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► **To cite this version:**

François-Xavier Le Dimet, Innocent Souopgui, M.Yussuf Hussaini, Ha Tran Thu. Sensitivity Analysis: A Variational Approach. 25th Biennial Conference on Numerical Analysis, Jun 2013, Glasgow, United Kingdom. 2013. hal-00932570

HAL Id: hal-00932570

<https://inria.hal.science/hal-00932570>

Submitted on 20 Jan 2014

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Sensitivity Analysis : A Variational Approach

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September 30, 2013

- General Sensitivity Analysis
- Sensitivity and Data Assimilation
- Second Order Analysis
- A 1-D Example
- An Application to a pollution problem

Sensitivity Analysis: Deterministic Approach

- Model: \mathcal{F} :

$$\mathcal{F}(\mathcal{X}, \mathcal{U}) = 0 \quad (1)$$

- Scalar Response Function \mathcal{G} :

$$\mathcal{G}(\mathcal{X}, \mathcal{U}) \quad (2)$$

- Sensitivity \mathcal{S} is by definition the gradient of \mathcal{G} with respect to \mathcal{U} :

$$\mathcal{S} = \nabla \mathcal{G}(\mathcal{X}(\mathcal{U}), \mathcal{U}) \quad (3)$$

- An adjoint variable \mathcal{P} is introduced as the solution of :

$$\left[\frac{\partial \mathcal{F}}{\partial \mathcal{X}} \right]^t \cdot \mathcal{P} = \left[\frac{\partial \mathcal{G}}{\partial \mathcal{X}} \right] \quad (4)$$

- Then we get :

$$\mathcal{S} = \left[\frac{\partial \mathcal{G}}{\partial \mathcal{U}} \right] - \left[\frac{\partial \mathcal{F}}{\partial \mathcal{U}} \right]^t \cdot \mathcal{P} \quad (5)$$

Data Assimilation for Pollution Modeling

- X is the state variable (velocity, surface elevation) governed by :

$$\begin{cases} \frac{dX}{dt} = F(X) \\ X(0) = U \end{cases} \quad (6)$$

- The concentration of pollutant C , produced by sources S verifies:

$$\begin{cases} \frac{dC}{dt} = G(X, C, S) \\ C(0) = V \end{cases} \quad (7)$$

- U and V are unknown. The VDA problem is to evaluate them from observation X_{obs} and C_{obs} , in order to minimize the cost function J defined by:

$$J(U, V) = \frac{1}{2} \int_0^T \|EX - X_{obs}\|^2 dt + \frac{1}{2} \int_0^T \|DC - C_{obs}\|^2 dt \quad (8)$$

- For sake of simplicity regularization terms, of great practical importance, are not displayed

Data Assimilation for Pollution Modeling: Optimality System

- P and Q adjoint variables are introduced as the solution of the system :

$$\begin{cases} \frac{dP}{dt} + \left[\frac{\partial F}{\partial X} \right]^t \cdot P + \left[\frac{\partial G}{\partial X} \right]^t \cdot Q = E^t(EX - X_{obs}) \\ P(T) = 0; \end{cases} \quad (9)$$

∴

$$\begin{cases} \frac{dQ}{dt} + \left[\frac{\partial G}{\partial C} \right]^t \cdot Q = D^t(DC - C_{obs}); \\ Q(T) = 0, \end{cases} \quad (10)$$

- Then the gradient of J with respect to U and V are given by :

$$\nabla J_U = -P(0) \quad (11)$$

$$\nabla J_V = -Q(0) \quad (12)$$

- If some response function \mathcal{S} is introduced, how to evaluate the sensitivity with respect to observations? For instance how to evaluate the impact of an error of observation on a prediction?
- What should be the "model" \mathcal{F} of the general sensitivity analysis?
- Because only the Optimality System contains the observation, the sensitivity analysis must be carried out on the O.S. considered as a Generalized Model
- Deriving the O.S. leads to carry out a **Second Order Analysis**.

Computing the sensitivity with respect to sources : second order adjoint.

- We need to introduce four second order adjoint variables Γ , Λ , Φ and Ψ as the solution of :

$$\left\{ \begin{array}{l} \frac{d\Gamma}{dt} + \left[\frac{\partial F}{\partial X} \right]^t \cdot \Gamma + \left[\frac{\partial F}{\partial X} \right]^t \cdot \Lambda + \left[\frac{\partial^2 F}{\partial X^2} P \right]^t \cdot \Phi \\ \quad + \left[\frac{\partial^2 G}{\partial X^2} Q \right]^t \cdot \Phi + \left[\frac{\partial^2 G}{\partial C \partial X} Q \right]^t \cdot \Psi - E^t E \Phi = 0; \\ \Gamma(0) = 0; \\ \Gamma(T) = 0, \end{array} \right. \quad (13)$$

$$\left\{ \begin{array}{l} \frac{d\Lambda}{dt} + \left[\frac{\partial F}{\partial C} \right]^t \cdot \Lambda + \left[\frac{\partial^2 G}{\partial C \partial X} Q \right]^t \cdot \Phi \\ \quad + \left[\frac{\partial^2 G}{\partial X^2} Q \right]^t \cdot \Psi - D^t D\Psi = \frac{\partial \varphi}{\partial C}; \\ \Lambda(0) = 0; \\ \Lambda(T) = 0, \end{array} \right. \quad (14)$$

$$\frac{d\Phi}{dt} + \left[\frac{\partial F}{\partial X} \right]^t \cdot \Phi = 0, \quad (15)$$

$$\frac{d\Psi}{dt} + \left[\frac{\partial G}{\partial C} \right]^t \cdot \Psi = 0, \quad (16)$$

- Then it comes :

$$\nabla \varphi = \left[\frac{\partial F}{\partial S} \right]^t \cdot \Lambda + \left[\frac{\partial^2 G}{\partial X^2} Q \right]^t \cdot \Phi + \left[\frac{\partial^2 G}{\partial C \partial S} Q \right]^t \cdot \Psi + \frac{\partial \varphi}{\partial S} \quad (17)$$

- The sensitivity is obtained by solving the coupled system of four equations
- The System involves second order terms.
- We found a **non-standard problem** : **two equations have two conditions an initial condition and a final condition, the other two equations have no condition**

Solving the Non-Standard problem

- The Non-Standard problem can be symbolically written :

$$\begin{cases} \frac{dX}{dt} = K(X, Y), t \in [0, T]; \\ \frac{dY}{dt} = L(X, Y), t \in [0, T] \end{cases} \quad (18)$$

- with :

$$\begin{cases} X(0) = 0; \\ X(T) = 0 \end{cases} \quad (19)$$

and no condition on Y .

NSP is transformed into a problem of optimal control by introducing the control U and a cost-function $J_P(U)$ with :

$$\begin{cases} X(0) = 0; \\ Y(0) = U. \end{cases} \quad (20)$$

Solving the Non-Standard problem 2

A cost function $J_P(U)$ is defined by:

$$J_P(U) = \frac{1}{2} \|X(T, U)\|^2 + \frac{1}{2} \|U\|^2 \quad (21)$$

If Z and W are defined as the solution of:

$$\frac{dW}{dt} + \left[\frac{\partial K}{\partial X} \right]^t \cdot W + \left[\frac{\partial L}{\partial X} \right]^t \cdot Z = 0; \quad (22)$$

$$\frac{dZ}{dt} + \left[\frac{\partial K}{\partial Y} \right]^t \cdot W + \left[\frac{\partial L}{\partial Y} \right]^t \cdot Z = 0; \quad (23)$$

$$Z(T) = 0; W(T) = X(T), \quad (24)$$

then we get

$$\nabla J_P(U) = -Z(0) + U \quad (25)$$

Solving the Non-Standard problem 3

This problem involved third derivatives of the original model.
Recent developments on the NSP have been recently carrier out by V. Shutyaev and F.-X. Le Dimet
The existence of a solution is demonstrated
Another method to solve NSP is proposed.

A 1-D Example 1

Let us assume that the one dimensional velocity field $u = u(x, t)$ evolves according to the Burgers equation given by :

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = f, x \in \Omega =] - 1, 1[, t \in [0, T]; \\ u = u_0, t = 0, \\ u = u_1, x \in \{-1, 1\}, \end{array} \right. \quad (26)$$

Evolution of the pollutant's concentration:

$$\left\{ \begin{array}{l} \frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = \eta \frac{\partial^2 c}{\partial x^2} + s, x \in] - 1, 1[, t \in [0, T] \\ c = c_0, t = 0; \\ c = c_1, x \in \{-1, 1\} \end{array} \right. \quad (27)$$

A 1-D Example: cost function

The cost function takes the form (with continuous observation in space and time):

$$J(u_0, c_0) = \frac{1}{2} \int_0^T \|u - u_{obs}\|_{\Omega}^2 dt + \frac{1}{2} \int_0^T \|c - c_{obs}\|_{\Omega}^2 dt. \quad (28)$$

where $\|f\|_{\Omega}^2 = \int_{\Omega} f(x)f(x)dx = \int_0^1 f(x)f(x)dx$.

A 1-D Example: adjoint model

The adjoint variables p and q are introduced as the solutions of :

$$\left\{ \begin{array}{l} \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \nu \frac{\partial^2 p}{\partial x^2} + q \frac{\partial c}{\partial x} = u - u_{obs} \\ p(t = T) = 0 \\ p = 0, x \in \{-1, 1\} \end{array} \right. \quad (29)$$

$$\left\{ \begin{array}{l} \frac{\partial q}{\partial t} + \frac{\partial uq}{\partial x} + \eta \frac{\partial^2 q}{\partial x^2} = c - c_{obs} \\ q(t = T) = 0 \\ q = 0, x \in \{-1, 1\} \end{array} \right. \quad (30)$$

And the gradient of the cost function is given by:

$$\begin{aligned} \nabla_{u_0} J &= -p(0) \\ \nabla_{c_0} J &= -q(0) \end{aligned}$$

Sensitivity of a response function

Let φ be a function of the concentration and the source functions, the response function is given by:

$$\Phi_A(t, s) = \int_{\Omega_A} \varphi(c, s) dx \quad (31)$$

where $\Omega_A \subset \Omega$ is the response region. Following the guidelines of the derivation of the gradient, we introduce the adjoint variables Γ , ϕ, ψ and Λ as the solution of:

Sensitivity of a response function

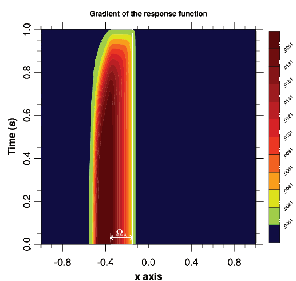
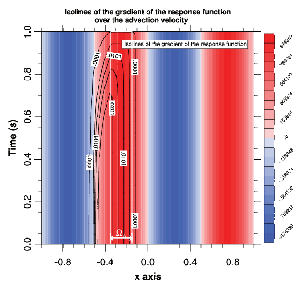
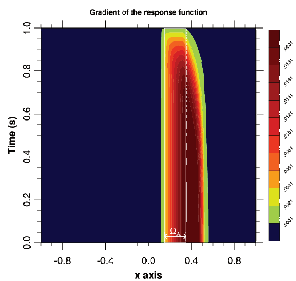
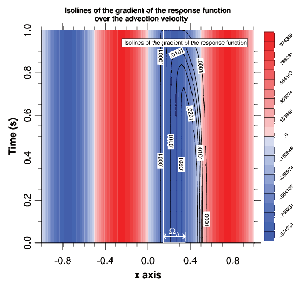
$$\left\{ \begin{array}{l} \frac{\partial \Gamma}{\partial t} + u \frac{\partial \Gamma}{\partial x} + v \frac{\partial^2 \Gamma}{\partial x^2} - \Lambda \frac{\partial c}{\partial x} \\ \quad - \phi \frac{\partial p}{\partial x} + q \frac{\partial \psi}{\partial x} - \phi = 0 \\ \Gamma = 0, t \in \{0, T\} \\ \Gamma = 0, x \in \{-1, 1\} \end{array} \right. \quad (32)$$

$$\left\{ \begin{array}{l} \frac{\partial \Lambda}{\partial t} + \frac{\partial u \Lambda}{\partial x} + \eta \frac{\partial^2 \Lambda}{\partial x^2} + \frac{\partial q \phi}{\partial x} - \psi = -\frac{\partial \varphi}{\partial c} \\ \Lambda = 0, t \in \{0, T\} \\ \Lambda = 0, x \in \{-1, 1\} \end{array} \right. \quad (33)$$

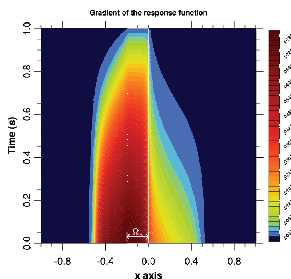
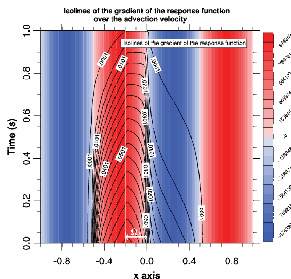
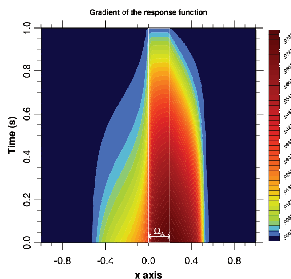
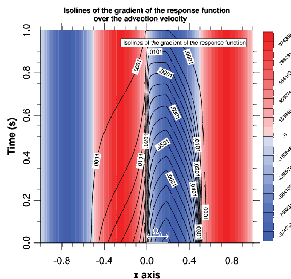
Where the function 1_{Ω_A} is:

$$1_{\Omega_A}(x) = \begin{cases} 1, & \text{if } x \in \Omega_A \\ 0, & \text{if not.} \end{cases}$$

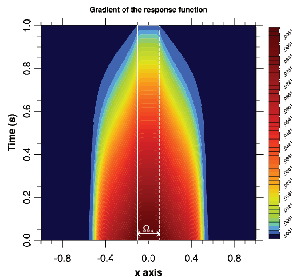
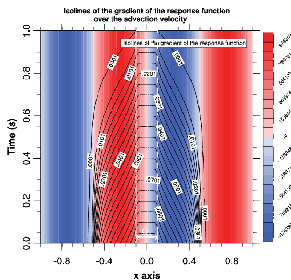
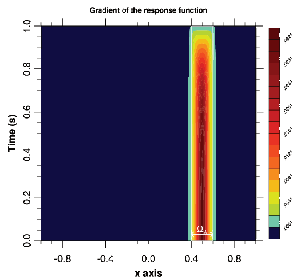
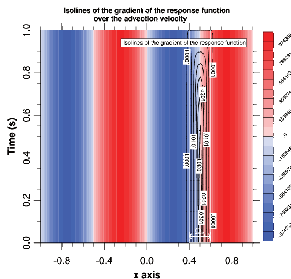
Numerical results



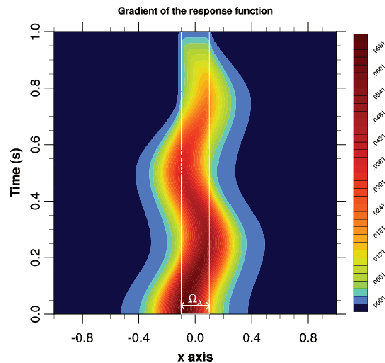
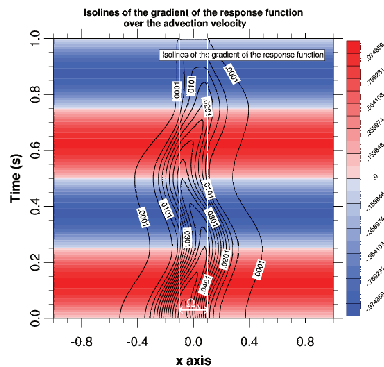
Numerical results



Numerical results



Numerical results



The End