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# Converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations\*

Ihab Haidar<sup>†</sup>, Paolo Mason<sup>‡</sup>, Mario Sigalotti<sup>§</sup>

## Abstract

In this article, we give a collection of converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations. We show that the existence of a weakly degenerate Lyapunov–Krasovskii functional is a necessary and sufficient condition for the global exponential stability of the linear retarded functional differential equations. This is carried out using the switched system transformation approach.

## 1 Introduction

The stability of *time-varying delay* systems is a problem of current interest. Two principal approaches in the stability analysis are the Lyapunov–Krasovskii method and Lyapunov–Razumikhin method [6]. A variety of stability criteria, based on these two approaches, have been developed in this context (see e.g. [2, 3, 12, 13, 15] and references therein). These criteria are often formulated as linear matrix inequalities (LMIs) which yield sufficient conditions for stability. Switched systems theory offers a complementary insight in this context. In [8] Hetel, Daafouz and Iung establish a theoretical link between the Lyapunov–Krasovskii approach and the switched system transformation in the context of discrete-time systems with time-varying delays. They prove that applying the multiple Lyapunov functions approach to the switched systems representation is equivalent to using a general, delay-dependent, Lyapunov–Krasovskii functional for the initial system. This paper shares the same spirit as [8] with the significant difference that it considers the case of continuous-time systems with time-varying delays. An important feature of this setting is that the switched system formulation, obtained by standard functional representation, describes an evolution in an infinite-dimensional space. Converse Lyapunov theorems for

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switched systems in Banach and Hilbert spaces were carried out in [7, 16]. In this paper we give a collection of converse Lyapunov–Krasovskii theorems for uncertain retarded differential equations, by exploiting similar ideas and some of the results in [7].

Several converse Lyapunov–Krasovskii theorems have been presented in the literature for systems described by retarded functional differential equations, in a general (possibly time-varying but not uncertain) setting [4, 9, 11, 14]. In [14], in particular, the authors show that the existence of a Lyapunov–Krasovskii functional is a necessary and sufficient condition for the uniform global asymptotic stability and the global exponential stability of autonomous systems described by neutral functional differential equations in Hale’s form. In [9], the author extends the well-known Lyapunov theorem given for linear ordinary differential equations to general linear time-delay systems. A method to construct a quadratic Lyapunov functional for such systems is proposed. Even though this Lyapunov functional is quadratic, it only admits a cubic lower bound. Another method to construct Lyapunov functionals for a class of linear time-delay systems is proposed in [4, 11]. These Lyapunov functionals admit a quadratic lower bound.

More in the spirit of the results presented here are the converse Lyapunov theorems for uncertain nonlinear retarded differential equations obtained by Karafyllis in [10]. The latter are obtained for a very general class of dynamics (which in particular are allowed to be non-autonomous). The conclusions are accordingly less detailed on the properties of the Lyapunov functionals with respect to ours, as explained below.

In this work, we consider a general linear retarded functional differential equation (RFDE)

$$\dot{x}(t) = L_{\sigma(t)}x_t, \quad t \geq 0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  represents the system state at time  $t$ ,  $x_t : [-r, 0] \rightarrow \mathbb{R}^n$  is the standard notation for the history function defined by  $x_t(\theta) = x(t + \theta)$ ,  $-r \leq \theta \leq 0$ . The operator  $L_\sigma$  associates with  $x_t$  a vector in  $\mathbb{R}^n$ : a typical example is  $L_\sigma x_t = A_0 x(t) + A_1 x(t - \tau(\sigma))$  for some  $n \times n$  matrices  $A_0$  and  $A_1$ , and some delay  $\tau(\sigma) \in [-r, 0]$ . Two phase spaces are considered:  $C([-r, 0], \mathbb{R}^n)$  and  $H^1([-r, 0], \mathbb{R}^n)$ . The signal  $\sigma(\cdot)$  is a piecewise constant (or, more generally, measurable) function taking values in an index set. We are interested in properties that are uniform with respect to  $\sigma(\cdot)$ , which plays the role of a switching signal. We prove that system (1) is *uniformly exponentially stable* in  $C([-r, 0], \mathbb{R}^n)$  if and only if its restriction to  $H^1([-r, 0], \mathbb{R}^n)$  is also uniformly exponentially stable (with respect to the  $H^1$ -norm). We then give a collection of converse Lyapunov–Krasovskii theorems. They characterize the uniform exponential stability in terms of the existence of functionals with suitable commensurability conditions and decreasing uniformly along the solutions of (1). One of the novelties of our results is that these functionals may be *weakly degenerate*, i.e., they may not have a strictly positive norm-dependent lower bound, in contrast with what is known in the literature.

The paper is organized as follows. Section 2 is devoted to the problem frame-

work. Namely, we discuss the well-posedness and the switched system representation of system (1). The statement of the main result (Theorem 2) is presented in Section 3. Next, some stability properties are proved in Section 4. The core of the proof of Theorem 2 is given in Section 5. Section 6 is devoted to the comparison of our results with some previously known Lyapunov–Krasovskii theorems.

While piecewise constant switching signals are considered throughout the paper, we show in the Appendix how this study can be extended to a larger class of switched systems by considering the case of measurable signals.

## 2 Problem framework

Let  $r > 0$  be a real number and let  $X = X([-r, 0], \mathbb{R}^n)$  be a linear real vector space of functions mapping  $[-r, 0]$  into  $\mathbb{R}^n$ . Assume that a norm  $\|\cdot\|_X$  is given in  $X$ , and assume that  $(X, \|\cdot\|_X)$  is a Banach space. Two different choices of Banach spaces are considered:  $X = (C([-r, 0], \mathbb{R}^n), \|\cdot\|_C)$  and  $X = (H^1([-r, 0], \mathbb{R}^n), \|\cdot\|_{H^1})$ . Let  $\mathbb{S}$  be an index set and consider a family of linear operators parameterized by  $\mathbb{S}$  as follows

$$Q := \{L_\sigma \in \mathcal{L}(X, \mathbb{R}^n) \mid \sigma \in \mathbb{S}\}.$$

Let us associate with  $Q$  the linear Retarded Functional Differential Equation (RFDE)

$$\dot{x}(t) = L_{\sigma(t)}x_t, \quad (2)$$

where  $x(t) \in \mathbb{R}^n$  represents the system state at time  $t$ ,  $x_t : [-r, 0] \rightarrow \mathbb{R}^n$  denotes the state defined by  $x_t(\theta) = x(t + \theta)$ ,  $-r \leq \theta \leq 0$ ,  $\sigma : [0, \infty) \rightarrow \mathbb{S}$  is piecewise constant. The function  $\sigma(\cdot)$  plays the role of a switching signal and we are interested in properties that are uniform with respect to  $\sigma(\cdot)$ .

Let the initial condition be

$$x(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-r, 0], \quad (3)$$

where  $\varphi(\cdot) \in X$  and  $t_0 \geq 0$ . If  $t \geq t_0$ , we adopt the standard notation  $x_t$  for the history function defined by

$$x_t : \theta \mapsto x(t + \theta), \quad \theta \in [-r, 0].$$

Of special interest is the case in which  $Q$  is a bounded subset of  $\mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$ , i.e. there exists a positive constant  $m$  such that

$$|L_\sigma \varphi| \leq m \|\varphi\|_C \quad \forall \varphi \in C([-r, 0], \mathbb{R}^n), s \in \mathbb{S}. \quad (4)$$

**Example 1** *A particular case of system (2) is the following type of linear time-varying delayed systems*

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^p A_{\sigma(t), i} x(t - \tau_i(\sigma(t))), & t \geq 0, \\ x_0 &= \varphi, \end{aligned} \quad (5)$$

where  $\varphi$  is the initial condition,  $\tau(\cdot) = (\tau_1(\cdot), \dots, \tau_p(\cdot))$ , with  $0 \leq \tau_i(\sigma(t)) \leq r$ , and  $A_{\sigma,i}$  is a  $n \times n$  matrix for every  $\sigma \in \mathbb{S}$ ,  $i \in \{1, \dots, p\}$ . If the set of matrices  $\{A_{\sigma,i} \mid \sigma \in \mathbb{S}, i \in \{1, \dots, p\}\}$  is bounded, then the set  $Q$  of the operators  $L_\sigma \varphi = \sum_{i=1}^p A_{\sigma,i} \varphi(t - \tau_i(\sigma))$ ,  $\sigma \in \mathbb{S}$ , is clearly bounded in  $\mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$ .

**Example 2** Another particular case of system (2) is the following integro-differential equation

$$\begin{aligned} \dot{x}(t) &= \int_0^r A_{\sigma(t),\theta} x(t-\theta) d\theta, & t \geq 0, \\ x_0 &= \varphi, \end{aligned}$$

where  $\varphi \in X$  is the initial condition and  $A_{\sigma,\theta}$  is a  $n \times n$  matrix uniformly bounded with respect to  $\sigma \in \mathbb{S}$  and  $\theta \in [0, r]$  and measurable with respect to  $\theta$ . In this case the set  $Q$  of the operators  $L_\sigma \varphi = \int_0^r A_{\sigma,\theta} \varphi(-\theta) d\theta$ ,  $\sigma \in \mathbb{S}$ , is clearly bounded in  $\mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$ .

## 2.1 Well-posedness and switched system representation

In this section we discuss the well-posedness problem for a RFDE, depending on the choice of the space  $X$ . Before that, we recall, following [6], that a function  $x$  is said to be a *solution* of system (2) on  $[-r, +\infty)$  with initial condition  $\varphi \in X([-r, 0], \mathbb{R}^n)$  if  $x$  is absolutely continuous on  $[0, +\infty)$ ,  $x_t \in X([-r, 0], \mathbb{R}^n)$  for every  $t \geq 0$  and  $x(t)$  satisfies system (2), for almost every  $t \geq 0$ . In the case  $X = C([-r, 0], \mathbb{R}^n)$ , under the hypothesis that  $Q$  is bounded, it is well-known that (2) has a unique solution defined on  $[0, +\infty)$  for each initial condition  $\varphi$ . The solution is defined under the more general *Carathéodory condition*, i.e. when  $t \mapsto L_{\sigma(t)} \varphi$  is measurable for every  $\varphi \in C([-r, 0], \mathbb{R}^n)$  (see [6]).

**Lemma 1** Consider the linear RFDE given by system (2). Let  $X$  be the Banach spaces  $C([-r, 0], \mathbb{R}^n)$  or  $H^1([-r, 0], \mathbb{R}^n)$ . Assume that condition (4) holds. For every  $\varphi \in X$  there exists a unique solution of (2) in  $X$  with initial condition  $\varphi$ .

*Proof.* When  $X = C([-r, 0], \mathbb{R}^n)$  and under the hypothesis given by equation (4), the existence and uniqueness of solutions for system (2) is guaranteed by [6, Theorem 1.1]. When  $X = H^1([-r, 0], \mathbb{R}^n)$ , one has to check that such solution, a priori defined in  $C([-r, 0], \mathbb{R}^n)$ , actually stays in  $X$ . This is true since both  $x$  and  $\dot{x}$  are locally bounded for positive times.  $\square$

In the case where  $Q$  is not bounded one can nevertheless associate with (2) an evolution operator obtained by concatenating the flows corresponding to constant values of  $\sigma$ . Recall that with any  $\sigma \in \mathbb{S}$  one can associate a  $C_0$ -semigroup  $T_\sigma(t) : X \rightarrow X$  defined by  $T_\sigma(t)(x_0) = x_t$ , with infinitesimal generator  $\mathcal{A}_\sigma$  given

by

$$\begin{aligned} D(\mathcal{A}_\sigma) &= \left\{ \varphi \in X : \frac{d\varphi}{d\theta} \in X, \frac{d\varphi}{d\theta}(0) = L_\sigma \varphi \right\}, \\ \mathcal{A}_\sigma \varphi &= \frac{d\varphi}{d\theta}. \end{aligned} \tag{6}$$

The evolution operator corresponding to a piecewise constant signal  $\sigma(t) = \sum_{k \geq 0} \mathbf{1}_{[t_k, t_{k+1})}(t) \sigma_k$  with  $t_0 \geq 0$ ,  $t_k < t_{k+1}$  for  $k \geq 0$  is given by

$$T_{\sigma(\cdot)}(t) = T_{\sigma_k}(t - t_k) T_{\sigma_{k-1}}(t_k - t_{k-1}) \dots T_{\sigma_0}(t_1 - t_0)$$

for each  $t \in [t_k, t_{k+1})$ . In order to define the evolution in this sense one just needs a family of  $C^0$ -semigroups. This is exactly the notion of switched system

$$\begin{aligned} x_t &= T_{\sigma(\cdot)}(t)x_0, \\ x_{t_0} &= \varphi \in X, \end{aligned} \tag{7}$$

considered in [7] for general Banach spaces.

Similarly, one can associate a switched system with (5) for initial conditions in the  $L^2$  setting. For every  $\sigma \in \mathbb{S}$  and  $(\varphi, z) \in L^2([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n$  there exists a function  $x(\cdot)$ , absolutely continuous on  $[0, +\infty)$ , satisfying  $x(0) = z$  and (5) almost everywhere. Moreover  $(x_t(\cdot), x(t)) = T_\sigma(t)(\varphi(\cdot), z)$  where  $T_\sigma(\cdot)$  is a  $C_0$ -semigroup on  $L^2([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n$  (see e.g. [1]). We discuss the stability properties of this switched system in Section 4.2.

### 3 Statement of the main result

The notion of uniform exponential stability is recalled in the following definition.

**Definition 1** *We say that system (2) is uniformly exponentially stable in  $X$  if there exist two constants  $\alpha \geq 1$  and  $\beta > 0$  such that for every initial condition  $\varphi \in X$  the solution  $x(t, \varphi)$  of (2)-(3) exists for every  $t \in [t_0, +\infty)$  and satisfies*

$$\|x_t\|_X \leq \alpha e^{-\beta(t-t_0)} \|\varphi\|_X, \quad t \geq t_0, \quad \sigma(\cdot)\text{-uniformly.}$$

Here and in the following,  $\sigma(\cdot)$ -uniformly means that the constants  $a, b$  do not depend on the choice of the piecewise constant  $\sigma : [0, \infty) \rightarrow \mathbb{S}$ .

**Remark 1** *Because of its linear nature, one can easily check that system (2) is uniformly exponentially stable if and only if it is uniformly asymptotically stable. We recall that system (2) is said to be uniformly asymptotically stable if for every  $R > 0$  there exists  $r > 0$  such that, for every solution  $x(\cdot)$  of (2)-(3),  $\|\varphi\|_X < r$  implies  $\|x_t\|_X < R$  for  $t \geq t_0$ , and if, moreover, the solutions of (2)-(3) converge to zero uniformly (i.e., there exists a constant  $\delta > 0$ , independent of  $t_0$  and  $\sigma(\cdot)$ , such that for every  $\epsilon > 0$  there exists a  $\zeta(\epsilon)$  such that  $\|\varphi\|_X < \delta$  implies  $\|x_t\|_X < \epsilon$  for every  $t \geq t_0 + \zeta$ ).*

**Remark 2** By definition of  $T_{\sigma(\cdot)}(t)$ , we have that system (2) is uniformly exponentially stable in  $X$  if and only if there exist constants  $\alpha \geq 1$  and  $\beta > 0$  such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq \alpha e^{-\beta(t-t_0)}, \quad t \geq t_0, \quad \sigma(\cdot)\text{-uniformly.}$$

From now on, without loss of generality, we assume that  $t_0 = 0$ .

In order to state our main result we introduce the generalized Dini derivatives of a function  $V : X \rightarrow [0, \infty)$  as follows

$$\begin{aligned} \overline{D}_\sigma V(x) &= \limsup_{t \rightarrow 0} \frac{V(T_\sigma(t)x) - V(x)}{t}, \\ \underline{D}_\sigma V(x) &= \liminf_{t \rightarrow 0} \frac{V(T_\sigma(t)x) - V(x)}{t}, \end{aligned}$$

noting that possibly  $\underline{D}_\sigma V(x), \overline{D}_\sigma V(x) = \infty$  for some  $x \in X$  and  $\sigma \in \mathbb{S}$ . Recall that  $V(\cdot)$  is said to be *directionally differentiable in the sense of Fréchet* at  $x \in X$  if there exists a continuous linear functional  $V'(x)$  such that for every  $\psi \in X$

$$V'(x)(\psi) = \lim_{t \rightarrow 0} \frac{V(x + t\psi) - V(x)}{t},$$

and, moreover,

$$\lim_{\psi \rightarrow 0} \frac{V(x + \psi) - V(x) - V'(x)(\psi)}{\|\psi\|_X} = 0.$$

The main result of this paper is the following theorem.

**Theorem 2** Let  $Q \subset \mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$  be bounded. The following statements are equivalent:

- (i) System (2) is uniformly exponentially stable in  $C([-r, 0], \mathbb{R}^n)$ .
- (ii) System (2) is uniformly exponentially stable in  $H^1([-r, 0], \mathbb{R}^n)$ .
- (iii) There exists a function  $V : C([-r, 0], \mathbb{R}^n) \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $C([-r, 0], \mathbb{R}^n)$ ,

$$\underline{c}\|\varphi\|_C^2 \leq V(\varphi) \leq \bar{c}\|\varphi\|_C^2$$

for some constants  $\underline{c}, \bar{c} > 0$  and

$$\overline{D}_\sigma V(\varphi) \leq -\|\varphi\|_C^2, \quad \sigma \in \mathbb{S}, \varphi \in C([-r, 0], \mathbb{R}^n).$$

- (iv) There exists a directionally Fréchet differentiable function  $V : H^1([-r, 0], \mathbb{R}^n) \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $H^1([-r, 0], \mathbb{R}^n)$ ,

$$\underline{c}\|\varphi\|_{H^1}^2 \leq V(\varphi) \leq \bar{c}\|\varphi\|_{H^1}^2$$

for some constants  $\underline{c}, \bar{c} > 0$  and

$$\overline{D}_\sigma V(\varphi) \leq -\|\varphi\|_{H^1}^2, \quad \sigma \in \mathbb{S}, \varphi \in H^1([-r, 0], \mathbb{R}^n).$$

(v) *There exists a continuous function*

$$V : C([-r, 0], \mathbb{R}^n) \rightarrow [0, \infty)$$

*such that*

$$V(\varphi) \leq c \|\varphi\|_C^2$$

*for some constant  $c > 0$  and*

$$\underline{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in C([-r, 0], \mathbb{R}^n).$$

(vi) *There exists a continuous function*

$$V : H^1([-r, 0], \mathbb{R}^n) \rightarrow [0, \infty)$$

*such that*

$$V(\varphi) \leq c \|\varphi\|_{H^1}^2$$

*for some constant  $c > 0$  and*

$$\underline{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in H^1([-r, 0], \mathbb{R}^n).$$

Clearly, a Lyapunov function  $V(\cdot)$  satisfying condition (v) (respectively (vi)) does not necessarily satisfy the stronger conditions appearing in condition (iii) (respectively (iv)). Hence, condition (v) (respectively (vi)) is better suited for proving the global uniform exponential stability of a RFDE, while condition (iii) (respectively (iv)) provides more information on a linear uncertain time-delay system that is known to be globally uniformly exponentially stable, by tightening the properties satisfied by  $V(\cdot)$ .

**Remark 3** *Concerning the regularity of the Lyapunov functions satisfying conditions (iii) and (iv), recall that being a squared norm is equivalent to be positive definite, homogeneous of degree 2, continuous and convex.*

## 4 Stability

We begin this section by showing that the solutions of (2) are  $\sigma(\cdot)$ -uniformly exponentially bounded, i.e, there exist  $M \geq 1$ ,  $w > 0$  such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq M e^{wt}, \quad t \geq 0, \quad \sigma(\cdot)\text{-uniformly}, \quad (8)$$

in both cases  $X = C([-r, 0], \mathbb{R}^n)$  and  $X = H^1([-r, 0], \mathbb{R}^n)$ . This property, which is a necessary condition for the  $\sigma(\cdot)$ -uniform exponential stability of (2), is not satisfied when  $X = L^2([-r, 0], \mathbb{R}^n)$ , as we show by means of a counterexample. Then, we prove the equivalence between the uniform exponential stability of (2) in the spaces  $C([-r, 0], \mathbb{R}^n)$  and  $H^1([-r, 0], \mathbb{R}^n)$ .



## 4.1 Uniform exponential boundedness

The  $\sigma(\cdot)$ -uniform exponential boundedness of the solutions of (2), both when  $X = C([-r, 0], \mathbb{R}^n)$  and  $H^1([-r, 0], \mathbb{R}^n)$ , is given by the following lemma.

**Lemma 3** *Let  $Q \subset \mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$  be bounded. If  $X = C([-r, 0], \mathbb{R}^n)$  or  $H^1([-r, 0], \mathbb{R}^n)$  then the solutions of (2) are  $\sigma(\cdot)$ -uniformly exponentially bounded.*

*Proof.* The proof is provided separately for the two cases.

Let  $X = C([-r, 0], \mathbb{R}^n)$  and  $\varphi \in X$ . By integrating system (2) and using equation (4), one has for every  $t \geq 0$  and every  $\mu \in [-r, 0]$

$$\begin{aligned} |x(t + \mu)| &\leq \max\{\|\varphi\|_C, |\varphi(0)| + \int_0^{\max\{0, t+\mu\}} m \|x_s\|_C ds\} \\ &\leq \|\varphi\|_C + m \int_0^{\max\{0, t+\mu\}} \|x_s\|_C ds, \end{aligned}$$

that is

$$\|x_t\|_C \leq \|\varphi\|_C + m \int_0^t \|x_s\|_C ds.$$

Thanks to Gronwall's Lemma, we have

$$\|x_t\|_C \leq \|\varphi\|_C e^{mt}, \quad (9)$$

which implies that the solutions of (2) are  $\sigma(\cdot)$ -uniformly exponentially bounded in  $C([-r, 0], \mathbb{R}^n)$ .

Let now  $X = H^1([-r, 0], \mathbb{R}^n)$  and  $\varphi \in X$ . For each  $t \geq 0$ , we have

$$\|x_t\|_{L^2}^2 = \int_{-r}^0 |x_t(s)|^2 ds \leq \int_{-r}^0 \|x_t\|_C^2 ds \leq r \|x_t\|_C^2. \quad (10)$$

Combining equations (9) and (10) and using the Poincaré inequality

$$\|\varphi\|_C \leq \lambda_r \|\varphi\|_{H^1}, \quad (11)$$

where  $\lambda_r$  only depends on  $r$ , gives

$$\|x_t\|_{L^2}^2 \leq r \lambda_r^2 e^{2mt} \|\varphi\|_{H^1}^2. \quad (12)$$

On the other hand, from (2), (4) and (9) we have the following inequalities

$$\begin{aligned}
\|\dot{x}_t\|_{L^2}^2 &= \int_{-r}^0 |\dot{x}_t(s)|^2 ds = \int_{t-r}^t |\dot{x}(s)|^2 ds \\
&\leq \|\dot{\varphi}\|_{L^2}^2 + \int_0^t |\dot{x}(s)|^2 ds \\
&\leq \|\varphi\|_{H^1}^2 + m^2 \int_0^t \|x_s\|_C^2 ds \\
&\leq \|\varphi\|_{H^1}^2 + m^2 \|\varphi\|_C^2 \int_0^t e^{2ms} ds \\
&\leq \left(1 + \frac{m\lambda_r^2}{2} (e^{2mt} - 1)\right) \|\varphi\|_{H^1}^2 \\
&\leq (1 + m\lambda_r^2) e^{2mt} \|\varphi\|_{H^1}^2.
\end{aligned} \tag{13}$$

Combining equations (12) and (13) gives

$$\|x_t\|_{H^1} \leq C_t \|\varphi\|_{H^1}$$

with

$$C_t^2 = (1 + m\lambda_r^2 + r\lambda_r^2) e^{2mt},$$

which ends the proof.  $\square$

## 4.2 Uniform exponential boundedness: case $X = L^2([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n$

The uniform exponential boundedness of the solutions of (2) is not, in general, an acquired property. In fact, even if this property holds when  $X = C([-r, 0], \mathbb{R}^n)$  or  $X = H^1([-r, 0], \mathbb{R}^n)$ , this is no longer true in the case  $X = L^2([-r, 0], \mathbb{R}^n) \times \mathbb{R}^n$ . This point is clarified by means of the following counterexample. Consider the one-dimensional switched delayed system in  $L^2([-1, 0], \mathbb{R}) \times \mathbb{R}$  defined by

$$\begin{aligned}
\dot{x}(t) &= -x(t - \sigma_p(t)) & t \geq 0 \\
x_0 &= \varphi \\
x(0) &= \varphi_0.
\end{aligned} \tag{14}$$

Let, for every  $p \geq 1$ ,

$$\sigma_p(t) = \begin{cases} \sum_{k=1}^p a_k 1_{\mathcal{A}_k}(t) + \frac{1}{2} & \text{for } 0 \leq t < 1/2, \\ 0 & \text{for } t \geq 1/2, \end{cases}$$

with  $\mathcal{A}_k = [a_k, a_{k+1})$ ,  $a_k = \frac{k-1}{2p}$ ,  $k = 1, \dots, p+1$ , and let

$$\varphi(s) = \left|s + \frac{1}{2}\right|^\alpha, \quad -1 < s < 0,$$

where  $\alpha$  is a real number such that  $\alpha \in (-1/2, 0)$ . The existence and uniqueness of an absolutely continuous function, solution of (14) for a fixed  $p$ , is guaranteed by [1, Theorem 2.4.1]. Such solutions are not uniformly exponentially bounded with respect to the choice of  $p$ . In fact, one can easily verify that, for  $t \leq 1/2$ , the solution of system (14) with  $\varphi_0 = 0$  is given by

$$x(t) = - \int_0^t \varphi(s - \sigma_p(s)) ds.$$

Since

$$-\frac{1}{2} \leq s - \sigma_p(s) < \frac{1}{2p} - \frac{1}{2} \quad \forall s \in \mathcal{A}_k,$$

we obtain the inequality

$$|x(t)| = \int_0^t \varphi(s - \sigma_p(s)) ds > \int_0^t \left(\frac{1}{2p}\right)^\alpha ds = \frac{t}{(2p)^\alpha}.$$

By consequence, we have that  $\|x_t\|_{L^2} \geq \frac{t}{(2p)^\alpha}$ . Letting  $p$  tend to  $+\infty$ , one clearly cannot find two constants  $M \geq 1$  and  $w > 0$  such that inequality (8) holds for every  $0 < t \leq 1/2$ .

Since the solutions of system (14) are not uniformly exponentially bounded, they cannot be uniformly exponentially stable in  $L^2([-1, 0], \mathbb{R}) \times \mathbb{R}$ . Notice, however, that system (14) is asymptotically stable in  $L^2([-1, 0], \mathbb{R}) \times \mathbb{R}$ . Indeed, by considering as initial condition at time  $t_0 = 1$  the continuous function  $x_1$ , we may interpret equation (14) as a delayed system in  $C([-1, 0], \mathbb{R})$ . For this system we can apply [6, Theorem V.5.2], which ensures the uniform asymptotic stability in  $C([-1, 0], \mathbb{R})$ .

### 4.3 Equivalence between exponential stability in $C([-r, 0], \mathbb{R}^n)$ and in $H^1([-r, 0], \mathbb{R}^n)$

The equivalence between the exponential stability in  $C([-r, 0], \mathbb{R}^n)$  and in  $H^1([-r, 0], \mathbb{R}^n)$  of system (2) is given by the following proposition.

**Proposition 4** *Let  $Q \subset \mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$  be bounded. The following two statements are equivalent:*

- (i) *System (2) is uniformly exponentially stable in  $C([-r, 0], \mathbb{R}^n)$ .*
- (ii) *System (2) is uniformly exponentially stable in  $H^1([-r, 0], \mathbb{R}^n)$ .*

*Proof.* Firstly, remark that we have the following norm comparison

$$\|x_t\|_{H^1} \leq \gamma \|x_t\|_{C[-2r, 0]}, \quad t \geq r, \quad (15)$$

with  $\gamma = \sqrt{r(1+m^2)}$ , which follows from the two estimates

$$\|x_t\|_{L^2}^2 = \int_{-r}^0 |x_t(s)|^2 ds \leq r \|x_t\|_{C[-r, 0]}^2$$

and

$$\begin{aligned}\|\dot{x}_t\|_{L^2}^2 &= \int_{t-r}^t |L_{\sigma(s)}x_s|^2 ds \\ &\leq rm^2 \|x_t\|_{C[-2r,0]}^2.\end{aligned}$$

Now, assume that system (2) is uniformly exponentially stable in  $C([-r, 0], \mathbb{R}^n)$ . Then, there exist two constants  $\alpha \geq 1$  and  $\beta > 0$  such that, for every  $\varphi \in C([-r, 0], \mathbb{R}^n)$ , the solution  $x(\cdot)$  to (2) satisfies

$$\|x_t\|_{C[-r,0]} \leq \alpha e^{-\beta t} \|\varphi\|_{C[-r,0]}. \quad (16)$$

Then, we have that

$$\|x_{t-r}\|_{C[-r,0]} \leq \alpha e^{-\beta(t-r)} \|\varphi\|_{C[-r,0]},$$

which can be rewritten as

$$\|x_t\|_{C[-2r,-r]} \leq \alpha e^{\beta r} e^{-\beta t} \|\varphi\|_{C[-r,0]}. \quad (17)$$

Combining (16) and (17) gives that

$$\|x_t\|_{C[-2r,0]} \leq \alpha e^{\beta r} e^{-\beta t} \|\varphi\|_{C[-r,0]}. \quad (18)$$

Then, the uniform exponential stability in  $H^1([-r, 0], \mathbb{R}^n)$  derives from equations (15) and (18) together with (11).

Conversely, assume that system (2) is uniformly exponentially stable in  $H^1([-r, 0], \mathbb{R}^n)$ . Then, there exist two constants  $\alpha \geq 1$  and  $\beta > 0$  such that, for every  $\eta \in H^1([-r, 0], \mathbb{R}^n)$ , the solution  $x(\cdot)$  of (2) with initial condition  $x_{t_0} = \eta$  satisfies

$$\|x_t\|_{H^1} \leq \alpha e^{-\beta(t-t_0)} \|\eta\|_{H^1}, \quad t \geq t_0, \sigma(\cdot)\text{-uniformly}. \quad (19)$$

Let  $x(\cdot)$  be the unique solution of (2) with initial condition  $\varphi \in C([-r, 0], \mathbb{R}^n)$ . From (15), one has that  $x_r \in H^1([-r, 0], \mathbb{R}^n)$ . For  $t \geq r$  it follows from (19) that

$$\|x_t\|_{H^1} \leq \alpha e^{-\beta(t-r)} \|x_r\|_{H^1}, \quad (20)$$

which implies, together with (15), that

$$\|x_t\|_{H^1} \leq \alpha \gamma e^{-\beta(t-r)} \|x_r\|_{C[-2r,0]}. \quad (21)$$

Furthermore, from (9), we have that

$$\|x_r\|_{C[-2r,0]} \leq e^{mr} \|\varphi\|_{C[-r,0]}. \quad (22)$$

Combining (21) and (22) together with (11) gives the inequalities

$$\begin{aligned}\|x_t\|_{C[-r,0]} &\leq \lambda_r \|x_t\|_{H^1} \\ &\leq \lambda_r e^{mr} \alpha \gamma e^{-\beta(t-r)} \|\varphi\|_{C[-r,0]},\end{aligned}$$

which end the proof.  $\square$

## 5 Sufficient conditions for stability: weakly degenerate Lyapunov–Krasovskii functional

In this section we establish two equivalence results for the global uniform exponential stability of system (2) in  $H^1([-r, 0], \mathbb{R}^n)$  and  $C([-r, 0], \mathbb{R}^n)$ . The first equivalence is a straightforward application of [7, Theorem 3], which gives a necessary and sufficient condition for the global uniform exponential stability of switched systems of the type (7). The second one uses a Lyapunov–Krasovskii functional satisfying weakened conditions adapted to the RFDE setting.

Due to the  $\sigma(\cdot)$ -uniform exponential boundedness of the solutions of (2), in  $C([-r, 0], \mathbb{R}^n)$  and  $H^1([-r, 0], \mathbb{R}^n)$ , a direct application of [7, Theorem 3] gives a first necessary and sufficient condition for the global uniform exponential stability of (2). This is given by the following theorem.

**Theorem 5** *Let  $Q \subset \mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$  be bounded. System (2) is uniformly exponentially stable in  $X$ , if and only if there exists a function  $V : X \rightarrow [0, \infty)$  such that  $\sqrt{V}(\cdot)$  is a norm on  $X$ ,*

$$V(\varphi) \leq c\|\varphi\|_X^2,$$

for some constant  $c > 0$  and

$$\underline{D}_\sigma V(\varphi) \leq -\|\varphi\|_X^2, \quad \sigma \in \mathbb{S}, \varphi \in X.$$

*Proof.* The proof results directly from Lemma 3 and [7, Theorem 3].  $\square$

The crucial step for the second equivalence is given by the following lemma proved in [7], which extends a result obtained in [17] in the framework of strongly continuous semigroups to switched system of the form (7).

**Lemma 6** *Assume that*

(i) *there exist  $M \geq 1$  and  $w > 0$  such that*

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{wt}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly,}$$

(ii) *there exist  $c \geq 0$  and  $p \in [1, +\infty)$  such that*

$$\int_0^{+\infty} \|T_{\sigma(\cdot)}(t)x\|_X^p \leq c\|x\|_X^p, \quad \sigma(\cdot)\text{-uniformly,}$$

for every  $x \in X$ .

Then there exist  $K \geq 1$  and  $\mu > 0$  such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly.}$$

The second equivalence result, in the case  $X = H^1([-r, 0], \mathbb{R}^n)$ , is given by the following theorem.

**Theorem 7** Let  $Q \subset \mathcal{L}(C([-r, 0], \mathbb{R}^n), \mathbb{R}^n)$  be bounded. Then system (2) is uniformly exponentially stable in  $H^1([-r, 0], \mathbb{R}^n)$  if and only if there exists a continuous function  $V : H^1([-r, 0], \mathbb{R}^n) \rightarrow [0, +\infty)$  such that

$$V(\varphi) \leq c \|\varphi\|_{H^1}^2, \quad (23)$$

for some constant  $c > 0$  and

$$\underline{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \sigma \in \mathbb{S}, \varphi \in H^1([-r, 0], \mathbb{R}^n). \quad (24)$$

*Proof.* Suppose that there exist a continuous function  $V : H^1([-r, 0], \mathbb{R}^n) \rightarrow [0, +\infty)$  such that conditions (23)-(24) hold. From Lemma 3, we know that the solutions of system (2) are  $\sigma$ -uniformly exponentially bounded. Then, thanks to Lemma 6, it suffices to prove that there exists a constant  $c_0 > 0$  such that

$$\int_0^{+\infty} \|x_t\|_X^2 ds \leq c_0 \|\varphi\|_X^2, \varphi \in X, \sigma(\cdot)\text{-uniformly}, \quad (25)$$

to conclude the global exponential stability of the solution of (2). For all piecewise constant  $\sigma(\cdot)$  with values in  $\mathbb{S}$ ,  $\varphi \in H^1([-r, 0], \mathbb{R}^n)$  and  $t \geq 0$ , we have

$$V(x_t) - V(x_0) \leq - \int_0^t |x_s(0)|^2 ds, \quad (26)$$

as it follows from (24) and the properties of the Dini derivative (see, e.g., [5]). Using the fact that  $V$  is non-negatively valued, one deduces from (23) and (26) the following inequality

$$\int_0^{+\infty} |x_s(0)|^2 ds \leq c \|\varphi\|_{H^1}^2. \quad (27)$$

In the sequel, we deduce from (27) that equation (25) holds. Firstly, remark that

$$\begin{aligned} \int_0^t \|x_s\|_{L^2}^2 ds &= \int_0^t ds \int_{-r}^0 |x_s(\theta)|^2 d\theta \\ &= \int_0^t ds \int_{-r}^0 |x_{s+\theta}(0)|^2 d\theta \\ &= \int_{-r}^0 d\theta \int_0^t |x_{s+\theta}(0)|^2 ds \\ &\leq \int_{-r}^0 d\theta \int_{-r}^t |x_s(0)|^2 ds \\ &= r \int_{-r}^t |x_s(0)|^2 ds. \end{aligned} \quad (28)$$

Furthermore, we have

$$\begin{aligned}
\int_0^t \|\dot{x}_s\|_{L^2}^2 ds &\leq r \int_{-r}^t |\dot{x}_s(0)|^2 ds \\
&= r \int_{-r}^t |\dot{x}(s)|^2 ds \\
&= r \int_{-r}^0 |\dot{\varphi}(s)|^2 ds + r \int_0^t |\dot{x}(s)|^2 ds \\
&\leq r \|\varphi\|_{H^1}^2 + r \int_0^t |\dot{x}(s)|^2 ds. \tag{29}
\end{aligned}$$

Let  $s \geq 0$ . There exists  $\tau \in [-r, 0]$  such that  $\|x_s\|_C = |x(s + \tau)|$ . Let  $N \geq 1$  be a natural number. There exists  $1 \leq j(s) \leq N$  such that

$$0 \leq \tau + j(s) \frac{r}{N} \leq \frac{r}{N}.$$

We have that

$$\begin{aligned}
|x(s + \tau)| &\leq \left| x \left( s - j(s) \frac{r}{N} \right) \right| + \int_{s - j(s) \frac{r}{N}}^{s + \tau} |\dot{x}(\theta)| d\theta \\
&\leq \left| x \left( s - j(s) \frac{r}{N} \right) \right| + \sqrt{\frac{r}{N}} \|\dot{x}\|_{L^2[s - j(s) \frac{r}{N}, s + \tau]} \\
&\leq \left| x \left( s - j(s) \frac{r}{N} \right) \right| + \sqrt{\frac{r}{N}} \|\dot{x}_s\|_{L^2[-r, 0]} \\
&\leq \left| x \left( s - j(s) \frac{r}{N} \right) \right| + \sqrt{\frac{r}{N}} \|x_s\|_{H^1}.
\end{aligned}$$

Therefore, from equation (4), we have

$$\begin{aligned}
|\dot{x}(s)| &\leq m \left| x \left( s - j(s) \frac{r}{N} \right) \right| + m \sqrt{\frac{r}{N}} \|x_s\|_{H^1} \\
&\leq m \sum_{j=1}^N \left| x \left( s - j \frac{r}{N} \right) \right| + m \sqrt{\frac{r}{N}} \|x_s\|_{H^1},
\end{aligned}$$

from which we conclude that

$$|\dot{x}(s)|^2 \leq 2Nm^2 \sum_{j=1}^N \left| x \left( s - j \frac{r}{N} \right) \right|^2 + 2m^2 \frac{r}{N} \|x_s\|_{H^1}^2.$$

By integrating the previous equation over  $[0, t]$ , we obtain

$$\begin{aligned}
\int_0^t |\dot{x}(s)|^2 ds &\leq 2Nm^2 \sum_{j=1}^N \int_0^t |x(s - j\frac{r}{N})|^2 ds \\
&\quad + 2m^2 \frac{r}{N} \int_0^t \|x_s\|_{H^1}^2 ds \\
&\leq 2N^2m^2 \int_{-r}^t |x(s)|^2 ds \\
&\quad + 2m^2 \frac{r}{N} \int_0^t \|x_s\|_{H^1}^2 ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_0^t |\dot{x}(s)|^2 ds &\leq 2N^2m^2 \|\varphi\|_{H^1}^2 + 2N^2m^2 \int_0^t |x(s)|^2 ds \\
&\quad + 2m^2 \frac{r}{N} \int_0^t \|x_s\|_{H^1}^2 ds.
\end{aligned} \tag{30}$$

Equations (29) and (30) together imply that

$$\begin{aligned}
\int_0^t \|\dot{x}(s)\|_{L^2}^2 ds &\leq r(1 + 2N^2m^2) \|\varphi\|_{H^1}^2 \\
&\quad + 2rN^2m^2 \int_0^t |x(s)|^2 ds \\
&\quad + 2m^2 \frac{r^2}{N} \int_0^t \|x_s\|_{H^1}^2 ds.
\end{aligned} \tag{31}$$

Furthermore, equations (28) and (31) together yield

$$\begin{aligned}
\int_0^t \|x_s\|_{H^1}^2 ds &\leq 2r(1 + N^2m^2) \|\varphi\|_{H^1}^2 \\
&\quad + r(1 + 2N^2m^2) \int_0^t |x(s)|^2 ds \\
&\quad + 2m^2 \frac{r^2}{N} \int_0^t \|x_s\|_{H^1}^2 ds.
\end{aligned}$$

For sufficiently large  $N$ , the quantity  $1 - 2m^2 \frac{r^2}{N}$  is positive and we have

$$\int_0^t \|x_s\|_{H^1}^2 ds \leq c_1 \int_0^t |x(s)|^2 ds + c_2 \|\varphi\|_{H^1}^2, \tag{32}$$

with

$$c_1 = \frac{r(1 + 2N^2m^2)}{1 - 2m^2 \frac{r^2}{N}} \quad \text{and} \quad c_2 = \frac{2r(1 + N^2m^2)}{1 - 2m^2 \frac{r^2}{N}}.$$



Equations (27) and (32) together imply that

$$\int_0^t \|x_s\|_{H^1}^2 ds \leq c_0 \|\varphi\|_{H^1}^2,$$

with  $c_0 = cc_1 + c_2$ . When  $t$  tends to  $+\infty$  we deduce that

$$\int_0^{+\infty} \|x_t\|_{H^1}^2 ds \leq c_0 \|\varphi\|_{H^1}^2, \quad (33)$$

which concludes, thanks to Lemma 6, the proof of the global uniform exponential stability of system (2) in the case when  $X = H^1([-r, 0], \mathbb{R}^n)$ .

Conversely, suppose that system (2) is uniformly exponentially stable in  $H^1([-r, 0], \mathbb{R}^n)$ . Then, from Theorem 5, there exists a function  $V : H^1([-r, 0], \mathbb{R}^n) \rightarrow [0, \infty)$  such that  $\sqrt{V(\cdot)}$  is a norm on  $H^1([-r, 0], \mathbb{R}^n)$ ,

$$V(\varphi) \leq c \|\varphi\|_{H^1}^2$$

for some constant  $c > 0$  and

$$\underline{D}_\sigma V(\varphi) \leq -\|\varphi\|_{H^1}^2, \quad \sigma \in \mathbb{S}, \varphi \in H^1([-r, 0], \mathbb{R}^n). \quad (34)$$

Thanks to the Poincaré inequality (given by equation (11)) and the fact that  $|\varphi(0)| \leq \|\varphi\|_C$ , equation (34) yields

$$\underline{D}_\sigma V(\varphi) \leq -\lambda_r |\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in H^1([-r, 0], \mathbb{R}^n),$$

which ends the proof.  $\square$

The extension of this equivalence result to the case when  $X = C([-r, 0], \mathbb{R}^n)$  is given by the following corollary.

**Corollary 8** *System (2) is uniformly exponentially stable in  $C([-r, 0], \mathbb{R}^n)$ , if and only if there exists a continuous function  $V : C([-r, 0], \mathbb{R}^n) \rightarrow [0, +\infty)$  such that*

$$V(\varphi) \leq c \|\varphi\|_C^2, \quad (35)$$

for some constant  $c > 0$  and

$$\underline{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in C([-r, 0], \mathbb{R}^n). \quad (36)$$

*Proof.* Suppose that system (2) is uniformly exponentially stable in  $C([-r, 0], \mathbb{R}^n)$ .

Then, following the same reasoning as in Theorem 7, we know that there exists a function  $V : C([-r, 0], \mathbb{R}^n) \rightarrow [0, \infty)$  that satisfies conditions (35)-(36).

Conversely, suppose that conditions (35)-(36) are satisfied. Let  $\varphi \in C([-r, 0], \mathbb{R}^n)$  and let  $x(\cdot)$  be the unique solution of system (2) with initial condition  $x_0 = \varphi$ . From equation (15), we know that  $x_t \in H^1([-r, 0], \mathbb{R}^n)$ , for  $t \geq r$ . Furthermore, from equation (35), we have

$$V(x_t) \leq c \|x_t\|_C^2,$$

from which we deduce, using the Poincaré inequality, that

$$V(x_t) \leq c_1 \|x_t\|_{H^1}^2, \tag{37}$$

for some constant  $c_1 > 0$ . Equations (36) and (37) together with Theorem 7 imply that system (2) is uniformly exponentially stable in  $H^1([-r, 0], \mathbb{R}^n)$ . Finally, we deduce, from Proposition 4, that system (2) is uniformly exponentially stable in  $C([-r, 0], \mathbb{R}^n)$ , concluding the proof.  $\square$

### 5.1 Proof of Theorem 2

Theorem 2 is a direct consequence of our previous results together with [7, Theorem 6]. In fact, from Proposition 4 we obtain the equivalence (i)  $\Leftrightarrow$  (ii). Theorem 7 and Corollary 8 give the equivalence (ii)  $\Leftrightarrow$  (vi) and (i)  $\Leftrightarrow$  (v), respectively. Finally, with [7, Theorem 6], we have the equivalences (i)  $\Leftrightarrow$  (iii) and (ii)  $\Leftrightarrow$  (iv). The regularity of  $V(\cdot)$  in (iv) actually follows from [7, Corollary 9].  $\square$

## 6 Discussion

We compare here the results obtained in the previous section with the Lyapunov–Krasovskii theorems given in [6] and [10]. We start by recalling a Lyapunov–Krasovskii theorem given in [6, Theorem V.2.1] for general retarded functional differential equation of the form

$$\dot{x} = f(t, x_t).$$

**Theorem 9** *Suppose that  $f : \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$  is a continuous function. Suppose that for any bounded set  $B$ ,  $f$  maps  $\mathbb{R} \times B$  into a bounded set of  $\mathbb{R}^n$ , and  $u, v, w : [0, +\infty) \rightarrow [0, +\infty)$  are continuous nondecreasing functions,  $u(s)$  and  $v(s)$  are positive for  $s > 0$ , and  $u(0) = v(0) = 0$ . If there exists a continuous function  $V : \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} u(|\varphi(0)|) &\leq V(t, \varphi) \leq v(\|\varphi\|_C) \\ \overline{D}V(t, \varphi) &\leq -w(|\varphi(0)|) \end{aligned}$$

*then the solution  $x = 0$  of equation (9) is uniformly stable. If  $u(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ , the solutions of equation (9) are uniformly bounded. If  $w(s) > 0$  for  $s > 0$ , then the solution  $x = 0$  is uniformly asymptotically stable.*

This theorem is not given for switched systems and the uniformity property mentioned is with respect to the initial condition. In [14] the authors show that the existence of a Lyapunov–Krasovskii functional is a necessary and sufficient condition for the uniform global asymptotic stability and the global exponential stability of autonomous systems described by neutral functional differential equations in Hale’s form.

The proof of Theorem 9 can be straightforwardly modified to consider system (2), and this is reformulated by the following theorem.

**Theorem 10** Consider system (2). If there exists a continuous function  $V : C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$  such that

$$\underline{c}|\varphi(0)|^2 \leq V(\varphi) \leq \bar{c}\|\varphi\|_C^2 \quad (38)$$

for constants  $\underline{c}, \bar{c} > 0$  and

$$\overline{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in C([-r, 0], \mathbb{R}^n) \quad (39)$$

then the solution to (2) is uniformly exponentially stable in  $C([-r, 0], \mathbb{R}^n)$ .

Thanks to our previous results, we can give a converse version for Theorem 10. More precisely, Theorem 2 implies that if system (2) is uniformly exponentially stable in  $C([-r, 0], \mathbb{R}^n)$ , then there exists a continuous function  $V : C([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}$  such that equations (38)-(39) hold.

Let us conclude this section by commenting on [10, Theorem 2.10]. When restricted to linear systems of the form (2), such result establishes the equivalence between the uniform exponential stability in  $C([-r, 0], \mathbb{R}^n)$  and the existence of a Lyapunov–Krasovskii functional either as in the statement (iii) of Theorem 2 or with a lower bound as the one in (38). In the latter case, however, the Lyapunov–Krasovskii functional is defined on a space of the type  $C([-r - \tau, 0], \mathbb{R}^n)$ , for some  $\tau \geq 0$ , on which system (2) is lifted.

## 7 Conclusion

In this work we give a collection of converse Lyapunov–Krasovskii theorems for uncertain retarded functional differential equations. These results, which are summarized by Theorem 2, are essentially given by Theorems 5 and 7. They are obtained using the switched system transformation approach. By Theorem 5, we show that the existence of a squared norm  $V(\cdot)$  on  $C([-r, 0], \mathbb{R}^n)$ , with suitable commensurability conditions, is a necessary and sufficient condition for the uniform exponential stability of system (2). Theorem 7 represents our main result, where the assumption that  $V(\cdot)$  is a squared norm on  $C([-r, 0], \mathbb{R}^n)$  is dropped. One of the novelties of our results is that these functionals may not have a strictly positive norm-dependent lower bound, in contrast with what is known in the literature.

Concerning the differences between conditions (iii) (respectively (iv)) and (v) (respectively (vi)) appearing in the statement of Theorem 2, we already noticed that a Lyapunov function  $V(\cdot)$  satisfying condition (v) (respectively (vi)) does not necessarily satisfy the stronger condition (iii) (respectively (iv)). Hence, condition (v) (respectively (vi)) is better suited for proving the global uniform exponential stability of a linear uncertain retarded differential equations, while condition (iii) (respectively (iv)) provides more information on a linear uncertain retarded differential equations that is known to be globally uniformly exponentially stable, by tightening the properties satisfied by  $V(\cdot)$ .

Finally, a comparison of our results with the Lyapunov–Krasovskii theorems given in [6] and [10] is discussed in Section 6.

## A Appendix: extension to the measurable case

Let  $Q$  be a bounded subset of the Banach space  $\mathcal{L}(C([-r, 0], \mathbb{R}^d), \mathbb{R}^d)$ , and fix  $m > 0$  such that

$$|L\varphi| \leq m\|\varphi\|_C \quad \forall \varphi \in C([-r, 0], \mathbb{R}^d). \quad (40)$$

Let  $T > 0$  and  $\Gamma : [0, T] \rightarrow Q$  be such that  $\Gamma(\cdot)\varphi$  is measurable for any  $\varphi \in C([-r, 0], \mathbb{R}^d)$ . We associate with  $\Gamma$  the linear retarded functional differential equation

$$\begin{aligned} \dot{x}(t) &= \Gamma(t)x_t, \\ x_0 &= \varphi. \end{aligned} \quad (41)$$

For every  $\varphi \in C([-r, 0], \mathbb{R}^d)$ , we know that under assumption (40), system (41) has a unique solution on  $[-r, +\infty)$  (Lemma 1).

For  $n \geq 1$ , consider the function  $\Gamma(\cdot)x_{j\frac{T}{n}} \in L^\infty(I_j, \mathbb{R}^d)$  with  $I_j = [(j-1)\frac{T}{n}, j\frac{T}{n}]$ ,  $1 \leq j \leq n$ . Remark that  $\frac{n}{T} \int_{I_j} \Gamma(s)x_{j\frac{T}{n}} ds \in \text{conv}(E)$ , where

$$E = \{Lx_{j\frac{T}{n}} : L \in Q\} \subset \mathbb{R}^d.$$

Carathéodory's theorem implies that  $\frac{n}{T} \int_{I_j} \Gamma(s)x_{j\frac{T}{n}} ds$  is a convex combination of  $d+1$  points of  $E$ , i.e.

$$\frac{n}{T} \int_{I_j} \Gamma(s)x_{j\frac{T}{n}} ds = \sum_{i=1}^{d+1} \alpha_{ij} L_{ij} x_{j\frac{T}{n}},$$

where  $L_{ij} \in Q$ ,  $\alpha_{ij}$  is nonnegative and  $\sum_{i=1}^{d+1} \alpha_{ij} = 1$ . We use the coefficients of this convex combination to define the following function

$$\Gamma_n(t) = L_{ij} \quad t \in I_{ij},$$

where the intervals  $I_{ij}$  are such that  $\bigcup_j I_{ij} = I_j$  and  $|I_{ij}| = \alpha_{ij} \frac{T}{n}$ . We clearly

have that  $\Gamma_n(\cdot)$  belongs to the set of piecewise constant functions with values in  $Q$ , which we denote by  $PC([0, T], Q)$ .

Associate with  $\Gamma_n$  the linear retarded functional differential equation

$$\begin{aligned} \dot{x}_n(t) &= \Gamma_n(t)x_{n,t}, \\ x_{n,0} &= x_0, \end{aligned} \quad (42)$$

where  $x_{n,t} : [-r, 0] \rightarrow \mathbb{R}^n$  denotes the state defined by  $x_{n,t}(\theta) = x_n(t + \theta)$ ,  $-r \leq \theta \leq 0$ . We have the following lemma.

**Lemma 11** *Suppose that assumption (40) holds. Take  $T > 0$  and let  $x(\cdot)$  and  $x_n(\cdot)$  be the unique solutions of systems (41) and (42), respectively. Then*

$$\lim_{n \rightarrow +\infty} \|x_{n,T} - x_T\|_C = 0. \quad (43)$$

*Proof.* Let  $y_n(\cdot) = x_n(\cdot) - x(\cdot)$  and  $y_{n,t} = x_{n,t} - x_t$ . By integrating systems (41) and (42) we have that, for every  $\theta \in [-r, 0]$  such that  $T + \theta > 0$ ,

$$y_n(T + \theta) = \int_0^{T+\theta} (\Gamma_n(s)x_{n,s} - \Gamma(s)x_s) ds.$$

Adding and subtracting  $\int_0^{T+\theta} \Gamma_n(s)x_s ds$  to the previous equation, and using equation (40), we obtain

$$|y_n(T + \theta)| \leq m \int_0^{T+\theta} \|y_{n,s}\|_C ds + |f_n(T + \theta)|,$$

with  $f_n(t) = \int_0^t (\Gamma_n(s) - \Gamma(s)) x_s ds$ . This leads to

$$\|y_{n,T}\|_C \leq m \int_0^T \|y_{n,s}\|_C ds + \|f_n\|_{C[0,T]},$$

from which we obtain, thanks to Gronwall's Lemma, that

$$\|y_{n,T}\|_C \leq e^{mT} \|f_n\|_{C[0,T]}. \quad (44)$$

We are going to show that  $f_n$  converges uniformly to zero on the interval  $[0, T]$ . Observe that

$$\begin{aligned} f_n(t) &= \sum_{j=1}^n \int_{I_j \cap [0,t]} (\Gamma_n(s) - \Gamma(s)) (x_s - x_{j\frac{T}{n}}) ds \\ &\quad - \underbrace{\sum_{j=1}^n \int_{I_j \cap [0,t]} (\Gamma_n(s) - \Gamma(s)) x_{j\frac{T}{n}} ds}_{K_j}. \end{aligned}$$

We distinguish three different cases:

- if  $j < \lceil \frac{nt}{T} \rceil$ , then by definition of  $\Gamma_n(\cdot)$ , we have

$$\begin{aligned} K_j &= \int_{I_j} \Gamma_n(s) x_{j\frac{T}{n}} ds - \int_{I_j} \Gamma(s) x_{j\frac{T}{n}} ds \\ &= \sum_{i=1}^{d+1} \int_{I_{ij}} L_{ij} x_{j\frac{T}{n}} ds - \frac{T}{n} \sum_{i=1}^{d+1} \alpha_{ij} L_{ij} x_{j\frac{T}{n}} \\ &= \sum_{i=1}^{d+1} \alpha_{ij} \frac{T}{n} L_{ij} x_{j\frac{T}{n}} - \frac{T}{n} \sum_{i=1}^{d+1} \alpha_{ij} L_{ij} x_{j\frac{T}{n}} \\ &= 0, \end{aligned}$$

- if  $j = \lfloor \frac{nt}{T} \rfloor$ , then

$$\begin{aligned}
K_j &= \int_{[(j-1)\frac{T}{n}, t]} (\Gamma_n(s) - \Gamma(s)) x_{j\frac{T}{n}} ds \\
&\leq \int_{[(j-1)\frac{T}{n}, t]} |(\Gamma_n(s) - \Gamma(s)) x_{j\frac{T}{n}}| ds \\
&\leq 2m \int_{I_j} \|x_{j\frac{T}{n}}\|_C ds \\
&= 2m \frac{T}{n} \|x_{j\frac{T}{n}}\|_C,
\end{aligned}$$

- if  $j > \lfloor \frac{nt}{T} \rfloor$ , then  $I_j \cap [0, t] = \emptyset$ , and by consequence  $K_j = 0$ .

Since  $x(\cdot)$  is uniformly continuous on  $[0, T]$ , we deduce that for every  $\epsilon > 0$ , there exists  $n_0$  such that if  $n \geq n_0$  then  $\sum_{j=1}^n |K_j| < \epsilon$ . Moreover, there exists  $n_1$  such that if  $n \geq n_1$  then  $\|x_s - x_{j\frac{T}{n}}\|_C < \epsilon$  for every  $s \in I_j$  and every  $j = 1, \dots, n$ . As a consequence, if  $n \geq \max\{n_0, n_1\}$  we have

$$|f_n(t)| \leq (2mT + 1) \epsilon.$$

Note that this is true for every  $t \in [0, T]$ . We conclude that

$$\|f_n\|_{C[0, T]} \leq (2mT + 1) \epsilon. \quad (45)$$

Equation (45), together with (44), concludes the proof of (43).  $\square$

**Theorem 12** *System (41) is uniformly exponentially stable for  $\Gamma : [0, +\infty) \rightarrow Q$  such that  $\Gamma(\cdot)\varphi$  is measurable for any  $\varphi \in C([-r, 0], \mathbb{R}^d)$  if and only if it is uniformly exponentially stable for  $\Gamma \in PC([0, +\infty), Q)$ .*

*Proof.* Let  $T > 0$  and  $\Gamma \in L^\infty([0, T], Q)$ . Thanks to Lemma 11, we know that

$$\|x_T\|_C = \lim_{n \rightarrow +\infty} \|x_{n, T}\|_C, \quad (46)$$

where  $x(\cdot)$  and  $x_n(\cdot)$  are the solutions of systems (41) and (42), respectively. Suppose that system (41) is uniformly exponentially stable for  $\Gamma \in PC([0, \infty], Q)$ . Then, there exist constants  $a \geq 1$  and  $b > 0$  such that for every initial condition  $\varphi \in C([-r, 0], \mathbb{R}^n)$  we have

$$\|x_{n, T}\|_C \leq a e^{-bT} \|\varphi\|_C. \quad (47)$$

Combining equations (46) and (47) gives

$$\|x_T\|_C \leq a e^{-bT} \|\varphi\|_C.$$

Remark that the property given in the previous equation is true for all  $T > 0$ , concluding the proof.  $\square$

**Remark 4** *Theorem 12 is still true when the phase space  $C([-r, 0], \mathbb{R}^d)$  is replaced by  $H^1([-r, 0], \mathbb{R}^d)$ . This follows from the same arguments as in Lemma 1 and Proposition 4.*

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