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Impulsive observer-based control for linear systems using irregularly sampled measurements

Y. Khaled[1,2], J-P. Barbot[1,3], K. Busawon[4] and D. Benmerzouk[2]
[1]ECS-Lab ENSEA, 6 Avenue du Ponceau, 95014 Cergy-Pontoise, France
[2]Department of Mathematics, Tlemcen University, Algeria
[3]EPI Non-A, INRIA Lille, Nord-Europe
[4]Northumbria University, Faculty of Engineering and Environment, Newcastle upon Tyne, UK
Email: krishna.busawon@unn.ac.uk

***Abstract*—In this paper, we consider the issue of stabilizing a class of linear systems using irregular sampled output measurements. For this purpose, we design a standard linear state feedback controller and an impulsive observer to provide an estimate the non-measured states, which are subsequently fed back in the control algorithm. We consider linear systems that can be decomposed, via a change of coordinates, into their respective measured and unmeasured dynamics. We consider the two cases whereby the unmeasured subspace is stable and unstable respectively. In the case where the unmeasured subspace is stable, we employ a standard impulsive observer coupled with a continuous linear feedback control. Next, when the unmeasured subspace is unstable, we employ two cascaded observers - an impulsive and a Luenberger observer - in conjunction with a linear feedback control. In order to prove the stability of the overall closed-loop system we proposed a practical stability result for a class of linear impulsive systems. Some simulation results are presented to show the performance of the observer-based control. Finally, some conclusions are drawn.**

***Index Terms*—observer-based control, discrete measurement, impulsive systems**

I. Introduction

In many practical control applications, we are faced with the issue of controlling a continuous-time system with measurements available at discrete instants of time. When faced with such a situation, one can attempt to discretise the system if the output is sampled fast enough; so that the discretised differential equation describing the system gives a good representation of the original continuous system. One can then employ a discrete output feedback controller or a discrete observer-based controller to control the system. However, in the case where the measurements are irregularly spaced such discretisation is not possible. Such is the case for chemical processes in general, whereby the output measurements are available in discrete time and the time intervals between the samples are not necessarily regularly spaced or constant. In such a case, one has to find alternative means to control these systems.

In this paper, we propose an observer-based control design methodology for a class of linear systems with irregularly sampled output measurements. More precisely, we consider the following class of systems:

$$\begin{cases} \dot{x}(t) = Fx(t) + Gu(t) \\ y(t_k) = Hx(t_k) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input and $y(t_k) \in \mathbb{R}^p$ is the output. The matrices F, G and H are constant and of appropriate dimension. Moreover, the system is assumed to be detectable and stabilizable. We assume that there exists a linear transformation $z = Tx$ that permits to transform system (1), into the following system:

$$\begin{cases} \dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t) + B_1u(t) \\ \dot{z}_2(t) = A_{22}z_2(t) + B_2u(t) \\ y(t_k) = z_1(t_k) \end{cases} \tag{2}$$

where $z(t) = (z_1^T, z_2^T)^T$ with $A_{11} \in \mathbb{R}^{p\times p}$, $A_{12} \in \mathbb{R}^{p\times(n-p)}$, $A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$, $B_1 \in \mathbb{R}^{p\times m}$, $B_2 \in \mathbb{R}^{(n-p)\times m}$ and A_{22} Hurwitz continuous.

We consider the two cases where the matrix A_{22} is stable and unstable respectively. In the case where A_{22} is stable, we design an impulsive observer, which is basically a copy of the system with an appropriate reset function. Additionally, we employ a continuous linear feedback in order to design an observer-based control for the system. This effectively shows that a separation principle is possible in that case. In the case where A_{22} is not stable the observer design is not trivial. For this, we employ two cascaded observers composed of a Luenberger and an impulsive observer. The reason for including a Luenberger observer is to able to modify the eigenvalues of A_{22} via its output correction term.

The result obtained is also valid in the case where output sampling frequency is smaller than the Nyquist-Shannon sampling frequency; i.e. in the case of sparse measurements (see eg. [1] [2]). Effectively, this work shows that it is possible to guaranty closed loop stability under sparse measurement (see e.g. [3]).

The paper is organized as follows: In the next section a stability result with respect to impulsive systems is presented.This stability result is then used in the subsequent sections to prove the stability of the overall closed-loop system. Additional results on different types of stability of impulsive systems can

be found in [4], [5], [6], [7] and [2]. In Section 3, an observer-based control is proposed for the case where the unmeasured subspace is stable; i.e. A_{22} is stable. The case where the unmeasured subspace is unstable is considered in Section 4. In Section 5, an example is given in order to highlight the efficiency of the proposed observer based control scheme. Finally, the paper ends with some conclusions and perspectives for future research work.

II. Practical stability for a class of impulsive system

In this section, we are going to give some conditions under which a class of linear impulsive systems is practically stable. More precisely, we consider the following class of linear impulsive systems:

$$\begin{cases} \dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t); \quad t \neq t_k \\ \dot{x}_2(t) = A_{22}x_2(t) \\ x_1(t_k^+) = Rx_1(t_k) \\ x_2(t_k^+) = x_2(t_k) \end{cases} \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $A_{11} \in \mathbb{R}^{p\times p}$, $A_{12} \in \mathbb{R}^{p\times(n-p)}$, $A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $R \in \mathbb{R}^{p\times p}$. The sampling sequence $t_k \in T = \{t_i : i \in \mathbb{N}\} \subset \mathbb{R}$. We assume that there exists τ_{max} and τ_{min} with $\tau_{max} > \tau_{min} > 0$ such that $\forall i > 0$

$$t_i + \tau_{min} < t_{i+1} < t_i + \tau_{max}. \tag{4}$$

The above strict inequality permits to exclude the Zeno phenomena since system (3) is essentially a particular class of hybrid systems [8].

Additionally, we define:

$$\begin{aligned} x(t_k^+) &\triangleq \lim_{h\to 0} x(t_k + h) \\ x(t_k^-) &\triangleq \lim_{h\to 0} x(t_k - h) = x(t_k) \end{aligned}$$

Throughout the paper, we assume that the system (3) verifies the sufficient conditions for the existence of solution [7].

Before stating the main stability result of this section, we recall some important results and definitions. First, recall that if $A \in \mathbb{R}^{n\times n}$ is a discrete Hurwitz matrix (that is, all eigenvalues are strictly inside the unit circle of the complex plane) then $\lim_{k\to\infty} A^k = 0$. Additionally, there exist at least a matrix norm $\| \bullet \|$, such that $\|A\| < 1$.

Next, we recall that:

Definition 1. [4] *The system (3) is said to be:*

1) *Practically stable with respect to (δ, ϵ) if given (δ, ϵ) with $0 < \delta < \epsilon$, we have that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \epsilon$, $t > t_0$ for some $t_0 \in R$.*
2) *Uniformly practically stable with respect to (δ, ϵ), if 1) holds for every $t_0 \in R$.*
3) *Practically asymptotically stable with respect to (δ, ϵ), if 1) holds and $\lim_{t\to\infty} \|x(t)\| = 0$.*

We can now state the following:

Theorem 1. *Assume that system (3) verifies the following conditions:*

1) *A_{22} is a continuous Hurwitz matrix; that is its eigenvalues have strictly negative real part.*
2) *$\|Re^{A_{11}t}\|^2 < 1$, for all $t \in [0, \tau_{max}[> 0$.*

Then, $\forall \epsilon > 0$, the state of system (3) converges to a ball of radius ϵ (practical stability).

Proof: First, the solution of (3) for $t \in [t_k^+, t_{k+1}^-]$ with the initial time $t_0 = t_k^+$ is given by

$$x_1(t) = e^{A_{11}(t-t_k)}x_1(t_k^+) + \int_{t_k}^{t} e^{A_{11}(t-\tau)}A_{12}x_2(\tau)d\tau \tag{5}$$

and

$$x_2(t) = e^{A_{22}(t-t_k)}x_2(t_k^+) \tag{6}$$

Replacing (6) in (5), we obtain, at $t = t_{k+1}$:

$$\begin{aligned} x_1(t_{k+1}) &= e^{A_{11}\theta_k}x_1(t_k^+) \\ &\quad + \int_{t_k}^{t_{k+1}} e^{A_{11}(t_{k+1}-\tau)}A_{12}e^{A_{22}(\tau-t_k)}x_2(t_k^+)d\tau \end{aligned}$$

where $\theta_k = t_{k+1} - t_k$.

Consider the following Lyapunov function:

$$V(x) = V_1(x_1) + V_2(x_2)$$

with

$$\begin{aligned} V_1(x_1) &= x_1^T x_1 = \|x_1\|^2 \\ V_2(x_2) &= x_2^T x_2 = \|x_2\|^2 . \end{aligned}$$

Starting with $V_2(x_2)$, we get

$$\begin{aligned} \triangle V_2(x_2(t_k)) &= V_2(x_2(t_{k+1}^+)) - V_2\left(x_2(t_k^+)\right) \\ &= V_2(x_2(t_{k+1})) - V_2\left(x_2(t_k^+)\right) \\ &= \|e^{A_{22}\theta_k}x_2(t_k^+)\|^2 - \|x_2(t_k^+)\|^2 \\ &\leq \left(\|e^{A_{22}\theta_k}\|^2 - 1\right) V(x_2(t_k^+)) \end{aligned}$$

Since the matrix A_{22} is Hurwitz continuous, this implies that $e^{A_{22}\theta_k}$ is a discrete Hurwitz matrix and therefore

$$\|e^{A_{22}\theta_k}\|^2 < 1$$

Consequently,

$$\triangle V_2(x_2(t_k)) < 0 \tag{7}$$

This inequality ensures the uniform asymptotic convergence of x_2 to zero (uniformly with respect to x_1).

Therefore, $\forall \delta > 0$ there exists h such that:

$$\forall k > h \ , \quad x_2(t_k) < \delta. \tag{8}$$

Now, for the Lyapunov function V_1, we have

$$\begin{aligned} \triangle V_1(x_1(t_k)) &= V_1(x(t_{k+1}^+)) - V_1(x(t_k^+)) \\ &= V_1(Rx_1(t_{k+1})) - V_1(x_1(t_k^+)) \end{aligned}$$

By replacing the above expression of $x_1(t_{k+1})$, we get:

$$\begin{aligned} \triangle V_1 &\leq \|Re^{A_{11}\theta_k}x_1\|^2 - \|x_1\|^2 \\ &\quad + \|R\textstyle\int_{t_k}^{t_{k+1}} e^{A_{11}(t_{k+1}-\tau)}A_{12}e^{A_{22}(\tau-t_k)}d\tau x_2\|^2 \\ &\leq (\|Re^{A_{11}\theta_k}\|^2 - 1)V_1(x_1) \\ &\quad + \|A_{12}\|^2\|x_2\|^2 \textstyle\int_{t_k}^{t_{k+1}} \|Re^{A_{11}(t_{k+1}-\tau)}\|^2\|e^{A_{22}(\tau-t_k)}\|^2 d\tau \end{aligned}$$

Since A_{22} is Huzwitz continuous and that from the second condition $\|Re^{A_{11}(t_{k+1}-\tau)}\|^2 < 1$ for all $\tau \in [t_k, t_{k+1}[$, we have:

$$\begin{aligned}\triangle V_1 &\leq (\|Re^{A_{11}\theta_k}\|^2 - 1)V_1(x_1) + \|A_{12}\|^2\|x_2\|^2 \int_{t_k}^{t_{k+1}} d\tau \\ &\leq (\|Re^{A_{11}\theta_k}\|^2 - 1)V_1(x_1) + \theta_k\|A_{12}\|^2\|x_2\|^2\end{aligned}$$

So, from (8) we have
$\forall\delta > 0, \exists h > 0, \forall k > h : \ \|x_2(t_k)\| < \delta$.
Now, choose $\epsilon > 0$ and define $V_{1,\min} = \min\limits_{\|x_1\|=\epsilon} V_1(x_1)$. Then, for all $\|x_1(k)\| > \epsilon$ and $\theta_k < \dfrac{(\|Re^{A_{11}\theta_k}\|^2 - 1)V_{1\min}}{\|A_{12}\|\delta}$, we have:

$$\triangle V_1(x_1(t_k)) < 0$$

Owing to the fact that $\delta > 0$ can be arbitrarily chosen, the only constraint on θ_k is $\|Re^{A_{11}\theta_k}\|^2 < 1$. Consequently (7) and the previous inequality imply that $\exists h$ such that $\forall k > h$ we have $\triangle V(x_1(t_k), x_2(t_k)) < 0$ if $\|x_1\| > \epsilon$.

This completes the proof of Theorem 1.

III. Case 1: stable unmeasured subspace

Consider system (2) and assume that the matrix A_{22} is Hurwitz continuous.

In a compact form system (2) is written as:

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) \\ y(t_k) = Cz(t_k) \end{cases} \tag{9}$$

where $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ and $C = \begin{pmatrix} I_p & 0 \end{pmatrix}$ with I_p being the p-dimensional identity matrix.

Remark 1: *Note that the above assumption that there exists a linear transformation $z = Tx$ that permits to transform system (1), into (2) is not too restrictive, because if $\ker\{H\}$ is included on the stable subspace of F, it is always possible to select a part of the output $y(t_k)$ as the considered output, in order to obtain the system (2) under the transformation $z = Tx$.*

We design an observer for system (2) as follows:

$$\begin{cases} \dot{\hat{z}}_1(t) = A_{11}\hat{z}_1(t) + A_{12}\hat{z}_2(t) + B_1u(t) \\ \dot{\hat{z}}_2(t) = A_{22}\hat{z}_2(t) + B_2u(t) \\ \hat{z}_1(t_k^+) = R\hat{z}_1(t_k) + (I_p - R)z_1(t_k) \\ \hat{y}(t_k) = \hat{z}_1(t_k) \end{cases} \tag{10}$$

where $R \in \mathbb{R}^{m\times n}$ is the reset matrix, such that
$R = diag\{r_1, .., r_p\}$, with $r_i \in]-1, 1[$, for $i = 1, .., p$.

As a result, a linear observer based feedback is given by:

$$u(t) = -K\hat{z} \triangleq (-K_1, -K_2)\begin{pmatrix} \hat{z}_1(t) \\ \hat{z}_2(t) \end{pmatrix} \tag{11}$$

where $K \in \mathbb{R}^{m\times n}$ is chosen such that $(A - BK)$ is stable.

Theorem 2. *Assume that there exists τ_{max} and τ_{min} with $\tau_{max} > \tau_{min} > 0$ such that $\forall i > 0$, $t_i + \tau_{min} < t_{i+1} < t_i + \tau_{max}$. Then, the closed-loop system:*

$$\dot{z}(t) = Az(t) - BK\hat{z}(t) = (A - BK)\,z(t) + BKe(t) \tag{12}$$

is practically asymptotically stable.

Proof: By setting $e_1 = z_1 - \hat{z}_1$, $e_2 = z_2 - \hat{z}_2$, we obtain:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

On the other hand, by expanding the closed-loop system *(12)* we obtain

$$\dot{z} = (A - BK)\,z + BKe$$

where

$$\begin{aligned} A_{pp} &= (A - BK) = \begin{pmatrix} A_{11} - B_1K_1 & A_{12} - B_1K_2 \\ -B_2K_1 & A_{22} - B_2K_2 \end{pmatrix} \\ B_{pp} &= BK = \begin{pmatrix} B_2K_1 & B_1K_2 \\ B_2K_1 & B_2K_2 \end{pmatrix} \end{aligned}$$

The reset function becomes:

$$e_1(t_k^+) = Re_1(t_k)$$

The overall state dynamics and observation error dynamics can be written as

$$\begin{pmatrix} \dot{e} \\ \dot{z} \end{pmatrix} = \left(\begin{array}{c|c} A & 0 \\ \hline B_{pp} & A_{pp} \end{array}\right)\begin{pmatrix} e \\ z \end{pmatrix}$$

with the following reset function:

$$\begin{bmatrix} e_1(t_k^+) \\ e_2(t_k^+) \\ z_1(t_k^+) \\ z_2(t_k^+) \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & I_{2n-p} \end{bmatrix}\begin{bmatrix} e_1(t_k) \\ e_2(t_k) \\ z_1(t_k) \\ z_2(t_k) \end{bmatrix}$$

The observation error dynamics has exactly the same form as system (3) and is independent of z_i, so e is practically asymptotically stable. Now, again from the theorem 1 for all $\epsilon > 0$ $\exists h > 0$ such that $\|e_i(k)\| < \epsilon$. Now setting $V_3(z) \triangleq z^Tz = \|z\|^2$ and proceeding in the same way as for the proof of Theorem 1 , we obtain:

$$\begin{aligned}\triangle V_3(z(t_k)) &= V_3\Big(e^{A_{pp}\theta_k}z(t_k) \\ &\quad + \int_{t_k}^{t_{k+1}} e^{A_{pp}(t_{k+1}-\tau)}B_{pp}e(\tau)d\tau\Big) \\ &\quad - V_3(x_1(t_k))\end{aligned}$$

As $k > h : \|e\|^2 < \epsilon$, we have:

$$\begin{aligned}\triangle V_3 &\leq \|e^{A_{pp}\theta_k}z\|^2 \\ &\quad + \|\int_{t_k}^{t_{k+1}} e^{A_{pp}(t_{k+1}-\tau)}B_{pp}d\tau\epsilon\|^2 - \|z\|^2 \\ &\leq (\|Re^{A_{pp}\theta_k}\|^2 - 1)V_3(x_1) \\ &\quad + \int_{t_k}^{t_{k+1}} \|Re^{A_{pp}(t_{k+1}-\tau)}B_{pp}e^{A_{22}(\tau-t_k)}\|^2 d\tau\|x_2\|^2 \\ &\leq (\|e^{A_{pp}\theta_k}\|^2 - 1)V_3(z(k)) \\ &\quad + \|B_{pp}\|^2\epsilon\int_{t_k}^{t_{k+1}} \|e^{A_{pp}(t_{k+1}-\tau)}\|^2 d\tau\end{aligned}$$

This ensures the practical stability and completes the proof of Theorem 2.

IV. Case 2: unstable unmeasured subspace

Consider again system (2) and assume that the matrix A_{22} is unstable.

Remark 2. *As in the previous section, that there exists a linear transformation $z = Tx$ that permits to transform system (1), into (2) is not too restrictive in this case either. This is because it is always possible to consider only one part of the output y as a considered output, in order to obtain the system (2) under the transformation $z = Tx$. However, in some cases, this output restriction can destroy the system detectability and then the basic assumption for considering only detectable and stabilizable system is violated.*

The observer-based control for system (2) is quite different to the one proposed in the previous section as far as the observer part is concerned. In fact, we employ two cascaded observers - an impulsive and a Luenberger observer. More precisely,

$$\begin{cases} \dot{\hat{z}}_1(t) = A_{11}\hat{z}_1(t) + A_{12}\hat{z}_2(t) + B_1u(t) \\ \dot{\hat{z}}_2(t) = A_{22}\hat{z}_2(t) + M(\xi_2(t) - \hat{z}_2(t)) + B_2u(t) \\ \dot{\xi}_1(t) = A_{11}\xi_1(t) + A_{12}\xi_2(t) + L_1(\hat{z}_1(t) - \xi_1(t)) + B_1u(t) \\ \dot{\xi}_2(t) = A_{22}\xi_2(t) + L_2(\hat{z}_1(t) - \xi_1(t)) + B_2u(t) \\ \hat{z}_1(t_k^+) = \bar{R}\hat{z}_1(t_k) + (I_d - \bar{R})z_1(t_k) \end{cases} \tag{13}$$

where $K \in \mathbb{R}^{m\times n}$, with $\bar{R} = diag\{\bar{r}_1, .., \bar{r}_{n_1}\}$ and $-1 < \bar{r}_i < 1$, for $i = 1, \ldots, n_1$ and M, L_1 and L_2 are matrices of appropriate dimensions.

As for the controller, we employ a linear control as before:

$$u(t) = -K\hat{z} \triangleq (K_1, K_2)\begin{pmatrix} \hat{z}_1(t) \\ \hat{z}_2(t) \end{pmatrix} \tag{14}$$

whereby the estimated state variables $\hat{z}_1$and $\hat{z}_2$ are issued from the generalized cascaded impulsive observer.
Denote $\chi \triangleq (e_1^T, e_2^T, \varepsilon_1^T, \varepsilon_2^T, z_1^T, z_2^T)^T$ with $e_i = z_i - \hat{z}_i$ and $\varepsilon_i = z - \xi$. Then, one can show that:

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - M \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 0 \\ M\varepsilon_2 \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \end{pmatrix} = \begin{pmatrix} A_{11} - L_1C & A_{12} \\ -L_2C & A_{22} \end{pmatrix}\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} + \begin{pmatrix} L_1e_1 \\ L_2e_1 \end{pmatrix}$$

Additionally, the *the closed-loop system via the observer is given by:*

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} B_2K_1 & B_1K_2 \\ B_2K_1 & B_2K_2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} A_{11} - B_1K_1 & A_{12} - B_1K_2 \\ -B_2K_1 & A_{22} - B_2K_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{15}$$

We are now able to state the main result of this section:

Theorem 3. *Assume that system (2) satisfies Assumption 2 and consider the closed-loop system (15). Then, for all $\beta_{max} > 0$ and $\epsilon > 0$ with $\beta_{max} > \epsilon$, there exist a maximum sampling period $\theta_{max} > 0$ and at least a reset matrix R such that for any initial condition $\chi(0)$ such that $\|\chi(0)\| < \beta_{max}$, the original system's state z converges to $\|z\| < \frac{\epsilon}{2}$ and the extended state at each sampling time t_k^+ converge into a set $B_\epsilon \triangleq \{\chi : \|\chi\| < \epsilon\}$, if the following conditions are verified:*

i) the matrix $\Gamma_1 = \begin{pmatrix} A_{22} - M & 0 & M \\ 0 & A_{11} - L_1 & A_{12} \\ 0 & -L_2 & A_{22} \end{pmatrix}$ is a continuous Hurwitz matrix.

ii) the control matrix $\Gamma_2 = \begin{pmatrix} A_{11} - B_1K_1 & A_{12} - B_1K_2 \\ -B_2K_1 & A_{22} - B_2K_2 \end{pmatrix}$ is a continuous Hurwitz matrix.

Proof: One can check that

$$e_1(t_k^+) = Re_1(t_k)$$

Now, we define a Lyapunov function $V_1(e_1) \triangleq e_1^Te_1 = \|e_1\|^2$, we obtain

$$\begin{aligned} &V_1(e_1(t_{k+1}^+)) \\ &\quad = V_1(Re^{A_{11}\theta_k}e_1(t_k^+) + \textstyle\int_{t_k}^{t_{k+1}} e^{A_{11}(t_{k+1}-\tau)}A_{12}e_2(\tau)d\tau) \end{aligned}$$

Assuming that $\forall\tau \in [t_k^+, t_{k+1}^+[$, $\|e_2(\tau)\| < \beta_{\max}$ (this will be proved later in the proof), with the same arguments as for Theorem 1, we have:

$$\begin{aligned} V_1(e_1(t_{k+1}^+)) &\le \|Re^{A_{11}\theta_k}e_1\|^2 \\ &\quad + \|R\textstyle\int_{t_k}^{t_{k+1}} e^{A_{11}(t_{k+1}-\tau)}A_{12}e^{A_{22}(\tau-t_k)}\beta_{\max}d\tau\|^2 \\ &\le (\|Re^{A_{11}\theta_k}\|^2)V_{1,\max} \\ &\quad + \|A_{12}\|^2\beta_{\max}\textstyle\int_{t_k}^{t_{k+1}} \|Re^{A_{11}(t_{k+1}-\tau)}\|^2\|e^{A_{22}(\tau-t_k)}\|^2d\tau \end{aligned}$$

with $V_{1,\max} \triangleq \max\limits_{\|e_1\|=\beta_{max}} \{V_1(e_1)\}$. It is always possible to find R such that:

$$V_1(x_1(t_{k+1}^+)) < \epsilon' \quad \text{or} \quad \|\chi(t_{k+1})\| < \left(\frac{\epsilon}{4}\right)^2 \tag{16}$$

Note that the above inequality is verified by $R = 0$ and, by continuity, also by a neighborhood of $R = 0$. Setting $\chi_2 \triangleq (e_2^T, \varepsilon_1^T, \varepsilon_2^T, z_1^T, z_2^T)^T$ and using the Lyapunov function $V_2 \triangleq \chi_2^TP_2\chi_2$, with P_2 a symmetric positive definite (SPD) matrix, we have:

$$\begin{aligned} \dot{V}_2 &= \left[A_2\chi_2 + \bar{B}_2e_1\right]^T P_2\chi_2 \\ &\quad + \chi_2^TP_2\left[A_2\chi_2 + \bar{B}_2e_1\right] \end{aligned}$$

with

$$A_2 = \left[\begin{array}{c|c} \Gamma_1 & 0 \\ \hline \Gamma_3 & \Gamma_2 \end{array}\right]$$

where

$$\Gamma_3 = \begin{pmatrix} \tilde{B}_1K_2 & 0 & 0 \\ \tilde{B}_2K_2 & 0 & 0 \end{pmatrix}$$

and $\bar{B}_2 = [0^T, L_1^T, L_2^T, (B_1K_1)^T, (B_2K_2)^T]^T$. Conditions i) and ii) of Theorem 3 ensure that A_2 is a continuous Hurwitz matrix. Consequently, there exists an SPD matrix P_2 such that $A_2^TP_2 + P_2A_2 = -I_d$. Consequently,

$$\dot{V}_2 = -\|\chi_2\|^2 + 2\|B_2\|\|e_1\|\|P_2\|\|\chi_2\|$$

Considering the fact that $\|\chi_2\| \geq \frac{\epsilon}{2}$ and for $\|e_1\| \leq (\frac{\epsilon}{4})^2$ by continuity of e_1 between $[t_k, t_{k+1}^-]$, then there exists $\theta_{\max}$ such that $\forall \tau \in [t_k, t_k + \theta_{\max}]$, we have:

$$\dot{V}_2 = -\|\chi_2\|^2 + 2\|B_2\|\|e_1\|\|P_2\|\|\chi_2\|$$

Therefore, X_2 converges to $\frac{\epsilon}{2}$ and $e_1(t_k)$ converges to $(\frac{\epsilon}{4})^2$ and hence $\chi(t_k)$ converges to $\left(\frac{\epsilon}{2} + \frac{\epsilon}{4}\right)^2$ and z converges to $\frac{\epsilon}{2}$.

This ends the proof of Theorem 3.

V. Simulation results

Let us consider the following linear triangular system:

$$\begin{cases} \dot{x}_1 = 2x_1 + x_2 + 3x_3 + u \\ \dot{x}_2 = x_2 + x_3 \\ \dot{x}_3 = -x_3 - u \\ y(t_k) = x_1(t_k) \end{cases} \tag{17}$$

Here, we have

$$\begin{aligned} A_{11} &= 2, \; A_{12} = \begin{pmatrix} 1 & 3 \end{pmatrix} \\ A_{22} &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

It can be seen that the matrix A_{22} is unstable. We are therefore in the Case 2 whereby the unmeasured subspace is unstable. The main objective is stabilise system (17) using an observer-based linear control law $u = -k_1\hat{x}_1 - k_2\hat{x}_2 - k_3\hat{x}_3$.

Here we choose $k_1 = -48$, $k_2 = -30$ and $k_3 = -55$. In this case, all the eigenvalues will be located at -2 and -1 in the left-half complex plane. We design a cascaded observer of the form (13), by choosing the parameters as follows: For the impulsive observer and in order to ensure the convergence of the observer the observer gains is designed as follows $R = 0$, $M = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$, $L_1 = 6$ and $L_2 = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$. The parameters of observer control satisfy the conditions of Theorem 3, with the corresponding matrix of observer is

$$\Gamma_1 = \begin{pmatrix} -4 & 1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 3 \\ 0 & 0 & -6 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and the corresponding matrix of control is

$$\Gamma_2 = \begin{pmatrix} -48 & 31 & 58 \\ 1 & 1 & 0 \\ -30 & -56 & -48 \end{pmatrix}$$

Fig. 1 and Fig. 2 shows the performance of the observer (13) with two different periods of measure; the first $\theta_k = 0.4s$ is fixed and the second $\theta_k \in [0.3s, 0.6s]$ is variable. The initial conditions are chosen as follows: $x(0) = (0.1, -0.1, 0.1)$, $\hat{z}(0) = (1, 0.6, 0.5)$.

Fig 3 show that with a variable sampling period $\theta_k \in [0.3s, 0.6s]$, the estimated states $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ converges respectively to the states x_1, x_2 and x_3 of the system. It should be noted that the coupling matrix M, plays a very important role, permitting the stabilisation of the system (17).

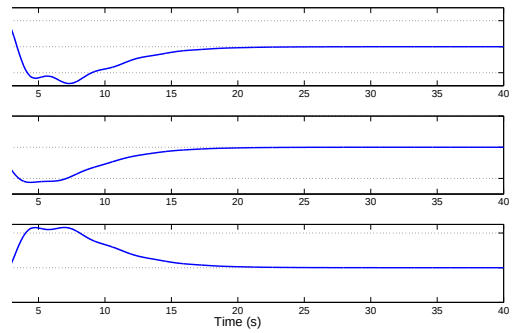

Fig. 1. System states with sampling period $\theta_k = 0.4$

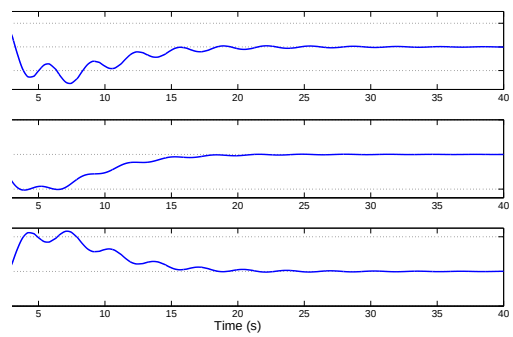

Fig. 2. System states with sampling period $\theta_k \in [0.3s, 0.6s]$

VI. Conclusion

In this paper, we have shown that it is possible to design an observer-based control for a class of linear systems using discrete measurements with a variable sampling period. The key idea is to employ an impulsive observer coupled with an appropriate linear feedback control whenever the unmeasured subspace is stable. Otherwise, one has to employ two cascaded observers in conjunction with a linear feedback control when the unmeasured subspace is unstable. A practical stability result has been proposed to prove the stability of the closed-loop systems. Simulation results on an academic example has shown good performance of the observer-based control. Future works will consist in extending the proposed results to some classes of nonlinear systems.

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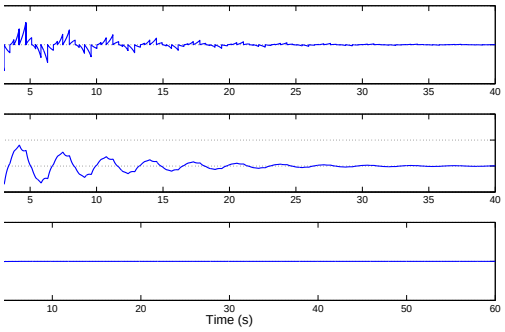

Fig. 3. The error of observation with sampling period $\theta_k \in [0.3s, 0.6s]$

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