# Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model 

Monique Dauge, Patrick Dular, Laurent Krähenbühl, Victor Péron, Ronan<br>Perrussel, Clair Poignard

## - To cite this version:

Monique Dauge, Patrick Dular, Laurent Krähenbühl, Victor Péron, Ronan Perrussel, et al.. Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model. JSA 2013 - Journées Singulières Augmentées, Aug 2013, Rennes, France. hal-00931735

HAL Id: hal-00931735
https://inria.hal.science/hal-00931735
Submitted on 19 Mar 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Corner Asymptotics of the Magnetic Potential in the Eddy-Current Model 

Monique Dauge ${ }^{\star}$ Patrick Dular ${ }^{\curvearrowright}$ Laurent Krähenbüh $\imath^{\diamond}$ Victor Péron ${ }^{\star}$ Ronan Perrussel ${ }^{\dagger}$ Clair Poignard ${ }^{\star}$<br>${ }^{\star}$ IRMAR (Rennes), ${ }^{\diamond}$ ACE (Liège), ${ }^{\diamond}$ Ampère (Lyon), ${ }^{\dagger}$ EPI Magique3D \& LMAP, UPPA (Pau),<br>${ }^{\dagger}$ LAPLACE (Toulouse), ${ }^{\boldsymbol{*} E P I}$ MC2 (Bordeaux)

JSA 2013,
Conférence en l'honneur de Martin Costabel pour ses 65 ans,
Rennes, August 26-30.

## The Configuration

$\Omega_{-}$: a conducting body ( $\sigma>0$ )
$\Omega_{+}$: a dielectric medium
c : a corner
$\omega \in(0,2 \pi)$ : the angle of the corner


We describe the Magnetic potential in the vicinity of the corner $\mathbf{c}$.

## The Magnetic Potential

## The eddy-current problem

The magnetic vector potential $\mathcal{A}$ satisfies

$$
\left\{\begin{array}{rlrl}
-\Delta \mathcal{A}^{-}+4 \mathrm{i} \zeta^{2} \mathcal{A}^{-} & =0 \text { in } \Omega_{-}, & {[\mathcal{A}]_{\Sigma}=0, \text { on } \Sigma,}  \tag{1}\\
-\Delta \mathcal{A}^{+} & =\mu_{0} J \text { in } \Omega_{+}, & {\left[\partial_{n} \mathcal{A}\right]_{\Sigma}=0, \text { on } \Sigma .} \\
\mathcal{A}^{+} & =0 \text { on } \Gamma,
\end{array}\right.
$$

Here $\zeta^{2}=\kappa \mu_{0} \sigma / 4>0$,
$J$ : a smooth data, vanishing near the corner c.

## The Magnetic Potential

## The eddy-current problem

The magnetic vector potential $\mathcal{A}$ satisfies

$$
\left\{\begin{array}{rlrl}
-\Delta \mathcal{A}^{-}+4 \mathrm{i} \zeta^{2} \mathcal{A}^{-} & =0 \text { in } \Omega_{-}, & {[\mathcal{A}]_{\Sigma}=0, \text { on } \Sigma,}  \tag{1}\\
-\Delta \mathcal{A}^{+} & =\mu_{0} J \text { in } \Omega_{+}, & {\left[\partial_{n} \mathcal{A}\right]_{\Sigma}=0, \text { on } \Sigma .} \\
\mathcal{A}^{+} & =0 \text { on } \Gamma,
\end{array}\right.
$$

Here $\zeta^{2}=\kappa \mu_{0} \sigma / 4>0$,
$J$ : a smooth data, vanishing near the corner c.

## The Magnetic Potential

## The eddy-current problem

The magnetic vector potential $\mathcal{A}$ satisfies

$$
\left\{\begin{array}{rlrl}
-\Delta \mathcal{A}^{-}+4 i \zeta^{2} \mathcal{A}^{-} & =0 \text { in } \Omega_{-}, & {[\mathcal{A}]_{\Sigma}=0, \text { on } \Sigma,}  \tag{1}\\
-\Delta \mathcal{A}^{+} & =\mu_{0} J \text { in } \Omega_{+}, & {\left[\partial_{n} \mathcal{A}\right]_{\Sigma}=0, \text { on } \Sigma .} \\
\mathcal{A}^{+} & =0 \text { on } \Gamma,
\end{array}\right.
$$

Here $\zeta^{2}=\kappa \mu_{0} \sigma / 4>0$,
$J$ : a smooth data, vanishing near the corner c.

## Proposition

There exists a unique solution $\mathcal{A}$ in $H_{0}^{1}(\Omega)$ to problem (1). Moreover, $\mathcal{A}$ belongs to $H^{\frac{5}{2}-\varepsilon}(\Omega)$ for any $\varepsilon>0$. In particular, $\mathcal{A}$ belongs to $\mathcal{C}^{1}(\bar{\Omega})$.
$\mathcal{A}$ possesses a corner asymptotic expansion near $\mathbf{c}$.

## Corner Asymptotics

- To generalize the Taylor Expansion
- Corner Asymptotics involve
(1) The singular functions (primal and dual): belong to the kernel of the considered operator in $\mathbb{R}^{2}$.
(2) The singular coefficients:
its calculation requires the knowledge of dual singular functions.
Aim : To explicit the Corner Asymptotics of $\mathcal{A}$ near the corner $\mathbf{c}$.


## Motivation : The eddy-current phenomenon

- The limit problem in large frequency/high conductivity:

$$
\left\{\begin{align*}
-\Delta \mathcal{A}_{0}^{+} & =\mu_{0} J \text { in } \Omega_{+}  \tag{2}\\
\mathcal{A}_{0}^{+} & =0 \text { on } \partial \Omega_{+}
\end{align*}\right.
$$

## Motivation : The eddy-current phenomenon

- The limit problem in large frequency/high conductivity :

$$
\left\{\begin{align*}
-\Delta \mathcal{A}_{0}^{+} & =\mu_{0} J \text { in } \Omega_{+}  \tag{2}\\
\mathcal{A}_{0}^{+} & =0 \text { on } \partial \Omega_{+}
\end{align*}\right.
$$



- If $\Omega_{-}$has a convex corner, i.e. $\omega \in(0, \pi)$, Problem (2) has non $\mathcal{C}^{1}$ singularities


## Motivation : The eddy-current phenomenon

- The limit problem in large frequency/high conductivity :

$$
\left\{\begin{align*}
-\Delta \mathcal{A}_{0}^{+} & =\mu_{0} J \text { in } \Omega_{+}  \tag{2}\\
\mathcal{A}_{0}^{+} & =0 \text { on } \partial \Omega_{+}
\end{align*}\right.
$$



- If $\Omega_{-}$has a convex corner, i.e. $\omega \in(0, \pi)$, Problem (2) has non $\mathcal{C}^{1}$ singularities
- Problem (1) has $\mathcal{C}^{1}$ singularities


## Motivation : Multi-scale analysis for eddy-currents

围 Buret et al (IEEE Trans. on Mag. '12)
Eddy currents and corner singularities

## Motivation : Multi-scale analysis for eddy-currents

皿 Buret et al (IEEE Trans. on Mag. '12)
Eddy currents and corner singularities

- When $\delta=\sqrt{\frac{2}{\kappa \mu_{0} \sigma}} \ll 1$, there holds

$$
\mathcal{A}_{\delta} \approx \mathcal{A}_{0}+a \delta^{\alpha} V\left(\frac{\dot{\delta}}{\delta}\right)+R_{\delta}
$$

Here $V$ is a profile defined in $\mathbb{R}^{2}$ such that

$$
-\Delta V+2 i v \mathbb{1}_{\Omega_{-}}=0 \quad \text { near } \mathbf{c}
$$

$$
\text { and } \alpha=\frac{\pi}{2 \pi-\omega} \text {. }
$$

## Motivation : Multi-scale analysis for eddy-currents

皿 Buret et al (IEEE Trans. on Mag. '12)
Eddy currents and corner singularities

- When $\delta=\sqrt{\frac{2}{\kappa \mu_{0} \sigma}} \ll 1$, there holds

$$
\mathcal{A}_{\delta} \approx \mathcal{A}_{0}+a \delta^{\alpha} V\left(\frac{\dot{\delta}}{\delta}\right)+R_{\delta}
$$

Here $V$ is a profile defined in $\mathbb{R}^{2}$ such that

$$
-\Delta V+2 i v \mathbb{1}_{\Omega_{-}}=0 \quad \text { near } \mathbf{c}
$$

$$
\text { and } \alpha=\frac{\pi}{2 \pi-\omega} \text {. }
$$

## Motivation : Multi-scale analysis for eddy-currents

皿 Buret et al (IEEE Trans. on Mag. '12)
Eddy currents and corner singularities

- When $\delta=\sqrt{\frac{2}{\kappa \mu_{0} \sigma}} \ll 1$, there holds

$$
\mathcal{A}_{\delta} \approx \mathcal{A}_{0}+a \delta^{\alpha} V\left(\frac{\dot{\delta}}{\delta}\right)+R_{\delta}
$$

Here $V$ is a profile defined in $\mathbb{R}^{2}$ such that

$$
-\Delta V+2 i v \mathbb{1}_{\Omega_{-}}=0 \quad \text { near } \mathbf{c},
$$

and $\alpha=\frac{\pi}{2 \pi-\omega}$.
$\Longrightarrow$ The knowledge of the singularities of $-\Delta+4 i \zeta^{2} \mathbb{1}_{\Omega_{-}}$is "crucial".

## Main difficulty

The considered operator is

$$
-\Delta+i \kappa \mu_{0} \sigma \mathbb{1}_{\Omega_{-}}= \begin{cases}-\Delta+i \kappa \mu_{0} \sigma & \text { in } \Omega_{-}, \\ -\Delta & \text { in } \Omega_{+} .\end{cases}
$$

The singularities are generated by the term $i \kappa \mu_{0} \sigma \mathbb{1}_{\Omega_{-}}$.

## Main difficulty

The considered operator is

$$
-\Delta+i \kappa \mu_{0} \sigma \mathbb{1}_{\Omega_{-}}= \begin{cases}-\Delta+i \kappa \mu_{0} \sigma & \text { in } \Omega_{-} \\ -\Delta & \text { in } \Omega_{+}\end{cases}
$$

The singularities are generated by the term $i \kappa \mu_{0} \sigma \mathbb{1}_{\Omega_{-}}$.
$\Longrightarrow$ the derivation of the Corner Asymptotics is not obvious.

## Our Reference

T. M. DAUGE et al (To appear in MMAS - INRIA RR 8204)

Corner asymptotics of the magnetic potential in the eddy-current model

## Our Reference

T M. DAUGE et al (To appear in MMAS - INRIA RR 8204)
Corner asymptotics of the magnetic potential in the eddy-current model

- $\mathcal{A}$ possesses a corner asymptotic expansion

$$
\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta)+\sum_{k \geqslant 1} \sum_{p \in\{0,1\}} \Lambda^{k, p} \mathfrak{S}^{k, p}(r, \theta),
$$

$(r, \theta)$ : polar coordinates centered at $\mathbf{c}$
$\mathfrak{S}^{k, p}$ : primal singular functions
$\Lambda^{k, p}$ : singular coefficients

## Our Reference

( M. DAUGE et al (To appear in MMAS - INRIA RR 8204)
Corner asymptotics of the magnetic potential in the eddy-current model

- $\mathcal{A}$ possesses a corner asymptotic expansion

$$
\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta)+\sum_{k \geqslant 1} \sum_{p \in\{0,1\}} \Lambda^{k, p} \mathfrak{S}^{k, p}(r, \theta),
$$

$(r, \theta)$ : polar coordinates centered at $\mathbf{c}$
$\mathfrak{S}^{k, p}$ : primal singular functions
$\Lambda^{k, p}$ : singular coefficients

- We provide a constructive procedure to determine the primal and dual singularities


## Our Reference

R. DAUGE et al (To appear in MMAS - INRIA RR 8204)

Corner asymptotics of the magnetic potential in the eddy-current model

- $\mathcal{A}$ possesses a corner asymptotic expansion

$$
\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta)+\sum_{k \geqslant 1} \sum_{p \in\{0,1\}} \Lambda^{k, p} \mathfrak{S}^{k, p}(r, \theta),
$$

$(r, \theta)$ : polar coordinates centered at $\mathbf{c}$
$\mathfrak{S}^{k, p}$ : primal singular functions
$\Lambda^{k, p}$ : singular coefficients

- We provide a constructive procedure to determine the primal and dual singularities
- We generalize the method of moments and we introduce the method of quasi-dual functions to determine $\Lambda^{\kappa, p}$


## Outline

(1) The case $\zeta=0$ :

We introduce the method of moments and the method of dual functions.
(2) The case $\zeta \neq 0$ :

We provide a constructive procedure to determine the singularities.
We introduce the method of quasi-dual singular functions.
(3) Numerical simulations

## Laplace Operator $(\zeta=0)$

- We consider the solution $\mathcal{A}$ to

$$
\left\{\begin{aligned}
-\Delta \mathcal{A} & =\mu_{0} J \text { in } \Omega \\
\mathcal{A} & =0 \text { on } \Gamma
\end{aligned}\right.
$$

- $\mathcal{A}$ admits the Taylor expansion at $\mathbf{c}$ :

$$
\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0}+\sum_{k \geqslant 1} \sum_{p \in\{0,1\}} \Lambda^{k, p} \underbrace{r^{k} \cos (k \theta-p \pi / 2)}_{=\mathfrak{s}^{k, p}(r, \theta)}
$$

$\mathfrak{s}^{k, p}$ : harmonic polynomials

## Methods to extract the coefficients $\wedge^{k, p}$

We use dual harmonic functions :

$$
\mathfrak{k}^{k, p}(r, \theta)= \begin{cases}-\frac{1}{2 \pi} \log r, & \text { if } k=0, p=0, \\ \frac{1}{2 k \pi} r^{-k} \cos (k \theta-p \pi / 2), & \text { if } k \geqslant 1, p=0,1 .\end{cases}
$$

- The method of moments
- The dual function method


## The Method of Moments

For $R>0$, we introduce the form $\mathcal{M}_{R}$ :

$$
\mathcal{M}_{R}(K, A)=\frac{1}{R} \int_{r=R} K A R \mathrm{~d} \theta .
$$

Assume $J$ has a support outside the ball $\mathcal{B}(\mathbf{c}, R)$.

## Proposition

Let $\mathcal{A}$ be the solution to the Laplace equation. Then

$$
\mathcal{M}_{R}(1, \mathcal{A})=2 \pi \Lambda^{0,0} \quad \text { and } \quad \mathcal{M}_{R}\left(\mathfrak{k}^{k, p}, \mathcal{A}\right)=\frac{1}{2 k} \Lambda^{\kappa, p}, \quad k \geqslant 1, p=0,1 .
$$

## The Dual Function Method

R V.G. Maz'Ya, B.A. Plamenevskil (Amer. Math. Soc. Trans. (2) '84) On the coefficients in the asymptotic of solutions of the elliptic boundary problem in domains with conical points

- For $R>0$, let us introduce the bilinear form

$$
\mathcal{J}_{R}(K, A)=\int_{r=R}\left(K \partial_{r} A-\partial_{r} K A\right) R \mathrm{~d} \theta .
$$

## Proposition

Let $\mathcal{A}$ be the solution to the Laplace equation. Then

$$
\mathcal{J}_{R}\left(\mathfrak{k}^{0,0}, \mathcal{A}\right)=\Lambda^{0,0} \quad \text { and } \quad \mathcal{J}_{R}\left(\mathfrak{k}^{k, p}, \mathcal{A}\right)=\Lambda^{k, p}, k \geqslant 1, p=0,1 .
$$

## The singularities

The singularities $\mathfrak{U}$ of $-\Delta+4 i \zeta^{2} \mathbb{1}_{\mathcal{S}_{-}}$:

$$
\begin{cases}-\Delta \mathfrak{U}+4 i \zeta^{2} \mathfrak{U}=0 & \text { in } \mathcal{S}_{-} \\ -\Delta \mathfrak{U}=0 & \text { in } \mathcal{S}_{+}\end{cases}
$$



## Description of the Singularities

婳 V.A. Kondratev (Trudy Moskov. Mat. Obšč. '67)
Boundary value problems for elliptic equations in domains with conical or angular points

## Description of the Singularities

圊 V.A. Kondratev (Trudy Moskov. Mat. Obšč. '67)
Boundary value problems for elliptic equations in domains with conical or angular points
-

$$
\mathfrak{U}=\underbrace{\mathfrak{u}_{0}}_{\text {leading part }}+\sum_{j \geqslant 1}\left(\mathrm{i} \zeta^{2}\right)^{j} \underbrace{\mathfrak{u}_{j}}_{\text {shadow }}
$$

## Description of the Singularities

囯 V.A. Kondratev (Trudy Moskov. Mat. Obšč. '67)
Boundary value problems for elliptic equations in domains with conical or angular points
-

$$
\mathfrak{U}=\underbrace{\mathfrak{u}_{0}}_{\text {leading part }}+\sum_{j \geqslant 1}\left(\mathrm{i} \zeta^{2}\right)^{j} \underbrace{\mathfrak{u}_{j}}_{\text {shadow }}
$$

- We solve

$$
\begin{aligned}
& \Delta \mathfrak{u}_{0}=0, \quad \Delta \mathfrak{u}_{1}=4 \mathfrak{u}_{0} \mathbb{1}_{\mathcal{S}_{-}}, \quad \ldots, \quad \Delta \mathfrak{u}_{j}=4 \mathfrak{u}_{j-1} \mathbb{1}_{\mathcal{S}_{-}}, \\
& \mathfrak{u}_{j} \in \mathrm{~S}^{\lambda}=\operatorname{Span}\left\{r^{\lambda} \log ^{q} r \Phi(\theta), \quad q \in \mathbb{N}, \quad \Phi \in \mathcal{C}^{1}(\mathbb{T}), \Phi^{ \pm} \in \mathcal{C}^{\infty}\left(\overline{\mathbb{T}_{ \pm}}\right)\right\} \\
& \text {Here } \lambda \in \mathbb{Z}, \\
& \mathbb{T}=\mathbb{R} /(2 \pi \mathbb{Z}), \mathbb{T}_{-}=(-\omega / 2, \omega / 2) \text { and } \mathbb{T}_{+}=\mathbb{T} \backslash \overline{\mathbb{T}_{-}} .
\end{aligned}
$$

## Existence of the shadows

## Lemma

Let $\lambda \in \mathbb{Z}$ and $\mathfrak{f} \in T^{\lambda-2}$. Then, there exists $\mathfrak{u} \in \mathrm{S}^{\lambda}$ such that $\Delta \mathfrak{u}=\mathfrak{f}$.
Moreover

$$
\begin{aligned}
& \text { (i) If } \lambda \in \mathbb{Z} \backslash\{0\}, \operatorname{deg} \mathfrak{u} \leqslant \operatorname{deg} \mathfrak{f}+1 \text {, } \\
& \text { (ii) If } \lambda=0 \text {, } \operatorname{deg} \mathfrak{u} \leqslant \operatorname{deg} \mathfrak{f}+2
\end{aligned}
$$

Here,

$$
\mathrm{T}^{\lambda}=\operatorname{Span}\left\{r^{\lambda} \log ^{a} r \Psi(\theta), \quad q \in \mathbb{N}, \quad \Psi \in L^{2}(\mathbb{T}), \Psi^{ \pm} \in \mathcal{C}^{\infty}\left(\overline{\mathbb{T}}_{ \pm}\right)\right\}
$$

The degree of $\mathfrak{g}$ is its degree as polynomial of $\log r$.

## Proof of Lemma

M. Dauge Elliptic Boundary Value Problems in Corner Domains Smoothness and Asymptotics of Solutions, Lecture Notes in Mathematics, Vol. 1341, Berlin 1988
We introduce the Mellin symbol $\mathscr{M}$ associated with $\Delta$

$$
\mathscr{M}(\mu)=\left(\mu^{2}+\partial_{\theta}^{2}\right), \theta \in \mathbb{T} .
$$

Setting

$$
\mathfrak{f}=\sum_{q=0}^{\operatorname{deg} \mathfrak{f}} r^{\lambda-2} \log ^{q} r \Psi_{q} \in T^{\lambda-2},
$$

and

$$
\mathfrak{u}=\operatorname{Res}_{\mu=\lambda}\left\{r^{\mu} \mathscr{M}(\mu)^{-1} \sum_{q=0}^{\operatorname{deg} \mathfrak{f}} \frac{q!\Psi_{q}}{(\mu-\lambda)^{q+1}}\right\} \in S^{\lambda}
$$

we check that $\Delta \mathfrak{u}=\mathfrak{f}$.

## Primal and Dual Singularities

- Primal singular functions $\mathfrak{S}$ belong to $H^{1}$ in any bounded neighborhood $\mathcal{B}$ of $\mathbf{c}$.
- Dual singular functions $\mathfrak{K}$ do not belong to $H^{1}$ in such a neighborhood $\mathcal{B}$.

The functions $\mathfrak{K}$ are needed for the determination of the coefficients $\Lambda$ involved in the asymptotics.

## 1. A basis for Primal Singularities

For $(k, p)=(0,0)$ and for $(k, p) \in \mathbb{N}^{*} \times\{0,1\}$,

$$
\mathfrak{S}^{k, p}=\underbrace{\mathfrak{s}^{k, p}}_{\text {leading part }}+\sum_{j \geqslant 1}\left(\mathrm{i}^{2} \zeta^{j}\right)^{j} \underbrace{\mathfrak{s}_{j}^{k, p}}_{\text {shadow }} .
$$

## Lemma

For any $j \geqslant 1$, there exists $\mathfrak{s}_{j}^{k, p} \in \mathrm{~S}^{k+2 j}$, with deg $\mathfrak{s}_{j}^{k, p} \leqslant j$, satisfying

$$
\Delta \mathfrak{s}_{j}^{k, p}=4 \mathfrak{s}_{j-1}^{k, p} \mathbb{1}_{\mathcal{S}_{-}} .
$$

Application: $\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta)+\sum_{k \geqslant 1} \sum_{p \in\{0,1\}} \Lambda^{k, p} \mathfrak{S}^{k, p}(r, \theta)$

## 2. A basis for Dual Singularities

For $(k, p)=(0,0)$ and for $(k, p) \in \mathbb{N}^{*} \times\{0,1\}$,

$$
\mathfrak{K}^{k, p}=\underbrace{\mathfrak{f}^{k, p}}_{\text {leading part }}+\sum_{i \geqslant 1}\left(i \zeta^{2}\right)^{j} \underbrace{\mathfrak{k}_{j}^{k, p}}_{\text {shadow }} .
$$

## Lemma

For any $j \geqslant 1$, there exists $\mathfrak{k}_{j}^{k, p} \in \mathrm{~S}^{-k+2 j}$, with deg $\mathfrak{k}_{j}^{k, p} \leqslant j+1$, satisfying

$$
\Delta \mathfrak{e}_{j}^{k, p}=4 \mathfrak{e}_{j-1}^{k, p} 1_{\mathcal{S}_{-}} .
$$

## 3. The Quasi-Dual Function Method

- M. Costabel et al (SIAM '04)

A quasidual function method for extracting edge stress intensity functions
囯 S. Shannon et al (Preprint ${ }^{\prime} 12$ )
Extracting generalized edge flux intensity functions by the quasidual function method along circular 3-D edges

- We use quasi-dual functions $\mathfrak{K}_{m}^{k, p}$ :

$$
\mathfrak{K}_{m}^{k, p}=\mathfrak{k}^{k, p}+\sum_{j=1}^{m}\left(\mathrm{i} \zeta^{2}\right)^{j} \underbrace{\mathfrak{k}_{j}^{k, p}}_{\text {shadow }}
$$

## The Quasi-Dual Function Method

## Extraction of coefficients

## Theorem

Let $\mathcal{A}$ be the solution to problem (1). Let $k \in \mathbb{N}$ and $p \in\{0,1\}$ ( $p=0$ if $k=0$ ). Then for all $m$ such that $2 m+2>k$,

$$
\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{k, p}, \mathcal{A}\right) \underset{R \rightarrow 0}{=} \Lambda^{k, p}+\sum_{\ell=1}^{[k / 2]} \mathcal{J}^{k, p ; k-2 \ell, p} \Lambda^{k-2 \ell, p}+\mathcal{O}\left(R^{-k} R_{0}^{2 m+2} \log R\right),
$$

$$
\text { where } R_{0}=\zeta R(1+\sqrt{|\log R|}) \text {. }
$$

## The Quasi-Dual Function Method

## Examples

- For $k=0$

$$
\Lambda_{R \rightarrow 0}^{0,0}=\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{0,0}, \mathcal{A}\right)+\mathcal{O}\left(R_{0}^{2+2 m} \log R\right)
$$

- For $k=1$

$$
\Lambda^{1,0} \underset{R \rightarrow 0}{=} \mathcal{J}_{R}\left(\mathfrak{K}_{m}^{1,0}, \mathcal{A}\right)+\mathcal{O}\left(R^{-1} R_{0}^{2 m+2}\right)
$$

- For $k=2$, we need $m \geqslant 1$

$$
\Lambda_{R \rightarrow 0}^{2,0}=\mathcal{J}_{R}\left(\mathfrak{K}_{1}^{2,0}, \mathcal{A}\right)-\mathcal{J}^{2,0 ; 0,0} \Lambda^{0,0}+\mathcal{O}\left(R^{-2} R_{0}^{4} \log R\right)
$$

## The Quasi-Dual Function Method

We use quasi-primal singularities $\mathfrak{S}_{m}^{k, p}$ :

$$
\mathfrak{S}_{m}^{k, p}=\mathfrak{s}^{k, p}+\sum_{j=1}^{m}\left(\mathrm{i} \zeta^{2}\right)^{j} \underbrace{\mathfrak{s}_{j}^{k, p}}_{\text {shadow }}
$$

For $\varepsilon \in(0, R)$, we evaluate

$$
\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{k}, \mathfrak{S}_{m^{\prime}}^{k^{\prime}}\right)-\mathcal{J}_{\varepsilon}\left(\mathfrak{K}_{m}^{k}, \mathfrak{S}_{m^{\prime}}^{k^{\prime}}\right)_{R \rightarrow 0}^{=} \mathcal{O}\left(R^{-k+k^{\prime}} R_{0}^{2 m+2} \log R\right)
$$

with any chosen $k^{\prime}$, and $m^{\prime} \geqslant m$.

## 4. Calculation of singularities

- M. Costabel, M. Dauge (Math. Nachr. '93)

Construction of corner singularities for Agmon-Douglis-Nirenberg elliptic systems.
We use appropriate complex variables $z_{ \pm}$instead of the polar coordinates:
$z_{-}=z\left(\right.$ when $\left.z \in \mathcal{S}_{-}\right)$and $z_{+}=-z\left(\right.$ when $\left.z \in \mathcal{S}_{+}\right)$

- We use an Ansatz involving only integer powers of

$$
z_{ \pm}, \bar{z}_{ \pm}, \log z_{ \pm}, \text {and } \log \bar{z}_{ \pm} .
$$

## The first shadow of primal singularities

## Example : expression of $\mathfrak{s}_{1}^{0,0}$

$$
\begin{aligned}
& \mathfrak{s}_{1}^{0,0}(r, \theta)=\frac{\sin \omega}{\pi} r^{2}(\log r \cos 2 \theta-\theta \sin 2 \theta)+r^{2}\left(1-\frac{\cos \omega \cos 2 \theta}{2}\right) \text { in } \mathcal{S}_{-} \\
& \mathfrak{s}_{1}^{0,0}(r, \theta)=\frac{\sin \omega}{\pi} r^{2}\left(\log r \cos 2 \theta-\theta_{+} \sin 2 \theta\right)+r^{2} \frac{\cos \omega \cos 2 \theta}{2} \text { in } \mathcal{S}_{+}
\end{aligned}
$$

with $\theta_{+}=\theta-\pi \operatorname{sgn} \theta$

## The first shadow of primal singularities

$$
\begin{aligned}
\mathfrak{s}_{1}^{1,0}(r, \theta) & =\frac{2 \sin \omega+\sin 2 \omega}{6 \pi} r^{3}(\log r \cos 3 \theta-\theta \sin 3 \theta) \\
& +r^{3}\left(\frac{\cos \theta}{2}-\frac{\cos \omega \cos 3 \theta}{3}\right), \text { in } \mathcal{S}_{-} \\
\mathfrak{s}_{1}^{1,0}(r, \theta) & =\frac{2 \sin \omega+\sin 2 \omega}{6 \pi} r^{3}\left(\log r \cos 3 \theta-\theta_{+} \sin 3 \theta\right) \\
& +r^{3} \frac{\cos 2 \omega \cos 3 \theta}{6}, \text { in } \mathcal{S}_{+}
\end{aligned}
$$

## Framework

- $\Omega$ : disk of radius 50 mm
- Conducting sector : $\omega=\pi / 4$
- $\zeta=1 /(5 \sqrt{2}) \mathrm{mm}^{-1}$
- Source : $\mathcal{A}^{+}=\frac{|\theta|}{2 \pi}$ on $\partial \Omega$



## Finite Element Solution

We plot the real part and the imaginary part of the FE solution.



## Accuracy for the computation of $\Lambda^{0,0}$ as a function of $R$

- There holds

$$
\Lambda_{R \rightarrow 0}^{0,0}=\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{0,0}, \mathcal{A}\right)+\mathcal{O}\left(R_{0}^{2+2 m} \log R\right)
$$

- We plot $\left|\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{0,0}, \mathcal{A}\right)-\mathcal{A}\right|_{\mathrm{c}}\left|/|\mathcal{A}|_{\mathrm{c}}\right|$ as a function of $R(m=0,1)$.



## Accuracy for the computation of $\Lambda^{1,0}$ as a function of $R$

- Recall that

$$
\Lambda_{R \rightarrow 0}^{1,0}=\mathcal{J}_{R}\left(\mathfrak{K}_{m}^{1,0}, \mathcal{A}\right)+\mathcal{O}\left(R^{-1} R_{0}^{2 m+2}\right)
$$

- We plot $\left|\mathcal{J}_{\mathcal{R}}\left(\mathfrak{K}_{m}^{1,0}, \mathcal{A}\right)-\mathcal{J}_{\mathcal{R}_{s}}\left(\mathfrak{K}_{1}^{1,0}, \mathcal{A}\right)\right| /\left|\mathcal{J}_{\mathcal{R}_{s}}\left(\mathfrak{K}_{1}^{1,0}, \mathcal{A}\right)\right|$.



## Comparison of the FE solution and of the local expansion

In the expansion, we collect the terms which behave as constant, $r, r^{2}$ and $r^{2} \log r$.

$$
\begin{aligned}
\mathcal{A} \underset{r \rightarrow 0}{\sim} \mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{0,0}, \mathcal{A}\right)(1 & \left.+\mathrm{i} \zeta^{2} \mathfrak{s}_{1}^{0,0}\right)+\mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{1,0}, \mathcal{A}\right) \mathfrak{s}^{1,0} \\
& +\left(\mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{2,0}, \mathcal{A}\right)-\mathcal{J}^{2,0 ; 0,0} \mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{0,0}, \mathcal{A}\right)\right) \mathfrak{s}^{2,0}
\end{aligned}
$$



## Comparison of the FE solution and of the local expansion

Then we add the terms which behave as $r^{3}$ and $r^{3} \log r$.

$$
\begin{array}{r}
\mathcal{A} \underset{r \rightarrow 0}{\sim} \mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{0,0}, \mathcal{A}\right)\left(1+\mathrm{i} \zeta^{2} \mathfrak{s}_{1}^{0,0}\right)+\mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{1,0}, \mathcal{A}\right)\left(\mathfrak{s}^{1,0}+\mathrm{i} \zeta^{2} \mathfrak{s}_{1}^{1,0}\right)+ \\
\left(\mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{2,0}, \mathcal{A}\right)-\mathcal{J}^{2,0 ; 0,0} \mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{0,0}, \mathcal{A}\right)\right) \mathfrak{s}^{2,0}+ \\
\left(\mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{3,0}, \mathcal{A}\right)-\mathcal{J}^{3,0 ; 1,0} \mathcal{J}_{R_{\mathrm{S}}}\left(\mathfrak{K}_{1}^{1,0}, \mathcal{A}\right)\right) \mathfrak{s}^{3,0}
\end{array}
$$



## Thank you for your attention

