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An asymptotic two layers monodomain model of cardiac electrophysiology in the atria

Yves Coudière · Jacques Henry · Simon Labarthe

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Abstract Numerical simulations of the cardiac electrophysiology in the atria are often based on the standard bidomain or monodomain equations stated on a two-dimensional manifold. These simulations take advantage of the thinness of the atrial tissue, and their computational costs is reduce, as compared to three-dimensional simulations. However, these models do not take into account the heterogeneities located in the thickness of the tissue, like discontinuities of the fiber direction, although they can be a substrate for atrial arrhythmia [Hocini et al., 2002, Ho et al., 2002, Nattel, 2002]. We investigate a two-dimensional model with two coupled, superimposed layers that allows to introduce three-dimensional heterogeneities, but retains a reasonable computational cost. We introduce the mathematical derivation of this model, show its convergence toward the three-dimensional model and give some numerical illustration of its interest. Our model would be an efficient tool to test the influence of three-dimensional fiber direction heterogeneities in reentries or atrial arrhythmia without using three-dimensional models.

Keywords Cardiac modeling · Atrial model · Surface model · Asymptotic analysis

Mathematics Subject Classification (2000) MSC 92C30 · MSC 92C50 · 35Q92

1 Introduction

Modeling the electrophysiology of cardiac tissues is considered an investigation tool for clinical and fundamental research. Theoretical studies and numerical simulations have been recognized an efficient way to improve our knowledge of arrhythmia genesis [Zemlin et al., 2009, Haissaguerre et al., 2007], perpetuation [Cain, 2007], and ablation efficiency [Rotter et al., 2007, Nagaiah et al., 2013].

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In three-dimensional tissues, the propagation of the action potential is modeled by the monodomain or bidomain systems of equations. These are reaction-diffusion equations, where the reaction represents the ionic function of the tissue and the diffusion accounts for the diffusion of electrical charges.

Models of the propagation of the action potential through the atria are often formulated as monodomain or bidomain systems of equations on bi-dimensional manifolds [Haissaguerre et al., 2007, Rotter et al., 2007, Dang et al., 2005]. Such surface models take advantage of the thinness of the atria with respect to the length scale of the heart, and drastically reduce the numerical cost of their resolution, hence allowing thorough in-silico investigations of atrial arrhythmias. These surface models approximate the first term of the asymptotic expansion (with respect to the thickness) of the three-dimensional equations, as presented in section 4. The thickness itself is not part of the final model, since it is assumed to vanish.

On the other hand, there exists structural and functional heterogeneities within the atria, such as fiber direction discontinuities through the wall. For instance, several superimposed layers with different fiber directions were described in the atrial wall or in the ostium zones of the pulmonary veins [Ho et al., 2001, 1999, Hocini et al., 2002, Saito et al., 2000]. In these anatomical regions, the transition of fibers directions between layers can be very abrupt. These abrupt transitions can trigger complex propagation patterns [Vetter et al., 2005] (and also figure 3), and are suspected to be a substrate for re-entries or arrhythmia [Hocini et al., 2002, Ho et al., 2002, Nattel, 2002]. Functional transmural heterogeneities have also been demonstrated during atrial fibrillation [Eckstein et al., 2011]. The usual surface models do not account for such heterogeneities. The model proposed in [Chapelle et al., 2013] is a surface bidomain system of equations with anisotropic conductivity derived from fiber direction varying smoothly through the atrial wall. This derivation is based on a rigorous mathematical study, including curvature effects. Although it seems to be an improvement over the usual models, it still does not account for the complex propagation pattern mentioned above.

In order to overcome this difficulty, models with several surface layers have been proposed [Weber dos Santos and Dickstein, 2003, Jacquemet, 2004, Gharaviri et al., 2012]. The main idea is to include the structural heterogeneities of leading importance while keeping the convenience of a surface model. However, the mathematical foundations of these models have not been studied in depth, and the physiological assumptions on which they rely are still unclear.

In this paper, we show that a special attention must be paid to the physiological scaling of the equations. We notably show that the balance between the reaction terms and transverse diffusion (conductivity through the wall) entirely determines the regime of the solutions. If transverse diffusion dominates, then the usual surface models are sufficient to represent the average behavior of the action potential. But if transverse diffusion and reaction have the same order of magnitude, then complex patterns (figure 3) are observed, and the transmural effects must not be neglected. This is typically the case during electrical depolarization for human atria, as discussed in section 3.2. To account for this regime, we first introduce the second order term in the asymptotic expansion of the solutions, instead of simply the first one. But this would not yield a model that clearly differentiates between different layers.

Hence we propose a model that represents the evolution of the transverse averages of the transmembrane voltage in each of the tissue's layers. Consequently, we design a surface model with two layers having distinct conductivity tensor and electrophysiology source function. With this new vision, we can see how the thickness appears in the equations, and we account for the main structural and functional observations described above.

For a slab of tissue with two distinct layers, the model is formally derived from the three-dimensional monodomain equations in two steps: the second order completion of the usual surface model is first computed, and then the averages of this second order model are shown to solve a set of two coupled surface monodomain equations, associated to the two layers of the tissue. The error between this two-surfaces model and the transverse averages of the three-dimensional model is rigorously proved to be of order ε^3 , where ε is the aspect ratio of the atria. Some numerical results illustrate the good behavior of this model.

The paper is organized as follows. Section 2 briefly recalls the monodomain equations. Section 3 introduces the geometrical framework of our paper, introduces the small parameter ε , explains the scaling of the equations and discusses the different regime observed. It justifies the fact that the thickness cannot be neglected during the depolarization process in the atria. Section 4 presents a derivation of the second order surface model similar (but one order more accurate) to the usual surface models, and gives the corresponding error estimate. Section 5 proposes a derivation of our two layers model with two systems of monodomain equations formulated on a surface, and also proves the corresponding error estimate. Section 6 is devoted to the numerical illustrations. The conclusions are discussed in section 7.

2 The monodomain equations

The bidomain equations, first introduced in [Clerc, 1976, Tung, 1978], is a degenerate system of two anisotropic reaction-diffusion equation coupled to a set of ordinary differential equations. They describe the evolution of the intra- and extra-cellular potential at the tissue scale [Krassowska and Neu, 1993], and they can mimic complex phenomena such as virtual electrodes [Sepulveda et al., 1989]. Under the equal anisotropy ratio assumption the system can be reduced to the single monodomain reaction-diffusion equation. In the vast majority of applications, the solutions to this monodomain equation are very close to the solution to the bidomain equation [Potse et al., 2006]. The models proposed and discussed in this paper are based on the monodomain approximation although it is straightforward to derive a bidomain version of them.

Let Ω be an open subset of \mathbb{R}^3 and (u, w) be the solution of the monodomain equations, that read

$$A(C\partial_t u + f(u, w)) = \operatorname{div}(\sigma \nabla u) \quad \text{in } (0, +\infty) \times \Omega, \quad (1)$$

$$\partial_t w + g(u, w) = 0 \quad \text{in } (0, +\infty) \times \Omega \quad (2)$$

where the parameters A , C and σ are, respectively, the ratio of surface of membrane to total volume (in cm^{-1}), the membrane capacitance (in μFcm^{-2}) and the conduc-

tivity (in mScm^{-2}). The unknowns are the transmembrane potential $u(t, x) \in \mathbb{R}$ (in mV) and the m variables $w(t, x) \in \mathbb{R}^m$ that describe the electrophysiological state of the membrane. The evolution of the electrophysiological state w is modeled by the functions $f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$.

The domain Ω represents the myocardial domain. In this domain, the cardiomyocytes organize into fibers, that in turn organize into laminae. This cardiac tissue is modeled as a homogenized continuous medium with heterogeneous and anisotropic electrical conductivity σ that reads:

$$\forall x \in \bar{\Omega}, \quad \sigma(x) = \sum_{i=1}^3 \sigma_i v_i(x) v_i^T(x)$$

where (v_1, v_2, v_3) is the orthonormal basis in \mathbb{R}^3 aligned on the fiber and laminae directions and $0 < \underline{\sigma} := \sigma_3 \leq \sigma_2 \leq \sigma_1 := \bar{\sigma}$ are constant parameters. As a consequence, the diffusion operator is bounded and uniformly elliptic:

$$\forall x \in \bar{\Omega}, \quad \underline{\sigma} |\xi|^2 \leq \xi^T \sigma(x) \xi = \sum_{i=1}^3 \sigma_i |\xi_i|^2 \leq \bar{\sigma} |\xi|^2.$$

Equations (1) and (2) are supplemented with the Neumann boundary condition

$$\sigma \nabla u \cdot n = 0 \quad \text{in } (0, +\infty) \times \partial\Omega, \quad (3)$$

that models an electrically insulated heart, together with the initial condition

$$u(0, x) = u^0(x), \quad w(0, x) = w^0(x) \quad \text{a.e. } x \in \Omega. \quad (4)$$

We will assume that the monodomain equations (1) and (2) with the boundary and initial conditions (3) and (4) have a unique solution u and v defined for a.e. $x \in \Omega$ and for all $t > 0$. We will furthermore assume that this solution is regular enough to carry out all our proofs.

For instance, assuming that f and g are uniform Lipschitz functions, the unique weak solution to the monodomain equations is given by some functions u and w such that: $u \in L^2(0, T; H^1(\Omega))$ and $\partial_t u \in L^2(0, T; (H^1(\Omega))')$, and $w \in L^2(0, T; L^2(\Omega))$ and $\partial_t w \in L^2(0, T; L^2(\Omega))$, for any $T > 0$ (see [Bendahmane and Karlsen, 2006]).

3 Transmural discontinuities of the fiber structure and dimensional analysis

3.1 A two-layers slab of myocardium

We consider an ideal slab of cardiac tissue Ω composed of two layers with the same thickness $h > 0$ and distinct fiber directions, specifically $\Omega = \omega \times (-h, h) \subset \mathbb{R}^3$, where ω is an open bounded subset of \mathbb{R}^2 and the two layers are $\Omega^{(1)} = \omega \times (0, h)$ and $\Omega^{(2)} = \omega \times (-h, 0)$. The boundary of each layer is split into the external boundary and the interface between the layers, namely $\Gamma^{(k)} = \partial\Omega^{(k)} \cap \partial\Omega$ for $k = 1, 2$ and $\Sigma = \omega \times \{0\}$. The coordinates of a point $x \in \Omega$ are written $x = (x', z)$ with $x' \in \omega$

and $-h < z < h$. We assume that the fibers have a constant direction in each layer, specifically, for $k = 1, 2$,

$$\mathbf{v}_1^{(k)} = \begin{pmatrix} \cos(\theta^{(k)}) \\ \sin(\theta^{(k)}) \\ 0 \end{pmatrix}, \quad \mathbf{v}_2^{(k)} = \begin{pmatrix} -\sin(\theta^{(k)}) \\ \cos(\theta^{(k)}) \\ 0 \end{pmatrix}, \quad \mathbf{v}_3^{(k)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (5)$$

where $\theta^{(1)}$ and $\theta^{(2)}$ are fixed angles. Consequently, the conductivity tensors in the layers read

$$\boldsymbol{\sigma}^{(k)} = \underbrace{\sum_{j=1}^2 \sigma_j^{(k)} \mathbf{v}_j^{(k)} \mathbf{v}_j^{(k)T}}_{=\boldsymbol{\sigma}'^{(k)}} + \sigma_3^{(k)} \mathbf{v}_3^{(k)} \mathbf{v}_3^{(k)T} \quad \text{for } k = 1, 2. \quad (6)$$

where $\boldsymbol{\sigma}'^{(k)}$ is the 2×2 longitudinal conductivity tensor associated to the layer number k .

We denote by $\boldsymbol{\sigma}$ the conductivity tensor in Ω , defined by $\boldsymbol{\sigma}(x) = \boldsymbol{\sigma}^{(1)}(x)$ if $x \in \Omega^{(1)}$ and $\boldsymbol{\sigma}(x) = \boldsymbol{\sigma}^{(2)}(x)$ if $x \in \Omega^{(2)}$. Hence, given initial data u^0 and w^0 in Ω , the propagation of the action potential in Ω is uniquely described by the monodomain equations (1) and (2) in Ω with the piecewise constant conductivity $\boldsymbol{\sigma}$, the boundary condition (3) on $\partial\Omega$, and the initial condition (4). We denote by $(u^{(k,0)}, w^{(k,0)})$ the restriction of the initial data (u^0, w^0) to the layer $\Omega^{(k)}$, and by $(u^{(k)}, w^{(k)})$ the restriction of the solution (u, w) to $\Omega^{(k)}$. Each of these restrictions solves the monodomain equations written in the subdomain $\Omega^{(k)}$ ($k = 1, 2$), that read

$$A \left(C \partial_t u^{(k)} + f(u^{(k)}, w^{(k)}) \right) = \operatorname{div}_{x'} \left(\boldsymbol{\sigma}'^{(k)} \nabla_{x'} u^{(k)} \right) + \sigma_3^{(k)} \partial_{zz} u^{(k)} \quad (7)$$

$$\partial_t w^{(k)} + g(u^{(k)}, w^{(k)}) = 0 \quad (8)$$

for $t > 0$ and $x \in \Omega^{(k)}$ with the boundary and transmission conditions

$$\boldsymbol{\sigma}^{(k)} \nabla u^{(k)} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma^{(k)}, k = 1, 2 \quad (9)$$

$$\boldsymbol{\sigma}^{(1)} \nabla u^{(1)} \cdot \mathbf{n}_\Sigma = \boldsymbol{\sigma}^{(2)} \nabla u^{(2)} \cdot \mathbf{n}_\Sigma, \quad u^{(1)} = u^{(2)} \quad \text{on } \Sigma \quad (10)$$

where \mathbf{n} is the unit normal to $\partial\Omega$ outward of Ω and \mathbf{n}_Σ is the unit normal to Σ pointing from $\Omega^{(2)}$ to $\Omega^{(1)}$, and the initial conditions

$$u^{(k)}(0, x) = u^{(k,0)}(x), \quad w^{(k)}(0, x) = w^{(k,0)}(x) \quad \text{a.e. } x \in \Omega^{(k)}, k = 1, 2. \quad (11)$$

3.2 Dimensionless scaling of the equations

In order to discuss the relative importance of the various terms, the monodomain equations are recast in a dimensionless system of equations following the strategies proposed in [Colli Franzone et al., 1990, Keener, 1991, Rioux, 2012] for cardiac

tissues. Given some time and space characteristic lengths t_0 and x_0 and given the thickness $h > 0$ of the two layers, we first define the dimensionless variables

$$\bar{x} = \frac{x'}{x_0}, \quad \bar{t} = \frac{t}{t_0} \quad \text{and} \quad \bar{z} = \frac{z}{h}.$$

For instance, to observe the whole action potential, we will set the length scale to $x_0 = 1\text{cm}$ and the time scale to $t_0 = 400\text{ms}$, because they are the typical extent of atrial structures such as pulmonary veins and the duration of an action potential. The two-layers domain $\Omega = \omega \times (-h, h)$ is mapped to $\bar{\omega} \times (-1, 1)$ where $\bar{\omega} = \frac{1}{x_0}\omega \subset \mathbb{R}^2$. The physical quantities are rescaled as follows:

$$\begin{aligned} \bar{u}^{(k)}(\bar{t}, \bar{x}, \bar{z}) &= \frac{u^{(k)}(t, x', z) - u_r}{\delta u}, \\ \bar{\sigma}^{(k)} &= \frac{1}{\sigma_0} \sigma'^{(k)}, \quad \bar{\sigma}_3^{(k)} = \frac{\sigma_3^{(k)}}{\sigma_0}, \\ \bar{f}(\bar{u}^{(k)}, \bar{w}^{(k)}) &= \frac{1}{f_0} f(u^{(k)}, w^{(k)}), \quad \text{and} \quad \bar{g}(\bar{u}^{(k)}, \bar{w}^{(k)}) = t_0 g(u^{(k)}, w^{(k)}) \end{aligned}$$

where we use the typical amplitude of an action potential δu , the resting potential u_r , the maximum value of the sodium current f_0 (specifically the maximum value reached by the function f) and the characteristic conductivity scale σ_0 . The monodomain equations for the two-layers domain finally read

$$\begin{aligned} AC \frac{x_0^2}{\sigma_0 t_0} \left(\partial_{\bar{t}} \bar{u}^{(k)} + \frac{f_0 t_0}{C \delta u} \bar{f}(\bar{u}^{(k)}, \bar{w}^{(k)}) \right) &= \text{div}_{\bar{x}} \left(\bar{\sigma}^{(k)} \nabla_{\bar{x}} \bar{u}^{(k)} \right) + \frac{x_0^2}{h^2} \bar{\sigma}_3^{(k)} \partial_{\bar{z}\bar{z}} \bar{u}^{(k)} \\ \partial_{\bar{t}} \bar{w}^{(k)} + \bar{g}(\bar{u}^{(k)}, \bar{w}^{(k)}) &= 0 \end{aligned}$$

for $t > 0$ and (\bar{x}, \bar{z}) in the domains $\bar{\Omega}^{(1)} := \bar{\omega} \times (0, 1)$ and $\bar{\Omega}^{(2)} := \bar{\omega} \times (-1, 0)$. The boundary and transmission conditions (9) and (10) remain unchanged but are now stated on $\bar{\Gamma}^{(k)} = \partial \bar{\Omega}^{(k)} \cap \partial \bar{\Omega}$ and $\bar{\Sigma} = \bar{\omega} \times \{0\}$.

As a consequence, the dimensionless numbers

$$\alpha = AC \frac{x_0^2}{\sigma_0 t_0}, \quad \beta = \frac{f_0 t_0}{C \delta u}, \quad \text{and} \quad \varepsilon = \frac{h}{x_0}$$

characterize the solutions. The order of magnitude of each of the physical quantities are set according to values usual for cardiac electrophysiology (see table 3.2 for some values of the conductivities):

$$\begin{aligned} A &= 1000\text{cm}^{-1}, \quad C = 1\mu\text{Fcm}^{-2}, \quad \delta u = 100\text{mV}, \\ f_0 &= 100\mu\text{Acm}^{-2}, \quad \sigma_0 = 1.5\text{mScm}^{-1}. \end{aligned}$$

The orders of magnitude of the observation scales (t_0 and x_0) are then set to $x_0 = 1\text{cm}$, $t_0 = 400\text{ms}$, and consequently, the dimensionless parameters α and β are $\alpha = 1.67$, and $\beta = 400$. Since we consider layers of cardiac tissue, the important quantity

| | Article | 1 | 2 | 3 | | | 4 | |
|---------------|-----------------------|------|------|-------|------|------|--------|-------|
| Intracellular | σ_1 | 3.0 | 3.0 | 1.741 | 1.7 | 2.4 | 2.0 | 3.0 |
| | $\sigma_2 = \sigma_3$ | 0.3 | 0.3 | 0.475 | 0.19 | 0.24 | 0.4167 | 0.315 |
| Extracellular | σ_1 | 3.0 | 3.0 | 3.906 | 6.2 | 4.8 | 2.5 | 2.0 |
| | $\sigma_2 = \sigma_3$ | 1.2 | 1.2 | 1.97 | 2.4 | 2.2 | 1.25 | 1.35 |
| Monodomain | σ_1 | 1.5 | 1.5 | 1.204 | 1.33 | 1.6 | 1.11 | 1.2 |
| | $\sigma_2 = \sigma_3$ | 0.24 | 0.24 | 0.383 | 0.18 | 0.22 | 0.31 | 0.25 |

Table 1 Electrical conductivities taken from the literature (mScm^{-1} – monodomain conductivities are the half of the harmonic averages of bidomain ones). Article 1: [Boulakia et al., 2010], article 2: [Potse et al., 2006], article 3: [Clements et al., 2004] (review), and article 4: [Colli Franzone et al., 1990].

is the aspect ratio $\varepsilon = h/x_0$, assumed to be small. The dimensionless monodomain equations in each layer finally read:

$$\begin{aligned} \alpha \left(\partial_t \bar{u}^{(k)} + \beta \bar{f}(\bar{u}^{(k)}, \bar{w}^{(k)}) \right) &= \text{div}_{\bar{x}} \left(\bar{\sigma}^{(k)} \nabla_{\bar{x}} \bar{u}^{(k)} \right) + \frac{\bar{\sigma}_3^{(k)}}{\varepsilon^2} \partial_{\bar{z}\bar{z}} \bar{u}^{(k)} \\ \partial_t \bar{w}^{(k)} + \bar{g}(\bar{u}^{(k)}, \bar{w}^{(k)}) &= 0 \end{aligned}$$

and every parameter is fixed and of order 1 except ε and β . According to our analysis, the transverse diffusion $\partial_{\bar{z}\bar{z}} \bar{u}^{(k)}$ dominates the reaction terms whenever $\beta \varepsilon^2 \ll 1$. In the next section, we will derive an asymptotic model for $\varepsilon \rightarrow 0$, consequently in the dominant transverse diffusion regime.

Remark 1 *The documented thickness of human atrial tissues is small, but it remains of order $h = 0.1\text{cm}$. Hence, the reaction and diffusion terms actually balance during the depolarization phase ($\beta \varepsilon^2 \simeq 1$) and the behavior of the asymptotic model may be different from the observation in atrial tissues. For this reason, we develop a higher order asymptotic model in section 5.*

During the repolarization, the ionic currents has far less intense, so that the asymptotic regime applies quite well for human atria (β decreases, so that $\beta \varepsilon^2 \leq 1$).

4 The asymptotic model with one layer

4.1 Formal derivation of the equations

In this section, we recall how to derive the usual two-dimensional surface model of the atria, that has been used in several numerical studies (see e.g. [Haissaguerre et al., 2007, Rotter et al., 2007]). According to the previous section (and dropping the $\bar{\cdot}$ above the dimensionless quantities), the three-dimensional dimensionless problem reads

$$\alpha \left(\partial_t u_\varepsilon^{(k)} + \beta f \left(u_\varepsilon^{(k)}, w_\varepsilon^{(k)} \right) \right) = \text{div}_x \left(\sigma^{(k)} \nabla_x u_\varepsilon^{(k)} \right) + \frac{\sigma_3^{(k)}}{\varepsilon^2} \partial_{zz} u_\varepsilon^{(k)} \quad (12)$$

$$\partial_t w_\varepsilon^{(k)} + g \left(u_\varepsilon^{(k)}, w_\varepsilon^{(k)} \right) = 0 \quad (13)$$

for $k = 1, 2$, $t > 0$ and $(x, z) \in \Omega^{(k)}$. The boundary and transmission conditions become

$$\sigma^{(1)} \nabla_x u_\varepsilon^{(1)} \cdot n = 0 \quad \text{in } \partial\omega \times (0, 1) \quad \text{and} \quad \sigma_3^{(1)} \partial_z u_\varepsilon^{(1)} = 0 \quad \text{in } \omega \times \{1\}, \quad (14)$$

$$\sigma^{(2)} \nabla_x u_\varepsilon^{(2)} \cdot n = 0 \quad \text{in } \partial\omega \times (-1, 0) \quad \text{and} \quad \sigma_3^{(2)} \partial_z u_\varepsilon^{(2)} = 0 \quad \text{in } \omega \times \{-1\}, \quad (15)$$

$$\sigma_3^{(1)} \partial_z u_\varepsilon^{(1)} = \sigma_3^{(2)} \partial_z u_\varepsilon^{(2)}, \quad u_\varepsilon^{(1)} = u_\varepsilon^{(2)} \quad \text{in } \omega \times \{0\}, \quad (16)$$

and we assume, for the sake of simplicity, that the initial data are independent of $k \in \{1, 2\}$ and of $z \in (-1, 1)$, specifically:

$$u_\varepsilon^{(k)}(0, x, z) = u^0(x), \quad w_\varepsilon^{(k)}(0, x, z) = w^0(x) \quad \text{in } \Omega^{(k)}, \quad k = 1, 2, \quad (17)$$

where the functions $u^0(x)$ and $w^0(x)$ are given functions of $x \in \omega$. We consider the following expansion of $u_\varepsilon^{(k)}$ and $w_\varepsilon^{(k)}$:

$$u_\varepsilon^{(k)} = u_0^{(k)} + \varepsilon^2 u_1^{(k)} + \varepsilon^4 u_2^{(k)} + o(\varepsilon^4), \quad w_\varepsilon^{(k)} = w_0^{(k)} + \varepsilon^2 w_1^{(k)} + o(\varepsilon^2),$$

for all $t \geq 0$, so that, in particular,

$$\begin{aligned} u_0^{(k)}(0, x, z) &= u^0(x), & u_j^{(k)}(0, x, z) &= 0 \quad \text{for } j \geq 1, \\ w_0^{(k)}(0, x, z) &= w^0(x), & w_j^{(k)}(0, x, z) &= 0 \quad \text{for } j \geq 1. \end{aligned}$$

We introduce this expansion in the system of eqs. (12) and (13) and identify the coefficients having the same order with respect to ε^2 . The coefficient of order $1/\varepsilon^2$ yields the equation $\partial_{zz} u_0^{(k)} = 0$ for $k = 1, 2$ together with the boundary and transmission conditions for $u_0^{(k)}$, in the direction z ,

$$\begin{aligned} \partial_z u_0^{(1)}(t, x, 1) &= \partial_z u_0^{(2)}(t, x, -1) = 0, \\ \sigma_3^{(1)} \partial_z u_0^{(1)}(t, x, 0) &= \sigma_3^{(2)} \partial_z u_0^{(2)}(t, x, 0), \quad u_0^{(1)}(t, x, 0) = u_0^{(2)}(t, x, 0). \end{aligned}$$

These equations easily show that $u_0^{(1)}(t, x, z) = u_0^{(2)}(t, x, z) := u_0(t, x)$ is a function independent of z , defined for $t \geq 0$ and a.e. $x \in \omega$. We can write a second order approximation of the functions f and g :

$$\begin{aligned} f(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) &= f(u_0, w_0^{(k)}) + \varepsilon^2 \nabla f(u_0, w_0^{(k)}) \cdot (u_1^{(k)}, w_1^{(k)}) + o(\varepsilon^2), \\ g(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) &= g(u_0, w_0^{(k)}) + \varepsilon^2 \nabla g(u_0, w_0^{(k)}) \cdot (u_1^{(k)}, w_1^{(k)}) + o(\varepsilon^2). \end{aligned}$$

We then get the following equation on $u_1^{(k)}$, for $k = 1, 2$, identifying the coefficients of the terms of order ε^0 .

$$\sigma_3^{(k)} \partial_{zz} u_1^{(k)} = \alpha \left(\partial_t u_0 + \beta f(u_0, w_0^{(k)}) \right) - \operatorname{div}_x \left(\sigma^{(k)} \nabla_x u_0 \right), \quad (18)$$

$$\partial_t w_0^{(k)} + g(u_0, w_0^{(k)}) = 0, \quad (19)$$

with the boundary and interface conditions (14) to (16) on $u_1^{(k)}$. The evolution of $w_0^{(k)}$ only depends on the function $(t, x, z) \mapsto g(u_0(t, x), \cdot)$, which is independent of z . Since $w_\varepsilon^{(k)}(0, x, z) = w^0(x)$ is independent of z and of $k = 1, 2$, the functions $w_0^{(k)}$ are independent of z for all time $t > 0$ and have the same value, solution to equation (19), that we denote by $w_0(t, x) := w_0^{(1)}(t, x, z) = w_0^{(2)}(t, x, z)$ for all $t \geq 0$ and a.e. $x \in \omega$. Afterwards, we integrate the equation (18) on $u_1^{(k)}$ along z and use equations (14) and (15) to get

$$-\sigma_3^{(1)} \partial_z u_1^{(1)}(\cdot, 0) = \alpha (\partial_t u_0 + \beta f(u_0, w_0)) - \operatorname{div}_x (\sigma^{(1)} \nabla_x u_0), \quad (20)$$

$$\sigma_3^{(2)} \partial_z u_1^{(2)}(\cdot, 0) = \alpha (\partial_t u_0 + \beta f(u_0, w_0)) - \operatorname{div}_x (\sigma^{(2)} \nabla_x u_0). \quad (21)$$

Finally, we add these two equations and use the transmission condition (16) to obtain the following system of equations on (u_0, w_0) :

$$\alpha (\partial_t u_0 + \beta f(u_0, w_0)) = \operatorname{div}_x (\sigma^m \nabla_x u_0), \quad (22)$$

$$\partial_t w_0 + g(u_0, w_0) = 0, \quad (23)$$

with the boundary condition $\sigma^m \nabla_x u_0 \cdot n = 0$ on $\partial \omega$ and for $t > 0$ and initial condition $u_0(0, x) = u^0(x)$ and $w_0(0, x) = w^0(x)$ in ω . We denoted by $\sigma^m = \frac{\sigma^{(1)} + \sigma^{(2)}}{2}$ the arithmetic average of the conductivity matrices in both layers. For the sake of simplicity in the computations below, we also introduce the notation $\sigma^d = \frac{\sigma^{(1)} - \sigma^{(2)}}{2}$.

Remark 2 Equations (22) and (23), defined on the surface ω , independent of ε , and with a conductivity matrix averaged in the thickness, are the usual surface model for the atria. It has been rigorously derived for instance in [Chapelle et al., 2013].

Now, note that the right-hand side of equations (18) is independent of z . Using equation (22), it is equal to $\operatorname{div}_x (\sigma^m \nabla_x u_0) - \operatorname{div}_x (\sigma^{(k)} \nabla_x u_0) = \pm \operatorname{div}_x (\sigma^d \nabla_x u_0)$, and to the right-hand side of equations (20) and (21). Consequently, the functions $z \mapsto u_1^{(k)}(t, x, z)$ are quadratic in z . There exists $a^{(k)}, b^{(k)}$ and $c^{(k)}$ such that

$$u_1^{(k)}(t, x, z) = a^{(k)}(t, x) z^2 + b^{(k)}(t, x) z + c^{(k)}(t, x)$$

and the left-hand side of equations (18), (20), and (21) are, respectively, $2a^{(k)} \sigma_3^{(k)}$, $-\sigma_3^{(1)} b^{(1)}$ and $\sigma_3^{(2)} b^{(2)}$. Consequently, we can define

$$b := \operatorname{div}_x (\sigma^d \nabla_x u_0) = -2a^{(1)} \sigma_3^{(1)} = \sigma_3^{(1)} b^{(1)} = 2a^{(2)} \sigma_3^{(2)} = \sigma_3^{(2)} b^{(2)}.$$

The transmission condition on $u_1^{(k)}$ on ω yields $c^{(1)} = c^{(2)} := c$ and we can summarize the results as follows: we have

$$u_1^{(k)} = \frac{b}{\sigma_3^{(k)}} z \left(1 - \frac{|z|}{2} \right) + c \quad \text{where } b = \operatorname{div}_x (\sigma^d \nabla_x u_0), \quad (24)$$

and $c = c(t, x)$ is an unknown function. We denote by $\bar{u}_1^{(k)}$ the averages of $u_1^{(k)}$ through each layer $k = 1, 2$ and find that

$$\bar{u}_1^{(1)} := \int_0^1 u_1^{(1)}(\cdot, z) dz = c + \frac{1}{3} \frac{b}{\sigma_3^{(1)}} \quad \text{and} \quad \bar{u}_1^{(2)} := \int_{-1}^0 u_1^{(2)}(\cdot, z) dz = c - \frac{1}{3} \frac{b}{\sigma_3^{(2)}}.$$

As a consequence, we have the relations

$$\bar{u}_1 := \frac{\bar{u}_1^{(1)} + \bar{u}_1^{(2)}}{2} = c + \frac{1}{6} b \frac{\sigma_3^{(2)} - \sigma_3^{(1)}}{\sigma_3^{(2)} \sigma_3^{(1)}}, \quad \frac{\bar{u}_1^{(1)} - \bar{u}_1^{(2)}}{2} = \frac{1}{6} b \frac{\sigma_3^{(2)} + \sigma_3^{(1)}}{\sigma_3^{(2)} \sigma_3^{(1)}} := \frac{1}{3} \frac{b}{\sigma_3^h},$$

where \bar{u}_1 denotes the average of $u_1^{(k)}$ through the whole thickness of the tissue and $\sigma_3^h = 2 \frac{\sigma_3^{(1)} \sigma_3^{(2)}}{\sigma_3^{(1)} + \sigma_3^{(2)}}$ is the harmonic average of the transverse conductivities. Hence, c can be found if we know \bar{u}_1 . We also denote by \bar{w}_1 the average of $w_1^{(k)}$ through the whole thickness of the tissue, defined by

$$\bar{w}_1 = \frac{1}{2} \left(\int_0^1 w_1^{(1)}(\cdot, z) dz + \int_{-1}^0 w_1^{(2)}(\cdot, z) dz \right).$$

Now, the functions (\bar{u}_1, \bar{w}_1) are solution to the system of equations obtained by identifying the coefficients of order ε^2 in the expansion of $(u^{(k)}, w^{(k)})$. We first get the equations

$$\sigma_3^{(k)} \partial_{zz} u_2^{(k)} = \alpha \left(\partial_t u_1^{(k)} + \beta \nabla f(u_0, w_0) \cdot (u_1^{(k)}, w_1^{(k)}) \right) - \operatorname{div}_x \left(\sigma^{(k)} \nabla_x u_1^{(k)} \right), \quad (25)$$

$$\partial_t w_1^{(k)} + \nabla g(u_0, w_0) \cdot (u_1^{(k)}, w_1^{(k)}) = 0, \quad (26)$$

with the boundary and transmission conditions (14) to (16) on $u_2^{(k)}$. For $k = 1, 2$ and for each $(x, z) \in \Omega^{(k)}$, the equation (26) on $w_1^{(k)}$ is a first order linear Cauchy problem of the form $w'(t) + a(t)w(t) = -b(t)$ with $a(t) = \partial_2 g(u_0(t, x), w_0(t, x))$ and $b(t) = \partial_1 g(u_0(t, x), w_0(t, x)) u_1^{(k)}(t, x, z)$ and with $w(0) = 0$ because $w_1^{(k)}(0, x, z) = 0$ (eq. (17)). Its solution is computed explicitly: $w(t) = -\int_0^t b(s) \exp\left(-\int_s^t a(\tau) d\tau\right) ds$. We note that the function $a(t)$ is independent of z and the functions $z \mapsto b(t)$ ($k = 1, 2$) are C^∞ (polynomial) functions for all $t \geq 0$ and a.e. $x \in \omega$. As a consequence, the functions $z \mapsto w_1^{(k)}(t, x, z)$ are C^∞ for all $t \geq 0$ and a.e. $x \in \omega$ (in fact, they are also 2nd order polynomials). Furthermore, the function w is continuous through the interface Σ , meaning that $w_1^{(1)}(t, x, 0) = w_1^{(2)}(t, x, 0)$ for all $t \geq 0$ and a.e. $x \in \omega$ (because it is true for $u_1^{(k)}$).

Afterwards, we integrate again equation (25) for $z \in (0, 1)$ and $z \in (-1, 0)$, add the resulting equations, use the transmission conditions (14) and (15) on $u_2^{(k)}$. We remark that

$$\frac{1}{2} \operatorname{div}_x \left(\sigma^{(1)} \nabla_x \bar{u}_1^{(1)} + \sigma^{(2)} \nabla_x \bar{u}_1^{(2)} \right) = \operatorname{div}_x \left(\sigma^m \nabla_x \bar{u}_1 + \sigma^d \nabla_x \frac{\bar{u}_1^{(1)} - \bar{u}_1^{(2)}}{2} \right),$$

recall that $\frac{\bar{u}_1^{(1)} - \bar{u}_1^{(2)}}{2} = \frac{1}{3} \frac{b}{\sigma_3^h}$, and finally obtain the following equations for (\bar{u}_1, \bar{w}_1) :

$$\alpha (\partial_t \bar{u}_1 + \beta \nabla f(u_0, w_0) \cdot (\bar{u}_1, \bar{w}_1)) = \operatorname{div}_x (\sigma^m \nabla_x \bar{u}_1) + \operatorname{div}_x \left(\frac{1}{3\sigma_3^h} \sigma^d \nabla_x b \right) \quad (27)$$

$$\partial_t \bar{w}_1 + \nabla g(u_0, w_0) \cdot (\bar{u}_1, \bar{w}_1) = 0 \quad (28)$$

with the boundary condition $\sigma^m \nabla_x \bar{u}_1 \cdot n + \frac{1}{2\sigma_3^h} \sigma^d \nabla_x b \cdot n = 0$ on $\partial\omega$ and for $t > 0$, and the initial conditions $\bar{u}_1(0, x) = 0$ and $\bar{w}_1(0, x) = 0$ for a.e. $x \in \omega$.

4.2 Definition of the first order solution and associated asymptotic problem

Consider some data (u^0, w^0) defined for $x \in \omega$ and the solutions (u_0, w_0) and (\bar{u}_1, \bar{w}_1) to the bi-dimensional systems of equations (22), (23), and (27), (28), respectively, for $t > 0$ and $x \in \omega$, and with the boundary conditions

$$\sigma^m \nabla_x u_0 \cdot n = 0, \quad \sigma^m \nabla_x \bar{u}_1 \cdot n + \frac{1}{3\sigma_3^h} \sigma^d \nabla_x b \cdot n = 0,$$

and the initial condition

$$u_0(0, x) = u^0(x), \quad w_0(0, x) = w^0(x), \quad \text{and} \quad \bar{u}_1(0, x) = 0, \quad \bar{w}_1(0, x) = 0,$$

for a.e. $x \in \omega$. In equation (27) and (28), the functions u_0 and w_0 are the solutions to (22), (23), and the function b is defined on $(0, +\infty) \times \omega$ by $b = \operatorname{div}_x (\sigma^d \nabla_x u_0)$. Afterwards, we define the three-dimensional functions $u_1^{(k)}$ and $w_1^{(k)}$ on $(0, +\infty) \times \Omega^{(k)}$ for $k = 1, 2$ as follows: the function $u_1^{(k)}$ is explicitly given by

$$u_1^{(k)} = \frac{b}{\sigma_3^{(k)}} z \left(1 - \frac{|z|}{2} \right) + c, \quad \text{with } c = \bar{u}_1 - \frac{1}{6} b \frac{\sigma_3^{(2)} - \sigma_3^{(1)}}{\sigma_3^{(1)} \sigma_3^{(2)}} \quad (29)$$

(from eq. (24)) and the function $w_1^{(k)}$ is the solution to the system of equations (26) for $k = 1, 2$ that also reads:

$$w_1^{(k)}(t, x, z) = - \int_0^t \partial_1 g(u_0(s, x), w_0(s, x)) u_1^{(k)}(s, x, z) \exp \left(- \int_s^t \partial_2 g(u_0(\tau, x), w_0(\tau, x)) d\tau \right) ds. \quad (30)$$

Remark 3 (Averages of $u_1^{(k)}, w_1^{(k)}$) . Given the solutions \bar{u}_1 and \bar{w}_1 to (27) and (28), consider the functions $u_1^{(k)}, w_1^{(k)}$ defined by the equalities (29) and (30) above. Some straightforward computations show that, conversely, the averages in the thickness of $u_1^{(k)}$ and $w_1^{(k)}$ are exactly $\bar{u}_1 = c + \frac{1}{6} b \frac{\sigma_3^{(2)} - \sigma_3^{(1)}}{\sigma_3^{(2)} \sigma_3^{(1)}}$ and \bar{w}_1 , the given solutions to the equations (27) and (28).

While (u_0, w_0) is the solution of order 0, we define the first order approximate solution $\tilde{u}_\varepsilon^{(k)}$ and $\tilde{w}_\varepsilon^{(k)}$ on $(0, +\infty) \times \Omega^{(k)}$ by:

$$\tilde{u}_\varepsilon^{(k)}(t, x, z) = u_0(t, x) + \varepsilon^2 u_1^{(k)}(t, x, z), \quad \tilde{w}_\varepsilon^{(k)}(t, x, z) = w_0(t, x) + \varepsilon^2 w_1^{(k)}(t, x, z),$$

and the corresponding errors with respect to the complete three-dimensional solution, $e_\varepsilon^{(k)}$ and $f_\varepsilon^{(k)}$, by

$$e_\varepsilon^{(k)} := u_\varepsilon^{(k)} - \tilde{u}_\varepsilon^{(k)}, \quad f_\varepsilon^{(k)} := w_\varepsilon^{(k)} - \tilde{w}_\varepsilon^{(k)}. \quad (31)$$

4.3 Error estimate for the one-layer model

We consider some bounded initial data $u^0(x) \in L^\infty(\omega)$, $w^0(x) \in [L^\infty(\omega)]^m$, we define the functions $u_\varepsilon^{(k)}$ and $w_\varepsilon^{(k)}$ solutions to the original three-dimensional system of equations (12) and (13) with the boundary and interface conditions (14) to (16) and the initial condition $u_\varepsilon^{(k)}(0, x, z) = u^0(x)$ and $w_\varepsilon^{(k)}(0, x, z) = w^0(x)$ for $(x, z) \in \Omega^{(k)}$, $k = 1, 2$.

In Theorem 1 below, we prove that the errors $e_\varepsilon^{(k)} := u_\varepsilon^{(k)} - \tilde{u}_\varepsilon^{(k)}$ and $f_\varepsilon^{(k)} := w_\varepsilon^{(k)} - \tilde{w}_\varepsilon^{(k)}$ are of order ε^3 in $L^2(\Omega^{(k)})$ for all time $t > 0$, and in $L^2(0, t; H^1(\Omega^{(k)}))$ also for all time $t > 0$.

Theorem 1 (Error estimates for the one-layer model) *Assume that the functions f and g are $C^2(\mathbb{R} \times \mathbb{R}^m)$; and that the solutions (u_0, w_0) and $(u_1^{(k)}, w_1^{(k)})$ are bounded in $\mathbb{R} \times \mathbb{R}^m$, uniformly in time; and that the solutions $(u_\varepsilon^{(k)}, w_\varepsilon^{(k)})_{\varepsilon > 0}$ are bounded uniformly with respect to ε and time $t > 0$. Specifically, we require that there exists $M > 0$ such that, for all $t > 0$ and $(x, z) \in \Omega^{(k)}$ ($k = 1, 2$),*

$$|(u_0(t, x), w_0(t, x))| \leq M, \quad \left| \left(u_1^{(k)}(t, x, z), w_1^{(k)}(t, x, z) \right) \right| \leq M,$$

and for all $\varepsilon > 0$, for all $t > 0$ and $(x, z) \in \Omega^{(k)}$ ($k = 1, 2$),

$$\left| \left(u_\varepsilon^{(k)}(t, x, z), w_\varepsilon^{(k)}(t, x, z) \right) \right| \leq 2M.$$

We now define the functions $\psi^{(1)}(t, x, z) = -\int_z^1 \phi^{(1)}(t, x, \zeta) d\zeta$ and $\psi^{(2)}(t, x, z) = \int_{-1}^z \phi^{(2)}(t, x, \zeta) d\zeta$ where the functions $\phi^{(k)}$ are given by equation (36) below. We assume that $d_1(s) := \bar{\sigma} \sum_{k=1,2} \|\psi^{(k)}(s)\|_{L^2(\Omega^{(k)})}^2 ds$ belongs to $L^1_{loc}(\mathbb{R}^+)$.

Then, we have the following estimates, for all $0 < \varepsilon \leq 1$, for all $t > 0$, and for $k = 1, 2$,

$$\left\| e_\varepsilon^{(k)}(t) \right\|_{L^2(\Omega^{(k)})} \leq \varepsilon^3 k_0(t) \exp\left(\frac{c_0}{2}t\right), \quad (32)$$

$$\left\| f_\varepsilon^{(k)}(t) \right\|_{L^2(\Omega^{(k)})} \leq \varepsilon^3 k_0(t) \exp\left(\frac{c_0}{2}t\right), \quad (33)$$

$$\left\| \nabla_x e_\varepsilon^{(k)} \right\|_{L^2(0,t; L^2(\Omega^{(k)}))} \leq \varepsilon^3 k_1(t) \exp\left(\frac{c_0}{2}t\right), \quad (34)$$

$$\left\| \partial_z e_\varepsilon^{(k)} \right\|_{L^2(0,t; L^2(\Omega^{(k)}))} \leq \sqrt{2}\varepsilon^4 k_1(t) \exp\left(\frac{c_0}{2}t\right), \quad (35)$$

with $k_0(t) = \frac{1}{\sqrt{\alpha}} \left(\frac{d_0}{c_0} + \int_0^t d_1(s) ds \right)^{1/2}$ and $k_1(t) = \left(\frac{d_0}{2\sigma c_0} (1 + c_0 t) + \frac{1}{\sigma} \int_0^t d_1(s) ds \right)^{1/2}$ and where $c_0 = 2\Lambda_0 + \frac{1}{2}\Lambda_1$ and $d_0 = \alpha\Lambda_1 M^4 |\omega|$. The constant Λ_0 and Λ_1 are the maximum norm of the gradient and Hessian of the function $(u, w) \mapsto (\beta f(u, w), g(u, w))$ in the compact set $\{(u, w) : |(u, w)| \leq 2M\} \subset \mathbb{R} \times \mathbb{R}^m$.

Along the proofs of Theorems 1, 2, and 3, we will need the following technical lemmas.

Lemma 1 Consider a function $h : u \in \mathbb{R}^p \mapsto h(u) \in \mathbb{R}^q$ of class C^2 in \mathbb{R}^p . For any u_0 and u in \mathbb{R}^p , we define $I_h(u_0, u) := h(u) - h(u_0) - \nabla h(u_0) \cdot (u - u_0) = \int_0^1 (1-t) \nabla^2 h(tu + (1-t)u_0) (u - u_0) \cdot (u - u_0) dt$, the integral remainder in the Taylor expansion. We have

$$|u_0| \leq M \text{ and } |u| \leq M \Rightarrow |I_h(u_0, u)| \leq \frac{1}{2} \sup_{|u| \leq M} |\nabla^2 h(u)| |u - u_0|^2$$

where $\nabla^2 h$ is the Hessian matrix of h .

Lemma 2 Consider a function $u : z \in [a, b] \mapsto u(z) \in \mathbb{R}$ of class C^1 in $[a, b]$ with $b - a = 1$.

$$\forall z \in [a, b], \quad |\bar{u} - u(z)| \leq \int_a^b |u'(t)| dt$$

where $\bar{\cdot}$ denotes the integral on $[a, b]$.

Lemma 3 (Gronwall) Suppose that $y(t) \geq 0$ is a C^1 function of $t > 0$, and $h(t) \geq 0$, $d_1(t) \geq 0$ are functions in $L^1_{loc}((0, +\infty))$. Consider some constants $c_0 > 0$ and $d_0 > 0$. For all $\varepsilon > 0$, if

$$y'(t) + h(t) \leq c_0 y(t) + \varepsilon^6 (d_0 + d_1(t)),$$

for all $t > 0$, and $y(0) = 0$, then

$$y(t) \leq \varepsilon^6 \left(\frac{d_0}{c_0} + \int_0^t d_1(s) ds \right) \exp(c_0 t),$$

$$\int_0^t h(s) ds \leq \varepsilon^6 \left(\frac{d_0}{c_0} (1 + c_0 t) + 2 \int_0^t d_1(s) ds \right) \exp(c_0 t).$$

Proof of lemmas 1 to 3. Lemmas 1 and 2 are straightforward derivations of Taylor expansions with integral remainders. Lemma 3 is a direct consequence of the usual inequality of Gronwall $y(t) \leq \exp(c_0 t) y(0) + \int_0^t \varepsilon^6 (d_0 + d_1(s)) \exp(c_0(t-s)) ds$. We obtain the first inequality because $y(0) = 0$ and $\int_0^t \exp(c_0(t-s)) ds \leq \frac{1}{c_0} \exp(c_0 t)$ and $\exp(c_0(t-s)) \leq \exp(c_0 t)$. Hence, we integrate this inequality to obtain $\int_0^t y(s) ds \leq \varepsilon^6 \left(\frac{d_0}{c_0} + \int_0^t d_1(s) ds \right) \frac{1}{c_0} \exp(c_0 t)$. The final result is a direct consequence of the integrated inequality: $\int_0^t h(s) ds \leq c_0 \int_0^t y(s) ds + \varepsilon^6 (d_0 t + \int_0^t d_1(s) ds)$.

Proof of Theorem 1. We define the functions $\phi^{(k)}$ on $(0, +\infty) \times \Omega^{(k)}$ for $k = 1, 2$ by

$$\sigma_3^{(k)} \phi^{(k)} = \alpha \left(\partial_t u_1^{(k)} + \beta \nabla f(u_0, w_0) \cdot (u_1^{(k)}, w_1^{(k)}) \right) - \operatorname{div}_x \left(\sigma^{(k)} \nabla_x u_1^{(k)} \right). \quad (36)$$

The functions $\phi^{(k)}$ play the role of the functions $\partial_{zz} u_2^{(k)}$ in the formal expansion of the previous section. The functions $\psi^{(1)}(t, x, z) = -\int_z^1 \phi^{(1)}(t, x, \zeta) d\zeta$ and $\psi^{(2)}(t, x, z) = \int_{-1}^z \phi^{(2)}(t, x, \zeta) d\zeta$ are such that $\psi^{(1)}(\cdot, 1) = 0$, $\psi^{(2)}(\cdot, -1) = 0$, and $\partial_z \psi^{(k)} = \phi^{(k)}$ for $k = 1, 2$. We average the equation (36) in the total thickness; use the result from remark 3 and the tricks used to establish equation (27) to prove that, for all $t > 0$ and a.e. $x \in \omega$,

$$\begin{aligned} & \frac{1}{2} \left(-\sigma_3^{(1)} \psi^{(1)}(\cdot, 0) + \sigma_3^{(2)} \psi^{(2)}(\cdot, 0) \right) \\ &= \alpha \left(\partial_t \bar{u}_1 + \beta \nabla f(u_0, w_0) \cdot (\bar{u}_1, \bar{w}_1) \right) - \operatorname{div}_x \left(\sigma^m \nabla_x \bar{u}_1 \right) - \operatorname{div}_x \left(\frac{1}{3\sigma_3^h} \sigma^d \nabla_x b \right) = 0 \end{aligned}$$

because \bar{u}_1 and \bar{w}_1 are solutions to the system (27), (28). It shows that $\sigma_3^{(1)} \psi^{(1)}(\cdot, 0) = \sigma_3^{(2)} \psi^{(2)}(\cdot, 0)$, that is the transmission condition (16).

A linear combination of the equations that define $u_\varepsilon^{(k)}$, $w_\varepsilon^{(k)}$, u_0 , w_0 and $u_1^{(k)}$, $w_1^{(k)}$, and the definition of $\psi^{(k)}$ immediately shows that

$$\begin{aligned} & \alpha \left(\partial_t \left(u_\varepsilon^{(k)} - u_0 - \varepsilon^2 u_1^{(k)} \right) \right. \\ & \quad \left. + \beta \left(f(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) - f(u_0, w_0) - \varepsilon^2 \nabla f(u_0, w_0) \cdot (u_1^{(k)}, w_1^{(k)}) \right) \right) \\ &= \operatorname{div}_x \left(\sigma^{(k)} \nabla_x u_\varepsilon^{(k)} - \sigma^m \nabla_x u_0 - \varepsilon^2 \sigma^{(k)} \nabla_x u_1^{(k)} \right) + \frac{\sigma_3^{(k)}}{\varepsilon^2} \partial_{zz} u_\varepsilon^{(k)} - \varepsilon^2 \sigma_3^{(k)} \partial_z \psi^{(k)}, \end{aligned}$$

and (see also remark 3)

$$\begin{aligned} & \partial_t \left(w_\varepsilon^{(k)} - w_0 - \varepsilon^2 w_1^{(k)} \right) \\ & \quad + \left(g(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) - g(u_0, w_0) - \varepsilon^2 \nabla g(u_0, w_0) \cdot (u_1^{(k)}, w_1^{(k)}) \right) = 0. \end{aligned}$$

Now, remark that

$$\begin{aligned} & \operatorname{div}_x \left(\sigma^{(k)} \nabla_x u_\varepsilon^{(k)} - \sigma^m \nabla_x u_0 - \varepsilon^2 \sigma^{(k)} \nabla_x u_1^{(k)} \right) \\ &= \operatorname{div}_x \left(\sigma^{(k)} \nabla_x \left(u_\varepsilon^{(k)} - u_0 - \varepsilon^2 u_1^{(k)} \right) \right) + \operatorname{div}_x \left((\sigma^{(k)} - \sigma^m) \nabla_x u_0 \right) \\ & \quad = \operatorname{div}_x \left(\sigma^{(k)} \nabla_x e_\varepsilon^{(k)} \right) - \sigma_3^{(k)} \partial_{zz} u_1^{(k)} \end{aligned}$$

because we have $\sigma^{(k)} - \sigma^m = \frac{\sigma^{(1)} - \sigma^{(2)}}{2} = \sigma^d$ if $k = 1$ and $\sigma^{(k)} - \sigma^m = \frac{\sigma^{(2)} - \sigma^{(1)}}{2} = -\sigma^d$ if $k = 2$, so that

$$\operatorname{div}_x \left((\sigma^{(k)} - \sigma^m) \nabla_x u_0 \right) = \begin{cases} b = -\sigma_3^{(1)} \partial_{zz} u_1^{(1)} & \text{if } k = 1, \\ -b = -\sigma_3^{(2)} \partial_{zz} u_1^{(2)} & \text{if } k = 2. \end{cases}$$

Remark at last that $\partial_{zz}u_\varepsilon^{(k)} - \varepsilon^2\partial_{zz}u_1^{(k)} = \partial_{zz}\left(u_\varepsilon^{(k)} - u_0 - \varepsilon^2u_1^{(k)}\right) = \partial_{zz}e_\varepsilon^{(k)}$ because $\partial_{zz}u_0 = 0$. We obtain the following equations on the errors $e_\varepsilon^{(k)}$ and $f_\varepsilon^{(k)}$:

$$\alpha\left(\partial_t e_\varepsilon^{(k)} + \beta E_\varepsilon^{(k)}(f)\right) = \operatorname{div}_x\left(\sigma^{(k)}\nabla_x e_\varepsilon^{(k)}\right) + \frac{\sigma_3^{(k)}}{\varepsilon^2}\partial_{zz}e_\varepsilon^{(k)} - \varepsilon^2\sigma_3^{(k)}\partial_z\psi^{(k)}, \quad (37)$$

$$\partial_t f_\varepsilon^{(k)} + E_\varepsilon^{(k)}(g) = 0, \quad (38)$$

where $E_\varepsilon^{(k)}(f) = f(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) - f(u_0, w_0) - \varepsilon^2\nabla f(u_0, w_0) \cdot (u_1^{(k)}, w_1^{(k)})$ and $E_\varepsilon^{(k)}(g) = g(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) - g(u_0, w_0) - \varepsilon^2\nabla g(u_0, w_0) \cdot (u_1^{(k)}, w_1^{(k)})$. For $k = 1, 2$, we multiply the first equation by $e_\varepsilon^{(k)}$ and the second by $\alpha f_\varepsilon^{(k)}$, integrate on $\Omega^{(k)}$ and add the resulting equations in order to obtain the following energy estimate:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}y(t) + \sum_{k=1,2}\int_{\Omega^{(k)}}\left(\underline{\sigma}\left|\nabla_x e_\varepsilon^{(k)}\right|^2 + \frac{\sigma_3^{(k)}}{\varepsilon^2}\left|\partial_z e_\varepsilon^{(k)}\right|^2\right)dzdx \\ \leq \sum_{k=1,2}\int_{\Omega^{(k)}}\alpha\left|\beta E_\varepsilon^{(k)}(f)e_\varepsilon^{(k)} + E_\varepsilon^{(k)}(g)\cdot f_\varepsilon^{(k)}\right|dzdx \\ + \sum_{k=1,2}\int_{\Omega^{(k)}}\sigma_3^{(k)}\left|\varepsilon^2\psi^{(k)}\partial_z e_\varepsilon^{(k)}\right|dzdx, \end{aligned}$$

where $y(t) = \alpha\sum_{k=1,2}\left(\|e_\varepsilon^{(k)}\|_{L^2(\Omega^{(k)})}^2 + \|f_\varepsilon^{(k)}\|_{[L^2(\Omega^{(k)})]^m}^2\right)$ is the total L^2 norm of the error. Remark that $e_\varepsilon^{(k)}$ and $\psi^{(k)}$ verify the boundary and transmission conditions (14) to (16). Consequently, the boundary terms vanish and the interface terms cancel each other in the energy estimate. Specifically, we have that $\sigma_\varepsilon^{(k)}\nabla_x e_\varepsilon^{(k)} \cdot n = 0$ for $x \in \partial\omega$ and $z \in (0, 1)$ (for $k = 1$) or $z \in (-1, 0)$ (for $k = 2$); $\sigma_3^{(k)}\partial_z e_\varepsilon^{(k)} = 0$ and $\psi^{(k)} = 0$ for $x \in \omega$ and $z = 1$ (for $k = 1$) or $z = -1$ (for $k = 2$). The terms on the interface cancel each other because $\sigma_3^{(1)}\partial_z e_\varepsilon^{(1)} = \sigma_3^{(2)}\partial_z e_\varepsilon^{(2)}$, $\sigma_3^{(1)}\psi^{(1)} = \sigma_3^{(2)}\psi^{(2)}$ and $e_\varepsilon^{(1)} = e_\varepsilon^{(2)}$ for $x \in \omega$ and $z = 0$ (just remark that u_0 is independent of z , recall the expression (24) of $u_1^{(k)}$ and the transmission condition for $u_\varepsilon^{(k)}$).

Using the inequality $\left|\varepsilon^2\psi^{(k)}\partial_z e_\varepsilon^{(k)}\right| \leq \frac{\varepsilon^6|\psi^{(k)}|^2}{2} + \frac{|\partial_z e_\varepsilon^{(k)}|^2}{2\varepsilon^2}$, we obtain the inequality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}y(t) + \underline{\sigma}\sum_{k=1,2}\int_{\Omega^{(k)}}\left(\left|\nabla_x e_\varepsilon^{(k)}\right|^2 + \frac{1}{2\varepsilon^2}\left|\partial_z e_\varepsilon^{(k)}\right|^2\right)dzdx \\ \leq \alpha\sum_{k=1,2}\int_{\Omega^{(k)}}\left|E_\varepsilon^{(k)}(\beta f, g)\cdot\left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)}\right)\right|dzdx + \frac{\bar{\sigma}}{2}\varepsilon^6\sum_{k=1,2}\int_{\Omega^{(k)}}\left|\psi^{(k)}\right|^2dzdx, \end{aligned}$$

where $E_\varepsilon^{(k)}(\beta f, g) = \left(\beta E_\varepsilon^{(k)}(f), E_\varepsilon^{(k)}(g) \right)$. We now define the function $\mathbf{f} : (u, w) \in \mathbb{R} \times \mathbb{R}^m \mapsto (\beta f(u, w), g(u, w)) \in \mathbb{R} \times \mathbb{R}^m$, and use lemma 1, so that, for $k = 1, 2$,

$$\begin{aligned} \left| E_\varepsilon^{(k)}(\beta f, g) \right| &= \left| \mathbf{f}(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) - \mathbf{f}(u_0, w_0) - \varepsilon^2 \nabla \mathbf{f}(u_0, w_0) \cdot (u_1^{(k)}, w_1^{(k)}) \right| \\ &\leq \left| \mathbf{f}(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) - \mathbf{f}(\tilde{u}_\varepsilon^{(k)}, \tilde{w}_\varepsilon^{(k)}) \right| + \left| \mathbf{f}(u_0, w_0; \tilde{u}_\varepsilon^{(k)}, \tilde{w}_\varepsilon^{(k)}) \right| \\ &\leq \Lambda_0 \left| (e_\varepsilon^{(k)}, f_\varepsilon^{(k)}) \right| + \frac{1}{2} \Lambda_1 \varepsilon^4 \left| (u_1^{(k)}, w_1^{(k)}) \right|^2 \leq \Lambda_0 \left| (e_\varepsilon^{(k)}, f_\varepsilon^{(k)}) \right| + \frac{1}{2} \Lambda_1 \varepsilon^4 M^2 \quad (39) \end{aligned}$$

where $\Lambda_0 = \sup_{|(u,v)| \leq 2M} |\nabla \mathbf{f}(u, v)|$ and $\Lambda_1 = \sup_{|(u,v)| \leq M} |\nabla^2 \mathbf{f}(u, v)|$. Afterwards, remark that $\left| (\tilde{u}_\varepsilon^{(k)}, \tilde{w}_\varepsilon^{(k)}) \right| \leq M(1 + \varepsilon^2) \leq 2M$ for $\varepsilon \leq 1$, because of the assumptions on $(u_\varepsilon^{(k)}, w_\varepsilon^{(k)})$, (u_0, w_0) and $(u_1^{(k)}, w_1^{(k)})$. Now, we can write the last estimate:

$$\begin{aligned} &\alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \left| E_\varepsilon^{(k)}(\beta f, g) \cdot (e_\varepsilon^{(k)}, f_\varepsilon^{(k)}) \right| dz dx \\ &\leq \alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \Lambda_0 \left| (e_\varepsilon^{(k)}, f_\varepsilon^{(k)}) \right|^2 dx dz + \alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \frac{1}{2} \Lambda_1 \varepsilon^4 \left| (e_\varepsilon^{(k)}, f_\varepsilon^{(k)}) \right| M^2 \\ &\leq \left(\Lambda_0 + \frac{1}{4} \Lambda_1 \right) y(t) + \frac{1}{2} \alpha \Lambda_1 \varepsilon^8 M^4 |\omega| \end{aligned}$$

with the inequality of Young $\varepsilon^4 \left| (e_\varepsilon^{(k)}, f_\varepsilon^{(k)}) \right| M^2 \leq \frac{1}{2} \left(\left| (e_\varepsilon^{(k)}, f_\varepsilon^{(k)}) \right|^2 + \varepsilon^8 M^4 \right)$ and because $\sum_{k=1,2} |\Omega^{(k)}| = 2|\omega|$. The final energy estimate reads:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} y(t) + \underline{\sigma} \sum_{k=1,2} \int_{\Omega^{(k)}} \left(\left| \nabla_x e_\varepsilon^{(k)} \right|^2 + \frac{1}{2\varepsilon^2} \left| \partial_z e_\varepsilon^{(k)} \right|^2 \right) dz dx \\ &\leq \left(\Lambda_0 + \frac{1}{4} \Lambda_1 \right) y(t) + \frac{1}{2} \alpha \Lambda_1 \varepsilon^8 M^4 |\omega| + \frac{\bar{\sigma}}{2} \varepsilon^6 \sum_{k=1,2} \|\psi^{(k)}(t)\|_{L^2(\Omega^{(k)})}^2. \end{aligned}$$

We assume that $\varepsilon \leq 1$, so that $\varepsilon^8 \leq \varepsilon^6$ and simplify the inequality by noting $c_0 = 2\Lambda_0 + \frac{1}{2}\Lambda_1$, $d_0 = \alpha \Lambda_1 M^4 |\omega|$ and $d_1(t) = \bar{\sigma} \sum_{k=1,2} \|\psi^{(k)}(t)\|_{L^2(\Omega^{(k)})}^2$. Hence, we have the inequality $y'(t) + h(t) \leq c_0 y(t) + (d_0 + d_1(t)) \varepsilon^6$ for any $t \geq 0$ (and for $h(t) = \underline{\sigma} \sum_{k=1,2} \int_{\Omega^{(k)}} \left(\left| \nabla_x e_\varepsilon^{(k)} \right|^2 + \frac{1}{2\varepsilon^2} \left| \partial_z e_\varepsilon^{(k)} \right|^2 \right) dz dx$), and can apply lemma 3 to complete the proof.

Remark 4 An asymptotic expansion of the initial data could be carried out without additional difficulties, if the initial data were not constant along z . The final error estimate is of order ε^3 , whereas a continuation of the formal study in the beginning of the section 4.1 would show that the coefficient of order ε^3 vanishes, and that the convergence order is of order ε^4 .

Theorem 2 (Additional error estimates) *Under the assumptions of Theorem 1, and with the following additional assumptions: for all $t > 0$ and a.e. $x \in \omega$, the functions $z \mapsto u_1^{(k)}(t, x, z)$ and $z \mapsto w_1^{(k)}(t, x, z)$ belong to $H^2(0, 1)$ if $k = 1$ and $H^2(-1, 0)$ if $k = 2$, and for all $t > 0$ and a.e. $(x, z) \in \Omega^{(k)}$,*

$$\left| \left(\partial_z u_1^{(k)}(t, x, z), \partial_z w_1^{(k)}(t, x, z) \right) \right| \leq M, \quad \left| \left(\partial_z u_\varepsilon^{(k)}(t, x, z), \partial_z w_\varepsilon^{(k)}(t, x, z) \right) \right| \leq 2M,$$

with the bound M from Theorem 1. Assume that $\sum_{k=1,2} \|\partial_z \Psi^{(k)}(s)\|_{L^2(\Omega^{(k)})}^2$ belongs to $L_{loc}^1(\mathbb{R}^+)$. Then, we have the following estimates, for all $0 < \varepsilon \leq 1$, for all $t > 0$, and for $k = 1, 2$,

$$\left\| \partial_z e_\varepsilon^{(k)}(t) \right\|_{L^2(\Omega^{(k)})} \leq \varepsilon^3 k_2(t) \exp\left(\frac{c_1}{2}t\right), \quad (40)$$

$$\left\| \partial_z f_\varepsilon^{(k)}(t) \right\|_{L^2(\Omega^{(k)})} \leq \varepsilon^3 k_2(t) \exp\left(\frac{c_1}{2}t\right), \quad (41)$$

$$\left\| \nabla_x \partial_z e_\varepsilon^{(k)} \right\|_{L^2(0,t;L^2(\Omega^{(k)}))} \leq \varepsilon^3 k_3(t) \exp\left(\frac{c_1}{2}t\right), \quad (42)$$

$$\left\| \partial_{zz} e_\varepsilon^{(k)}(t) \right\|_{L^2(0,t;L^2(\Omega^{(k)}))} \leq \sqrt{2} \varepsilon^4 k_3(t) \exp\left(\frac{c_1}{2}t\right), \quad (43)$$

with $k_2(t) = \frac{1}{\sqrt{\alpha\bar{\sigma}}} \left(\frac{d_2}{c_1} + \int_0^t d_3(s) ds \right)^{1/2}$, $k_3(t) = \frac{1}{\bar{\sigma}\sqrt{2}} \left(\frac{d_2}{c_1} (1 + c_1 t) + 2 \int_0^t d_3(s) ds \right)^{1/2}$, and where $c_1 = 2(\Lambda_0 + \Lambda_1)$, $d_2 = 2\alpha\bar{\sigma}\Lambda_1 M^4 |\omega| = 2\bar{\sigma}d_0$, and

$$d_3(t) = \alpha\Lambda_1 \bar{\sigma} M^2 k_0(t)^2 \exp(c_0 t) + \bar{\sigma} \sum_{k=1,2} \|\partial_z \Psi^{(k)}(t)\|_{L^2(\Omega^{(k)})}^2.$$

We use Lemma 4 below, which proof is reported to the appendix.

Lemma 4 *For all $t > 0$, and a.e. $x \in \omega$, the functions $w_\varepsilon^{(k)}$ verify exactly the transmission condition (16):*

$$w_\varepsilon^{(1)}(t, x, 0) = w_\varepsilon^{(2)}(t, x, 0), \quad \text{and} \quad \sigma_3^{(1)} \partial_z w_\varepsilon^{(1)}(t, x, 0) = \sigma_3^{(2)} \partial_z w_\varepsilon^{(2)}(t, x, 0).$$

Proof of Theorem 2. We test the equations (37) and (38) on the errors against the functions $-\sigma_3^{(k)} \partial_{zz} e_\varepsilon^{(k)}$ and $-\alpha\sigma_3^{(k)} \partial_{zz} f_\varepsilon^{(k)}$. In order to complete the integration by parts in z , note that

$$-\sum_{k=1,2} \int_{\Omega^{(k)}} \sigma_3^{(k)} \partial_{zz} e_\varepsilon^{(k)} \partial_t e_\varepsilon^{(k)} dx dz = \frac{1}{2} \frac{d}{dt} \sum_{k=1,2} \int_{\Omega^{(k)}} \sigma_3^{(k)} \left| \partial_z e_\varepsilon^{(k)} \right|^2,$$

and after integration by parts in x and z ,

$$\begin{aligned} & -\sum_{k=1,2} \int_{\Omega^{(k)}} \operatorname{div}_x \left(\sigma^{(k)} \nabla_x e_\varepsilon^{(k)} \right) \sigma_3^{(k)} \partial_{zz} e_\varepsilon^{(k)} dz dx \\ & = -\sum_{k=1,2} \int_{\Omega^{(k)}} \left(\sigma^{(k)} \nabla_x \partial_z e_\varepsilon^{(k)} \right) \cdot \left(\sigma_3^{(k)} \nabla_x \partial_z e_\varepsilon^{(k)} \right) dz dx, \end{aligned}$$

both because $e_\varepsilon^{(k)}$ ($k = 1, 2$) verify the boundary and transmission conditions (14) to (16), so that we finally have:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} y(t) + \sum_{k=1,2} \int_{\Omega^{(k)}} \left(\underline{\sigma}^2 \left| \nabla_x \partial_z e_\varepsilon^{(k)} \right|^2 + \frac{\sigma_3^{(k)2}}{\varepsilon^2} \left| \partial_{zz} e_\varepsilon^{(k)} \right|^2 \right) dz dx \\ \leq \alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \sigma_3^{(k)} E_\varepsilon^{(k)}(\beta f, g) \cdot \left(\partial_{zz} e_\varepsilon^{(k)}, \partial_{zz} f_\varepsilon^{(k)} \right) dx dz \\ + \varepsilon^2 \sum_{k=1,2} \int_{\Omega^{(k)}} \sigma_3^{(k)2} \partial_z \psi^{(k)} \partial_{zz} e_\varepsilon^{(k)} dx dz \end{aligned}$$

with now $y(t) = \alpha \sum_{k=1,2} \sigma_3^{(k)} \left(\|\partial_z e_\varepsilon^{(k)}\|_{L^2(\Omega^{(k)})}^2 + \|\partial_z f_\varepsilon^{(k)}\|_{L^2(\Omega^{(k)})}^2 \right)$. After integration by parts in the first term of the right-hand side of this inequality, and the inequality of Young for the second one, we find that:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} y(t) + \underline{\sigma}^2 \sum_{k=1,2} \int_{\Omega^{(k)}} \left(\left| \nabla_x \partial_z e_\varepsilon^{(k)} \right|^2 + \frac{1}{2\varepsilon^2} \left| \partial_{zz} e_\varepsilon^{(k)} \right|^2 \right) dz dx \\ \leq \alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \sigma_3^{(k)} \left| \partial_z E_\varepsilon^{(k)}(\beta f, g) \cdot \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right| dx dz \\ + \frac{\bar{\sigma}^2}{2} \varepsilon^6 \sum_{k=1,2} \int_{\Omega^{(k)}} \left| \partial_z \psi^{(k)} \right|^2 dx dz. \end{aligned}$$

Afterwards, we remark that

$$\begin{aligned} \partial_z \left(E_\varepsilon^{(k)}(\beta f, g) \right) = \nabla \mathbf{f}(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) \cdot \left(\partial_z u_\varepsilon^{(k)}, \partial_z w_\varepsilon^{(k)} \right) - \\ \varepsilon^2 \nabla \mathbf{f}(u_0, w_0) \cdot \left(\partial_z u_1^{(k)}, \partial_z w_1^{(k)} \right), \end{aligned}$$

and that $\left(\partial_z u_\varepsilon^{(k)}, \partial_z w_\varepsilon^{(k)} \right) = \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) + \varepsilon^2 \left(\partial_z u_1^{(k)}, \partial_z w_1^{(k)} \right)$. Then

$$\begin{aligned} \left| \partial_z \left(E_\varepsilon^{(k)}(\beta f, g) \right) \right| &\leq \left| \nabla \mathbf{f}(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) \right| \left| \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right| \\ &\quad + \varepsilon^2 \left| \nabla \mathbf{f}(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) - \nabla \mathbf{f}(u_0, w_0) \right| \left| \left(\partial_z u_1^{(k)}, \partial_z w_1^{(k)} \right) \right| \\ &\leq \Lambda_0 \left| \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right| \\ &\quad + \varepsilon^2 \Lambda_1 \left| \left(\partial_z u_1^{(k)}, \partial_z w_1^{(k)} \right) \right| \left(\left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right| + \varepsilon^2 \left| \left(u_1^{(k)}, w_1^{(k)} \right) \right| \right) \end{aligned}$$

because $(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) - (u_0, w_0) = \varepsilon^2(u_1^{(k)}, w_1^{(k)}) + (e_\varepsilon^{(k)}, f_\varepsilon^{(k)})$. We complete the estimate by writing:

$$\begin{aligned}
& \alpha \sum_{k=1,2} \sigma_3^{(k)} \int_{\Omega^{(k)}} \left| \partial_z E_\varepsilon^{(k)}(\beta f, g) \cdot \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right| dz dx \\
& \leq \alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \Lambda_0 \sigma_3^{(k)} \left| \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right|^2 dx dz \\
& \quad + \alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \sigma_3^{(k)} \Lambda_1 \varepsilon^2 M \left(\left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right| + \varepsilon^2 M \right) \left| \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right| \\
& \leq \alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \Lambda_0 \sigma_3^{(k)} \left| \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right|^2 dx dz \\
& \quad + \alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \sigma_3^{(k)} \Lambda_1 \left(\varepsilon^2 M \left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right| \right) \left| \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right| \\
& \quad \quad \quad + \varepsilon^4 M^2 \left| \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right| \\
& \leq \Lambda_0 y(t) \\
& \quad + \alpha \sum_{k=1,2} \sigma_3^{(k)} \Lambda_1 \frac{1}{2} \int_{\Omega^{(k)}} \left(\varepsilon^4 M^2 \left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right|^2 + \varepsilon^8 M^4 + 2 \left| \left(\partial_z e_\varepsilon^{(k)}, \partial_z f_\varepsilon^{(k)} \right) \right|^2 \right) \\
& \leq (\Lambda_0 + \Lambda_1) y(t) + \frac{1}{2} \Lambda_1 \bar{\sigma} M^2 \varepsilon^{10} \alpha k_0(t)^2 \exp(c_0 t) + \alpha \bar{\sigma} \Lambda_1 M^4 |\omega| \varepsilon^8,
\end{aligned}$$

where we use the bounds $\left| \left(\partial_z u_1^{(k)}, \partial_z w_1^{(k)} \right) \right| \leq M$ and $\left| \left(u_1^{(k)}, w_1^{(k)} \right) \right| \leq M$, some inequalities of Young, the inequality $\alpha \sum_{k=1,2} \int_{\Omega^{(k)}} \left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right|^2 \leq \alpha k_0(t)^2 \exp(c_0 t)$ from the proof of Theorem 1, and the definition of $y(t)$. The final estimate reads:

$$\begin{aligned}
& \frac{d}{dt} y(t) + 2\bar{\sigma}^2 \sum_{k=1,2} \int_{\Omega^{(k)}} \left(\left| \nabla_x \partial_z e_\varepsilon^{(k)} \right|^2 + \frac{1}{2\varepsilon^2} \left| \partial_{zz} e_\varepsilon^{(k)} \right|^2 \right) dz dx \\
& \leq c_1 y(t) + \varepsilon^6 (d_2 + d_3(t))
\end{aligned}$$

where the constants c_1 , d_2 , and the function $d_3(t)$ are given in the statement of Theorem 2. We conclude again with lemma 3.

5 The asymptotic model with two layers

5.1 Equations on the averages through the thickness of the layers of tissue

In this section, we prove that the averages in $z \in (0, 1)$ and $z \in (-1, 0)$ for each of the two layers of the solution $(u_\varepsilon^{(k)}, w_\varepsilon^{(k)})$ verifies a system of two surface monodomain equations, linearly coupled, up to an error of order ε^3 . It is the basis for the definition of our model with two layers. Consequently, unlike in the previous case, the source function is computed from a different value of the transmembrane potential in

each layer, and can have different dynamics in each layer. This is sufficient to trigger complex propagation patterns.

We integrate Equations (31) defining $e_\varepsilon^{(k)}$ and $f_\varepsilon^{(k)}$ for $z \in (0, 1)$ and $z \in (-1, 0)$, use the properties of $u_1^{(k)}$, and find that

$$\bar{u}_\varepsilon^{(k)}(t, x) = u_0(t, x) + \varepsilon^2 \left(c - (-1)^k \frac{1}{3} \frac{b}{\sigma_3^{(k)}} \right) + \bar{e}_\varepsilon^{(k)}(t, x, z)$$

We recall that $\bar{\cdot}$ denotes the integration for $z \in (0, 1)$ for $k = 1$ and $z \in (-1, 0)$ for $k = 2$. In particular, we have that

$$\frac{\bar{u}_\varepsilon^{(1)} - \bar{u}_\varepsilon^{(2)}}{2} = \varepsilon^2 \frac{1}{3} \frac{b}{\sigma_3^h} + \frac{\bar{e}_\varepsilon^{(1)} - \bar{e}_\varepsilon^{(2)}}{2}. \quad (44)$$

We derive (31) in z and obtain

$$\partial_z u_\varepsilon^{(k)} = \varepsilon^2 \partial_z u_1^{(k)} + \partial_z e_\varepsilon^{(k)} = \varepsilon^2 \frac{b}{\sigma_3^{(k)}} \left(1 + (-1)^k z \right) + \partial_z e_\varepsilon^{(k)}.$$

Now recall that $\sigma_3^{(1)} \partial_z u_\varepsilon^{(1)}(t, x, 0) = \sigma_3^{(2)} \partial_z u_\varepsilon^{(2)}(t, x, 0)$ according to the transmission condition (16), and we can define the residual

$$Su_\varepsilon(t, x) := \sigma_3^{(1)} \partial_z e_\varepsilon^{(1)}(t, x, 0) = \sigma_3^{(2)} \partial_z e_\varepsilon^{(2)}(t, x, 0) \quad (45)$$

so that

$$\sigma_3^{(1)} \partial_z u_\varepsilon^{(1)}(t, x, 0) = \sigma_3^{(2)} \partial_z u_\varepsilon^{(2)}(t, x, 0) = \varepsilon^2 b(t, x) + Su_\varepsilon(t, x).$$

Now, we are ready to integrate the monodomain equations (12) and (13) for $z \in (0, 1)$ and $z \in (-1, 0)$. This yields:

$$\begin{aligned} \alpha \left(\partial_t \bar{u}_\varepsilon^{(k)} + \beta f \left(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)} \right) \right) &= \operatorname{div}_x \left(\sigma^{(k)} \nabla_x \bar{u}_\varepsilon^{(k)} \right) + (-1)^k \left(b + \frac{1}{\varepsilon^2} Su_\varepsilon \right), \\ \partial_t \bar{w}_\varepsilon^{(k)} + g \left(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)} \right) &= 0, \end{aligned}$$

Like in the proof of Theorem 1, we denote by $\bar{E}_\varepsilon^{(k)}(f) := f \left(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)} \right) - \overline{f \left(u_\varepsilon^{(k)}, w_\varepsilon^{(k)} \right)}$ and $\bar{E}_\varepsilon^{(k)}(g) := g \left(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)} \right) - \overline{g \left(u_\varepsilon^{(k)}, w_\varepsilon^{(k)} \right)}$ the nonlinear errors. Afterwards, we substitute b according to Equation (44), so that the averages $(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)})$ are solutions to the monodomain system of equations, for $k = 1, 2$

$$\begin{aligned} \alpha \left(\partial_t \bar{u}_\varepsilon^{(k)} + \beta f \left(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)} \right) \right) &= \operatorname{div}_x \left(\sigma^{(k)} \nabla_x \bar{u}_\varepsilon^{(k)} \right) \\ &+ (-1)^k \left(\frac{3}{2} \sigma_3^h \frac{\bar{u}_\varepsilon^{(1)} - \bar{u}_\varepsilon^{(2)}}{\varepsilon^2} - \frac{3}{2} \sigma_3^h \frac{\bar{e}_\varepsilon^{(1)} - \bar{e}_\varepsilon^{(2)}}{\varepsilon^2} + \frac{1}{\varepsilon^2} Su_\varepsilon \right) + \alpha \beta \bar{E}_\varepsilon^{(k)}(f), \quad (46) \end{aligned}$$

and

$$\partial_t \bar{w}_\varepsilon^{(k)} + g \left(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)} \right) = \bar{E}_\varepsilon^{(k)}(g). \quad (47)$$

5.2 Definition of the two layers model

Consider some initial data $\hat{u}_0^{(k)}(x)$ and $\hat{w}_0^{(k)}(x)$ defined on ω . The two layers model is given by the following coupled systems of monodomain equations, for $k = 1, 2$:

$$\alpha \left(\partial_t \hat{u}_\varepsilon^{(k)} + \beta f \left(\hat{u}_\varepsilon^{(k)}, \hat{w}_\varepsilon^{(k)} \right) \right) = \operatorname{div}_x \left(\sigma^{(k)} \nabla_x \hat{u}_\varepsilon^{(k)} \right) + (-1)^k \frac{3}{2} \sigma_3^h \frac{\hat{u}_\varepsilon^{(1)} - \hat{u}_\varepsilon^{(2)}}{\varepsilon^2}, \quad (48)$$

$$\partial_t \hat{w}_\varepsilon^{(k)} + g \left(\hat{u}_\varepsilon^{(k)}, \hat{w}_\varepsilon^{(k)} \right) = 0, \quad (49)$$

with boundary conditions $\sigma^{(k)} \nabla_x \hat{u}^{(k)} \cdot n = 0$ on $\partial\omega$ for $t > 0$ and initial condition $\hat{u}^{(k)}(0, x) = \hat{u}_0^{(k)}(x)$ and $\hat{w}^{(k)}(0, x) = \hat{w}_0^{(k)}(x)$. It is a perturbation of the original problem (46)-(47). We are going to show that this perturbation is actually small.

5.3 Error estimate for the two layer model

We estimate the error between the solution $(\hat{u}_\varepsilon^{(k)}, \hat{w}_\varepsilon^{(k)})$ of the equations (48) and (49), and the averages, per layer, $(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)})$ of the solutions of the equations (12) and (13).

Theorem 3 (Error estimates for the two-layers model) *Assume that, for $k = 1, 2$, we have $\hat{u}_0^{(k)}(x) = u^0(x)$, and $\hat{w}_0^{(k)}(x) = w^0(x)$. Under the assumptions of Theorems 1 and 2, we have the following estimates: for all $0 < \varepsilon \leq 1$, for all $t > 0$, for $k = 1, 2$,*

$$\|\bar{u}_\varepsilon^{(k)}(t) - \hat{u}_\varepsilon^{(k)}(t)\|_{L^2(\Omega^{(k)})} \leq \varepsilon^3 k_4(t) \exp\left(\frac{c_2}{2}t\right), \quad (50)$$

$$\|\bar{w}_\varepsilon^{(k)}(t) - \hat{w}_\varepsilon^{(k)}(t)\|_{[L^2(\Omega^{(k)})]^m} \leq \varepsilon^3 k_4(t) \exp\left(\frac{c_2}{2}t\right), \quad (51)$$

$$\|\nabla_x(\bar{u}_\varepsilon^{(k)} - \hat{u}_\varepsilon^{(k)})\|_{L^2(0,T;L^2(\Omega))} \leq \varepsilon^3 k_5(t) \exp\left(\frac{c_2}{2}t\right), \quad (52)$$

with $k_4(t) = \frac{1}{\sqrt{\alpha}} \left(\frac{d_4}{c_2} + \int_0^t d_5(s) ds \right)^{1/2}$, $k_5(t) = \frac{1}{\sqrt{2\sigma}} \left(\frac{d_4}{c_2} (1 + c_2 t) + 2 \int_0^t d_5(s) ds \right)^{1/2}$, and the constants $c_2 = 1 + 2\Lambda_0$, $d_4 = 4\alpha\Lambda_1^2 M^4 |\omega|$, and

$$d_5(t) = \frac{2\bar{\sigma}^2}{3\sigma_3^h \varepsilon^8} \sum_{k=1,2} \|\partial_{zz} e_\varepsilon^{(k)}\|_{L^2(\Omega^{(k)})}^2 + \frac{6\sigma_3^h}{\varepsilon^2} \sum_{k=1,2} \|\partial_z e_\varepsilon^{(k)}\|_{L^2(\Omega^{(k)})}^2 + 8\Lambda_0^2 \alpha k_0(t)^2 \exp(c_0 t).$$

The function $d_5(t)$ is such that $k_4(t)$ and $k_5(t)$ are bounded uniformly with respect to ε (see inequality (53)).

Proof We follow closely the method presented in the previous proofs. We subtract Equations (48) and (49) to Equations (46) and (47) and obtain, for $k = 1, 2$,

$$\begin{aligned} \alpha \left(\partial_t \hat{e}_\varepsilon^{(k)} + \beta \left(f(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)}) - f(\hat{u}_\varepsilon^{(k)}, \hat{w}_\varepsilon^{(k)}) \right) \right) &= \operatorname{div}_x \left(\sigma^{(k)} \nabla_x \hat{e}_\varepsilon^{(k)} \right) \\ &+ (-1)^k \frac{3}{2} \sigma_3^h \frac{\hat{e}_\varepsilon^{(1)} - \hat{e}_\varepsilon^{(2)}}{\varepsilon^2} + (-1)^k \frac{R u_\varepsilon}{\varepsilon^2} + \alpha \beta \bar{E}_\varepsilon^{(k)}(f), \end{aligned}$$

and

$$\partial_t \hat{f}_\varepsilon^{(k)} + g(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)}) - g(\hat{u}_\varepsilon^{(k)}, \hat{w}_\varepsilon^{(k)}) = \bar{E}_\varepsilon^{(k)}(g),$$

where $Ru_\varepsilon = Su_\varepsilon - \frac{3}{2}\sigma_3^h(\bar{e}_\varepsilon^{(1)} - \bar{e}_\varepsilon^{(2)})$ and with the notations $\hat{e}_\varepsilon^{(k)} = \bar{u}_\varepsilon^{(k)} - \hat{u}_\varepsilon^{(k)}$ and $\hat{f}_\varepsilon^{(k)} = \bar{w}_\varepsilon^{(k)} - \hat{w}_\varepsilon^{(k)}$. We multiply the equations by $\hat{e}_\varepsilon^{(k)}$ and $\alpha \hat{f}_\varepsilon^{(k)}$, sum for $k = 1, 2$, and finally obtain the energy estimate,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} y(t) + \underline{\sigma} \sum_{k=1,2} \int_\omega |\nabla_x \hat{e}_\varepsilon^{(k)}|^2 dx + \frac{3}{2} \int_\omega \sigma_3^h \frac{|\hat{e}_\varepsilon^{(1)} - \hat{e}_\varepsilon^{(2)}|^2}{\varepsilon^2} dx \\ & \leq \alpha \sum_{k=1,2} \int_\omega \hat{E}_\varepsilon^{(k)}(\beta f, g) \cdot (\hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)}) dx + \int_\omega \frac{Ru_\varepsilon}{\varepsilon^2} (\hat{e}_\varepsilon^{(2)} - \hat{e}_\varepsilon^{(1)}) dx \\ & \quad + \alpha \sum_{k=1,2} \int_\omega \bar{E}_\varepsilon^{(k)}(\beta f, g) \cdot (\hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)}) dx \end{aligned}$$

where $y(t) = \alpha \sum_{k=1,2} (\|\hat{e}_\varepsilon^{(k)}\|_{L^2(\omega)}^2 + \|\hat{f}_\varepsilon^{(k)}\|_{L^2(\omega)}^2) = \alpha \sum_{k=1,2} \left\| (\hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)}) \right\|_{L^2(\omega)}^2$ and the nonlinear terms read $\hat{E}_\varepsilon^{(k)}(\beta f, g) = \mathbf{f}(\hat{u}_\varepsilon^{(k)}, \hat{w}_\varepsilon^{(k)}) - \mathbf{f}(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)})$, using the function $\mathbf{f} : (u, w) \in \mathbb{R} \times \mathbb{R}^m \mapsto (\beta f(u, w), g(u, w)) \in \mathbb{R} \times \mathbb{R}^m$. With the inequality of Young, we first remark that

$$\int_\omega \frac{Ru_\varepsilon}{\varepsilon^2} (\hat{e}_\varepsilon^{(2)} - \hat{e}_\varepsilon^{(1)}) dx \leq \frac{1}{2\gamma\varepsilon^2} \int_\omega |Ru_\varepsilon|^2 dx + \frac{\gamma}{2\varepsilon^2} \int_\omega (\hat{e}_\varepsilon^{(2)} - \hat{e}_\varepsilon^{(1)})^2 dx.$$

We take $\gamma = \frac{3}{2}\sigma_3^h$, in order to simplify this term with the left-hand side of the equation, and obtain the estimate:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} y(t) + \underline{\sigma} \sum_{k=1,2} \int_\omega |\nabla_x \hat{e}_\varepsilon^{(k)}|^2 dx + \frac{3}{4} \int_\omega \sigma_3^h \frac{|\hat{e}_\varepsilon^{(1)} - \hat{e}_\varepsilon^{(2)}|^2}{\varepsilon^2} dx \\ & \leq \frac{1}{3\sigma_3^h \varepsilon^2} \int_\omega |Ru_\varepsilon|^2 dx + \alpha \sum_{k=1,2} \int_\omega (\hat{E}_\varepsilon^{(k)}(\beta f, g) + \bar{E}_\varepsilon^{(k)}(\beta f, g)) \cdot (\hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)}) dx \end{aligned}$$

We are going to bound independently each element of the right hand side of this inequality. Firstly, we have:

$$\begin{aligned} \frac{1}{3\sigma_3^h \varepsilon^2} \int_\omega |Ru_\varepsilon|^2 dx & \leq \frac{1}{3\sigma_3^h \varepsilon^2} \int_\omega \left| Su_\varepsilon - \frac{3}{2}\sigma_3^h (\bar{e}_\varepsilon^{(1)} - \bar{e}_\varepsilon^{(2)}) \right|^2 dx \\ & \leq \frac{2}{3\sigma_3^h \varepsilon^2} \int_\omega |Su_\varepsilon|^2 dx + \frac{3}{2} \int_\omega \sigma_3^h \frac{|\bar{e}_\varepsilon^{(1)} - \bar{e}_\varepsilon^{(2)}|^2}{\varepsilon^2} dx. \end{aligned}$$

Now recall Equation (45) that defines Su_ε and note that, for all $t > 0$, for a.e. $x \in \omega$, we have $\partial_z e^{(1)}(t, x, 0) = -\int_0^1 \partial_{zz} e^{(1)}(t, x, z) dz$ so that

$$\int_\omega |Su_\varepsilon|^2 dx \leq \int_\omega \sigma_3^{(1)2} \left| \int_0^1 \partial_{zz} e_\varepsilon^{(1)} dz \right|^2 \leq \bar{\sigma}^2 \|\partial_{zz} e_\varepsilon^{(1)}\|_{L^2(\Omega^{(1)})}^2.$$

Here, the choice of $e_\varepsilon^{(1)}$ is arbitrary and could be replaced by $e_\varepsilon^{(2)}$ because of the continuity of the flux through $\Sigma = \omega \times \{0\}$. As a consequence, we also have $\int_\omega |Su_\varepsilon|^2 dx \leq \frac{1}{2} \bar{\sigma}^2 \sum_{k=1,2} \|\partial_{zz} e_\varepsilon^{(k)}\|_{L^2(\Omega^{(k)})}^2$. Afterwards, because of the continuity of $e_\varepsilon^{(k)}$ along the interface $\{z=0\}$, we have (using lemma 2):

$$\begin{aligned} & \int_\omega \left| \bar{e}_\varepsilon^{(1)} - \bar{e}_\varepsilon^{(2)} \right|^2 dx \\ & \leq 2 \int_\omega \left(\left| \bar{e}_\varepsilon^{(1)}(t, x) - e_\varepsilon^{(1)}(t, x, 0) \right|^2 + \left| \bar{e}_\varepsilon^{(2)}(t, x) - e_\varepsilon^{(2)}(t, x, 0) \right|^2 \right) dx \\ & \leq 2 \int_\omega \left(\left| \int_0^1 \partial_z e^{(1)} dz \right|^2 + \left| \int_0^1 \partial_z e^{(2)} dz \right|^2 \right) dx \leq 2 \sum_{k=1,2} \|\partial_z e^{(k)}\|_{L^2(\Omega^{(k)})}^2. \end{aligned}$$

We finally have:

$$\frac{1}{3\sigma_3^h \varepsilon^2} \int_\omega |Ru_\varepsilon|^2 dx \leq \frac{\bar{\sigma}^2}{3\sigma_3^h \varepsilon^2} \sum_{k=1,2} \|\partial_{zz} e_\varepsilon^{(k)}\|_{L^2(\Omega^{(k)})}^2 + \frac{3\sigma_3^h}{\varepsilon^2} \sum_{k=1,2} \|\partial_z e^{(k)}\|_{L^2(\Omega^{(k)})}^2.$$

Secondly, we have $\left| \hat{E}_\varepsilon^{(k)}(\beta f, g) \right| = \left| \mathbf{f}(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)}) - \mathbf{f}(\hat{u}_\varepsilon^{(k)}, \hat{w}_\varepsilon^{(k)}) \right| \leq \Lambda_0 \left| \left(\hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)} \right) \right|$, and then:

$$\left| \alpha \sum_{k=1,2} \int_\omega \hat{E}_\varepsilon^{(k)}(\beta f, g) \cdot \left(\hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)} \right) dx \right| \leq \alpha \Lambda_0 \sum_{k=1,2} \int_\omega \left| \left(\hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)} \right) \right|^2 dx = \Lambda_0 y(t).$$

Lastly, we recall that $(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)}) = (u_0, w_0) + \varepsilon^2 (u_1^{(k)}, w_1^{(k)})$, and then $(\bar{\bar{u}}_\varepsilon^{(k)}, \bar{\bar{w}}_\varepsilon^{(k)}) = (u_0, w_0) + \varepsilon^2 (\bar{\bar{u}}_1^{(k)}, \bar{\bar{w}}_1^{(k)})$ are the integrals of $(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)})$ in z , and we compute:

$$\begin{aligned} \bar{E}_\varepsilon^{(k)}(\beta f, g) &= \left(\mathbf{f}(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)}) - \mathbf{f}(\bar{\bar{u}}_\varepsilon^{(k)}, \bar{\bar{w}}_\varepsilon^{(k)}) \right) \\ &+ \left(\mathbf{f}(\bar{\bar{u}}_\varepsilon^{(k)}, \bar{\bar{w}}_\varepsilon^{(k)}) - \mathbf{f}(u_0, w_0) - \varepsilon^2 \nabla \mathbf{f}(u_0, w_0) \cdot \left(\bar{\bar{u}}_1^{(k)}, \bar{\bar{w}}_1^{(k)} \right) \right) \\ &+ \left(\mathbf{f}(u_0, w_0) + \varepsilon^2 \nabla \mathbf{f}(u_0, w_0) \cdot \left(\bar{\bar{u}}_1^{(k)}, \bar{\bar{w}}_1^{(k)} \right) - \mathbf{f}(u_\varepsilon^{(k)}, w_\varepsilon^{(k)}) \right). \end{aligned}$$

For these three terms, we have (Lipschitz continuity of \mathbf{f}):

$$\left| \mathbf{f}(\bar{u}_\varepsilon^{(k)}, \bar{w}_\varepsilon^{(k)}) - \mathbf{f}(\bar{\bar{u}}_\varepsilon^{(k)}, \bar{\bar{w}}_\varepsilon^{(k)}) \right| \leq \Lambda_0 \left| \left(\bar{e}_\varepsilon^{(k)}, \bar{f}_\varepsilon^{(k)} \right) \right| \leq \Lambda_0 \int_0^1 \left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right| dz$$

if $k=1$ and the same for $z \in (-1, 0)$ if $k=2$, and (lemma 1):

$$\begin{aligned} & \left| \mathbf{f}(\bar{\bar{u}}_\varepsilon^{(k)}, \bar{\bar{w}}_\varepsilon^{(k)}) - \mathbf{f}(u_0, w_0) - \varepsilon^2 \nabla \mathbf{f}(u_0, w_0) \cdot \left(\bar{\bar{u}}_1^{(k)}, \bar{\bar{w}}_1^{(k)} \right) \right| \\ &= \left| \mathbf{f} \left((u_0, w_0), \left(\bar{\bar{u}}_\varepsilon^{(k)}, \bar{\bar{w}}_\varepsilon^{(k)} \right) \right) \right| \leq \frac{1}{2} \Lambda_1 \varepsilon^4 \left| \left(\bar{\bar{u}}_1^{(k)}, \bar{\bar{w}}_1^{(k)} \right) \right|^2 \leq \frac{1}{2} \Lambda_1 \varepsilon^4 M^2, \end{aligned}$$

and, at last (for the case $k = 1$ – application of the estimate (39)):

$$\begin{aligned} & \left| \mathbf{f}(u_0, w_0) + \varepsilon^2 \nabla \mathbf{f}(u_0, w_0) \cdot \left(\bar{u}_1^{(k)}, \bar{w}_1^{(k)} \right) - \overline{\mathbf{f}(u_\varepsilon^{(k)}, w_\varepsilon^{(k)})} \right| \\ & \leq \int_0^1 \left| \mathbf{f}(u_0, w_0) + \varepsilon^2 \nabla \mathbf{f}(u_0, w_0) \cdot \left(u_1^{(1)}, w_1^{(1)} \right) - \mathbf{f}(u_\varepsilon^{(1)}, w_\varepsilon^{(1)}) \right| dz \\ & \leq \int_0^1 \Lambda_0 \left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right| dz + \frac{1}{2} \Lambda_1 \varepsilon^4 M^2, \end{aligned}$$

and the same estimate holds for the case $k = 2$. In consequence, we have (for $k = 1$ – the integral is on $z \in (-1, 0)$ for $k = 2$):

$$\left| \bar{E}_\varepsilon^{(k)}(\beta f, g) \right| \leq \Lambda_1 \varepsilon^4 M^2 + 2\Lambda_0 \int_0^1 \left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right| dz.$$

Hence, we find that:

$$\begin{aligned} & \left| \alpha \sum_{k=1,2} \int_\omega \bar{E}_\varepsilon^{(k)}(\beta f, g) \cdot \left(\hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)} \right) dx \right| \\ & \leq \frac{\alpha}{2} \sum_{k=1,2} \left(\int_\omega \left| \bar{E}_\varepsilon^{(k)}(\beta f, g) \right|^2 dx + \int_\omega \left| \hat{e}_\varepsilon^{(k)}, \hat{f}_\varepsilon^{(k)} \right|^2 dx \right) \\ & \leq \alpha \sum_{k=1,2} \left(\Lambda_1^2 \varepsilon^8 M^4 |\omega| + 4\Lambda_0^2 \int_{\Omega^{(k)}} \left| \left(e_\varepsilon^{(k)}, f_\varepsilon^{(k)} \right) \right|^2 dz dx \right) + \frac{1}{2} y(t) \\ & \leq 2\alpha \Lambda_1^2 \varepsilon^8 M^4 |\omega| + 4\Lambda_0^2 \alpha k_0(t)^2 \varepsilon^6 \exp(c_0 t) + \frac{1}{2} y(t) \end{aligned}$$

applying the results for Theorem 1. The final energy estimate reads (for $0 < \varepsilon \leq 1$):

$$\begin{aligned} \frac{d}{dt} y(t) + 2\bar{\sigma} \sum_{k=1,2} \int_\omega \left| \nabla_x \hat{e}_\varepsilon^{(k)} \right|^2 dx + \frac{3}{2} \sigma_3^h \int_\omega \sigma_3^h \frac{\left| \hat{e}_\varepsilon^{(1)} - \hat{e}_\varepsilon^{(2)} \right|^2}{\varepsilon^2} dx \\ \leq c_2 y(t) + \varepsilon^6 (d_4 + d_5(t)) \end{aligned}$$

where the constant c_2 , d_4 and the function $d_5(t)$ are given in the statement of Theorem 3. Using again lemma 3, this proves the result, because, for all $t > 0$ and $0 < \varepsilon \leq 1$, due to the inequalities (35) and (43) from Theorems 1 and 2,

$$\begin{aligned} \int_0^t d_5(s) ds &= 8\Lambda_0^2 \alpha \int_0^t k_0(s)^2 \exp(c_0 s) ds \\ &+ \frac{2\bar{\sigma}^2}{3\sigma_3^h \varepsilon^8} \sum_{k=1,2} \|\partial_{zz} e_\varepsilon^{(k)}\|_{L^2(0,t;L^2(\Omega^{(k)}))}^2 + \frac{6\sigma_3^h}{\varepsilon^8} \sum_{k=1,2} \|\partial_z e_\varepsilon^{(k)}\|_{L^2(0,t;L^2(\Omega^{(k)}))}^2 < +\infty, \quad (53) \end{aligned}$$

and then $k_4(t)$ and $k_5(t)$ are bounded uniformly with respect to ε .

Remark 5 *This theorem guarantees that the average by layer in the thickness of the three-dimensional potential converges toward the solution of the two-layers model. Furthermore, the accuracy of the two-layers model is limited by the precision of the approximation of the transverse diffusion which depends in our formulation on the precision of the asymptotic expansion of $u_\varepsilon^{(k)}$. If, as suggested in the remark 4, the accuracy of the second order asymptotic expansion is actually proportional to ε^4 , then this theorem can be adapted straightforwardly to get an ε^4 error*

Going back to the physical dimensions Let us give a version of the problem (48) in physical variables :

$$A(C\partial_t u^{(k)} + f(u^{(k)}, w^{(k)})) = \operatorname{div}_x(\sigma^{(k)} \nabla_x u^{(k)}) + (-1)^k \sigma_3^h (u^{(1)} - u^{(2)}) \quad (54)$$

$$\partial_t w^{(k)} + g(u^{(k)}, w^{(k)}) = 0 \quad (55)$$

where $\sigma_3^h = \frac{3}{h^2} \frac{\sigma_3^{(1)} \sigma_3^{(2)}}{\sigma_3^{(1)} + \sigma_3^{(2)}}$ is called the coupling coefficient and $A, C, \sigma^{(k)}, \sigma_3^{(k)}$ are the same parameters than in problem (1,2).

Remark 6 *The model also has a purely electric interpretation, as in [Jacquemet, 2004]: the two layers are coupled pointwise by a conductance. The coupling coefficient from our model, σ_3^h , is expressed in mScm^{-2} , which is consistent. Consequently, the term $\sigma_3^h (u^{(1)} - u^{(2)})$ is a density of current expressed in μAcm^{-2} .*

6 Numerical illustrations

6.1 The test cases

We want to simulate the propagation of the activation front on a three-dimensional slab of tissue $\Omega = \omega \times (-h, h)$ for various values of the thickness parameter $h > 0$, and with a discontinuity through the surface $\{z = 0\}$. We take $\omega = (0, x_0) \times (0, x_0)$ and define the conductivity matrices $\sigma^{(k)}$ in the layers $\Omega^{(k)}$ following equations (6) with fiber direction as in equations (5). We have the choice between three models:

3D: The complete three-dimensional equations (1) and (2) on Ω , or the equivalent systems of equations (7) and (8) on the layers $\Omega^{(k)}$, which solutions are denoted by $u_h^{(k)}$ and $w_h^{(k)}$;

2Dx1: The order 0 term of the asymptotic expansion, u_0 and w_0 , that solve the usual surface monodomain equation (22) and (23) on ω and does not depend on h ;

2Dx2: The solution of the two-layers model, $\hat{u}_h^{(k)}$ and $\hat{w}_h^{(k)}$, that solve the equations (54) and (55) on ω with the coupling parameter $\sigma_3^h = \frac{3}{h^2} \frac{\sigma_3^{(1)} \sigma_3^{(2)}}{\sigma_3^{(1)} + \sigma_3^{(2)}}$.

The functions f and g are defined by the Beeler-Reuter ionic model [Beeler and Reuter, 1977], for it is computationally simple but retains the essential features of the cardiac action potential. The angles $\theta^{(k)}$ of the fibers in layers $k = 1$ and 2, and the remaining parameters of the equations are given in table 2. We will explore values of

| $2h$ cm | x_0 cm | T ms | A cm^{-1} | C μFcm^{-2} | $\sigma_1^{(k)}$ | $\sigma_2^{(k)}$ mScm^{-1} | $\sigma_3^{(k)}$ | $\theta^{(1)}$ | $\theta^{(2)}$ |
|-------------|-------------|-----------|-------------------------|-----------------------------|------------------|--|------------------|----------------|----------------|
| [0.01, 0.4] | 1 | 400 | 500 | 1 | 1.5 | 0.2 | 0.2 | 0 | $\pi/2$ |

Table 2 Parameters of the equations

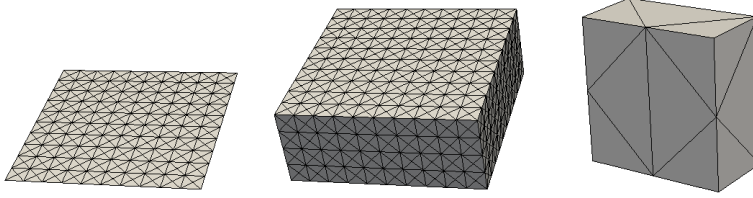


Fig. 1 Left: mesh of ω . Middle : mesh of Ω . Right : each hexahedron is split into 24 tetrahedra

the thickness parameter h ranging from the typical diameter of a cardiomyocyte (tens of μm) up to large atrial thickness (less than a cm) as found in humans.

An action potential is initiated by applying a transmembrane voltage of 20mV during a period of 2ms in the domain $S = (0.49x_0, 0.51x_0) \times (0.49x_0, 0.51x_0)$ for the surface models and in the cylinder $S \times (-h, h)$ for the 3D model.

6.2 Meshes, discretization, and resolution

We consider a Cartesian grid of Ω with space steps $\Delta x \times \Delta x \times \Delta z$ in the directions x , y , and z ; and the subgrid with space steps $\Delta x \times \Delta x$ of ω . These Cartesian grids are split into a simplicial mesh of Ω (that is with tetrahedra) and a simplicial submesh for ω (that is with triangles), by splitting each square into 4 triangles and each hexahedron into 24 tetrahedra (see figure 1). In order to guarantee the convergence of the discrete solution, we take $\Delta x = x_0/200 = 5.e - 3\text{cm}$ and dz is chosen as in table 3. This choice guarantees that there are at least 10 elements through the whole thickness of Ω (that is $2h$) and $\Delta z \leq 4\Delta x$.

The equations (1), (2) for the three-dimensional model, (54), (55) for the two-layers model, and (22), (23) for the usual surface model are discretized by using the standard P1-Lagrange finite elements in space with diagonal mass lumping (due to the numerical quadrature rule) and the Rush-Larsen time-stepping scheme [Rush and Larsen, 1978] with the fixed time-step $\Delta t = 0.05\text{ms}$. During the Rush-Larsen iteration, the diffusion terms are solved implicitly; so are the coupling terms in the coupled equations of the two-layers model. All the linear systems are solved with a conjugate gradient method, a Jacobi preconditionner, and a fixed tolerance equal to $1.e - 10$.

The objective of our study is not to investigate the numerical method but to study the impact of the modeling choices that we propose.

| | | | | | | |
|------------|-----------|-----------|-----------|-----------|-----------|-----------|
| $2h$ | 0.4 | 0.3 | 0.2 | 0.1 | 0.09 | 0.08 |
| Δz | 0.02 | 0.015 | 0.01 | 0.01 | 0.009 | 0.008 |
| #dof 2Dx1 | 80 401 | 80 401 | 80 401 | 80 401 | 80 401 | 80 401 |
| #dof 2Dx2 | 160 802 | 160 802 | 160 802 | 160 802 | 160 802 | 160 802 |
| #dof 3D | 3 296 421 | 3 296 421 | 3 296 421 | 1 688 411 | 1 688 411 | 1 688 411 |
| | | | | 0.07 | 0.06 | 0.05 |
| | | | | 0.007 | 0.006 | 0.005 |
| | | | | 80 401 | 80 401 | 80 401 |
| | | | | 160 802 | 160 802 | 160 802 |
| | | | | 1 688 411 | 1 688 411 | 1 688 411 |

Table 3 Meshes and numbers of degrees of freedom.

6.3 Numerical results

6.3.1 Qualitative behavior

For two values of the thickness parameter, corresponding to the asymptotic ($2h = 0.03\text{cm}$) and non-asymptotic ($2h = 0.1\text{cm}$) regimes, we display, for each layer, the average $\bar{u}_h^{(k)}(x)$ of the three-dimensional solution $u_h^{(k)}(x, z)$, the two-dimensional solution $\hat{u}_h^{(k)}(x)$, and the surface solution $u_0(x)$ of our three models (figures 2 and 3). All solutions are displayed at time $t = 9\text{ms}$. In the asymptotic regime, all solutions are qualitatively very similar (figure 2), as expected. In the non-asymptotic regime, the usual surface model clearly misses the important three-dimensional interplay of diffusion and reaction between the two distinct layers of tissue. This interplay induces the diamond-shaped wavefront seen on figure 3, that remains in our two-layers surface model, but not in the usual surface model. Note that such a wavefront was observed experimentally in [Vetter et al., 2005]. Our two-layers model clearly accounts for such phenomena.

6.3.2 Errors

In order to quantify the distances from the surface model or our two-layers model, to the three-dimensional one, and accordingly with the results from theorems 1 and 3, we define the following errors E_0 and E_1 :

$$E_0(h) = \frac{\max_n \|\bar{u}_h^n - u_0^n\|_{L^2(\omega)}}{\max_n \|\bar{u}_h^n\|_{L^2(\omega)}}, \quad E_1(h) = \frac{\max_n \|\bar{u}_h^{(k),n} - \hat{u}_h^{(k),n}\|_{L^2(\omega)}}{\max_n \|\bar{u}_h^{(k),n}\|_{L^2(\omega)}} \quad (56)$$

where $\bar{u}_h^n = \frac{1}{2} (\bar{u}_h^{(1),n} + \bar{u}_h^{(2),n})$, u_0^n , $\bar{u}_h^{(k),n}$, and $\hat{u}_h^{(k),n}$ refer to the discrete solutions of the models at time $t^n = n\Delta t$

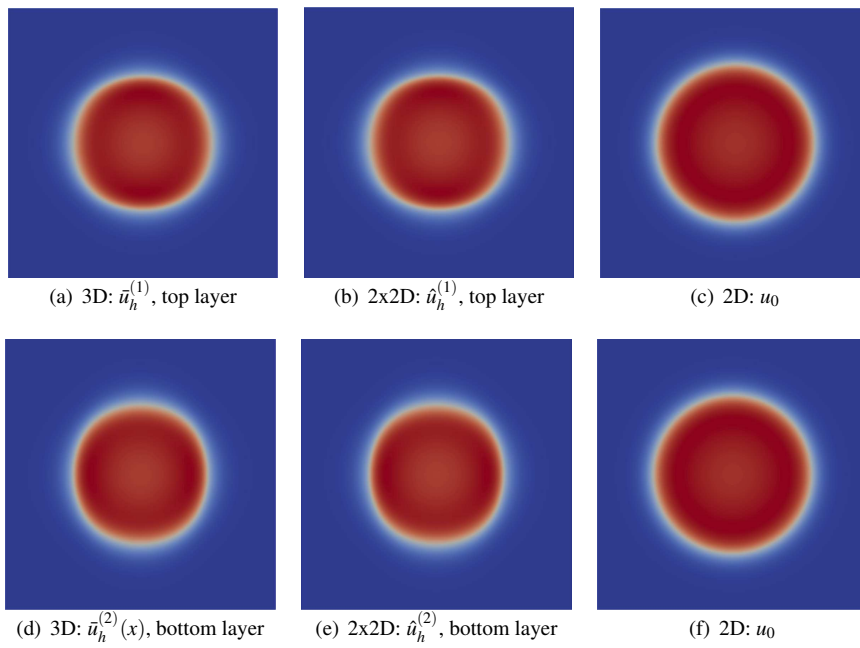


Fig. 2 Solutions in the asymptotic regime, $2h = 0.03\text{cm}$.

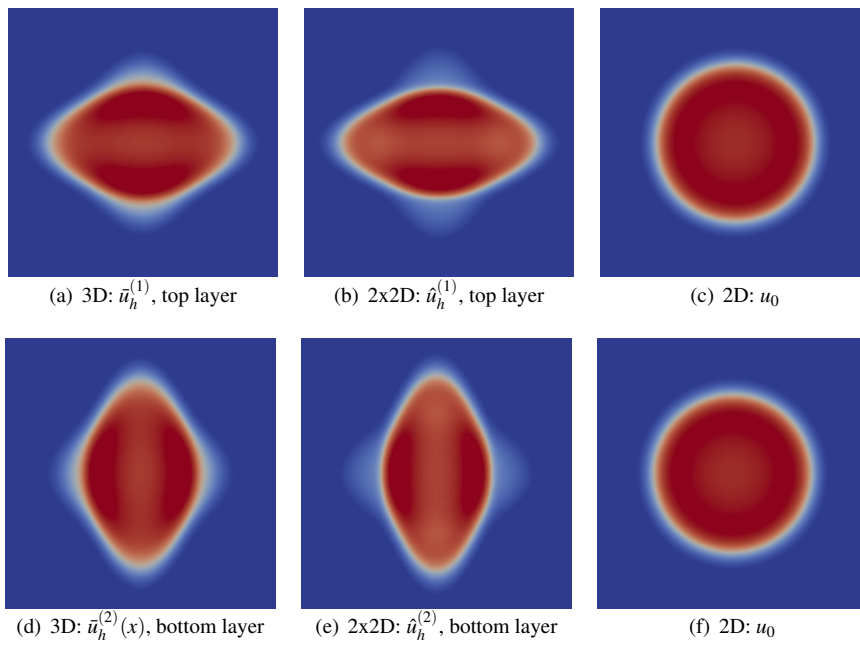


Fig. 3 Solutions in the physiologic regime, $2h = 0.1\text{cm}$.

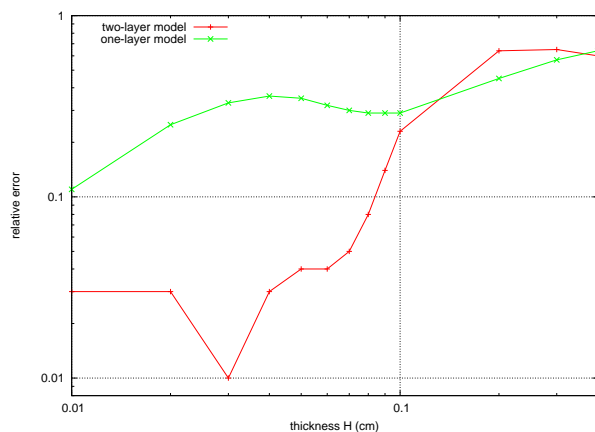


Fig. 4 Errors $E_0(h)$ and $E_1(h)$ defined in equation (56).

We expect that $E_0(h) = O(h^2)$ and $E_1(h) = O(h^3)$, although we presume that $E_1(h) = O(h^4)$ is the correct convergence rate (remark 5). The error graphs $(h, E_0(h))$ and $(h, E_1(h))$ are displayed on the figure 4. As expected, we observe the convergence of both models, and a higher convergence rate for the two-layers model ($E_1(h)$) than for the surface model ($E_0(h)$), but only for $h < 0.1$ cm.

Finally, table (3) shows the very strong improvement of the computational cost for the two-layers (25 to 70 times faster), and the usual surface (70 to 170 times faster) models.

| h | 0.4 | 0.3 | 0.2 | 0.1 | 0.09 | 0.08 | 0.07 | 0.06 |
|------|--------|--------|--------|--------|--------|--------|--------|--------|
| 2Dx1 | 0.0069 | 0.0066 | 0.0059 | 0.0127 | 0.0118 | 0.0119 | 0.0142 | 0.0142 |
| 2Dx2 | 0.0171 | 0.0162 | 0.0146 | 0.0310 | 0.0265 | 0.0271 | 0.0321 | 0.0324 |

| | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 |
|--|--------|--------|--------|--------|--------|
| | 0.0145 | 0.0152 | 0.0155 | 0.0130 | 0.0094 |
| | 0.0347 | 0.0365 | 0.0373 | 0.0313 | 0.0216 |

Table 4 Ratio of cpu time for the surface models with respect to the three-dimensional computation cpu time.

7 Conclusion

7.1 Interest of two-layers model

A similar two-layers model was proposed in [Jacquemet, 2004, Gharaviri et al., 2012], but its rigorous mathematical derivation, the error estimates proved in Theorem 3, and the numerical study from section 6 are new. In addition, the dimensional analysis from section 3.1 gives a good understanding of how the trade-off between reaction and diffusion could trigger complex three-dimensional propagation patterns.

The main advantage of the bilayer model over the usual surface model resides in the distinction of individual reaction terms and diffusion terms in the layers. This is important to trigger transmural gradients and dissociation of the electrical activity, together with complex anisotropic propagation patterns as observed in [Vetter et al., 2005]. The usual surface model, even enhanced by addition of the second order term $\varepsilon^2 u_1^{(k)}$ would not account for these phenomena. Although, we point out that these phenomena (dissociation, transmural gradients, complex propagation) are expected to play a major role in the initiation of arrhythmias. Hence, the two-layers model is a very efficient tool to investigate in depth the arrhythmogenic mechanisms in the atria.

At last, the computational costs of the two-layers model is divided by 50 (on average) with respect to the cost of three-dimensional model.

7.2 Limitations and perspectives

Although being a strong qualitative enhancement of the usual surface model, the two-layers model could still be improved, for instance by incorporating higher order terms in the Taylor expansions. The derivation of a more general bidomain version of the two layers model on more general geometries is also possible, following the work from [Chapelle et al., 2013].

Our numerical study illustrates the results from section 4.1 and 5 and we give errors in the L^2 norm. For medical issues, activation maps and numerical errors on the activation times have to be derived.

Mechanisms well rendered by the two-layers model (dissociation, transmural gradient...) occur only in small localized parts of the atria, with abrupt variation of the fiber orientation and thin tissues. These configurations are found in the pulmonary veins and in some areas of the atrial chambers such as the left atrium posterior and anterior walls or the pectinate muscles, and the Crista terminalis in the right atrium. Hence, coupling the two-layers model to the usual surface model might improve the computational efficiency of the overall method.

Beyond its mathematical derivation, this two-layers model is interesting to construct anatomically and structurally accurate models of the human atria: clinical images give the two-dimensional surfaces, that are completed by histological descriptions such as [Ho et al., 2002] or [Ho et al., 1999]. We believe that it will become an appreciated tool to study the structure-to-function relations in the atria, from a medical point of view. A first model therefore was presented in [Vigmond et al., 2013, Labarthe et al., 2013].

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A Proof of Lemma 4

For all $t > 0$ and a.e. $x \in \omega$, according to the equation (13) on $w_\varepsilon^{(k)}$, we have, for $k = 1, 2$,

$$\partial_t w_\varepsilon^{(k)}(t, x, 0) + g\left(u_\varepsilon^{(k)}(t, x, 0), w_\varepsilon^{(k)}(t, x, 0)\right) = 0.$$

We test these equations against the function $(w_\varepsilon^{(1)} - w_\varepsilon^{(2)})(t, x, 0)$ and subtract them one with another, in order to get

$$\frac{1}{2} \frac{d}{dt} \int_\omega \left| w_\varepsilon^{(1)} - w_\varepsilon^{(2)} \right|^2 dx \leq \Lambda_0 \int_\omega \left| w_\varepsilon^{(1)} - w_\varepsilon^{(2)} \right|^2 dx$$

because $u_\varepsilon^{(1)}(t, x, 0) = u_\varepsilon^{(2)}(t, x, 0)$ for $x \in \omega$, and where the constant Λ_0 is the one used in the proof of Theorem 1, that bounds also the Lipschitz constant of the function $w \mapsto g(u, w)$ for $|(u, w)| \leq 2M$. Since $w_\varepsilon^{(1)}(0, x, 0) = w_\varepsilon^{(2)}(0, x, 0)$ for a.e. $x \in \omega$, we see that $\int_\omega \left| w_\varepsilon^{(1)} - w_\varepsilon^{(2)} \right|^2 dx = 0$ for all $t > 0$, and hence $w_\varepsilon^{(1)} = w_\varepsilon^{(2)}$ on the interface $\Sigma = \omega \times \{0\}$.

After differentiation in z of equation (13), we find that, for $k = 1, 2$,

$$\partial_t \partial_z w_\varepsilon^{(k)}(t, x, 0) + \nabla g\left(u_\varepsilon^{(k)}(t, x, 0), w_\varepsilon^{(k)}(t, x, 0)\right) \cdot \left(\partial_z u_\varepsilon^{(k)}(t, x, 0), \partial_z w_\varepsilon^{(k)}(t, x, 0)\right) = 0.$$

We multiply these equations by $\sigma_3^{(k)}$, test them against $(\sigma_3^{(1)} w_\varepsilon^{(1)} - \sigma_3^{(2)} w_\varepsilon^{(2)})(t, x, 0)$ and subtract them one with another, in order to get, for all $t > 0$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\omega \left| \sigma_3^{(2)} \partial_z w_\varepsilon^{(2)} - \sigma_3^{(1)} \partial_z w_\varepsilon^{(1)} \right|^2 dx \\ & \leq - \int_\omega \left(\nabla g(u_\varepsilon^{(2)}, w_\varepsilon^{(2)}) \left(\sigma_3^{(2)} \partial_z u_\varepsilon^{(2)}, \sigma_3^{(2)} \partial_z w_\varepsilon^{(2)} \right) \right. \\ & \quad \left. - \nabla g(u_\varepsilon^{(1)}, w_\varepsilon^{(1)}) \left(\sigma_3^{(1)} \partial_z u_\varepsilon^{(1)}, \sigma_3^{(1)} \partial_z w_\varepsilon^{(1)} \right) \right) \left(\sigma_3^{(2)} \partial_z w_\varepsilon^{(2)} - \sigma_3^{(1)} \partial_z w_\varepsilon^{(1)} \right) dx \\ & \leq \Lambda_0 \int_\omega \left| \sigma_3^{(2)} \partial_z w_\varepsilon^{(2)} - \sigma_3^{(1)} \partial_z w_\varepsilon^{(1)} \right|^2 dx \end{aligned}$$

because $u_\varepsilon^{(1)} = u_\varepsilon^{(2)}$, $w_\varepsilon^{(1)} = w_\varepsilon^{(2)}$, and $\sigma_3^{(1)} \partial_z u_\varepsilon^{(1)} = \sigma_3^{(2)} \partial_z u_\varepsilon^{(2)}$ on $\Sigma = \omega \times \{0\}$, and because Λ_0 bounds $|\nabla g(u, w)|$ for $|(u, w)| \leq 2M$. Like above, $\sigma_3^{(2)} \partial_z w_\varepsilon^{(2)} = \sigma_3^{(1)} \partial_z w_\varepsilon^{(1)}$ on $\Sigma = \omega \times \{0\}$ for all $t > 0$ because it is true for $t = 0$.

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