^\delta$ and the corrector associated with $u_{T,k,\rho,\varepsilon}^{\delta,H_1}$. Since the solution $\tilde{\gamma}_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2,i}$ of (4.47) depends linearly in $H_0^1(Q_{H_2,i}^\delta \cap D)$ on $M_{H_2}(\nabla u_{T,k,\rho,\varepsilon}^{\delta,H_1})$, and M_{H_2} is a continuous linear map on $L^2(D)$, we have

$$\|\tilde{C}_{T,k,k',\rho,\varepsilon}^{\delta,H_1,H_2} - C_{T,k,k',\rho,\varepsilon}^{\delta,H_2}\|_{L^2(D)} \lesssim \|\nabla u_{T,k,\rho,\varepsilon}^{\delta,H_1} - \nabla u_{T,k,\rho,\varepsilon}^\delta\|_{L^2(D)}.$$

By Theorem 8, this second term converges uniformly to zero wrt to H_2 . For the third and last RHS terms we finally appeal to Theorem 9. This concludes the qualitative convergence analysis of the HMM with regularization and extrapolation under Hypothesis 1.

4.3. Numerical analysis in the case of periodic coefficients. In this last paragraph we make the qualitative convergence analysis of the HMM with regularization and extrapolation quantitative in the case of periodic coefficients. The choice of the periodic coefficients allows us to combine our quantitative analysis of Sections 2 and 3 with quantitative two-scale expansions. In particular, in order to complete the analysis in the case of random Poisson inclusions, we would need a quantitative two-scale expansion, which is not yet known to hold (see however the recent progress in that direction for discrete elliptic equations [21]).

Let the coefficients $\tilde{A} : D \times \mathbb{R} \rightarrow \mathcal{M}_{\alpha\beta}^{\text{sym}}$ of Hypothesis 1 be constant with respect to the first variable and Q -periodic with respect to the second variable. Let also f and D be smooth enough so that $u_{\text{hom}} \in W^{2,\infty}(D)$. Fix $\delta > 1$. Let V_H be a P1-finite element subspace of $H_0^1(D)$ of dimension I_H associated with a tessellation \mathcal{T}_H of D of meshsize $H > 0$. For every element $T_{H,i}$ of \mathcal{T}_H , let $\mathcal{T}_{H,i,h}^\delta$ be a tessellation of $T_{H,i}^\delta \cap D$ of meshsize $h > 0$ with $T_{H,i}^\delta = \{x \in \mathbb{R}^d \mid d(x, T_{H,i}) \leq (\delta - 1)H\}$, and let $V_{H,h,i}$ be the associated P1-finite element subspace of $H_0^1(T_{H,i}^\delta \cap D)$. For all $k \in \mathbb{N}$, all filters of order $p \geq 0$, $T > 0$ and $\varepsilon > 0$, we denote by $A_{T,k,H,\varepsilon}^{\delta,h}$ the (piecewise constant) approximation of $A_{T,k,\rho,\varepsilon}^\delta$ on \mathcal{T}_H (cf. (4.1)) obtained as follows. For all $1 \leq i \leq I_H$ such that $T_{H,i}^\delta \subset D$, we define $A_{T,k,H,\varepsilon}^{\delta,h}|_{T_{H,i}}$ as the approximation (4.1), with $T_{H,i}^\delta$ in place of $Q_{H,i}^\delta$ and $V_{H,h,i}$ in place of $H_0^1(Q_{H,i}^\delta \cap D)$. If $1 \leq i \leq I_H$ is such that $T_{H,i}^\delta \not\subset D$, we follow Remark 11 and set $A_{T,k,H,\varepsilon}^{\delta,h}|_{T_{H,i}} := A_{T,k,H,\varepsilon}^{\delta,h}|_{T_{H,j}}$ for some $1 \leq j \leq I_H$ such that $T_{H,j}^\delta \subset D$ and $d(T_{H,i}, T_{H,j}) \lesssim H$. Take now $\delta = \frac{3}{2}$ and

$T = \frac{H}{\varepsilon}$. We then have: For all $1 \leq i \leq I_H$ such that $T_{H,i}^\delta \subset D$, by Theorem 2 and a standard Galerkin approximation result:

$$\begin{aligned} |A_{T,k,H,\varepsilon}^{\delta,h} - A_{\text{hom}}| &\lesssim \left(\frac{\varepsilon}{H}\right)^{p+1} + \left(\frac{\varepsilon}{H}\right)^{2k} + \left(\frac{H}{\varepsilon}\right)^{1/4} \exp\left(-c \frac{1}{\sqrt{2^{k+1}}} \sqrt{\frac{H}{\varepsilon}}\right) + \left(\frac{h}{\varepsilon}\right)^2 \\ &\lesssim \left(\frac{\varepsilon}{H}\right)^{2k} + \left(\frac{h}{\varepsilon}\right)^2, \end{aligned} \quad (4.51)$$

where, for simplicity, we have assume that $p+1 = 2k$, and neglected the exponential term. Since A_{hom} is constant on D (and not only Lipschitz), this also yields for those $1 \leq i \leq I_H$ such that $T_{H,i}^\delta \not\subset D$,

$$|A_{T,k,H,\varepsilon}^{\delta,h} - A_{\text{hom}}| \lesssim \left(\frac{\varepsilon}{H}\right)^{2k} + \left(\frac{h}{\varepsilon}\right)^2. \quad (4.52)$$

(If A_{hom} were Lipschitz and non-constant, there would be an additional term of order H in (4.52).) Denote by $u_{T,k,H,\varepsilon}^{\delta,h}$ the Galerkin approximation in V_H of

$$-\nabla \cdot A_{T,k,H,\varepsilon}^{\delta,h} \nabla u_{T,k,H,\varepsilon}^{\delta,h} = f.$$

We have

$$\|u_{T,k,H,\varepsilon}^{\delta,h} - u_{\text{hom}}\|_{H^1(D)} \lesssim H + \left(\frac{\varepsilon}{H}\right)^{2k} + \left(\frac{h}{\varepsilon}\right)^2. \quad (4.53)$$

This estimate is a direct consequence of (4.51) and of the regularity of $u_{\text{hom}} \in H^2(D)$. Indeed, let $v_{T,k,H,\varepsilon}^{\delta,h}$ be the weak solution in $H_0^1(D)$ of

$$-\nabla \cdot A_{T,k,H,\varepsilon}^{\delta,h} \nabla v_{T,k,H,\varepsilon}^{\delta,h} = f.$$

We then appeal to (4.23) in the proof of Lemma 1, which takes the form

$$\|\nabla u_{\text{hom}} - \nabla v_{T,k,H,\varepsilon}^{\delta,h}\|_{L^2(D)} \lesssim \|A_{T,k,H,\varepsilon}^{\delta,h} - A_{\text{hom}}\|_{L^\infty(D)} \|\nabla u_{\text{hom}}\|_{L^2(D)}.$$

We then use (4.51) and (4.52),

$$\|\nabla u_{\text{hom}} - \nabla v_{T,k,H,\varepsilon}^{\delta,h}\|_{L^2(D)} \lesssim \left(\frac{\varepsilon}{H}\right)^{2k} + \left(\frac{h}{\varepsilon}\right)^2. \quad (4.54)$$

Finally, by C ea's lemma and the triangle inequality,

$$\begin{aligned} \|u_{T,k,H,\varepsilon}^{\delta,h} - u_{\text{hom}}\|_{H^1(D)} &\leq \|u_{T,k,H,\varepsilon}^{\delta,h} - v_{T,k,H,\varepsilon}^{\delta,h}\|_{H^1(D)} + \|v_{T,k,H,\varepsilon}^{\delta,h} - u_{\text{hom}}\|_{H^1(D)} \\ &\lesssim \inf_{v_H \in V_H} \|v_H - v_{T,k,H,\varepsilon}^{\delta,h}\|_{H^1(D)} + \|v_{T,k,H,\varepsilon}^{\delta,h} - u_{\text{hom}}\|_{H^1(D)} \\ &\leq \inf_{v_H \in V_H} \|v_H - u_{\text{hom}}\|_{H^1(D)} + 2\|v_{T,k,H,\varepsilon}^{\delta,h} - u_{\text{hom}}\|_{H^1(D)}, \end{aligned}$$

so that (4.53) follows from (4.54) and a Galerkin approximation result.

We turn to the numerical corrector result. Let $C_{T,k,\varepsilon}^{\delta,h,H}$ denote the numerical corrector $C_{T,k,k',\rho,\varepsilon}^{\delta,h,H_1,H_2}$ defined in (4.49), for the covering \mathcal{T}_H of D and with $\rho = H_1 = H_2 = H$ and $k = k'$. Let denote by ϕ^i the periodic corrector associated with \tilde{A} in direction e_i . On the one hand, we have the following quantitative two-scale expansion:

$$\|\nabla u_\varepsilon - \nabla u_{\text{hom}} - \sum_{i=1}^d \nabla_i u_{\text{hom}} \nabla \phi^i(\varepsilon^{-1} \cdot)\|_{L^2(D)} \lesssim \sqrt{\varepsilon}. \quad (4.55)$$

On the other hand, the combination of (4.54) with Remark 4 and a Galerkin approximation result yields for all $1 \leq i \leq I_H$ such that $T_{H,i}^\delta \subset D$,

$$\begin{aligned} & \|\nabla u_{\text{hom}} + \sum_{i=1}^d \nabla_i u_{\text{hom}} \nabla \phi^i(\varepsilon^{-1}\cdot) - C_{T,k,\varepsilon}^{\delta,h,H}\|_{L^2(T_{H,i})}^2 \\ & \lesssim |T_{H,i}| \left(|M_{H,i}(\nabla u_{\text{hom}} - \nabla u_{T,k,H,\varepsilon}^{\delta,h})| + \frac{h}{\varepsilon} + H + \left(\frac{\varepsilon}{H}\right)^k + \frac{h}{\varepsilon} \right)^2, \end{aligned} \quad (4.56)$$

and

$$\begin{aligned} & \|\nabla u_{\text{hom}} + \sum_{i=1}^d \nabla_i u_{\text{hom}} \nabla \phi^i(\varepsilon^{-1}\cdot) - C_{T,k,\varepsilon}^{\delta,h,H}\|_{L^2(T_{H,i})}^2 \\ & \lesssim |T_{H,i}| \left(|M_{H,i}(\nabla u_{\text{hom}} - \nabla u_{T,k,H,\varepsilon}^{\delta,h})| + \frac{h}{\varepsilon} + H + \frac{\varepsilon}{H} + \frac{h}{\varepsilon} \right)^2 \end{aligned} \quad (4.57)$$

otherwise (on such elements $T_{H,i}$ the resonance error remains a priori of order $\frac{\varepsilon}{H}$). The combination of (4.55), (4.54), and (4.57) on $N_H(\partial D) := \{x \in D \mid d(x, \partial D) \leq H\}$ and (4.56) on $D \setminus N_H(\partial D)$ then yields the last estimates of this article:

$$\|\nabla u_\varepsilon - C_{T,k,\varepsilon}^{\delta,h,H}\|_{L^2(D)} \lesssim \sqrt{\varepsilon} + H + \frac{h}{\varepsilon} + \sqrt{H} \frac{\varepsilon}{H} + \left(\frac{\varepsilon}{H}\right)^k \lesssim \sqrt{\varepsilon} + H + \frac{h}{\varepsilon} + \left(\frac{\varepsilon}{H}\right)^{\min\{2,k\}}, \quad (4.58)$$

and the associated ‘‘interior’’ estimate

$$\|\nabla u_\varepsilon - C_{T,k,\varepsilon}^{\delta,h,H}\|_{L^2(D \setminus N_H(\partial D))} \lesssim \sqrt{\varepsilon} + H + \frac{h}{\varepsilon} + \left(\frac{\varepsilon}{H}\right)^k. \quad (4.59)$$

Estimates (4.53), (4.58), and (4.59) are the best estimates available in the literature in terms of the resonance error $\frac{\varepsilon}{H}$ for a numerical homogenization method with an oversampling of fixed size $\delta > 1$ (taken $\frac{3}{2}$ for the reasoning). Note that estimate (4.58) may be overly pessimistic due to our currently poor understanding of the boundary layer.

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