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Numerical Reconstruction of Convex Polytopes from Directional Moments

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Abstract

We reconstruct an n-dimensional convex polytope from the knowledge of its directional moments up to a certain order. The directional moments are related to the projection of the polytope's vertices on a particular direction. To extract the vertex coordinates from the moment information we combine established numerical algorithms such as generalized eigenvalue computation and linear interval interpolation. Numerical illustrations are given for the reconstruction of 2-d and 3-d objects.

1 Introduction

The reconstruction of the boundary of a shape from its moments is a problem that has only partially been solved. For instance, when the shape is a polygon [21, 11], or when it defines a quadrature domain in the complex plane [14], it has been proved that its boundary can exactly be reconstructed from the knowledge of its moments. Both results only deal with particular 2-dimensional shapes. The reconstruction algorithm for polygons is based on Davis' exact integration formula [6] of a function in the complex plane. For polynomial functions, Davis' formula can be seen as a low dimensional case of a formula attributed to Brion [2]. Based on the latter N. Gravin, J. Lasserre, D. Pasechnik, and S. Robins proposed the reconstruction of a n-dimensional convex polytope in exact arithmetic [13]. The approach requires finding the roots of a polynomial, whose coefficients are the solution of a linear system, and testing equalities to zero.

Brion's integration formula over a polytope does not relate moment information directly to the vertices of the convex polytope, but rather to the projections of these vertices onto some 1-dimensional subspace. To recover the projections, we recognize an inverse problem that arises in several areas [10, 15, 17, 18, 22] and can be solved numerically as a generalized eigenvalue problem.

After recovering the projections of the vertices on various one-dimensional subspaces, remains the problem of matching different projections (in different directions) of the same vertex, with

that vertex. In this paper we describe how to solve this issue, without resorting to exact arithmetic.The problem cannot be solved with ordinary interpolation or least squares approximation. The former interpolates any set of n projections, and the latter approximates any set of more than n projections. But using an interval interpolation technique, we understand why we need $n+1$ projections (or more) and we can solve the matching.

Our method is the result of combining techniques from quite different mathematical disciplines: integer lattices, computer algebra, numerical linear algebra, interval methods, inverse problems. The complete algorithm, which we demonstrate in Section 6 and challenge in Section 7, consists of the following steps:

1. The exact number of vertices r is computed from an upper bound R and moments up to order $2R+1-n$, in a sample of directions.
2. For $n+1$, or more, directions, the projections of the vertices are obtained as the generalized eigenvalues of a structured pair of matrices whose entries are determined from the directional moments up to order $2r-n-1$
3. Each of these projections is then matched to the corresponding vertex and the coordinates are computed as the coefficients of n-dimensional interval interpolant.

The different steps in our algorithm involve structured matrices, in the singular value decomposition for the computation of r, as well as in the generalized eigenvalue problem delivering the vertex projections. Structured matrices with real elements have condition numbers that grow exponentially with their size [3], and the size of our matrices is determined by the number of vertices of the polytope. In Section 7 we are required to use higher precision floating-point computations for the polyhedra with many vertices that represents a brilliant diamond cut.

The paper is organized as follow. In Section 2 we introduce geometric, complex and directional moments together with Davis's and Brion's formulae. In Section 3 we review Prony's method and the related eigenvalue problem to determine the projections of the vertices from the directional moments. In Section 4 we discuss the determination of the number of vertices. In Section 5 we present an algorithm to solve the matching problem. Numerical illustrations are given in the Sections 6 and 7 where we reconstruct 2-d and 3-d objects.

2 Geometric and directional moments

In this section we present identities attributed to Brion. These identities are central in [1] for establishing the complexity of the computation of the moments of a polytope. Brion's identities are also at the core of the solution to the inverse problem proposed in [13]. They can actually be seen as a generalisation of Davis' integration formula that was used to solve the shape-from-moment in 2D [21, 11].

We consider a convex polytope in $\mathbb{R}^n$ determined by the set of its r vertices $\mathcal{V}$. Abusing the notation, $\mathcal{V}$ also denotes the polytope itself.

The *geometric moments* are

$$m_\alpha = \int_{\mathcal{V}} x^\alpha \, dx = \int \ldots \int_{\mathcal{V}} x_1^{\alpha_1} \ldots x_n^{\alpha_n} \, dx_1 \ldots dx_n, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.$$

The order of the geometric moment m_α is $|\alpha| = \alpha_1 + \ldots + \alpha_n$. These moments can be expressed as a multivariate polynomial in the coordinates of the vertices. A straightforward formula is given in [25] and a complexity conscious approach to computing those is offered in [1] for exact arithmetic.

The *moment in the direction* $\delta \in \mathbb{R}^n$ of order k is

$$m_k(\delta) = \int_{\mathcal{V}} \langle x, \delta \rangle^k \, dx, \quad k \in \mathbb{N},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^n$.

One can obtain any directional moment of order k from the geometric moments of order k with the multinomial formula

$$m_k(\delta) = \sum_{|\alpha|=k} \binom{k}{\alpha} m_\alpha \, \delta^\alpha.$$

Conversely, geometric moments of order k can be obtained from the directional moments of order k in $\binom{n+k-1}{k}$ distinct directions by solving a linear system of equations.

In the context of polygon retrieval ($n = 2$) from tomographic data in [21, 11], geometric moments and then complex moments are computed from directional moments. Complex moments can be understood as the moments in the direction $\delta = (1, \mathrm{i})$. At the core of this shape-from-moments problem is Davis's integration formula for an analytic function f on the polygon $\mathcal{V}$ in the complex plane [6]

$$\iint_{\mathcal{V}} f''(x + \mathrm{i}\, y) \, dxdy = \sum_{v \in \mathcal{V}} a_v \, f(v),$$

where the $v \in \mathcal{V}$ are here interpreted as complex numbers. Assuming that $\check{v}$ and $\hat{v}$ are the vertices adjacent to v, the coefficients in Davis's formula are

$$a_v = \frac{V_v}{(v - \check{v})(v - \hat{v})},$$

where V_v is the oriented area of the parallelogram defined by the vectors with vertices $\check{v} - v$ and $\hat{v} - v$.

This formula bears a generalisation to any dimension, known as Brion's identities. The formula relates the directional moments with the projections of the vertices. This allows us to work directly with directional moments, which are data that can be deduced from tomographic measurements.

Theorem 2.1. *[13] Provided that the orthogonal projections of the r vertices of the convex polytope $\mathcal{V}$ on the direction δ are distinct, we have the following equalities*

$$\begin{aligned} \tfrac{(k+n)!}{k!} \, m_k(\delta) &= \sum_{v \in \mathcal{V}} a_v(\delta) \, \langle v, \delta \rangle^{n+k} \qquad k \geq 0, \\ \textit{and} \qquad & \\ 0 &= \sum_{v \in \mathcal{V}} a_v(\delta) \, \langle v, \delta \rangle^{n-k} \quad 1 \leq k \leq n, \end{aligned} \tag{2.1}$$

where the $a_v(\delta)$ depend on δ and the adjacent vertices of v in a triangulation of $\mathcal{V}$.

Moreover

$$a_v(\delta) \neq 0, \quad v \in \mathcal{V}. \tag{2.2}$$

The formula for the coefficients $a_v(\delta)$ is given in [2, Section 10.3] when $\mathcal{V}$ is a simple convex polytope. That is, each vertex in the polytope has exactly n adjacent vertices. Let $\mathcal{V}_v$ be the set of n adjacent vertices of v. The volume V_v of the parallepiped determined by $\mathcal{V}_v$ is obtained through the determinant of the edge vectors of $\mathcal{V}_v$. Then

$$a_v(\delta) = \frac{V_v}{\prod_{u \in \mathcal{V}_v} \langle v - u, \delta \rangle}. \tag{2.3}$$

In particular, for a simplex $\triangle$ the vertices of which are $v_0, v_1, \ldots, v_n$

$$\frac{(k+n)!}{k!} \int_{\triangle} \langle x, \delta \rangle^k \, dx = V \sum_{i=0}^{n} \frac{\langle v_i, \delta \rangle^{k+n}}{\prod_{j \neq i} \langle v_i - v_j, \delta \rangle} = V \sum_{k_0 + \ldots + k_n = k} \langle v_0, \delta \rangle^{k_0} \ldots \langle v_n, \delta \rangle^{k_n} \tag{2.4}$$

where $V = V_{v_0} = \ldots = V_{v_n}$. Notice that this is actually a polynomial in δ though we shall use its more compact rational expression.

For a more general convex polytope, one has to consider a partition of the polytope into simplices that does not introduce any additional vertex [2, Theorem 3.1]. The coefficients $a_v(\delta)$ for the convex polytope is then a sum of its sibblings in the formulae for the simplices. That they do not vanish is proved in [13].

The directions $\delta \in \mathbb{R}^n$ to which the theorem applies are those for which $\langle u, \delta \rangle \neq \langle v, \delta \rangle$ for all distinct $u, v \in \mathcal{V}$. Those are the *generic directions*. We examine what happens when δ fails to be generic in this meaning.

On one hand Brion's identities are actually correct for any δ that do not make the denominators of a_v vanish. However, if δ is a direction for which the coefficients a_v are well defined but for which there are two distinct vertices $u, v \in \mathcal{V}$ such that $\langle \delta, u \rangle = \langle \delta, v \rangle$ we can write the formula with less than r terms. The linear recurrence introduced in Section 3 is then of order less than r and the associated Hankel matrix is of rank less than r.

In addition, when $\langle \delta, v_0 \rangle = \ldots = \langle \delta, v_p \rangle$ for distinct vertices $v_0, \ldots, v_p \in \mathcal{V}$ that belong to the same simplex of any triangulation, there is a formula similar to (2.1) where the $p+1$ terms $\langle \delta, v_i \rangle^{n+k}$ are replaced by terms $\langle \delta, v_0 \rangle^{n+k}$, $(n+k)\langle \delta, v_0 \rangle^{n+k-1}$, $\ldots$, $(n+k)\ldots(n+k-p+1)\langle \delta, v_0 \rangle^{n+k-p}$. This can be deduced from the rightmost expression in (2.4). The Hankel matrix constructed in Section 3 is still of rank r and admits $\langle v_0, \delta \rangle$ as a generalized eigenvalue of multiplicity $p+1$. See for instance [8, 20].

3 Recovering the projections of the vertices

In this section we address the problem of retrieving the projections $\mathcal{V}(\delta) = \{\langle v, \delta \rangle \mid v \in \mathcal{V}\}$ of the vertices of our convex polytope from its directional moments $m_k(\delta)$. We shall recognize an inverse problem that has appeared in several areas, as for instance exponential and sparse interpolation [10, 18, 22] or Padé approximants [15], beside shape-from-moments [21, 11, 13]. While [13] approaches the problem with Prony's method, which does not behave so well in floating point arithmetic, we shall favor a formulation in terms of generalized eigenvalues.

The standing assumption is that the projections of the vertices on the direction δ are pairwise distinct. Thus $|\mathcal{V}(\delta)| = |\mathcal{V}| = r$. Also, we assume in this section that the number of vertices is known. We discuss in next section how this number can be retrieved from only the knowledge of the moments.

From the directional moments $(m_k(\delta))_k$ we introduce the sequence $(\mu_k(\delta))_{k\in\mathbb{N}}$ of modified directional moments defined by

$$\begin{aligned} \mu_k(\delta) &= 0, & 0 \le k \le n-1, \\ \mu_k(\delta) &= \tfrac{k!}{(k-n)!}\, m_{k-n}(\delta), & k \ge n. \end{aligned}$$

By Theorem 2.1 there exist r non-zero real numbers $a_v(\delta)$ such that this sequence satisfies $\mu_k(\delta) = \sum_{v\in\mathcal{V}} a_v(\delta)\,\langle v,\delta\rangle^k, \quad k \in \mathbb{N}$. The goal is to retrieve the r elements $\langle v,\delta\rangle$ of $\mathcal{V}(\delta)$ from $(\mu_k(\delta))_k$ and hence $(m_k(\delta))_k$. This is just an instance of the following problem that appears in diverse areas listed above.

Inverse problem: Consider a sequence $(\mu_k)_{k\in\mathbb{N}}$ such that for some non zero real (or complex) numbers $a_1,\ldots,a_r$ and pairwise distinct real (or complex) numbers $w_1,\ldots w_r$,

$$\mu_k = \sum_{i=1}^{r} a_i\, w_i^k, \quad \forall k \in \mathbb{N}, \tag{3.1}$$

The problem is to find the w_i from the knowledge of r and $(\mu_k)_{0\le k\le 2r-1}$. The problem can be tackled by Prony's method or as a generalized eigenvalue problem. Besides, total least squares approaches to this problem in the context of exponential fitting are reviewed in [22].

First one observes that the sequence $(\mu_k)_k$ is a solution of a recurrence equation of order r, namely

$$\mu_{k+r} = p_{r-1}\,\mu_{k+r-1} + \ldots + p_0\,\mu_k, \tag{3.2}$$

where $(-p_0,\ldots,-p_{r-1},1)$ are the coefficients of the polynomial

$$p(z) = \prod_{i=1}^{r}(z-w_i) = z^r - p_{r-1}\,z^{r-1} - \ldots - p_1\,z - p_0.$$

Applying (3.2) to $(\mu_k)_{k\in\mathbb{N}}$ for $k = 0,\ldots,r-1$ leads to the linear system

$$\underbrace{\begin{pmatrix} \mu_0 & \mu_1 & \cdots & & \mu_{r-1} \\ \mu_1 & & & \iddots & \\ \vdots & & \iddots & & \vdots \\ & \iddots & & & \\ \mu_{r-1} & & \cdots & & \mu_{2r-2} \end{pmatrix}}_{H_r^{(0)}} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ \vdots \\ p_{r-1} \end{pmatrix} = \begin{pmatrix} \mu_r \\ \mu_{r+2} \\ \vdots \\ \vdots \\ \mu_{2r-1} \end{pmatrix}. \tag{3.3}$$

From $(\mu_k)_{0\le k\le 2r-1}$ we can retrieve the characteristic polynomial p of the underlying recurrence by solving the above linear system. The sought numbers $w_1,\ldots,w_r$ are the roots of this polynomial. Theses two steps (solving the linear system and computing the roots of the entailed

polynomial) is known as Prony's method. It was introduced in [23] and is used in the context of the shape-from-moments problem in [13, 21]. The authors of [11, 17] introduce a solution in terms of the generalized eigenvalues of a pencil of matrices. It is based on the following facts.

We can recast (3.3) into the matrix equality:

$$\underbrace{\begin{pmatrix} \mu_0 & \mu_1 & \cdots & & \mu_{r-1} \\ \mu_1 & & & \iddots & \\ \vdots & & \iddots & & \vdots \\ & \iddots & & & \\ \mu_{r-1} & & \cdots & & \mu_{2\,r-2} \end{pmatrix}}_{H_r^{(0)}} \underbrace{\begin{pmatrix} 0 & \cdots & \cdots & 0 & p_0 \\ 1 & \ddots & & \vdots & p_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & p_{r-1} \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} \mu_1 & \mu_2 & \cdots & & \mu_r \\ \mu_2 & & & \iddots & \\ \vdots & & \iddots & & \vdots \\ & \iddots & & & \\ \mu_r & & \cdots & & \mu_{2\,r-1} \end{pmatrix}}_{H_r^{(1)}}. \tag{3.4}$$

The last column of the matrix on the right hand side comes from Equation (3.3), while the other columns are simply shifts of the columns in the matrix on the left hand side.

Let us introduce the following notations from [15, Section 7.5] for the matrices arising in the above equality. The $r \times r$ Hankel matrix with first row given by $\begin{pmatrix} \mu_d & \ldots & \mu_{r+d-1} \end{pmatrix}$ is denoted by $H_r^{(d)}$. The companion matrix of the characteristic polynomial of the recurrence (3.2) is denoted by P. The matrix equality (3.4) becomes $H_r^{(0)}\, P = H_r^{(1)}$, and more generally we have $H_r^{(d)}\, P = H_r^{(d+1)}$ for $d \geq 0$.

Since $w_1, \ldots, w_r$ are the roots of $p(z) = z^r - p_{r-1}\, z^{r-1} - \ldots - p_1\, z - p_0$, we have

$$\underbrace{\begin{pmatrix} 1 & w_1 & \cdots & w_1^{r-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & w_r & \cdots & w_r^{r-1} \end{pmatrix}}_{W_r} \underbrace{\begin{pmatrix} 0 & \ldots & 0 & p_0 \\ 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & p_{r-2} \\ 0 & \ldots & 1 & p_{r-1} \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} w_1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & w_r \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & w_1 & \cdots & w_1^{r-1} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & w_r & \cdots & w_r^{r-1} \end{pmatrix}}_{W_r}.$$

Let D and W_r be respectively the diagonal and the Vandermonde matrices defined by $(w_1, \ldots, w_r)$ and appearing in the above equality. The latter can thus be written $W_r\, P = D W_r$. The w_i being pairwise distinct, W_r is invertible and $P\, W_r^{-1} = W_r^{-1}\, D$. That is, $w_1, \ldots, w_r$ are the eigenvalues of P and W_r^{-1} is a matrix of eigenvectors for P. From $H_r^{(1)} = H_r^{(0)}\, P$ in (3.4) we can deduce

$$H_r^{(1)}\, W_r^{-1} = H_r^{(0)}\, W_r^{-1}\, D,$$

and more generally, for $d \geq 0$,

$$H_r^{(d+1)}\, W_r^{-1} = H_r^{(d)}\, W_r^{-1}\, D, \qquad \text{for } d \in \mathbb{N}. \tag{3.5}$$

Thus $(w_1, \ldots, w_r)$ are generalized eigenvalues of the matrix pencils $\left(H_r^{(d+1)}, H_r^{(d)}\right)$, and W_r^{-1} is a matrix of associated generalized eigenvectors.

Computing generalized eigenvalues is a classical problem in numerical linear algebra [12, 7, 16]. The structured problem we consider here is unfortunately known to be potentially ill-conditioned. Following [4] we can give an upper bound for the conditioning of the generalized

eigenvalue problem as a constant multiplied by $\kappa(\delta)^2$, where $\kappa(\delta)$ is the condition number of the Vandermonde matrix $W_r(\delta)$.

To come back to our initial problem of retrieving $\mathcal{V}(\delta)$ from $(\mu_k(\delta))_k$ we shall introduce the pencil of Hankel matrices $\left(H_r^{(1)}(\delta),\, H_r^{(0)}(\delta)\right)$. Its generalized eigenvalues are the elements of $\mathcal{V}(\delta)$. From those we can construct a matrix of generalized eigenvectors, given by the inverse of the Vandermonde matrix $W_r(\delta)$.

4 Estimating the number of vertices

So far, the number r of vertices has been assumed to be given. But r can also be an unknown of the problem. In this section, we discuss how to numerically retrieve this number from the Hankel matrices $H_k^{(0)}(\delta)$ formed from the sequence of modified directional moments $(\mu_k(\delta))_k$ in a generic direction δ.

One first observe that $H_{r+\ell}^{(0)}(\delta)$ is at most of rank r for any $\ell \geq 0$. Indeed the sequence $(\mu_k(\delta))_{k\in\mathbb{N}}$ satisfies a recurrence equation (3.1) of order r. For any $\ell > 0$, each of the last ℓ columns of $H_{r+\ell}^{(0)}(\delta)$ is thus a linear combination of the previous r columns. Now, noting $w_1, \ldots, w_r$ the elements of $\mathcal{V}(\delta)$, we examine the Vandermonde factorisation of the Hankel matrix:

$$H_k^{(0)}(\delta) = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ w_1 & w_2 & \cdots & w_r \\ \vdots & \vdots & \ddots & \vdots \\ w_1^{k-1} & w_2^{k-1} & \cdots & w_r^{k-1} \end{bmatrix}}_{{}^tW_k(\delta)} \underbrace{\begin{bmatrix} a_1(\delta) & 0 & \cdots & 0 \\ 0 & a_2(\delta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_r(\delta) \end{bmatrix}}_{A(\delta)} \underbrace{\begin{bmatrix} 1 & w_1 & \cdots & w_1^{k-1} \\ 1 & w_2 & \cdots & w_2^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w_r & \cdots & w_r^{k-1} \end{bmatrix}}_{W_k(\delta)}.$$

For a generic direction δ, the r elements w_i of $\mathcal{V}(\delta)$ are paiwise distinct and therefore $\det H_r^{(0)}(\delta) \neq 0$. It follows that $H_{r+\ell}^{(0)}(\delta)$ is exactly of rank r, for all $\ell \geq 0$.

Based on this observation, if a strict upper bound R for the number of vertices r is given, then r can be determined as the rank of $H_R^{(0)}(\delta)$. In practice we determine the rank numerically with the singular value decomposition of $H_R^{(0)}(\delta)$. A caveat is that the matrix may be ill-conditioned. The conditioning of $H_r^{(0)}(\delta)$ is determined by the conditioning of $W_r(\delta)$ and $A(\delta)$ in (4.1). For this we also refer to the discussion in [10] and [19] that examine the situation in the context of sparse interpolation[1]. The conditioning of the Vandermonde matrix $W_r(\delta)$ depends on the distribution of the numbers in $\mathcal{V}(\delta)$ of the projections of the vertices in the direction δ. As for the matrix $A(\delta)$, having one of the $a_i(\delta)$ too small can also lead to an incorrect (numerical) rank for $H_R^{(0)}$. Since we can (even randomly) select multiple directions for the projections, we can retain only those directions for which not both $W_r(\delta)$ and $A(\delta)$ are too ill-conditioned. Alternatively we could apply the rank estimates for Hankel matrices of [5].

Therefore, if we have an overestimation R of the number of vertices we can recover the exact number from the analysis of the rank of $H_R^{(0)}(\delta)$. In practice this is read on the singular values

[1] In those references, and in the context of Pade approximants, one can equivalently study the ranks of $H_R^{(d)}$, for $d \geq 1$, because none of the w_i is zero. Here, one of the projections might very well be zero.

of $H_R^{(0)}(\delta)$ that are computed by a singular value decomposition [12, 7]. This is discussed on specific cases in Section 6 and 7.

5 Reconstruction of the vertices from their projections

In this section we show how to retrieve the set of vertices $\mathcal{V}$ from their projections $\mathcal{V}(\delta)$ onto several directions $\delta \in \Delta$. A difficulty to overcome is that the set of projections we start from are not ordered; we have to match them. This rules out classical interpolation or least square approximation. Our approach is based on the interval interpolation scheme presented in [24]. To simplify our line of arguments on the relative error we assume that the polytope is already contained in the unit ball, a situation to which we shall reduce in practice.

Let Δ be a set of $s > n$ vectors in the unit sphere S_n. The asumption is that for each $\delta \in \Delta$ we have a set of approximations $\tilde{\mathcal{V}}(\delta)$ for the values of $\mathcal{V}(\delta) = \{\langle v, \delta\rangle \,|\, v \in \mathcal{V}\}$, the projections of the vertices on the direction δ. The result should be a set $\tilde{\mathcal{V}}$ of r vectors in $\mathbb{R}^n$ that are approximations of the set $\mathcal{V}$ of vertices of the polytope.

The approximations in $\tilde{\mathcal{V}}(\delta)$ are obtained from the modified directional moments $(\mu_k(\delta))_{0\le k\le 2r-1}$ as discussed in Section 3. They are computed as generalized eigenvalues and the conditionning of this problem is given by the square of the condition number $\kappa(\delta)$ of the Vandermonde matrix $W_r(\delta)$ made of these generalized eigenvalues. Therefore, an element w of $\tilde{\mathcal{V}}(\delta)$ is actually understood as the center of an interval $[w^-, w^+]$ of size $2\epsilon(\delta)$ where $\epsilon(\delta)$ should be taken as

$$\epsilon(\delta) = \mathrm{O}\left(\kappa(\delta)^2\, \varepsilon_\mu(\delta)\right), \tag{5.1}$$

where

$$\varepsilon_\mu(\delta) = \max \frac{|\tilde{\mu}_k(\delta) - \mu_k(\delta)|}{|\mu_k(\delta)|}$$

is the bound on the inaccuracy of the input modified directional moments $(\tilde{\mu}_k(\delta))_{0\le k\le 2r-1}$.

The computation can then continue only when for at least $n+1$ directions δ the intervals of $\tilde{\mathcal{V}}(\delta)$ are disjoint. For simplicity, we assume in the sequel that in all s directions the intervals are disjoint. Note furthermore that with our asumptions, all the elements of $\mathcal{V}(\delta)$ are in the interval $[-1, 1]$.

For an element in $\tilde{\mathcal{V}}(\delta)$, we do not know which vertex projection it is an approximation of. So we need to find the correct *labelling* of the projections concurrently with the computation of the vertex coordinates.

An algorithm for the reconstruction of the vertex coordinates can be based on the computation of r linear interval interpolants in n variables, of the form

$$q_u : \begin{array}{ccc} S_n & \to & \mathbb{R} \\ \delta & \mapsto & \langle u, \delta\rangle \end{array} \quad \text{with } u = {}^t(u_1, \ldots, u_n) \in \mathbb{R}^n. \tag{5.2}$$

The set $\tilde{\mathcal{V}}$ of n-tuples of coefficients u used to define those r interpolants are the approximations to the set $\mathcal{V}$ of the polytope vertices. The interpolation condition has to reflect the fact that the r functions q_u interpolates exactly one w-value per $\tilde{\mathcal{V}}(\delta)$ but s of those values across the sets $\tilde{\mathcal{V}}(\delta)$. Formally, this can be written as

$$\forall u \in \tilde{\mathcal{V}},\ \forall \delta \in \Delta,\ \exists!\, w \in \tilde{\mathcal{V}}(\delta)\ s.t.\ q_u(\delta) \in [w^-, w^+]. \tag{5.3}$$

We remark at this point that any set of n projections can be interpolated by a function of the form $q_u(\delta)$, even if the width of the intervals is zero. At the same time any combination of s projections can be approximated in the least squares sense by a function of the form $q_u(\delta)$. So none of these classical approaches is very useful in figuring out which projections belong to the same vertex $v \in \mathcal{V}$. But an interval interpolant through at least $n+1$ disjoint intervals does the job: the nonzero interval width compensates for overdetermining the linear interpolant by at least one interpolation condition. The interval interpolant is stringing the intervals, containing the projections, like beads on the graph of the interpolating function. Any interval, meaning any projection, through which it passes, is marked as belonging to the same vertex.

Naturally, an interval interpolation problem does not need to have a unique solution: sufficiently small perturbations of the coefficients in the linear form may not violate the interval interpolation conditions. In [24] is described how the most robust interval interpolant of the form given in (5.2) is computed. By this we mean the interpolant that stays away as far as possible from violating the interpolation conditions imposed by the interval bounds. This solution is unique and its computation is based on the computation of Chebyshev centers.

Because of the labelling problem of the approximate projections, the interval interpolation algorithm becomes a 2-step procedure. In a first step we take the subset $\hat{\Delta}$ of the the n best conditioned directions in Δ. Consider the n-tuples in the Cartesian product $\prod_{\delta \in \hat{\Delta}} \tilde{\mathcal{V}}(\delta)$ and select those for which the unique (non-interval) interpolant intersects one interval of $\tilde{\mathcal{V}}(\delta)$ for each of the $s-n$ remaining directions $\delta \in \Delta \setminus \hat{\Delta}$. They reflect a correct labelling of the projections. These better conditioned direction $\hat{\Delta}$ are usually near to one another and that drastically cuts down the combinatorial aspect of the procedure by a continuity argument.

Since in this first step the coordinates of the vertices are computed from only n of the s available directions, the obtained values are not maximally accurate. After rearranging the s sets (organized per vertex) of s projections, the vertex coordinates can $\tilde{\mathcal{V}}(\delta)$, each containing r projections, into r sets be computed to maximal accuracy from the total of its s projections. To this end the interval interpolation method can be continued with the computation of the most robust interval interpolants satisfying (5.3).

6 Simulations

We now illustrate the proposed approach for the reconstruction of polytopes from their directional moments. For our simulations we consider centered and scaled polytopes: The origin is the center of gravity of the polytope and the vertices lie in the unit ball. This geometric normalisation corresponds to a transformation on the moments as described in [11].

The vertices of the polytope are to be reconstructed from directional moments. The latter are typically computed from measurements, as for instance in tomography. The proposed reconstruction of a convex polytope in dimension n requires directional moments in at least $n+1$ generic directions. The order of the moments required in each direction then depends on the number r of vertices: we need directional moments up to order $2r-n-1$. In dimension 2 this can be compared to [21], where the complex moments up to order $2r-3$ are obtained from directional moments up to the same order in $2r-2$ directions.

For a given direction δ, the directional moments $m_k(\delta)$ are used to form the entries of the pair

of Hankel matrices $\left(H_r^{(1)}(\delta),\, H_r^{(0)}(\delta)\right)$ described in Section 3. The algorithm consists of 3 main steps:

1. Determine the number r of vertices by analyzing the singular values of the Hankel matrix $H_k^{(0)}(\delta)$ for k big enough and a few number of random directions δ, as described in Section 4.
2. Compute the generalized eigenvalues for the pair of matrices $\left(H_r^{(1)}(\delta),\, H_r^{(0)}(\delta)\right)$ in at least $n+1$ directions δ as in Section 3. Determine the condition number $\kappa(\delta)$ of $W_r(\delta)$, the Vandermonde matrix formed with these generalized eigenvalues. The generalized eigenvalues are the projections of the vertices of the polytope with an error determined by a multiple of $\kappa(\delta)^2$.
3. Recover the vertices $\tilde{\mathcal{V}}$ from their approximate projections $\tilde{\mathcal{V}}(\delta)$ using the combinatorial interval interpolation technique described in Section 5.

The first two steps are performed using standard numerical linear algebra routines from the NAG library in Maple. In particular, Step 1 makes use of the implementation of the singular value decomposition and Step 2 the QZ-algorithm [26]. Step 3 is implemented in Matlab. Every computations in this Section are performed in double precision.

In the second step, we sample a number of directions and retain those for which the condition number is the smallest. The condition number indeed depends on the direction : $\kappa(\delta)$ depends on the distribution of the projected vertices [9]. In particular it increases when the projections of two vertices get closer to one another.

Furthermore, to cut down on the combinatorial complexity of the interpolation scheme in the last step, it makes sense to select directions reasonably close to one another. We make use of a generic reference direction $\hat{\delta}$ with reasonable conditioning and other directions in the neighborhood.

6.1 Reconstruction of polygons

We begin our simulations with the reconstruction of 2-dimensional polygons. A direction $\delta = (\cos\theta, \sin\theta)$ is represented by an angle $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}]$. The projection of vertex with coordinates (v_1, v_2) is given by $v_1\cos\theta + v_2\sin\theta$. Then the interval interpolation problem formulated in terms of θ is

$$v_1\cos\theta + v_2\sin\theta \in [w^-, w^+], \qquad \text{for a single } w \in \tilde{\mathcal{V}}(\theta).$$

where $w^- = w - \epsilon(\delta)$ and $w^+ = w + \epsilon(\delta)$ as in Section 5.

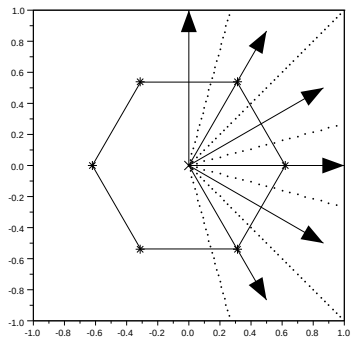
6.1.1 Reconstruction of a regular hexagon

The regular hexagon ($r = 6$) is presented in Figure 1(a). The condition number $\kappa(\delta)$ increases drastically when δ approaches an axis of symmetry. The axes of symmetry are represented by arrows in Figure 1(a). In Figure 1(b) the condition number $\kappa(\delta)$ of the Vandermonde matrix $W_6(\delta)$ is plotted for 300 equidistant generic directions. The condition number is minimal for directions bisecting two consecutive symmetry axes, represented by dotted lines in Figure 1(a).

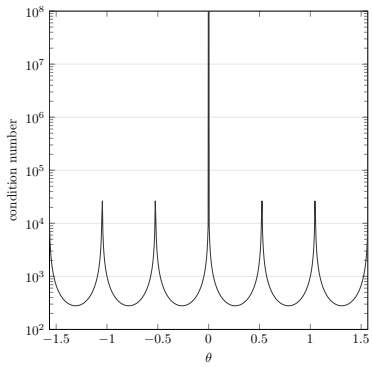
The modified directional moments $\mu_k(\delta)$ are computed in double precision. The number r of vertices is retrieved as the rank of $H_k^{(0)}(\delta)$, for k large enough and arbitrary direction δ. The singular values of $H_7^{(0)}(\delta)$ are plotted in Figure 1(c) for three directions picked at random.

We choose the direction $\hat{\delta}$ with angle $\hat{\theta} = \frac{5\pi}{12}$. It corresponds to one of the bisectors. We then take 4 nearby directions $\hat{\theta} \pm 0.05, \hat{\theta} \pm 0.10$. For each of the 5 directions we construct the pair of Hankel matrices $\left(H_6^{(1)}(\delta),\, H_6^{(0)}(\delta)\right)$ and compute their generalized eigenvalues. For these directions $\kappa(\delta)$ is around 300. We therefore take interval of size 2×10^{-10} according to (5.1).

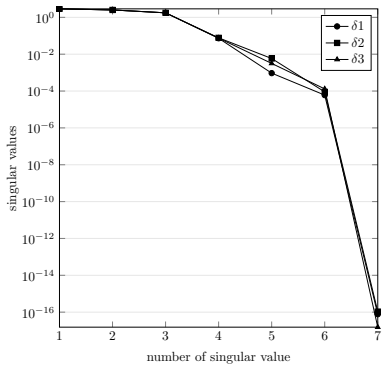
We compute the interval interpolants from the 5 sets of projections $\mathcal{V}(\hat{\theta} + k\ 0.05), k = -2,\ -1,\ 0,\ 1,\ 2$. The relative error on the computed projections is bounded overall by 6.4×10^{-13} for the 5 chosen directions. The coordinates of the 6 different vertices of the regular hexagon are recovered as the coefficients of the interpolants graphed in Figure 1(d). The relative error on each computed coordinate compared to its true value is bounded overall by 1.5×10^{-12}.



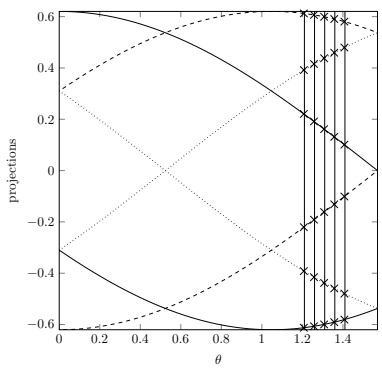
(a) Regular hexagon, its axes of symmetry and their bisectors



(b) Condition number $\kappa(\theta)$ with respect to the direction $(\cos\theta, \sin\theta)$



(c) Singular values of $H_7^{(0)}(\delta)$



(d) Interpolation with intervals of width 10^{-10}

Figure 1: Regular hexagon

6.1.2 Reconstruction of a random polygon with 12 vertices

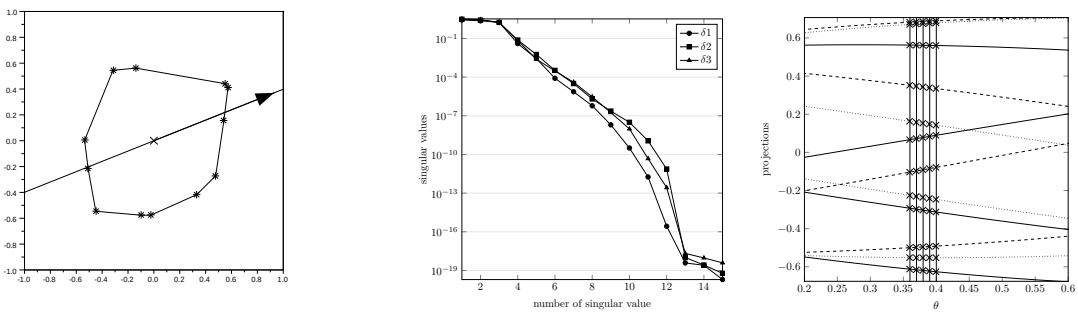
In this second simulation, we consider a random 12-gon, centered and scaled. It is drawn in Figure 2(a).

As in the case of the hexagon above, the number of vertices is retrieved by computing the singular values of $H_{15}^{(0)}(\delta)$ in 3 directions δ. From Figure 2(b) we deduce that the numerical rank is $r = 12$.

After inspecting some directions, we choose the reference direction $\hat{\delta}$ with $\hat{\theta} = 0.379521$ (arrow in Figure 2(a)) and 4 other nearby directions $\hat{\theta} \pm 0.05, \hat{\theta} \pm 0.10$. The projections of the vertices on these directions are obtained as the generalized eigenvalues of the pairs of Hankel matrices $\left(H_{12}^{(1)}(\delta),\ H_{12}^{(0)}(\delta)\right)$ whose entries are obtained from the respective modified directional moments $(\mu_k(\delta))_{0\leq k\leq 21}$. The condition number $\kappa(\delta)$ of the matrix of generalized eigenvectors $W_{12}(\delta)$ is around 7×10^5 for all 5 directions. The relative error on the computed projections compared to their true values is bounded by 1.8×10^{-7}.

From these 5 sets $\mathcal{V}(\hat{\theta} - 0.10), \mathcal{V}(\hat{\theta} - 0.05), \mathcal{V}(\hat{\theta}), \mathcal{V}(\hat{\theta} + 0.05), \mathcal{V}(\hat{\theta} + 0.10)$, we compute the 12 linear interval interpolants. According to (5.1) we take intervals of size $2\epsilon = 2 \times 10^{-10}$. The relative error in each coordinate is bounded by 3.9×10^{-6}.

Note that even the two very close vertices in the top right corner in Figure 2(a) are recovered with the accuracy mentioned above. The distance between them is only of the order of 10^{-2}.



(a) Centered and scaled polygon (b) Singular values of $H_{15}^{(0)}(\delta)$ (c) Interpolation with intervals of width 10^{-5}

Figure 2: Polygon with 12 vertices

6.2 Reconstruction of polyhedra

We now consider the reconstruction of convex polyhedra in dimension 3. The dimension does not introduce new difficulties in our method. A direction δ is represented by a unit vector $(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta))$ with (θ, ϕ) lying in $]-\frac{\pi}{2}, \frac{\pi}{2}]\times]-\frac{\pi}{2}, \frac{\pi}{2}]$. The projection of vertex v with coordinates (v_1, v_2, v_3) on $\delta = (\theta, \phi)$ equals $v_1 \cos(\theta)\cos(\phi) + v_2 \sin(\theta)\cos(\phi) + v_3 \sin(\phi)$. The tuples of coordinates (v_1, v_2, v_3), for $v \in \mathcal{V}$, are the unknowns in the linear interval interpolation problem $q_v(\delta) \in [w^-, w^+]$ for a single $w \in \tilde{\mathcal{V}}(\delta)$, as described in Section 5.

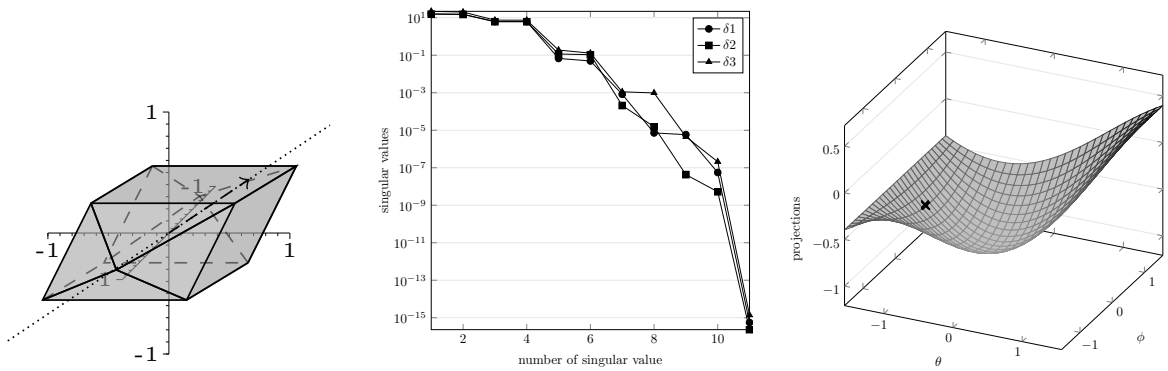
6.2.1 Reconstruction of a polyhedron with well-distributed vertices

We first consider an easy polyhedron with 10 vertices represented in Figure 3(a).

We retrieve the number of vertices of the polyhedron by computing the numerical rank of the Hankel matrix $H_{11}^{(0)}(\delta)$ in 3 different directions δ. The singular values of $H_{11}^{(0)}(\delta)$ are plotted in Figure 3(b) for three random directions.

After inspecting several directions, we select $(\hat{\theta}, \hat{\phi}) = (-1.256637, 0.261799)$ for the reference direction $\hat{\delta}$ and 4 other nearby directions where the condition number $\kappa(\delta)$ is of order 10^4. $\hat{\delta}$ is indicated by an arrow and a dotted line in Figure 3(a). We take the nearby directions as $(\hat{\theta}+\varepsilon, \hat{\phi})$, $(\hat{\theta}, \hat{\phi}-\varepsilon)$, $(\hat{\theta}-\varepsilon, \hat{\phi}+\varepsilon)$ and $(\hat{\theta}+\varepsilon, \hat{\phi}+\varepsilon)$ with $\varepsilon = 0.01$. For each direction δ, the pair of matrices $\left(H_{10}^{(1)}(\delta),\, H_{10}^{(0)}(\delta)\right)$ is built with directional moments up to order 16. The generalized eigenvalues of the pairs $\left(H_{10}^{(1)}(\delta),\, H_{10}^{(0)}(\delta)\right)$ provide the projections of the vertices with a relative error bounded by 1.3×10^{-8}.

We compute the 10 linear interval interpolants using intervals of width 2×10^{-7} for the projections. In Figure 3(c) we show one of those surfaces. The coordinates of the 10 vertices of the polyhedron are obtained as the coefficients of the interpolants. The relative error on this final result is bounded by 8.3×10^{-7}.



(a) Polyhedron (b) Singular values of $H_{11}^{(0)}(\delta)$ (c) Interpolant for one vertex

Figure 3: First polyhedron with 10 vertices

6.2.2 Reconstruction of a polyhedron with close vertices

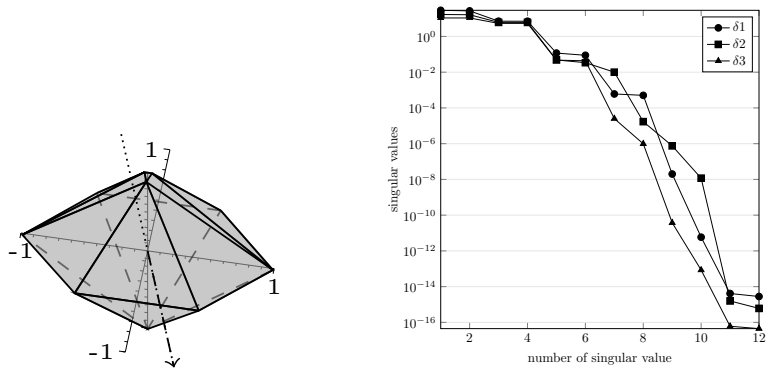
Here we examine a polyhedron with 10 vertices and a triangular facet of relatively small size (see the top of Figure 4(a)).

In Figure 4(b) we plot the singular values of $H_{12}^{(0)}(\delta)$ for three random directions δ. The gap between the tenth and the eleventh singular value appears more clearly in some of the directions.

We choose the reference direction $(\hat{\theta}, \hat{\phi}) = (-0.994838, -0.994838)$ for which the condition number $\kappa(\delta)$ is 4.2×10^4. It is indicated in Figure 4(a) by an arrow and a dotted line. We additionally pick the nearby directions $(\hat{\theta}+\varepsilon, \hat{\phi})$, $(\hat{\theta}, \hat{\phi}-\varepsilon)$, $(\hat{\theta}-\varepsilon, \hat{\phi}+\varepsilon)$ and $(\hat{\theta}+\varepsilon, \hat{\phi}+\varepsilon)$ with $\varepsilon = 0.01$. For each direction the pair of matrices $\left(H_{10}^{(1)}(\delta),\, H_{10}^{(0)}(\delta)\right)$ is built with directional

moments up to order 16. The projections $\tilde{\mathcal{V}}(\delta)$ of the vertices on those directions are retrieved as the generalized eigenvalues of $\left(H_{10}^{(1)}(\delta),\, H_{10}^{(0)}(\delta)\right)$ with a relative error bounded by 2.2×10^{-7}.

We determine the 10 interpolants for our sets of projections using intervals of width 2×10^{-6}. The coordinates of the 10 vertices appear as the coefficients of the interpolants. The relative error on these is bounded by 1.5×10^{-6}.

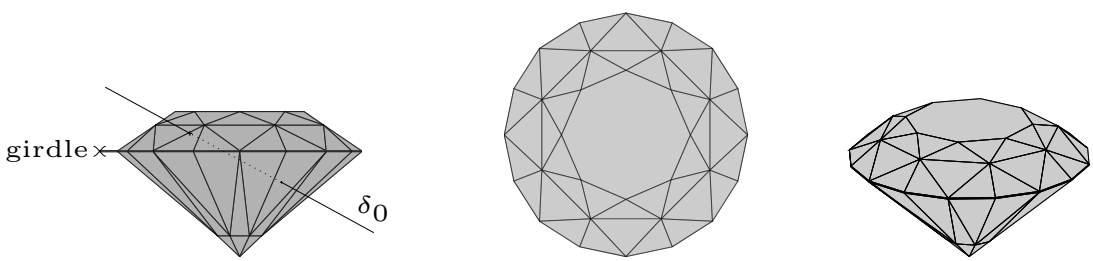


(a) Centered and scaled polyhedron (b) Singular values of $H_{12}^{(0)}$

Figure 4: Second polyhedron with 10 vertices

7 Diamond

As a challenge we choose an actual brilliant cut of a diamond. It is given as a convex polyhedron with 57 vertices and represented in Figure 5. The stone girdle consists of pairs of vertices very close to one another. The number of vertices and the small distance between the projections of the vertices severely impact the condition number $\kappa(\delta)$ of the Vandermonde matrix $W_{57}(\delta)$: double precision is no longer enough to retrieve sufficiently accurate values for the projections. We rely on the software floats of Maple to provide the needed number of digits for the computations.

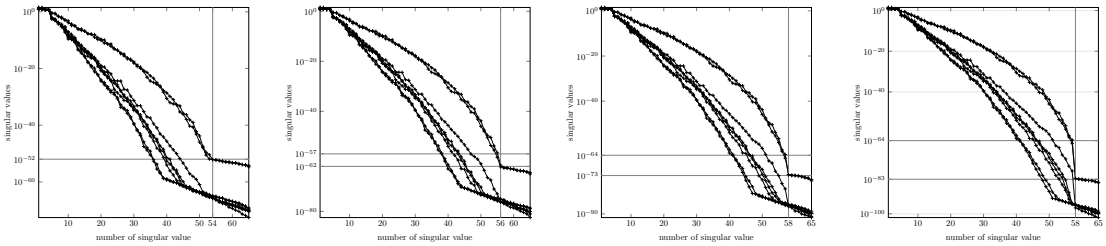


(a) Side view (b) Top view (c) In perspective

Figure 5: Diamond and reference direction δ_0

To reliably retrieve the number of vertices, we use a precision of 70 digits. Figure 6 tracks the singular values of $H_{65}^{(0)}(\delta)$ for 8 random directions δ and in different computational precisions.

After sampling a rather large number of directions, we select the reference direction $(\hat{\theta}, \hat{\phi}) =$



(a) 50 digits (b) 60 digits (c) 70 digits (d) 80 digits

Figure 6: Singular Values of $H_{65}^{(0)}(\delta)$ for several computational precision

$(0.261799, 1.047198)$ shown in Figure 5(a). The condition number $\kappa(\hat{\delta})$ for this direction is 1.67×10^{33}. We choose 4 nearby directions with a similar condition number, $(\hat{\theta}+\varepsilon, \hat{\phi})$, $(\hat{\theta}, \hat{\phi}-\varepsilon)$, $(\hat{\theta}-\varepsilon, \hat{\phi}+\varepsilon)$, $(\hat{\theta}+\varepsilon, \hat{\phi}+\varepsilon)$ where $\varepsilon = 0.0001$. Computing with 70 digits we expect to retrieve the projections of the vertices with an accuracy of at least 10^{-3}.

For each of the 5 selected directions δ, the pair of matrices $\left(H_{57}^{(1)}(\delta),\, H_{57}^{(0)}(\delta)\right)$ is built with the directional moments up to order 110. The projections of the vertices are obtained as the generalized eigenvalues. The relative error is actually bounded by 8.1×10^{-8}.

We compute the 57 interpolants in double precision using intervals of width 2×10^{-4}. The coordinates of the vertices are the coefficients in those interpolants. The relative error is bounded by 7.8×10^{-5}. We plot the error for all the vertices in Figure 7.

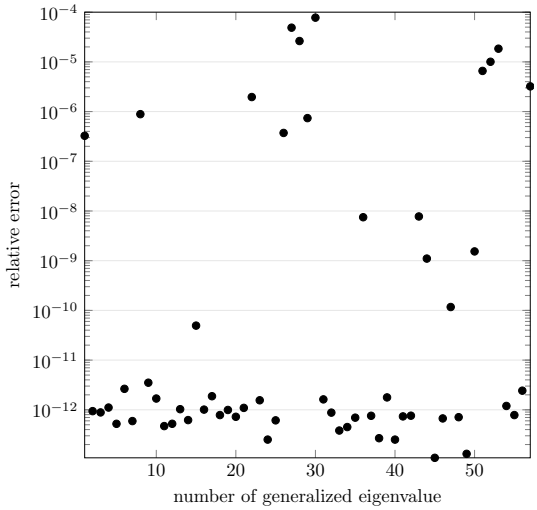
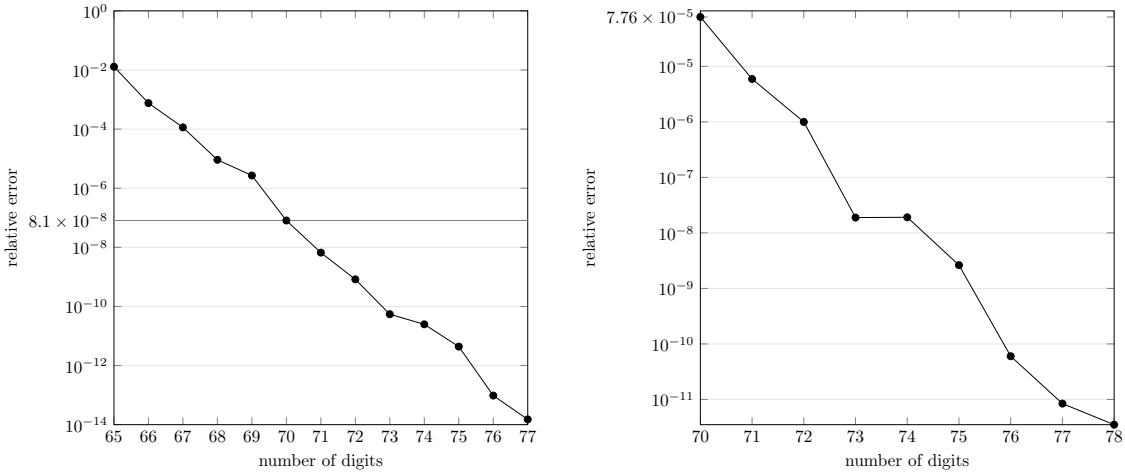


Figure 7: Error observed when operating with 70 digits

In Figure 8 we report on the error using various computational precisions. For computations with less than 65 digits we do not recover all the projections while the complete set of coordinates of the vertices can be retrieved only if we use at least 70 digits.



(a) Error on the retrieved projections

(b) Error on the retrieved vertex coordinates

Figure 8: Error for different computational precisions

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