



**HAL**  
open science

## Game theory approach for modeling competition over visibility on social networks

Alexandre Reiffers-Masson, Yezekael Hayel, Eitan Altman

► **To cite this version:**

Alexandre Reiffers-Masson, Yezekael Hayel, Eitan Altman. Game theory approach for modeling competition over visibility on social networks. Social Networking Workshop (in conjunction with COMSNETS 2014), Jan 2014, Bangalore, India. pp.1-6, 10.1109/COMSNETS.2014.6734939 . hal-00918036v1

**HAL Id: hal-00918036**

**<https://inria.hal.science/hal-00918036v1>**

Submitted on 12 Dec 2013 (v1), last revised 28 Mar 2016 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Game theory approach for modeling competition over visibility on social networks

Alexandre Reiffers-Masson<sup>◇ \*</sup>, Yezekael Hayel<sup>\*</sup> and Eitan Altman<sup>\*</sup>

**Abstract**—In Social Networks, such as facebook, linkedin, twitter, google+ and others, many members post messages to walls or to timelines of their friends, or to that of pages or of groups. There is permanent competition over visibility of content in these destination since timelines have finite capacity. As new content arrives, older content gets pushed away from the timeline. A selfish source that wishes to be more visible has to send from time to time new content thus preempting other content from the time-line. We assume that sending more content comes with some extra cost. We analyse the problem of selecting the rates of content creation as a non-cooperative game between several sources that share some common destination to which they send content. We identify conditions under which the problem can be reduced to the Kelly mechanism for which we compute explicitly the equilibrium. This is done in a very general probabilistic framework where time between arrival of content is only required to be stationary ergodic.

## I. INTRODUCTION

Timelines in social networks are limited due to capacity limits of the time line. It is also limited by visibility limitation due to a finite screen size and to scrolling habits of Internauts. Sources of content that wish to be visible on a given timeline have to take into account other flows of content that compete over space in the same timeline. Members of a social network may thus be pushed to resend their own content or other's content in order to make their opinions visible. In absence of such actions, the average number of their posots at the destination may be very small. The choice of rates of content can be formulated as a non cooperative game where a source's utility is the difference between some dissemination payoff and some cost for sendingn more content.

In a previous paper, Altman et al [12] have studied this game under the assumptions:

- of a smooth arrival process of contents modeled as having a Poisson distributed,
- the dissemination utility of a source was assumed to be the indicator of having at least one content of that source in the destination timeline,
- that one is interested in the stationary behavior.

In this paper we shall abandon the restrictive Poisson assumption. We allow for general dependence of times between creation of content by a given source and also across different sources. We shall only require that the times of content creation at each different source  $i$  is stationary ergodic point process, and that their superposition is a simple process (at a given time there may only be a single content arrival.

The assumption on stationary behavior is well suited to a subscriber's wall. Indeed, there is no reason to expect the subscriber to access his wall at times with a distribution which is not time uniform. In this paper, we consider in contrast the behavior of group subscribers. In facebook, these subscribers receive notifications whenever a new content arrives. We thus assume here that a subscriber does not observe the system at a stationary regime but rather at instants of arrivals of new contents. In the theory of point process, the system as observed at arrival instants of points in a stationary ergodic point process corresponds to the so called Palm probability.

Another difference with respect to [12] is that we shall consider in this paper the total number of contents at arrival times of contents as the dissemination payoff function.

Our first result of this paper is to show that the problem can be reduced to the well known generalized Kelly mechanism, which allows us to use many existing results to characterize the equilibrium. We moreover derive the equilibrium in explicit form and thus obtain new results for the generalized Kelly mechanism.

We then extend the model to study rerouting of posts, which corresponds to sharing content.

## II. RELATED WORK

We briefly discuss the generalised Kelly mechanism. Consider a resource  $K$  that is to be shared among  $J$  strategic users. A user makes propose a bid  $\lambda_j$ . Then the resource is shared proportionally to the bids, so that the amount of resource that player  $J$  receives is

$$K \frac{\lambda_j}{\sum_i \lambda_i}$$

and each player  $j$  has a proportionaly cost  $\gamma_j$  for his bid. Finally his objectif function is:

$$U_j(K \frac{\lambda_j}{\sum_i \lambda_i}) - \gamma_j \lambda_j$$

where  $U_j(\cdot)$  is a function that measure the satisfaction that receive the player  $j$  when he has  $K \frac{\lambda_j}{\sum_i \lambda_i}$  of the ressource. A study of this game was proposed in [6]. We can find some networking applications of this game in [2] and [3].

A similar analytical work like ours, has been recently proposed in [12]. In this paper, the authors propose to study the competition over a news feed between several information providers (sources). There are several difference with our study. First, they consider Poisson arrival rates whereas, in our model, we consider a more general point process. Second, they define the visibility over a news feed as the proportion

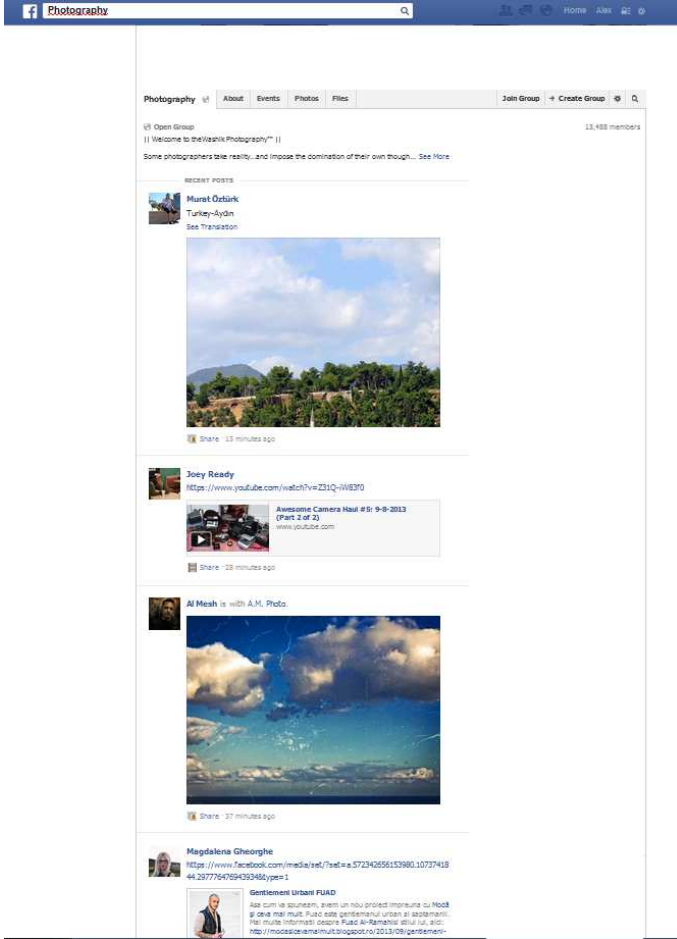


Fig. 1. A Facebook Group

of time one message is visible on the time line. Then, they consider in fact the case that only one message of each source can appear on the news feed. In our analysis we also generalize this point by considering that several messages, from the same source, can be visible on the news feed. Finally, we consider the propagation effect between news feed, which is not take into account in [12].

### III. THE GAME FORMULATION

Consider a single group of size  $K$ . It can thus contain upto  $K$  posts. Consider an arrival of content to that group which is a superposition of  $J$  stationary ergodic point processes  $N_i$  ( $i = 1, \dots, N$ ) assumed to be compatible with the flow  $\theta_t$ , each of which has a finite and non null intensity  $\lambda_i$ . We call by  $N$  their superposition.

We use the definition of [13] for the Palm probability which is the probability as seen at an arrival instant. Let  $P_N^o(A)$  be the Palm probability corresponding to the point process  $N$ , and define similarly the Palm process that corresponds to point process  $N_i$ . Assume that A1 (defined in the Introduction) holds. Then it is shown in page 37 or [13] that

$$P_N^o(N_i(\{0\} = 1)) = \frac{\lambda_i}{\lambda} \quad (1)$$

This formula states that when an arrival occurs, the probability that it is from source  $i$  is  $\lambda_i/\lambda$ .

We consider the dissemination utility for player  $i$  to be the sum of messages originating from it in the group at an arbitrary arrival instant of a content. This event can be written as

$$\sum_{j=0}^{K-1} 1(N_i^o T_{-j} = 1)$$

whose expectation with respect to the Palm probability measure is given by

$$\sum_{j=0}^{K-1} P_N^o(N_i^o T_{-j} = 1) = K \frac{\lambda_i}{\lambda}$$

due to (1.3.17) p 25 in [13] and eq (1).

We assume also that each source  $j$  has a cost for sending messages, which depends on his rate  $\lambda_j$ . This cost is expressed by  $\gamma_j \lambda_j$ . Finally, source's  $j$  objective function is defined by the difference between the average number of messages and the cost for his spamming activity:

$$U_j(\lambda_j, \lambda_{-j}) = K \frac{\lambda_j}{\sum_i \lambda_i} - \gamma_j \lambda_j, \quad (2)$$

where  $\lambda_{-j} = \{\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_J\}$  is the strategy vector of all the other sources.

### IV. COMPETITION BETWEEN SOURCES

We study the visibility competition between the sources. We are faced to a normal form non-cooperative game with  $J$  sources. The utility function of each source depends on his rate and also on the rates of the other sources. The utility function for source  $j$  is given by equation (2).

**Definition 1.** The decision vector  $\lambda^* = (\lambda_1^*, \dots, \lambda_J^*)$  is a nash equilibrium if for all  $j \in \{1, \dots, J\}$ ,

$$U_j(\lambda_j^*, \lambda_{-j}^*) = \max_{\lambda_j \in [0, \lambda_{\max}]} U_j(\lambda_j, \lambda_{-j}^*),$$

where  $\lambda_{\max}$  is the maximum source rate.

In other words, no user has an interest to change his decision unilaterally at the Nash equilibrium.

This game has been already originally proposed in [6] but not in the context of social networks. It has been also used to study competition over popularity in [1], but without considering a finite number of messages of interest in a group. Our game has several important properties. First, at equilibrium, at least one source send messages to the group. we give this property in the following proposition.

**Proposition 2.** The decision vector  $(0, \dots, 0)$  is not a Nash equilibrium.

*Proof.* If all sources  $\{1, \dots, J\} - \{j\}$  play 0,  $j$  can always find a very small  $\lambda_j^*$  such that  $\lambda_j^* < \frac{K}{\gamma_j}$ . Then, we have that  $U_j(\lambda_j^*, 0) > 0$  and the decision vector  $(0, \dots, 0)$  is not a Nash equilibrium.  $\square$

Second, we adapt in the following theorem the results obtained in [1] to our model. In this theorem, we obtain an explicit solution of the unique equilibrium of the game, when all the source send messages on the group.

**Theorem 3.** *If  $\sum_i \gamma_i - \max_j \{\gamma_j\}(J-1) > 0$ , we have that the Nash Equilibrium is uniquely defined by:*

$$\forall j, \quad \lambda_j^* = \frac{(J-1)K}{\sum_i \gamma_i} \left(1 - \frac{\phi \gamma_j}{K}\right) > 0.$$

$$\text{with } \phi = \frac{(J-1)K}{\sum_i \gamma_i}.$$

*Proof.* The first order condition:

$$\frac{\partial U_j}{\lambda_j}(\lambda_j, \lambda_{-j}) = 0$$

give us the best reply of the player  $j$ .

$$\begin{aligned} \frac{\partial U_j}{\lambda_j}(\lambda_j, \lambda_{-j}) &= 0 \\ \Leftrightarrow \left(\frac{1}{\sum_i \lambda_i} - \frac{\lambda_j}{(\sum_i \lambda_i)^2}\right) &= \frac{\gamma_j}{K} \\ \Leftrightarrow \left(\frac{\sum_i \lambda_i - \lambda_j}{(\sum_i \lambda_i)^2}\right) &= \frac{\gamma_j}{K} \quad (*) \end{aligned}$$

and we taking the sum over  $j$ , finally we have:

$$\begin{aligned} J(\sum_i \lambda_i) - \frac{1}{K} \sum_i \gamma_i (\sum_i \lambda_i)^2 &= (\sum_i \lambda_i) \\ \Leftrightarrow (J-1)(\sum_i \lambda_i) - \frac{1}{K} \sum_i \gamma_i (\sum_i \lambda_i)^2 &= 0 \\ \Leftrightarrow (\sum_i \lambda_i)((J-1) - \frac{1}{K} \sum_i \gamma_i (\sum_i \lambda_i)) &= 0 \end{aligned}$$

Because of proposition 2 we know that  $\sum_i \lambda_i > 0$ , and this is why

$$\sum_i \lambda_i = \frac{(J-1)K}{\sum_i \gamma_i}$$

and because of (\*) we have the result. The unicity come from that  $\lambda_j^*$  is strictly define by  $\sum_i \lambda_i$  and  $\sum_i \lambda_i$  is unique.  $\square$

In the previous theorem, their is a condition over the cost  $\gamma_i$  for all  $i$  such that  $\lambda_i > 0$  for all  $i$ . If this condition is not satisfied, some rates can be equal to 0 at equilibrium for some sources. his result is described in the next theorem. Given the explicit expression of the Nash equilibrium of the game, we are able to give some nice properties of the equilibrium depending on the parameters of the game like the costs  $\gamma_j$  and the size of the group  $K$ . We denote that For all source  $j$ , the rate  $\lambda_j^*$  at equilibrium is strictly decreasing depending on  $\gamma_j$ ,  $\lambda_j^*$  is a linear increasing in  $K$ .

With the precedent remark we can compute now a general form of the equilibrium and not just restrict our self to an equilibrium where  $\sum_i \gamma_i - \max_j \{\gamma_j\}(J-1) > 0$ , i.e when all  $\lambda_j^*$  are strictly positif. Assume we rearrange the sources, from smallest  $\gamma_j$  to largest one, then  $\gamma_1 > \gamma_2 > \dots > \gamma_J$

**Theorem 4.** *Assume that  $j'$  is such that  $j' = \max\{j \mid \frac{(J-1)K}{\sum_{i=j}^J \gamma_i} (1 - \frac{(J-1)\gamma_j}{\sum_{i=j}^J \gamma_i}) \leq 0\}$  i.e  $j'$  is the source with largest  $\gamma_j$  who doesn't send any messages in the group. In this case the Nash Equilibrium is uniquely defined by:*

$$\begin{aligned} \lambda_j^* &= \frac{(J'-1)K}{\sum_{i>j'} \gamma_i} \left(1 - \frac{(J'-1)\gamma_j}{\sum_{i>j'} \gamma_i}\right) \quad \forall j > j' \\ \lambda_j^* &= 0 \quad \forall j \leq j' \end{aligned}$$

where  $J' = J - |\{1, \dots, j'\}|$

*Proof.* The only thing we have to prove is that if  $j'$  plays 0 at equilibrium, all the other sources  $j < j'$  play also 0. This fact is obvious because  $\lambda_j^*$  is decreasing in  $\gamma_j$ .  $\square$

In order to study the inefficiency of the Nash Equilibrium we study now the price of anarchy [14]. First we define the social welfare of the game:

$$V(\lambda_1, \dots, \lambda_J) = \sum_i \left[ K \frac{\lambda_i}{\sum_i \lambda_i} - \gamma_j \lambda_i \right]$$

We maximize the social welfare:

$$(\lambda'_1, \dots, \lambda'_J) = \operatorname{argmax} V(\lambda_1, \dots, \lambda_J)$$

. The optimal solution  $(\lambda'_1, \dots, \lambda'_J)$  is  $(0, \dots, 0)$  and the price of anarchy is define by:

$$PoA = \frac{V(\lambda'_1, \dots, \lambda'_J)}{\min_i U_i(\lambda_i^*, \lambda_{-i}^*)}$$

**Theorem 5.** *If  $\sum_i \gamma_i - \max_j \{\gamma_j\}(J-1) > 0$  then the price of anarchy is:*

$$PoA = \frac{K \sum_i \gamma_i}{(\sum_i \gamma_i - (J-1)\gamma_{j'}) (\sum_i \gamma_i - \gamma_{j'}(J-1)K)}$$

where  $j'$  is the source such that  $\gamma_{j'} = \max_i \{\gamma_i\}$ .

*Proof.* We have to compute  $U_i(\lambda_i^*, \lambda_{-i}^*)$ . The source with the lower objectif function at equilibrium is the one with the higher  $\gamma_j$ . We call this source  $j'$

$$\begin{aligned} U_i(\lambda_i^*, \lambda_{-i}^*) &= \left(1 - \frac{(J-1)\gamma_{j'}}{\sum_i \gamma_i}\right) - \gamma_{j'} \left(\frac{(J-1)K}{\sum_i \gamma_i}\right) \left(1 - \frac{(J-1)\gamma_{j'}}{\sum_i \gamma_i}\right) \\ &= \left(1 - \frac{(J-1)\gamma_{j'}}{\sum_i \gamma_i}\right) \left(1 - \gamma_{j'} \frac{(J-1)K}{\sum_i \gamma_i}\right) \\ &= \frac{1}{\sum_i \gamma_i} (\sum_i \gamma_i - (J-1)\gamma_{j'}) (\sum_i \gamma_i - \gamma_{j'}(J-1)K) \end{aligned}$$

And we can easily deduce the form of the price of anarchy which is:

$$PoA = \frac{K \sum_i \gamma_i}{(\sum_i \gamma_i - (J-1)\gamma_{j'}) (\sum_i \gamma_i - \gamma_{j'}(J-1)K)}$$

$\square$

## V. PROPAGATION EFFECT WITHIN 2 GROUPS

We consider a generalization of our model to take into account the propagation effect between several groups. In fact, some messages posted on one group can be relayed (propagated) to another group. A message which is of interest for one community can be of interest for another community for which the center of interest is close.

We are interested in the particular question for the sources: "Knowing the propagation effect between groups (the probability that a message has to be copied and to appear in another group), how each source will decide to send his information flow to the groups, directly or indirectly using the propagation effect ?" Indeed, we have assumed that posting a message on a group has a cost for the source, whereas if a message is relayed, it is free for the source. Thus, in this section, we study this topological effect on the non-cooperative

game between the sources. In order to get interesting closed-form results, we reduce the complexity of the game analysis by considering two groups,  $L = 2$ , but our results will be extended to more complex topologies in future works.

First, in the next sub-section, we define the point process model that characterizes the arrival of the messages for the different groups.

#### A. point process model

Now assume that you have 2 groups  $l \in \{1, 2\}$ . Each source  $j \in \{1, \dots, J\}$  can send messages to each group 1 and 2. It means that each source  $j$  controls 2 point processes, which represent flow of messages of source  $j$  in each group, with intensity  $\lambda_j^l = E[N_j^l((0, 1])]$  for  $l = 1, 2$ . For each group  $l$ , any new message will be copied and posted, with a probability  $p_{vl}$ , to the other group  $l' \neq l$ . This process defines a new point process in each group  $l$  with intensity  $p_{vl} \sum_j \lambda_j^l$ . If we add all the point processes for a group  $l$  (direct and indirect messages), this new point process  $N^l$  has an intensity  $\sum_i (\lambda_i^l + p_{vl} \lambda_i^l)$ . We assume that  $K^l$  is the number of messages that subscribers of group  $l$  take care on it. We denote by  $T_n^l$  the arrival time of the  $n^{\text{th}}$  message in the group  $l$ . Let  $N_j^l((T_n^l, T_{n+K^l}^l])$  the number of messages from source  $j$  in group  $l$ , as long as the  $n^{\text{th}}$  message is visible on the group, i.e. is part of the first  $K^l$  messages. Source  $j$  wants to maximize  $\bar{N}_j$  that is the expected number of messages from him that are visible on all the groups:

$$\bar{N}_j = \lim_{N' \rightarrow +\infty} \sum_l \frac{1}{N'} \sum_{n=1}^{N'} N_j^l((T_n^l, T_{n+K^l}^l]).$$

By applying the similar analysis as in section for IV, we obtain the following closed-form expression:

$$\bar{N}_j = K^1 \frac{\lambda_j^1 + p_{21} \lambda_j^2}{\sum_i (\lambda_i^1 + p_{21} \lambda_i^2)} + K^2 \frac{\lambda_j^2 + p_{12} \lambda_j^1}{\sum_i (\lambda_i^2 + p_{12} \lambda_i^1)}.$$

Let  $\gamma_j^l$  be the cost from source  $j$  for sending one message to group  $l$ . We denote the control vector of source  $j$  by  $\lambda_j = \{\lambda_j^1, \lambda_j^2\}$ . The objective function of each source  $j$  is:

$$U_j(\lambda_j, \lambda_{-j}) = K^1 \frac{\lambda_j^1 + p_{21} \lambda_j^2}{\sum_i (\lambda_i^1 + p_{21} \lambda_i^2)} + K^2 \frac{\lambda_j^2 + p_{12} \lambda_j^1}{\sum_i (\lambda_i^2 + p_{12} \lambda_i^1)} - (\gamma_j^1 \lambda_j^1 + \gamma_j^2 \lambda_j^2). \quad (3)$$

#### B. Game analysis

For the analysis of (3), we consider an other variable for the controls of the players, and then we define a new game where the objective function for each player  $j$  is:

$$\max_{\Lambda_j^1, \Lambda_j^2} U_j(\Lambda_j, \Lambda_{-j}) = K^1 \frac{\Lambda_j^1}{\sum_i \Lambda_i^1} + K^2 \frac{\Lambda_j^2}{\sum_i \Lambda_i^2} - \Gamma_j^1 \Lambda_j^1 - \Gamma_j^2 \Lambda_j^2 \quad (4)$$

with  $(\Gamma_j^1, \Gamma_j^2) = (\frac{\gamma_j^1 - \gamma_j^2 p_{21}}{1 - p_{21} p_{12}}, \frac{\gamma_j^2 - \gamma_j^1 p_{12}}{1 - p_{21} p_{12}})$  for all  $j$ . We denote by  $(\Lambda_j^{1*}, \Lambda_j^{2*})$  the flow send by source  $j$  at equilibrium. We define a set  $A$ :

$$A := \{(a, b) \in [0, 1]^2 \mid \forall j, \Lambda_j^{2*} b - \Lambda_j^{1*} > 0 \text{ or } \Lambda_j^{1*} a - \Lambda_j^{2*} > 0\}$$

In the first proposition we give an other form of (3) in order to compute the equilibrium.

**Proposition 6.** *If  $(p_{12}, p_{21}) \in A$  (H1), then the game (3) is equivalent to the game (4) with  $(\Lambda_j^1, \Lambda_j^2) = (\lambda_j^1 + p_{21} \lambda_j^2, \lambda_j^2 + p_{12} \lambda_j^1)$ .*

*Proof.* We are going to prove this theorem in two steps.

Step 1: let us look at the limit of the average number of messages in the two groups for player  $j$  (i.e the first part of the objective function of the player  $j$ ). Easily we can notice that

$$K^1 \frac{\lambda_j^1 + p_{21} \lambda_j^2}{\sum_i (\lambda_i^1 + p_{21} \lambda_i^2)} + K^2 \frac{\lambda_j^2 + p_{12} \lambda_j^1}{\sum_i (\lambda_i^2 + p_{12} \lambda_i^1)} = K^1 \frac{\Lambda_j^1}{\sum_i \Lambda_i^1} + K^2 \frac{\Lambda_j^2}{\sum_i \Lambda_i^2}$$

with  $\Lambda_j^1 = \lambda_j^1 + p_{21} \lambda_j^2$  and  $\Lambda_j^2 = \lambda_j^2 + p_{12} \lambda_j^1$  for all  $j$ .

Step 2: We want to find  $\Gamma_j^1$  and  $\Gamma_j^2$  such that  $\Gamma_j^1 \Lambda_j^1 + \Gamma_j^2 \Lambda_j^2 = \gamma_j^1 \lambda_j^1 + \gamma_j^2 \lambda_j^2$ . Firstly  $(\lambda_j^1, \lambda_j^2) = (\frac{\Lambda_j^2 p_{21} - \Lambda_j^1}{p_{12} p_{21} - 1}, \frac{\Lambda_j^1 p_{12} - \Lambda_j^2}{p_{12} p_{21} - 1})$  is solution of

$$\begin{cases} \Lambda_j^1 &= \lambda_j^1 + p_{21} \lambda_j^2 \\ \Lambda_j^2 &= \lambda_j^2 + p_{12} \lambda_j^1. \end{cases}$$

Secondly with the result above we can compute  $\Gamma_j^1$  and  $\Gamma_j^2$ . In fact,

$$\gamma_j^1 \lambda_j^1 + \gamma_j^2 \lambda_j^2 = \frac{1}{p_{12} p_{21} - 1} [\Lambda_j^1 (\gamma_j^2 p_{12} - \gamma_j^1) + \Lambda_j^2 (\gamma_j^1 p_{21} - \gamma_j^2)].$$

Thus we can conclude that

$$(\Gamma_j^1, \Gamma_j^2) = (\frac{\gamma_j^2 p_{12} - \gamma_j^1}{p_{12} p_{21} - 1}, \frac{\gamma_j^1 p_{21} - \gamma_j^2}{p_{12} p_{21} - 1}).$$

□

Here, we define some new hypothesis.

$$(H2) \begin{cases} \{\Gamma_i^1\}_i \text{ are such that } \sum_i \Gamma_i^1 - \max_i \{\Gamma_i^1\} (J-1) > 0 \\ \{\Gamma_i^2\}_i \text{ are such that } \sum_i \Gamma_i^2 - \max_i \{\Gamma_i^2\} (J-1) > 0 \end{cases}$$

The next theorem gives us an explicit form of the equilibrium of the game when all sources send messages.

**Theorem 7.** *Assume that (H1) and (H2) are satisfied. In this case the Nash Equilibrium is uniquely defined by:*

$$(\lambda_j^{1*}, \lambda_j^{2*}) = \begin{cases} (a_j^1, 0) & \text{if } \Lambda_j^{1*} < \Lambda_j^{2*} \\ (0, a_j^2) & \text{if } \Lambda_j^{1*} > \Lambda_j^{2*} \end{cases}$$

with

$$\begin{aligned} a_j^1 &= p_{21} \frac{(J-1)K^2}{\sum_i \gamma_i^2 - \gamma_i^1 p_{12}} \left(1 - \frac{(J-1)(\gamma_j^2 - \gamma_j^1 p_{12})}{\sum_i \gamma_i^2 - \gamma_i^1 p_{12}}\right) \\ &\quad - \frac{(J-1)K^1}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}} \left(1 - \frac{(J-1)(\gamma_j^1 - \gamma_j^2 p_{21})}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}}\right) \\ a_j^2 &= p_{12} \frac{(J-1)K^1}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}} \left(1 - \frac{(J-1)(\gamma_j^1 - \gamma_j^2 p_{21})}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}}\right) \\ &\quad - \frac{(J-1)K^2}{\sum_i \gamma_i^2 - \gamma_i^1 p_{12}} \left(1 - \frac{(J-1)(\gamma_j^2 - \gamma_j^1 p_{12})}{\sum_i \gamma_i^2 - \gamma_i^1 p_{12}}\right) \\ \Lambda_j^{1*} &= \frac{(J-1)K^2(p_{12} p_{21} - 1)}{\sum_i \gamma_i^2 - \gamma_i^1 p_{12}} \left(1 - \frac{(J-1)(\gamma_j^2 - \gamma_j^1 p_{12})}{\sum_i \gamma_i^2 - \gamma_i^1 p_{12}}\right) \\ \Lambda_j^{2*} &= \frac{(J-1)K^1(p_{12} p_{21} - 1)}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}} \left(1 - \frac{(J-1)(\gamma_j^1 - \gamma_j^2 p_{21})}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}}\right) \end{aligned}$$

*Proof.* The proof follows two steps. Firstly we compute the equilibrium of (4) and secondly we make the change of variables in order to have the value of the direct flow of each sources at equilibrium.

Step 1: In order to find the equilibrium of (4) we have to compute the best reply. We do it just for one  $\Lambda_j^1$ . It works the same way as  $\Lambda_j^1$  for all the other  $j$  and group 2.

$$\begin{aligned} \frac{\partial U_j}{\partial \Lambda_j^1}(\Lambda_j, \Lambda_{-j}) &= 0 \\ K^1 \left( \frac{1}{\sum_i \Lambda_i^1} - \frac{\Lambda_j^1}{(\sum_i \Lambda_i^1)^2} \right) - \Gamma_j^1 &= 0 \end{aligned} \quad (5)$$

We remark that the best reply just depend of  $\{\lambda_j^1\}_j$  and it is equivalent to the best reply of the game (2) where the objectif function for each player  $j$  is:

$$\max_{\Lambda_j^1} U_j(\Lambda_j, \Lambda_{-j}) = K^1 \frac{\Lambda_j^1}{\sum_i \Lambda_i^1} - \Gamma_j^1 \Lambda_j^1$$

This is why by using theorem 3 we have that

$$\forall j, (\Lambda_j^{1*}, \Lambda_j^{2*}) = \left( \frac{(J-1)K^1}{\sum_i \Gamma_i^1} \left(1 - \frac{\Phi^1 \Gamma_j^1}{K^1}\right), \frac{(J-1)K^2}{\sum_i \Gamma_i^2} \left(1 - \frac{\Phi^2 \Gamma_j^2}{K^2}\right) \right) \quad (6)$$

with  $(\Phi^1, \Phi^2) = \left( \frac{(J-1)K^1}{\sum_i \Gamma_i^1}, \frac{(J-1)K^2}{\sum_i \Gamma_i^2} \right)$ .

Step 2: In the proof of proposition 10 we have prove that  $(\lambda_j^1, \lambda_j^2) = \left( \frac{\Lambda_j^2 p_{21} - \Lambda_j^1}{p_{12} p_{21} - 1}, \frac{\Lambda_j^1 p_{12} - \Lambda_j^2}{p_{12} p_{21} - 1} \right)$ . This is why by using this fact we have:

$$\begin{aligned} \lambda_j^1 &= p_{21} \frac{(J-1)K^2}{\sum_i \gamma_i^2 - \gamma_i^2 p_{12}} \left(1 - \frac{(J-1)(\gamma_j^2 - \gamma_j^1 p_{12})}{\sum_i \gamma_i^2 - \gamma_i^2 p_{12}}\right) \\ &\quad - \frac{(J-1)K^1}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}} \left(1 - \frac{(J-1)(\gamma_j^1 - \gamma_j^2 p_{21})}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}}\right) \\ \lambda_j^2 &= p_{12} \frac{(J-1)K^1}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}} \left(1 - \frac{(J-1)(\gamma_j^1 - \gamma_j^2 p_{21})}{\sum_i \gamma_i^1 - \gamma_i^2 p_{21}}\right) \\ &\quad - \frac{(J-1)K^2}{\sum_i \gamma_i^2 - \gamma_i^1 p_{12}} \left(1 - \frac{(J-1)(\gamma_j^2 - \gamma_j^1 p_{12})}{\sum_i \gamma_i^2 - \gamma_i^1 p_{12}}\right) \end{aligned}$$

Now in (H1) we have assume that  $\Lambda_j^{2*} p_{21} - \Lambda_j^{1*} > 0$  or  $\Lambda_j^{1*} p_{12} - \Lambda_j^{2*} > 0$ . We can remark that if  $\Lambda_j^{2*} > \Lambda_j^{1*}$  then  $\Lambda_j^{1*} p_{12} - \Lambda_j^{2*} < 0$  which implies that  $\lambda_j^{2*} = 0$ . And if  $\Lambda_j^{2*} < \Lambda_j^{1*}$  then  $\Lambda_j^{2*} p_{21} - \Lambda_j^{1*} < 0$  which implies that  $\lambda_j^{1*} = 0$ . This is why a player send messages just on one wall.  $\square$

The previous theorem shows that under condition (H1) and (H2) all the sources send messages. And a another important result is that each source just sends messages on one group. In the next section we study the effect of the parameter on the equilibrium.

## VI. NUMERICAL ILLUSTRATIONS

We make our numerical illustrations with  $\gamma_j^1 = \gamma_j^2 = \gamma$  for all  $j$ . We start fig. 2. at  $p_{12} = 0.75$  because with our parameters, when  $p_{12} < 0.75$  the condition (H1) is not verify. We can observe that  $\lambda_j^{1*}$  increasing in  $p_{12}$  and  $p_{21}$ .

In fig. 3. we can observe that sources (because symmetric sources have a symmetric behavior) prefer to send messages in group 2 when  $p_{12} < 0.4$  and  $p_{21} = 0.45$  or when  $p_{12} < 0.7$  and  $p_{21} = 0.7$ . And they prefer to send to group 1 in the other

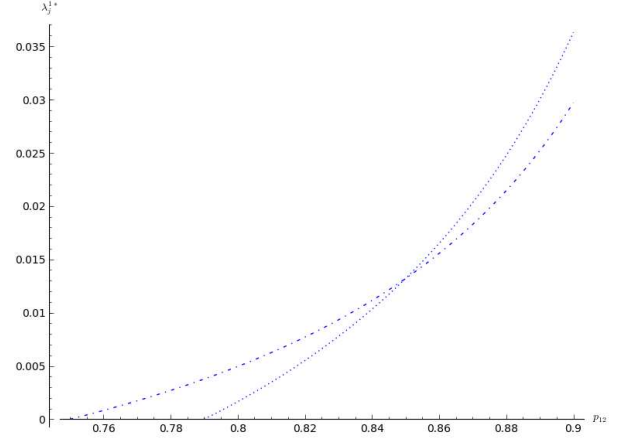


Fig. 2. Behavior of  $\lambda_j^{1*}$  under  $p_{12}$  with  $\gamma = 10$ ,  $K^1 = K^2 = 10$ ,  $J = 100$ ,  $p_{21} = 0.45$  for  $-$ . and  $p_{21} = 0.7$  for  $\dots$

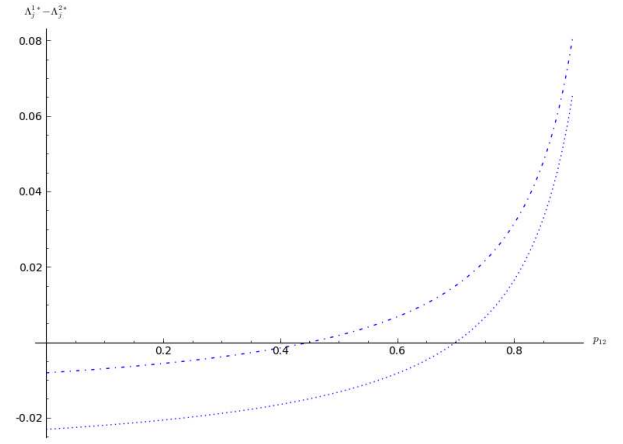


Fig. 3. Behavior of  $\Lambda_j^{1*} - \Lambda_j^{2*}$  under  $p_{12}$  with  $\gamma = 10$ ,  $K^1 = K^2 = 10$ ,  $J = 100$ ,  $p_{21} = 0.45$  for  $-$ . and  $p_{21} = 0.7$  for  $\dots$

case. You can see that when  $p_{12}$  increase you have a preference for sending message in group 1 and when  $p_{21}$  increase you have a preference for sending message in group 2.

## VII. CONCLUSIONS

In this paper, we model a competition between information sources that used a social network group in order to disseminate and to maximize his visibility, particularly the visibility of his messages inside a group. We take into consideration that the messages on a group (in fact a newsfeed of a group) are of interest of the subscribers only if the messages are in the first  $K$  positions (more  $K$  recent). We have modelled the competition between sources using a noncooperative game and we have proved that this non-cooperative game is a standard game well known in telecommunication which has been studied in different contexts (resource sharing, communication, etc). We have also considered an interesting feature taking into account several groups and the possibility for a message to be copied and relayed from one group to another one. Then we have generalized our result and prove the existence of the

equilibrium. In future works we plan to analyze more complex networks of groups in which messages can be propagated between several groups depending on the behavior of the group subscribers.

#### REFERENCES

- [1] E. Altman, "Game theoretic approaches for studying competition over popularity and over advertisement space in social networks", Valuetools, 2012.
- [2] R. T. Ma, D. M. Chiu, J. C. Lui, V. Misra, D. Rubenstein, " On resource management for cloud users: A generalized Kelly mechanism approach", Electrical Engineering, Tech. Rep., (2010).
- [3] J. Ramesh, J. N. Tsitsiklis. "A game theoretic view of efficiency loss in resource allocation." Advances in Control, Communication Networks, and Transportation Systems. Birkhuser Boston, 2005. 203-223.
- [4] Baccelli, F., Brmaud, P. (2003). Elements of queueing theory: Palm Martingale calculus and stochastic recurrences (Vol. 26). Springer.
- [5] Sinai, I. G. (1976). Introduction to ergodic theory (Vol. 18). Princeton University Press.
- [6] Tullock, G. (2001). Efficient rent seeking. In Efficient Rent-Seeking (pp. 3-16). Springer US.
- [7] Chu, S. C. (2011). Viral advertising in social media: Participation in Facebook groups and responses among college-aged users. Journal of Interactive Advertising, 12(1), 30-43.
- [8] Kushin, M. J., Kitchener, K. (2009). Getting political on social network sites: Exploring online political discourse on Facebook. First Monday, 14(11).
- [9] Small, T. (2008). The Facebook effect? on-line campaigning in the 2008 Canadian and US elections. POLICY, 85.
- [10] Tucker, C. (2012). Social advertising. Available at SSRN 1975897.
- [11] Monderer, D., Shapley, L. S. (1996). Potential games. Games and Economic Behavior, 14(1), 124-143.
- [12] Altman, E., Kumar, P., Venkatramanan, S., Kumar, A. Competition Over Timeline in Social Networks. ASSONAM 2013
- [13] Baccelli, F., Brmaud, P. (2003). Elements of queueing theory: Palm Martingale calculus and stochastic recurrences (Vol. 26). Springer.
- [14] Nisan, N. (Ed.). (2007). Algorithmic game theory. Cambridge University Press.