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Explicit coupling schemes for a fluid-fluid interaction problem arising in hemodynamics

Miguel A. Fernández^{*†}, Jean-Frédéric Gerbeau^{*†}, Saverio Smaldone^{*†}

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Abstract: In this work we propose a new approach to the loosely coupled time-marching of a fluid-fluid interaction problems involving the incompressible Navier-Stokes equations. The methods combine a specific explicit Robin-Robin treatment of the interface coupling with a weakly consistent interface pressure stabilization in time. A priori energy estimates guaranteeing stability of the splitting are obtained for a total pressure formulation of the coupled problem. The performance of the proposed schemes is illustrated on several numerical experiments related to simulation of aortic blood flow.

Key-words: fluid-structure interaction, incompressible fluid, time-discretization, loosely coupled scheme, Robin-Robin scheme, finite element method, Nitsche's method.

* Inria, REO project-team, Rocquencourt - B.P. 105, F-78153 Le Chesnay cedex, France

† UPMC Université Paris VI, REO project-team, UMR 7958 LJLL, F-75005 Paris, France

**RESEARCH CENTRE
PARIS – ROCQUENCOURT**

Domaine de Voluceau, - Rocquencourt
B.P. 105 - 78153 Le Chesnay Cedex

Schémas de couplage explicites pour un problème d'interaction fluide-fluide en hémodynamique

Résumé : Dans cet article, nous proposons une nouvelle approche pour le couplage faible de deux fluides incompressibles régis par les équations de Navier-Stokes. Ces méthodes combinent un traitement spécifique Robin-Robin explicite des conditions d'interface avec une stabilisation de la pression à l'interface faiblement consistante en temps. Une estimation d'énergie a priori garantissant la stabilité du découplage en temps est obtenue pour une formulation en pression totale du système couplé. Les propriétés des méthodes proposées sont illustrées dans des expériences numériques motivées par la simulation de la circulation du sang dans la aorte.

Mots-clés : interaction fluide-fluide, fluide incompressible, discrétisation en temps, schéma de couplage faible, schéma Robin-Robin, méthode des éléments finis, méthode de Nitsche.

1 Introduction

Global and local features in haemodynamics are generally described by coupling dimensionally heterogeneous models of blood flow. In this geometrical multiscale paradigm, the three-dimensional (3D) Navier-Stokes equations are coupled with one-dimensional (1D) and/or zero-dimensional (0D) models that account for the surrounding part of the 3D compartment (see, e.g., [4, 12, 25]).

In general, several physiological phenomena can coexist within a single 3D compartment. For instance, if a detailed modeling of blood flow in the left ventricle and the aorta is sought for, the 3D model must incorporate additional features, such as the deformation of the cardiac cavity and the opening-closing dynamics of the aortic valves. The underlying motivation of this work is the partitioning of the 3D computational domain into several complementary non-overlapping subdomains, in which these additional modeling features take place. We propose to advance in time the system via specific explicit coupling (or *loosely coupled*) strategies that allow the uncoupled time-stepping of the sub-compartments.

As a preliminary step in this direction, in this work we focus on the case of two sub-compartments described by the incompressible Navier-Stokes equations. The two sub-problems are coupled, across their shared (artificial) interface, by standard kinematic/kinetic conditions, viz., continuity of velocity and stress. We address the uncoupled time-marching of the two sub-compartments via explicit coupling schemes. Note that this approach clearly differs from [20], where partitioned iterative procedures acting on averaged/integrated interface quantities are proposed.

To this purpose, a major difficulty that has to be faced is related to the artificial energy transfers induced by the interface time-splitting, which can lead to numerical instabilities. For instance, if a standard Dirichlet-Neumann scheme is foreseen, the kinematic splitting generates two sources of instability, respectively related to the unbalance of static and dynamic pressure powers across the interface. It should be noted that this issue does not appear in fluid-fluid interaction models based on *rigid-lid* conditions, commonly used in oceanography modeling (see, e.g., [5, 9, 10, 23]), for which the normal velocity component vanishes on the interface.

In this work, the unbalanced static pressure power is controlled via a specific explicit Robin-Robin treatment of the interface coupling (based on Nitsche's method) and a weakly consistent interface pressure stabilization in time. Basically, this approach extends the loosely coupled schemes introduced in [6, 7] to fluid-fluid interaction problems. We also show that the second source of instabilities (unbalanced dynamic pressure power) can be treated, in a natural fashion, by applying this method to a total pressure formulation of the Navier-Stokes equations (see [18]).

Several numerical experiments, related to the simulation of aortic blood flow, illustrate the accuracy and performance of the schemes proposed.

The paper is organized as follows. In the next section, the incompressible Navier-Stokes equations are split into two sub-problems coupled across an artificial interface. A Nitsche's space semi-discrete formulation of this fluid-fluid interaction problem is presented in Section 3. Section 4 is devoted to the time-discretization. We present different implicit and explicit coupling strategies, in the static and total pressure formulations, and state their corresponding discrete energy balances. The numerical experiments are reported in Section 5. Finally, Section 6 summarizes the conclusions.

2 Coupled fluid-fluid problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded fixed domain and a fluid modelled by the incompressible Navier-Stokes equations. The velocity $\mathbf{u} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ and the pressure $p : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfy:

$$\begin{aligned} \rho \partial_t \mathbf{u} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, p) &= 0, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0, & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0}, & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with the initial conditions $\mathbf{u}(0) = \mathbf{u}_0$. The term $\boldsymbol{\sigma}(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\boldsymbol{\epsilon}(\mathbf{u})$, where $\boldsymbol{\epsilon}(\mathbf{u}) = (\nabla\mathbf{u} + \nabla^T\mathbf{u})/2$, is the fluid stress tensor, and ρ and μ stand for the fluid density and the dynamic viscosity respectively. For the sake of the analysis, homogeneous Dirichlet conditions are enforced on the whole boundary $\partial\Omega$. More realistic boundary conditions are considered in the numerical experiments of Section 5.

Problem (1) can be partitioned into two sub-problems defined in two non-overlapping sub-domains, $\Omega_1 \subset \mathbb{R}^d$ and $\Omega_2 \subset \mathbb{R}^d$, with $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. The two sub-domains are separated by an interface $\Sigma := \partial\Omega_1 \cap \partial\Omega_2$, and $\Gamma_1 := \partial\Omega_1 \setminus \Sigma$, $\Gamma_2 := \partial\Omega_2 \setminus \Sigma$.

The new formulation of the problem reads: find the velocity $\mathbf{u}_1 : \Omega_1 \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ and the pressure $p_1 : \Omega_1 \times \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \rho \partial_t \mathbf{u}_1 + \rho(\mathbf{u}_1 \cdot \nabla) \mathbf{u}_1 - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_1, p_1) &= 0, & \text{in } \Omega_1, \\ \nabla \cdot \mathbf{u}_1 &= 0, & \text{in } \Omega_1, \\ \mathbf{u}_1 &= \mathbf{0}, & \text{on } \Gamma_1, \end{aligned} \tag{2}$$

find the velocity $\mathbf{u}_2 : \Omega_2 \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ and the pressure $p_2 : \Omega_2 \times \mathbb{R}^+ \rightarrow \mathbb{R}$, such that

$$\begin{aligned} \rho \partial_t \mathbf{u}_2 + \rho(\mathbf{u}_2 \cdot \nabla) \mathbf{u}_2 - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}_2, p_2) &= 0, & \text{in } \Omega_2, \\ \nabla \cdot \mathbf{u}_2 &= 0, & \text{in } \Omega_2, \\ \mathbf{u}_2 &= \mathbf{0}, & \text{on } \Gamma_2. \end{aligned} \tag{3}$$

The solutions in the two sub-domains are coupled through the usual kinematic and kinetic conditions:

$$\begin{cases} \mathbf{u}_1 = \mathbf{u}_2, & \text{on } \Sigma, \\ \boldsymbol{\sigma}(\mathbf{u}_2, p_2) \cdot \mathbf{n}_2 = -\boldsymbol{\sigma}(\mathbf{u}_1, p_1) \cdot \mathbf{n}_1, & \text{on } \Sigma, \end{cases} \tag{4}$$

with $\mathbf{n}_1, \mathbf{n}_2$ the outward-pointing unit normal vectors on $\partial\Omega_1$ and $\partial\Omega_2$ respectively.

3 Space semi-discretization via interface Nitsche's method

The time splitting of problem (2)-(4) that will be presented in the following sections is based on a space discretization using Nitsche's formulation ([15, 22]). This method, used to impose weakly Dirichlet boundary conditions, has been used for many problems, in particular for Stokes ([8, 16]) and Navier-Stokes equations ([26]) as well as for fluid-structure interaction problems ([6, 7]).

3.1 Monolithic formulation

Let be $\{\mathcal{T}_{1,h}\}_{h>0}$ and $\{\mathcal{T}_{2,h}\}_{h>0}$ two families of regular finite element triangulation for Ω_1 and Ω_2 respectively, with typical diameter h . Let $W_{i,h} \times Q_{i,h}$ denote an inf-sup stable, conforming finite

element approximations of $[H^1(\Omega_i)]^d \times L^2(\Omega_i)$ for $i = 1, 2$. Let $V_{1,h} = W_{1,h} \cap [H_{\Gamma_1}^1(\Omega_1)]^d$ and $V_{2,h} = W_{2,h} \cap [H_{\Gamma_2}^1(\Omega_2)]^d$, where $[H_\gamma^1(\Omega_i)]^d$ denotes the space of $[H^1(\Omega_i)]^d$ functions vanishing on a part γ of $\partial\Omega_i$.

The considered space semi-discrete formulation of problem (2)-(4) reads: for all $t > 0$, find $(\mathbf{u}_{1,h}, p_{1,h}, \mathbf{u}_{2,h}, p_{2,h}) \in W_{1,h} \times Q_{1,h} \times W_{2,h} \times Q_{2,h}$ satisfying the essential boundary conditions and such that

$$\begin{aligned} & A_1[\mathbf{u}_{1,h}; (\mathbf{u}_{1,h}, p_{1,h}), (\mathbf{v}_{1,h}, q_{1,h})] + A_2[\mathbf{u}_{2,h}; (\mathbf{u}_{2,h}, p_{2,h}), (\mathbf{v}_{2,h}, q_{2,h})] \\ & + C_1[\mathbf{u}_{1,h}; (\mathbf{u}_{1,h}, \mathbf{u}_{2,h}), \mathbf{v}_{1,h}] + C_2[(\mathbf{u}_{2,h}, \mathbf{u}_{1,h}); (\mathbf{u}_{2,h}, \mathbf{u}_{1,h}), \mathbf{v}_{2,h}] \\ & + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{2,h} - \mathbf{u}_{1,h}) \cdot (\mathbf{v}_{2,h} - \mathbf{v}_{1,h}) - \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}, p_{2,h}) \cdot \mathbf{n}_2 \cdot (\mathbf{v}_{2,h} - \mathbf{v}_{1,h}) \\ & - \int_{\Sigma} (\mathbf{u}_{2,h} - \mathbf{u}_{1,h}) \cdot \mathbf{n}_2 q_{2,h} = 0, \end{aligned} \quad (5)$$

for all $(\mathbf{v}_{1,h}, q_{1,h}, \mathbf{v}_{2,h}, q_{2,h}) \in V_{1,h} \times Q_{1,h} \times V_{2,h} \times Q_{2,h}$, where

$$\begin{aligned} A_1[\mathbf{u}_{1,h}; (\mathbf{u}_{1,h}, p_{1,h}), (\mathbf{v}_{1,h}, q_{1,h})] := & \rho \int_{\Omega_1} \partial_t \mathbf{u}_{1,h} \cdot \mathbf{v}_{1,h} + \int_{\Omega_1} 2\mu \boldsymbol{\epsilon}(\mathbf{u}_{1,h}) : \boldsymbol{\epsilon}(\mathbf{v}_{1,h}) \\ & - \int_{\Omega_1} p_{1,h} \nabla \cdot \mathbf{v}_{1,h} + \int_{\Omega_1} q_{1,h} \nabla \cdot \mathbf{u}_{1,h} \\ & + \rho \int_{\Omega_1} \mathbf{u}_{1,h} \cdot \nabla \mathbf{u}_{1,h} \cdot \mathbf{v}_{1,h}, \end{aligned} \quad (6)$$

$$\begin{aligned} A_2[\mathbf{u}_{2,h}; (\mathbf{u}_{2,h}, p_{2,h}), (\mathbf{v}_{2,h}, q_{2,h})] := & \rho \int_{\Omega_2} \partial_t \mathbf{u}_{2,h} \cdot \mathbf{v}_{2,h} + \int_{\Omega_2} 2\mu \boldsymbol{\epsilon}(\mathbf{u}_{2,h}) : \boldsymbol{\epsilon}(\mathbf{v}_{2,h}) \\ & - \int_{\Omega_2} p_{2,h} \nabla \cdot \mathbf{v}_{2,h} + \int_{\Omega_2} q_{2,h} \nabla \cdot \mathbf{u}_{2,h} \\ & + \rho \int_{\Omega_2} \mathbf{u}_{2,h} \cdot \nabla \mathbf{u}_{2,h} \cdot \mathbf{v}_{2,h}, \end{aligned} \quad (7)$$

and the two terms C_1 and C_2 are defined by:

$$C_1[\mathbf{u}; (\mathbf{w}, \mathbf{z}), \mathbf{v}] := \frac{\rho}{2} \int_{\Omega_1} (\nabla \cdot \mathbf{u}) \mathbf{w} \cdot \mathbf{v} + \frac{\rho}{2} \int_{\Sigma} \mathbf{u} \cdot \mathbf{n}_1 (\mathbf{z} - \mathbf{w}) \cdot \mathbf{v}, \quad (8)$$

$$\begin{aligned} C_2[(\mathbf{u}, \boldsymbol{\xi}); (\mathbf{w}, \mathbf{z}), \mathbf{v}] := & \frac{\rho}{2} \int_{\Omega_2} (\nabla \cdot \mathbf{u}) \mathbf{w} \cdot \mathbf{v} - \frac{\rho}{2} \int_{\Sigma} (\mathbf{u} \cdot \mathbf{n}_2) (\mathbf{w} \cdot \mathbf{v}) \\ & + \frac{\rho}{2} \int_{\Sigma} (\boldsymbol{\xi} \cdot \mathbf{n}_2) (\mathbf{z} \cdot \mathbf{v}). \end{aligned} \quad (9)$$

The first integral in (8) and (9) is nothing but the standard Temam's trick (see, e.g., [27]). The remaining interface terms are also strongly consistent terms, which are introduced to handle the convective energy contributions related to the discontinuous nature of the discrete solution on the interface.

The fifth and sixth terms in the left-hand side of (5) result from Nitsche's method. The term $\gamma > 0$ is the Nitsche's penalty parameter, it is dimensionless and is chosen to ensure the stability. Note that $\mathbf{v}_{1,h}$ and $\mathbf{v}_{2,h}$ are independent, and that $\mathbf{u}_{1,h}$ and $\mathbf{u}_{2,h}$ do not exactly match on the interface.

The seventh term of (5) perturbs the divergence free constraint in Ω_2 in a strongly consistent way (in the sense that it vanishes for solutions continuous at the interface). It is introduced to control the pressure contribution of the stress terms appearing on the interface.

Remark 1 The sixth term in (5), involving the stress tensor on the interface, is computed with the Fluid 2 variables. It could be replaced by $-\int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{1,h}, p_{1,h}) \cdot \mathbf{n}_1(\mathbf{v}_{1,h} - \mathbf{v}_{2,h})$, i.e. computed with Fluid 1 variables, without compromising the stability of the method. In this case the seventh term (5) would be $\int_{\Omega_1} q_{1,h} \nabla \cdot \mathbf{u}_{1,h} = -\int_{\Sigma} (\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) \cdot \mathbf{n}_1 q_{1,h}$.

3.1.1 Stability analysis

The following lemma shows the stability in the energy norm of the semi-discrete problem (5). It makes use of the standard local inverse trace inequality (see [28] e.g.): for all $K \in \{\mathcal{T}_h\}_{h>0}$,

$$\|\mathbf{u}_h\|_{\partial K}^2 \leq C_T h^{-1} \|\mathbf{u}_h\|_K^2 \quad \forall \mathbf{u}_h \in W_h, \quad (10)$$

where $\|\cdot\|_K$ denotes the $L^2(K)$ -norm.

Lemma 1 *Let $(\mathbf{u}_{1,h}, p_{1,h}, \mathbf{u}_{2,h}, p_{2,h})$ be solution of (5). If $\gamma < 2C_T$, where $C_T > 0$ is defined in (10), the following estimate holds*

$$\begin{aligned} & \frac{\rho}{2} (\|\mathbf{u}_{1,h}(t)\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}(t)\|_{\Omega_2}^2) + \frac{3\mu}{4} \int_0^t (\|\boldsymbol{\epsilon}(\mathbf{u}_{1,h})\|_{\Omega_1}^2 + \|\boldsymbol{\epsilon}(\mathbf{u}_{2,h})\|_{\Omega_2}^2) \\ & + \frac{3\gamma\mu}{4h} \int_0^t \|\mathbf{u}_{2,h} - \mathbf{u}_{1,h}\|_{\Sigma}^2 \leq \frac{\rho}{2} (\|\mathbf{u}_{1,h}(0)\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}(0)\|_{\Omega_2}^2). \end{aligned}$$

Proof. Taking $(\mathbf{v}_{1,h}, q_{1,h}, \mathbf{v}_{2,h}, q_{2,h}) = (\mathbf{u}_{1,h}, p_{1,h}, \mathbf{u}_{2,h}, p_{2,h})$ in A_1 and A_2 yields

$$\begin{aligned} A_1[\mathbf{u}_{1,h}; (\mathbf{u}_{1,h}, p_{1,h}), (\mathbf{u}_{1,h}, p_{1,h})] &= \frac{\rho}{2} \frac{d}{dt} \|\mathbf{u}_{1,h}\|_{\Omega_1}^2 + 2\mu \|\boldsymbol{\epsilon}(\mathbf{u}_{1,h})\|_{\Omega_1}^2 \\ &\quad - \frac{\rho}{2} \int_{\Omega_1} (\nabla \cdot \mathbf{u}_{1,h}) |\mathbf{u}_{1,h}|^2 \\ &\quad + \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{1,h} \cdot \mathbf{n}_{1,h}) |\mathbf{u}_{1,h}|^2, \end{aligned} \quad (11)$$

$$\begin{aligned} A_2[\mathbf{u}_{2,h}; (\mathbf{u}_{2,h}, p_{2,h}), (\mathbf{u}_{2,h}, p_{2,h})] &= \frac{\rho}{2} \frac{d}{dt} \|\mathbf{u}_{2,h}\|_{\Omega_2}^2 + 2\mu \|\boldsymbol{\epsilon}(\mathbf{u}_{2,h})\|_{\Omega_2}^2 \\ &\quad - \frac{\rho}{2} \int_{\Omega_2} (\nabla \cdot \mathbf{u}_{2,h}) |\mathbf{u}_{2,h}|^2, \\ &\quad + \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{2,h} \cdot \mathbf{n}_{1,h}) |\mathbf{u}_{2,h}|^2. \end{aligned} \quad (12)$$

In addition, for the terms C_1 and C_2 we have

$$\begin{aligned} C_1[\mathbf{u}_{1,h}; (\mathbf{u}_{1,h}, \mathbf{u}_{2,h}), \mathbf{u}_{1,h}] &= \frac{\rho}{2} \int_{\Omega_1} (\nabla \cdot \mathbf{u}_{1,h}) |\mathbf{u}_{1,h}|^2 \\ &\quad + \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{1,h} \cdot \mathbf{n}_1) (\mathbf{u}_{2,h} \cdot \mathbf{u}_{1,h} - |\mathbf{u}_{1,h}|^2), \end{aligned} \quad (13)$$

$$\begin{aligned} C_2[(\mathbf{u}_{2,h}, \mathbf{u}_{1,h}); (\mathbf{u}_{2,h}, \mathbf{u}_{1,h}), \mathbf{u}_{2,h}] &= \frac{\rho}{2} \int_{\Omega_2} (\nabla \cdot \mathbf{u}_{2,h}) |\mathbf{u}_{2,h}|^2 \\ &\quad - \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{2,h} \cdot \mathbf{n}_2) |\mathbf{u}_{2,h}|^2 \\ &\quad + \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{1,h} \cdot \mathbf{n}_2) (\mathbf{u}_{1,h} \cdot \mathbf{u}_{2,h}). \end{aligned} \quad (14)$$

Hence by inserting (11)–(14) into (5) we get the following energy equality

$$\begin{aligned} & \frac{\rho}{2} \frac{d}{dt} (\|\mathbf{u}_{1,h}\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}\|_{\Omega_2}^2) + 2\mu (\|\boldsymbol{\epsilon}(\mathbf{u}_{1,h})\|_{\Omega_1}^2 + \|\boldsymbol{\epsilon}(\mathbf{u}_{2,h})\|_{\Omega_2}^2) \\ & + \frac{\gamma\mu}{h} \|\mathbf{u}_{2,h} - \mathbf{u}_{1,h}\|_{\Sigma}^2 = 2\mu \int_{\Sigma} \boldsymbol{\epsilon}(\mathbf{u}_{2,h}) \cdot \mathbf{n}_2 (\mathbf{u}_{2,h} - \mathbf{u}_{1,h}). \end{aligned} \quad (15)$$

The right-hand side of (15) is first treated with the Young's inequality and with the local inverse trace inequality (10):

$$2\mu \int_{\Sigma} \boldsymbol{\epsilon}(\mathbf{u}_{2,h}) \mathbf{n}_2 \cdot (\mathbf{u}_{2,h} - \mathbf{u}_{1,h}) \leq \frac{4\mu C_T}{\gamma} \|\boldsymbol{\epsilon}(\mathbf{u}_{2,h})\|_{\Omega_2}^2 + \frac{\mu\gamma}{4h} \|\mathbf{u}_{2,h} - \mathbf{u}_{1,h}\|_{\Sigma}^2.$$

Using this inequality in (15) and integrating over $(0, t)$, it gives:

$$\begin{aligned} & \frac{\rho}{2} (\|\mathbf{u}_{1,h}(t)\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}(t)\|_{\Omega_2}^2) + 2\mu \int_0^t \|\boldsymbol{\epsilon}(\mathbf{u}_{1,h})\|_{\Omega_1}^2 + 2\mu \left(1 - \frac{2C_T}{\gamma}\right) \int_0^t \|\boldsymbol{\epsilon}(\mathbf{u}_{2,h})\|_{\Omega_2}^2 \\ & + \frac{3\gamma\mu}{4h} \int_0^t \|\mathbf{u}_{2,h} - \mathbf{u}_{1,h}\|_{\Sigma}^2 \leq \frac{\rho}{2} (\|\mathbf{u}_{1,h}(0)\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}(0)\|_{\Omega_2}^2). \end{aligned}$$

The last result, with condition $\gamma < 2C_T$, completes the proof. \square

3.2 Partitioned formulation

Our goal is to devise an algorithm that yields separate solutions in domain Ω_1 and Ω_2 via suitable transmission conditions on the interface Σ . Taking first $(\mathbf{v}_{2,h}, q_{2,h}) = (\mathbf{0}, 0)$ and then $(\mathbf{v}_{1,h}, q_{1,h}) = (\mathbf{0}, 0)$, the monolithic formulation (5) can be split in two coupled sub-problems:

1. Fluid sub-problem 1: Given $(\mathbf{u}_{2,h}, p_{2,h}) \in W_{2,h} \times Q_{2,h}$, find $(\mathbf{u}_{1,h}, p_{1,h}) \in W_{1,h} \times Q_{1,h}$ such that

$$\begin{aligned} & A_1[\mathbf{u}_{1,h}; (\mathbf{u}_{1,h}, p_{1,h}), (\mathbf{v}_{1,h}, q_{1,h})] + C_1[\mathbf{u}_{1,h}; (\mathbf{u}_{1,h}, \mathbf{u}_{2,h}), \mathbf{v}_{1,h}] \\ & + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{1,h} - \mathbf{u}_{2,h}) \cdot \mathbf{v}_{1,h} + \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}, p_{2,h}) \cdot \mathbf{n}_2 \mathbf{v}_{1,h} = 0 \end{aligned}$$

for all $(\mathbf{v}_{1,h}, p_{1,h}) \in V_{1,h} \times Q_{1,h}$.

2. Fluid sub-problem 2: Given $(\mathbf{u}_{1,h}, p_{1,h}) \in W_{1,h} \times Q_{1,h}$, find $(\mathbf{u}_{2,h}, p_{2,h}) \in W_{2,h} \times Q_{2,h}$ such that

$$\begin{aligned} & A_2[\mathbf{u}_{2,h}; (\mathbf{u}_{2,h}, p_{2,h}), (\mathbf{v}_{2,h}, q_{2,h})] + C_2[(\mathbf{u}_{2,h}, \mathbf{u}_{1,h}); (\mathbf{u}_{2,h}, \mathbf{u}_{1,h}), \mathbf{v}_{2,h}] \\ & + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{2,h} - \mathbf{u}_{1,h}) \cdot \mathbf{v}_{2,h} - \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}, p_{2,h}) \cdot \mathbf{n}_2 \mathbf{v}_{2,h} \\ & - \int_{\Sigma} (\mathbf{u}_{2,h} - \mathbf{u}_{1,h}) \cdot \mathbf{n}_2 q_{2,h} = 0 \end{aligned}$$

for all $(\mathbf{v}_{2,h}, p_{2,h}) \in W_{2,h} \times Q_{2,h}$.

This partitioned formulation is the basis of the staggered algorithms presented in the next section.

4 Time-discretization: fully discrete formulations

The present section is devoted to the time discretization of formulation (5). A backward Euler scheme is used and the non-linear terms are linearized with a standard semi-implicit approach. The time derivatives in (6) and (7) are replaced by $\partial_{\delta t} x^n := (x^n - x^{n-1})/\delta t$, where $\delta t := T/N$ denotes the step size, the interval of interest is $(0, T)$, $N \in \mathbb{N}^+$ is a given integer and $x^n \approx x(n\delta t)$ with $0 \leq n \leq N$. The two discrete counterparts of operators A_1 and A_2 are defined by:

$$\begin{aligned} A_{1,\delta t}[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, p_{1,h}^n), (\mathbf{v}_{1,h}, q_{1,h})] &:= \rho \int_{\Omega_1} \partial_{\delta t} \mathbf{u}_{1,h}^n \cdot \mathbf{v}_{1,h} + 2\mu \int_{\Omega_1} \boldsymbol{\epsilon}(\mathbf{u}_{1,h}^n) : \boldsymbol{\epsilon}(\mathbf{v}_{1,h}) \\ &\quad - \int_{\Omega_1} p_{1,h}^n \nabla \cdot \mathbf{v}_{1,h} + \int_{\Omega_1} q_{1,h} \nabla \cdot \mathbf{u}_{1,h}^n \\ &\quad + \rho \int_{\Omega_1} \mathbf{u}_{1,h}^{n-1} \cdot \nabla \mathbf{u}_{1,h}^n \cdot \mathbf{v}_{1,h}, \end{aligned} \quad (16)$$

$$\begin{aligned} A_{2,\delta t}[\mathbf{u}_{2,h}^{n-1}; (\mathbf{u}_{2,h}^n, p_{2,h}^n), (\mathbf{v}_{2,h}, q_{2,h})] &:= \rho \int_{\Omega_2} \partial_{\delta t} \mathbf{u}_{2,h}^n \cdot \mathbf{v}_{2,h} + 2\mu \int_{\Omega_2} \boldsymbol{\epsilon}(\mathbf{u}_{2,h}^n) : \boldsymbol{\epsilon}(\mathbf{v}_{2,h}) \\ &\quad - \int_{\Omega_2} p_{2,h}^n \nabla \cdot \mathbf{v}_{2,h} + \int_{\Omega_2} q_{2,h} \nabla \cdot \mathbf{u}_{2,h}^n \\ &\quad + \rho \int_{\Omega_2} \mathbf{u}_{2,h}^{n-1} \cdot \nabla \mathbf{u}_{2,h}^n \cdot \mathbf{v}_{2,h}. \end{aligned} \quad (17)$$

4.1 Implicit coupling

Before addressing the case of a staggered time-stepping, we start with the study of a monolithic scheme where the two sub-problems are solved simultaneously at each time instant t^n : find $(\mathbf{u}_{1,h}^n, p_{1,h}^n, \mathbf{u}_{2,h}^n, p_{2,h}^n) \in W_{1,h} \times Q_{1,h} \times W_{2,h} \times Q_{2,h}$ satisfying the essential boundary conditions and such that

$$\begin{aligned} &A_{1,\delta t}[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, p_{1,h}^n), (\mathbf{v}_{1,h}, q_{1,h})] + A_{2,\delta t}[\mathbf{u}_{2,h}^{n-1}; (\mathbf{u}_{2,h}^n, p_{2,h}^n), (\mathbf{v}_{2,h}, q_{2,h})] + \\ &\quad + C_1[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, \mathbf{u}_{2,h}^n), \mathbf{v}_{1,h}] + C_2[(\mathbf{u}_{2,h}^{n-1}, \mathbf{u}_{1,h}^{n-1}); (\mathbf{u}_{2,h}^n, \mathbf{u}_{1,h}^n), \mathbf{v}_{1,h}] \\ &\quad + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n) \cdot (\mathbf{v}_{2,h} - \mathbf{v}_{1,h}) - \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}^n, p_{2,h}^n) \cdot \mathbf{n}_2 \cdot (\mathbf{v}_{2,h} - \mathbf{v}_{1,h}) \\ &\quad - \int_{\Sigma} (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n) \cdot \mathbf{n}_2 q_{2,h} = 0, \end{aligned} \quad (18)$$

for all $(\mathbf{v}_{1,h}, q_{1,h}, \mathbf{v}_{2,h}, q_{2,h}) \in V_{1,h} \times Q_{1,h} \times V_{2,h} \times Q_{2,h}$.

For the sake of conciseness, the strain rate tensors at time t^n are denoted by $\boldsymbol{\epsilon}_{1,h}^m := \boldsymbol{\epsilon}(\mathbf{u}_{1,h}^m)$ and $\boldsymbol{\epsilon}_{2,h}^m := \boldsymbol{\epsilon}(\mathbf{u}_{2,h}^m)$ and the following quantities are introduced:

$$\begin{aligned} E^n &:= \frac{\rho}{2} (\|\mathbf{u}_{1,h}^n\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}^n\|_{\Omega_2}^2) + \frac{3\gamma\mu}{4h} \delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 \\ &\quad + \frac{2\mu}{3} \delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{1,h}^m\|_{\Omega_1}^2 + \frac{2\mu}{3} \delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{2,h}^m\|_{\Omega_2}^2, \\ E^0 &:= \frac{\rho}{2} (\|\mathbf{u}_{1,h}^0\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}^0\|_{\Omega_2}^2). \end{aligned} \quad (19)$$

The next proposition shows that this monolithic algorithm is stable without any condition on the discretization steps.

Proposition 2 Let $(\mathbf{u}_{1,h}^n, p_{1,h}^n, \mathbf{u}_{2,h}^n, p_{2,h}^n)$ be the solution of the monolithic scheme (18). If $\gamma \geq 2C_T$, where $C_T > 0$ is the constant of the inverse inequality (10), then $E^n \leq E^0$.

Proof. The proof follows the same steps as the proof of Lemma 1. Choosing $(\mathbf{v}_{1,h}, q_{1,h}, \mathbf{v}_{2,h}, q_{2,h}) = (\mathbf{u}_{1,h}^n, p_{1,h}^n, \mathbf{u}_{2,h}^n, p_{2,h}^n)$ in (18), the interface terms resulting from the integration by parts of the advection in (16) and (17) cancel with interface stabilization terms (8) and (9). Then summing over $1 \leq m \leq n$ and multiplying by δt , we obtain:

$$\begin{aligned} & \frac{\rho}{2} \sum_{m=1}^n \left(\|\mathbf{u}_{1,h}^m\|_{\Omega_1}^2 - \|\mathbf{u}_{1,h}^{m-1}\|_{\Omega_1}^2 \right) + \frac{\rho}{2} \sum_{m=1}^n \|\mathbf{u}_{1,h}^m - \mathbf{u}_{1,h}^{m-1}\|_{\Omega_1}^2 + 2\mu\delta t \sum_{m=1}^n \|\epsilon_{1,h}^m\|_{\Omega_1}^2 \\ & + \frac{\rho}{2} \sum_{m=1}^n \left(\|\mathbf{u}_{2,h}^m\|_{\Omega_2}^2 - \|\mathbf{u}_{2,h}^{m-1}\|_{\Omega_2}^2 \right) + \frac{\rho}{2} \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}\|_{\Omega_2}^2 + 2\mu\delta t \sum_{m=1}^n \|\epsilon_{2,h}^m\|_{\Omega_2}^2 \\ & + \frac{\gamma\mu}{h} \delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 \\ \leq & \frac{4\mu C_T}{\gamma} \delta t \sum_{m=1}^n \|\epsilon_{2,h}^m\|_{\Omega_2}^2 + \frac{\mu\gamma}{4h} \delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2, \end{aligned}$$

which can be reformulated as

$$\begin{aligned} & \frac{\rho}{2} \left(\|\mathbf{u}_{1,h}^n\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}^n\|_{\Omega_2}^2 \right) + 2\mu\delta t \sum_{m=1}^n \|\epsilon_{1,h}^m\|_{\Omega_1}^2 \\ & + 2\mu \left(1 - \frac{2C_T}{\gamma} \right) \delta t \sum_{m=1}^n \|\epsilon_{2,h}^m\|_{\Omega_2}^2 + \frac{3\gamma\mu}{4h} \delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 \\ & \leq -\frac{\rho}{2} \delta t \left(\sum_{m=1}^n \|\mathbf{u}_{1,h}^m - \mathbf{u}_{1,h}^{m-1}\|_{\Omega_1}^2 + \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}\|_{\Omega_2}^2 \right) \\ & + \frac{\rho}{2} \left(\|\mathbf{u}_{1,h}^0\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}^0\|_{\Omega_2}^2 \right), \end{aligned}$$

which completes the proof. \square

4.2 Explicit coupling: a static pressure formulation

The monolithic scheme (18) involves the simultaneous computation of $(\mathbf{u}_{1,h}^n, p_{1,h}^n)$ and $(\mathbf{u}_{2,h}^n, p_{2,h}^n)$. In this section, a first staggered scheme (Algorithm 4.2) is proposed to solve problem (5). This method generalizes the explicit coupling schemes introduced in [6, 7] for incompressible fluid-structure interaction.

As shown in Proposition 3, we are not able to prove its stability. We nevertheless present it because it allows us to introduce the main ideas that will be useful in the next section. In addition, this scheme was stable in most of the numerical simulations where it was tested.

This scheme only yields one resolution of each sub-problem at each time step. The terms involving the two forms $A_{1,\delta t}$ and $A_{2,\delta t}$ take the same expressions (16) and (17) as in the monolithic scheme. The differences between the two algorithms only lie in the interface terms. First, the interface stabilization term $C_1[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, \mathbf{u}_{2,h}^n), \mathbf{v}_{1,h}]$ of the monolithic algorithm (18) is replaced by $C_1[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, \mathbf{u}_{2,h}^{n-1}), \mathbf{v}_{1,h}]$ in order to uncouple the two sub-problems. Second, the term

$$S(p_{2,h}^n, q_{2,h}) := \frac{\gamma_0 h}{\gamma \mu} \int_{\Sigma} (p_{2,h}^n - p_{2,h}^{n-1}) q_{2,h}, \quad (22)$$

-
1. Find $(\mathbf{u}_{1,h}^n, p_{1,h}^n) \in W_{1,h} \times Q_{1,h}$ satisfying the essential boundary conditions and such that

$$\begin{aligned} & A_{1,\delta t}[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, p_{1,h}^n), (\mathbf{v}_{1,h}, q_{1,h})] + C_1[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, \mathbf{u}_{2,h}^{n-1}), \mathbf{v}_{1,h}] \\ & + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^{n-1}) \cdot \mathbf{v}_{1,h} + \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}^{n-1}, p_{2,h}^{n-1}) \cdot \mathbf{n}_2 \mathbf{v}_{1,h} = 0, \end{aligned} \quad (20)$$

for all $(\mathbf{v}_{2,h}, p_{2,h}) \in V_{2,h} \times Q_{2,h}$.

2. Find $(\mathbf{u}_{2,h}^n, p_{2,h}^n) \in W_{2,h} \times Q_{2,h}$ satisfying the essential boundary conditions and such that

$$\begin{aligned} & A_{2,\delta t}[\mathbf{u}_{2,h}^{n-1}; (\mathbf{u}_{2,h}^n, p_{2,h}^n), (\mathbf{v}_{2,h}, q_{2,h})] + S(p_{2,h}^n, q_{2,h}) \\ & + C_2[(\mathbf{u}_{2,h}^{n-1}, \mathbf{u}_{1,h}^{n-1}); (\mathbf{u}_{2,h}^n, \mathbf{u}_{1,h}^n), \mathbf{v}_{2,h}] + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n) \cdot \mathbf{v}_{2,h} \\ & - \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}^{n-1}, p_{2,h}^{n-1}) \cdot \mathbf{n}_2 \mathbf{v}_{2,h} - \int_{\Sigma} (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n) \cdot \mathbf{n}_2 q_{2,h} = 0 \end{aligned} \quad (21)$$

for all $(\mathbf{v}_{1,h}, p_{1,h}) \in V_{1,h} \times Q_{1,h}$.

3. Go to next time-step.
-

is added in sub-domain Ω_2 in order to control spurious pressure fluctuations which appear at the interface because of the explicit coupling between the two sub-domains. The dimensionless parameter $\gamma_0 > 0$ will be fixed to ensure the energy stability of the method.

It is interesting to note that this scheme, obtained from Nitsche's penalty formulation, can be seen as an explicit Robin-Robin scheme associated with the following boundary conditions:

$$\begin{cases} \boldsymbol{\sigma}(\mathbf{u}_1^n, p_1^n) \cdot \mathbf{n}_1 + \frac{\gamma\mu}{h} \mathbf{u}_1^n = \frac{\gamma\mu}{h} \mathbf{u}_2^{n-1} - \boldsymbol{\sigma}(\mathbf{u}_2^{n-1}, p_2^{n-1}) \cdot \mathbf{n}_2, & \text{on } \Sigma, \\ \boldsymbol{\sigma}(\mathbf{u}_2^n, p_2^n) \cdot \mathbf{n}_2 + \frac{\gamma\mu}{h} \mathbf{u}_2^n = \frac{\gamma\mu}{h} \mathbf{u}_1^n + \boldsymbol{\sigma}(\mathbf{u}_2^{n-1}, p_2^{n-1}) \cdot \mathbf{n}_2, & \text{on } \Sigma. \end{cases} \quad (23)$$

We attempt to prove the stability in the energy norm of the explicit formulation (20)-(21). Although the result is not conclusive, it is useful since it pinpoints the difficulty that will be addressed in Algorithm 4.2. In addition, most of the arguments of the proof are reusable for the stability analysis of Algorithm 4.2.

Proposition 3 *Let $(\mathbf{u}_{1,h}^n, p_{1,h}^n, \mathbf{u}_{2,h}^n, p_{2,h}^n)$ be the solution of the staggered scheme (20)-(21). Under the conditions:*

$$\text{a) } \gamma_0 > 4, \quad \text{b) } \gamma\delta t \leq C_{\Sigma} h, \quad \text{c) } \gamma \geq 4C_T, \quad (24)$$

where $C_T > 0$ is the constant of the inverse inequality (10) and $C_{\Sigma} > 0$ is given, the following estimate holds:

$$\begin{aligned} E^n + \frac{\gamma\mu}{2h} \delta t \|\mathbf{u}_{2,h}^n\|_{\Sigma}^2 + \frac{\gamma_0 h}{2\gamma\mu} \delta t \|p_{2,h}^n\|_{\Sigma}^2 &\leq 3E^0 + \frac{3C_{\Sigma}\mu}{2} \|\mathbf{u}_{2,h}^0\|_{\Sigma}^2 \\ &+ \frac{3\gamma_0 h}{2\gamma\mu} \delta t \|p_{2,h}^0\|_{\Sigma}^2 + 24\delta t \|\epsilon_{2,h}^0\|_{\Omega_2}^2 \\ &+ \frac{3}{2} \rho \delta t \sum_{m=1}^n \int_{\Sigma} \mathbf{u}_{1,h}^{m-1} \cdot \mathbf{n}_1 \mathbf{u}_{1,h}^m \cdot (\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}). \end{aligned} \quad (25)$$

Proof. Let be $(\mathbf{v}_{1,h}, q_{1,h}, \mathbf{v}_{2,h}, q_{2,h}) = (\mathbf{u}_{1,h}^n, p_{1,h}^n, \mathbf{u}_{2,h}^n, p_{2,h}^n)$, the two forms $A_{1,\delta t}$ and $A_{2,\delta t}$ give

$$\begin{aligned} A_{1,\delta t}[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, p_{1,h}^n), (\mathbf{u}_{1,h}^n, p_{1,h}^n)] &= \frac{\rho}{2\delta t} \left(\|\mathbf{u}_{1,h}^n\|_{\Omega_1}^2 - \|\mathbf{u}_{1,h}^{n-1}\|_{\Omega_1}^2 + \|\mathbf{u}_{1,h}^n - \mathbf{u}_{1,h}^{n-1}\|_{\Omega_1}^2 \right) \\ &\quad + 2\mu \|\boldsymbol{\epsilon}_{1,h}^n\|_{\Omega_1}^2 - \frac{\rho}{2} \int_{\Omega_1} (\nabla \cdot \mathbf{u}_{1,h}^{n-1}) |\mathbf{u}_{1,h}^n|^2 \\ &\quad + \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{1,h}^{n-1} \cdot \mathbf{n}_{1,h}) |\mathbf{u}_{1,h}^n|^2, \end{aligned}$$

$$\begin{aligned} A_{2,\delta t}[\mathbf{u}_{2,h}^{n-1}; (\mathbf{u}_{2,h}^n, p_{2,h}^n), (\mathbf{u}_{2,h}^n, p_{2,h}^n)] &= \frac{\rho}{2\delta t} \left(\|\mathbf{u}_{2,h}^n\|_{\Omega_2}^2 - \|\mathbf{u}_{2,h}^{n-1}\|_{\Omega_2}^2 + \|\mathbf{u}_{2,h}^n - \mathbf{u}_{2,h}^{n-1}\|_{\Omega_2}^2 \right) \\ &\quad + 2\mu \|\boldsymbol{\epsilon}_{2,h}^n\|_{\Omega_2}^2 - \frac{\rho}{2} \int_{\Omega_2} (\nabla \cdot \mathbf{u}_{2,h}^{n-1}) |\mathbf{u}_{2,h}^n|^2 \\ &\quad + \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{2,h}^{n-1} \cdot \mathbf{n}_{2,h}) |\mathbf{u}_{2,h}^n|^2. \end{aligned}$$

Similarly, for the terms C_1 and C_2 we get

$$\begin{aligned} C_1[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, \mathbf{u}_{2,h}^{n-1}), \mathbf{u}_{1,h}^n] &= \frac{\rho}{2} \int_{\Omega_1} (\nabla \cdot \mathbf{u}_{1,h}^{n-1}) |\mathbf{u}_{1,h}^n|^2 - \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{1,h}^{n-1} \cdot \mathbf{n}_1) |\mathbf{u}_{1,h}^n|^2 \\ &\quad + \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{1,h}^{n-1} \cdot \mathbf{n}_1) (\mathbf{u}_{2,h}^{n-1} \cdot \mathbf{u}_{1,h}^n), \end{aligned}$$

$$\begin{aligned} C_2[(\mathbf{u}_{2,h}^{n-1}, \mathbf{u}_{1,h}^{n-1}); (\mathbf{u}_{2,h}^n, \mathbf{u}_{1,h}^n), \mathbf{u}_{2,h}^n] &= \frac{\rho}{2} \int_{\Omega_2} (\nabla \cdot \mathbf{u}_{2,h}^{n-1}) |\mathbf{u}_{2,h}^n|^2 - \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{2,h}^{n-1} \cdot \mathbf{n}_2) |\mathbf{u}_{2,h}^n|^2 \\ &\quad + \frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{1,h}^{n-1} \cdot \mathbf{n}_2) (\mathbf{u}_{1,h}^n \cdot \mathbf{u}_{2,h}^n). \end{aligned}$$

By inserting these expressions into (20)-(21), summing over $1 \leq m \leq n$ and multiplying by δt ,

we obtain

$$\begin{aligned}
& \frac{\rho}{2} \sum_{m=1}^n \left(\|\mathbf{u}_{1,h}^m\|_{\Omega_1}^2 - \|\mathbf{u}_{1,h}^{m-1}\|_{\Omega_1}^2 \right) + \frac{\rho}{2} \sum_{m=1}^n \|\mathbf{u}_{1,h}^m - \mathbf{u}_{1,h}^{m-1}\|_{\Omega_1}^2 + 2\mu\delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{1,h}^m\|_{\Omega_1}^2 \\
& + \frac{\rho}{2} \sum_{m=1}^n \left(\|\mathbf{u}_{2,h}^m\|_{\Omega_2}^2 - \|\mathbf{u}_{2,h}^{m-1}\|_{\Omega_2}^2 \right) + \frac{\rho}{2} \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}\|_{\Omega_2}^2 + 2\mu\delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{2,h}^m\|_{\Omega_2}^2 \\
& + \frac{\rho}{2} \delta t \sum_{m=1}^n \int_{\Sigma} \mathbf{u}_{1,h}^{m-1} \cdot \mathbf{n}_1 \mathbf{u}_{1,h}^m \cdot (\mathbf{u}_{2,h}^{m-1} - \mathbf{u}_{2,h}^m) \\
= & \underbrace{\delta t \sum_{m=1}^n \int_{\Sigma} (p_{2,h}^m - p_{2,h}^{m-1}) \mathbf{n}_2 \cdot (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m) - \frac{\gamma_0 h}{\gamma \mu} \delta t \sum_{m=1}^n \int_{\Sigma} (p_{2,h}^m - p_{2,h}^{m-1}) p_{2,h}^m}_{T_1} \\
& - \underbrace{\frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m) \cdot \mathbf{u}_{2,h}^m + \frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^{m-1} - \mathbf{u}_{1,h}^m) \cdot \mathbf{u}_{1,h}^m}_{T_2} \\
& + \underbrace{2\mu\delta t \sum_{m=1}^n \int_{\Sigma} \boldsymbol{\epsilon}_{2,h}^{m-1} \cdot \mathbf{n}_2 (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m)}_{T_3}.
\end{aligned} \tag{26}$$

We now proceed by adapting the arguments reported in [6]. Term T_1 involves the pressure fluctuations at the interface. Applying Young's inequality, it can be written as

$$\begin{aligned}
T_1 & \leq \frac{2h}{\gamma \mu} \delta t \sum_{m=1}^n \|p_{2,h}^m - p_{2,h}^{m-1}\|_{\Sigma}^2 + \frac{\gamma \mu}{8h} \delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 \\
& - \frac{1}{2} \frac{\gamma_0 h}{\gamma \mu} \delta t \|p_{2,h}^n\|_{\Sigma}^2 + \frac{1}{2} \frac{\gamma_0 h}{\gamma \mu} \delta t \|p_{2,h}^0\|_{\Sigma}^2 - \frac{\gamma_0 h}{\gamma \mu} \frac{\delta t}{2} \sum_{m=1}^n \|p_{2,h}^m - p_{2,h}^{m-1}\|_{\Sigma}^2 \\
& = \frac{\gamma \mu}{8h} \delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 - \frac{1}{2} \frac{\gamma_0 h}{\gamma \mu} \delta t \|p_{2,h}^n\|_{\Sigma}^2 + \frac{1}{2} \frac{\gamma_0 h}{\gamma \mu} \delta t \|p_{2,h}^0\|_{\Sigma}^2 \\
& + \frac{h}{2\gamma \mu} (4 - \gamma_0) \delta t \sum_{m=1}^n \|p_{2,h}^m - p_{2,h}^{m-1}\|_{\Sigma}^2
\end{aligned}$$

the last term being negative (assumption of (24a)), it will be removed from the upper bound.

Term T_2 concerns the velocity fluctuations at the interface. Adding and subtracting $\mathbf{u}_{2,h}^m$ in the second integral of T_2 , we have

$$\begin{aligned}
T_2 & = -\frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m) \cdot \mathbf{u}_{2,h}^m + \frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^{m-1} - \mathbf{u}_{2,h}^m) \cdot \mathbf{u}_{1,h}^m \\
& + \frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m) \cdot \mathbf{u}_{1,h}^m \\
& = -\frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 - \frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}) \cdot \mathbf{u}_{1,h}^m.
\end{aligned}$$

Adding and subtracting $\mathbf{u}_{2,h}^m$ in the second term of this relation, we obtain

$$\begin{aligned}
 & -\frac{\gamma\mu}{h}\delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}) \cdot \mathbf{u}_{1,h}^m \\
 &= -\frac{\gamma\mu}{h}\delta t \sum_{m=1}^n \left[\int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}) \cdot \mathbf{u}_{2,h}^m - \int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}) \cdot (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m) \right] \\
 &= -\frac{\gamma\mu}{h}\delta t \sum_{m=1}^n \left[\int_{\Sigma} \frac{|\mathbf{u}_{2,h}^m|^2}{2} - \frac{|\mathbf{u}_{2,h}^{m-1}|^2}{2} + \frac{|\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}|^2}{2} \right. \\
 &\quad \left. - \int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}) \cdot (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m) \right] \\
 &\leq -\frac{\gamma\mu}{2h}\delta t \sum_{m=1}^n \left[\|\mathbf{u}_{2,h}^m\|_{\Sigma}^2 - \|\mathbf{u}_{2,h}^{m-1}\|_{\Sigma}^2 - \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 \right].
 \end{aligned}$$

Using assumption (24b) we finally get

$$T_2 \leq -\frac{\gamma\mu}{2h}\delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 - \frac{\gamma\mu}{2h}\delta t \|\mathbf{u}_{2,h}^n\|_{\Sigma}^2 + \frac{\mu C_{\Sigma}}{2} \|\mathbf{u}_{2,h}^0\|_{\Sigma}^2.$$

Applying Young's inequality and the local trace inequality (10), the term T_3 can be bounded as follows:

$$\begin{aligned}
 T_3 &\leq \frac{8\mu C_T}{\gamma}\delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{2,h}^{m-1}\|_{\Omega_2}^2 + \frac{\gamma\mu}{8h}\delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 \\
 &= \frac{8\mu C_T}{\gamma}\delta t \sum_{m=1}^{n-1} \|\boldsymbol{\epsilon}_{2,h}^m\|_{\Omega_2}^2 + \frac{8\mu C_T}{\gamma}\delta t \|\boldsymbol{\epsilon}_{2,h}^0\|_{\Omega_2}^2 + \frac{\gamma\mu}{8h}\delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2.
 \end{aligned}$$

Inserting T_1 , T_2 and T_3 in (26), we have:

$$\begin{aligned}
 & \frac{\rho}{2} (\|\mathbf{u}_{1,h}^n\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}^n\|_{\Omega_2}^2) + \frac{\gamma\mu}{4h}\delta t \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m\|_{\Sigma}^2 + 2\mu\delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{1,h}^m\|_{\Omega_1}^2 \\
 &+ 2\mu \left(1 - \frac{4C_T}{\gamma}\right) \delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{2,h}^m\|_{\Omega_2}^2 + \frac{\gamma\mu}{2h}\delta t \|\mathbf{u}_{2,h}^n\|_{\Sigma}^2 + \frac{1}{2} \frac{\gamma_0 h}{\gamma\mu} \delta t \|p_{2,h}^n\|_{\Sigma}^2 \\
 &\leq \frac{\rho}{2} (\|\mathbf{u}_{1,h}^0\|_{\Omega_1}^2 + \|\mathbf{u}_{2,h}^0\|_{\Omega_2}^2) + \frac{\mu C_{\Sigma}}{2} \|\mathbf{u}_{2,h}^0\|_{\Sigma}^2 + \frac{1}{2} \frac{\gamma_0 h}{\gamma\mu} \delta t \|p_{2,h}^0\|_{\Sigma}^2 + \frac{8\mu C_T}{\gamma} \delta t \|\boldsymbol{\epsilon}_{2,h}^0\|_{\Omega_2}^2 \\
 &+ \frac{\rho}{2} \delta t \sum_{m=1}^n \int_{\Sigma} \mathbf{u}_{1,h}^{m-1} \cdot \mathbf{n}_1 \mathbf{u}_{1,h}^m \cdot (\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}).
 \end{aligned}$$

Finally under the condition (24c) the assertion is proved. \square

4.3 Explicit coupling: a stable but inconsistent formulation

Energy stability can be guaranteed if, instead of C_1 and C_2 , we consider the following alternative terms in Algorithm 4.2:

$$c_1[\mathbf{u}_{1,h}^{n-1}; \mathbf{u}_{1,h}^n, \mathbf{v}_{1,h}] := -\frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{1,h}^{n-1} \cdot \mathbf{n}_1) (\mathbf{u}_{1,h}^n \cdot \mathbf{v}_{1,h}), \quad (27)$$

$$c_2[\mathbf{u}_{2,h}^{n-1}; \mathbf{u}_{2,h}^n, \mathbf{v}_{2,h}] := -\frac{\rho}{2} \int_{\Sigma} (\mathbf{u}_{2,h}^{n-1} \cdot \mathbf{n}_2) (\mathbf{u}_{2,h}^n \cdot \mathbf{v}_{2,h}). \quad (28)$$

The corresponding staggered scheme is presented in Algorithm 4.3.

-
1. Find $(\mathbf{u}_{1,h}^n, p_{1,h}^n) \in W_{1,h} \times Q_{1,h}$ such that

$$\begin{aligned} & A_{1,\delta t}[\mathbf{u}_{1,h}^n; (\mathbf{u}_{1,h}^n, p_{1,h}^n)(\mathbf{v}_{1,h}, q_{1,h})] + c_1[\mathbf{u}_{1,h}^{n-1}; \mathbf{u}_{1,h}^n, \mathbf{v}_{1,h}] \\ & + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^{n-1}) \cdot \mathbf{v}_{1,h} + \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}^{n-1}, p_{2,h}^{n-1}) \cdot \mathbf{n}_2 \mathbf{v}_{1,h} = 0 \end{aligned}$$

for all $(\mathbf{v}_{2,h}, p_{2,h}) \in W_{2,h} \times Q_{2,h}$.

2. Find $(\mathbf{u}_{2,h}^n, p_{2,h}^n) \in W_{2,h} \times Q_{2,h}$ such that

$$\begin{aligned} & A_{2,\delta t}[\mathbf{u}_{2,h}^n; (\mathbf{u}_{2,h}^n, p_{2,h}^n)(\mathbf{v}_{2,h}, q_{2,h})] + S(p_{2,h}^n, q_{2,h}) \\ & + c_2[\mathbf{u}_{2,h}^{n-1}; \mathbf{u}_{2,h}^n, \mathbf{v}_{2,h}] + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n) \cdot \mathbf{v}_{2,h} \\ & - \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}^{n-1}, p_{2,h}^{n-1}) \cdot \mathbf{n}_2 \mathbf{v}_{2,h} - \int_{\Sigma} (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n) \cdot \mathbf{n}_2 q_{2,h} = 0 \end{aligned}$$

for all $(\mathbf{v}_{1,h}, p_{1,h}) \in V_{1,h} \times Q_{1,h}$.

3. Go to next time-step.
-

Note that this amounts to consider the skew-symmetric formulation of the convective term in each sub-domain, namely,

$$\frac{1}{2} \int_{\Omega_i} \mathbf{u}_i \cdot \nabla \mathbf{u}_i \cdot \mathbf{v}_i - \frac{1}{2} \int_{\Omega_i} \mathbf{u}_i \cdot \nabla \mathbf{v}_i \cdot \mathbf{u}_i. \quad (29)$$

Indeed, using integration by parts we have

$$\frac{1}{2} \int_{\Omega_i} \mathbf{u}_i \cdot \nabla \mathbf{u}_i \cdot \mathbf{v}_i - \frac{1}{2} \int_{\Omega_i} \mathbf{u}_i \cdot \nabla \mathbf{v}_i \cdot \mathbf{u}_i = \int_{\Omega_i} \mathbf{u}_i \cdot \nabla \mathbf{u}_i \cdot \mathbf{v}_i - \frac{1}{2} \int_{\Sigma} (\mathbf{u}_i \cdot \mathbf{n}_i) (\mathbf{u}_i \cdot \mathbf{v}_i). \quad (30)$$

A straightforward adaptation of Proposition 3 hence shows that Algorithm 4.3 is energy stable.

The skew-symmetrized form (29) is commonly used in the staggered time-marching of fluid-fluid interaction models based on *rigid-lid* coupling condition conditions (see, e.g., [10] and the references therein), for which $\mathbf{u}_i \cdot \mathbf{n}_i = 0$ on Σ . The expression (29) is thus consistent, since the interface term in the right-hand side of (30) vanishes. Clearly, this does not hold for the coupled

problem (2)-(4) and, therefore, Algorithm 4.3 involves an inconsistent treatment of (4). In fact, the following (inconsistent) Robin-Robin splitting is enforced

$$\begin{cases} \boldsymbol{\sigma}(\mathbf{u}_1^n, p_1^n) \cdot \mathbf{n}_1 - \frac{1}{2}(\mathbf{u}_1^{n-1} \cdot \mathbf{n}_1)\mathbf{u}_1^n + \frac{\gamma\mu}{h}\mathbf{u}_1^n = \frac{\gamma\mu}{h}\mathbf{u}_2^{n-1} - \boldsymbol{\sigma}(\mathbf{u}_2^{n-1}, p_2^{n-1}) \cdot \mathbf{n}_2, & \text{on } \Sigma, \\ \boldsymbol{\sigma}(\mathbf{u}_2^n, p_2^n) \cdot \mathbf{n}_2 - \frac{1}{2}(\mathbf{u}_2^{n-1} \cdot \mathbf{n}_2)\mathbf{u}_2^n + \frac{\gamma\mu}{h}\mathbf{u}_2^n = \frac{\gamma\mu}{h}\mathbf{u}_1^n + \boldsymbol{\sigma}(\mathbf{u}_2^{n-1}, p_2^{n-1}) \cdot \mathbf{n}_2, & \text{on } \Sigma. \end{cases}$$

The numerical experiments reported in Section 5 confirm the expected poor accuracy of this method.

4.4 Explicit coupling: a total pressure formulation

We propose in this Section a provably stable staggered algorithm. The difficulty to establish the stability of Algorithm 4.2 came from the last term of (25) which resulted from the integration by parts of the advection. To get rid of this term, we suggest to use a formulation for which the advection cancels in the energy equation without any integration by parts. This can be achieved by introducing the total pressure $\pi := p + \rho \frac{|\mathbf{u}|^2}{2}$ and by reformulating the Navier-Stokes equation as:

$$\begin{aligned} \rho \partial_t \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \rho \mathbf{u} \cdot (\nabla \mathbf{u})^T - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, \pi) &= 0, & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0, & \text{in } \Omega, \\ \mathbf{u} &= 0, & \text{on } \partial\Omega. \end{aligned} \quad (31)$$

This formulation was in particular discussed in [18, Page 337]. It simply results from the relation:

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, p) = -\frac{\rho}{2} \nabla |\mathbf{u}|^2 - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, \pi) = -\rho \mathbf{u} \cdot (\nabla \mathbf{u})^T - \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, \pi).$$

Note that the pressure unknown is now the total pressure π . This is not the case of the *curl $\mathbf{u} \times \mathbf{u}$ formulation* used in many works to enforce the total pressure in the boundary conditions, but which keeps the static pressure p as unknown (see, e.g., [2, 13, 14, 19, 24, 29]). The variational formulation derived from (31) naturally leads to replace the two forms $A_{1,\delta t}$ and $A_{2,\delta t}$ used in the Algorithm 4.2 by the new forms $A_{1,\delta t}^\pi$ and $A_{2,\delta t}^\pi$ defined by

$$\begin{aligned} A_{1,\delta t}^\pi[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, \pi_{1,h}^n), (\mathbf{v}_{1,h}, q_{1,h})] &:= \rho \int_{\Omega_1} \partial_{\delta t} \mathbf{u}_{1,h}^n \cdot \mathbf{v}_{1,h} + 2\mu \int_{\Omega_1} \boldsymbol{\epsilon}(\mathbf{u}_{1,h}^n) : \boldsymbol{\epsilon}(\mathbf{v}_{1,h}) \\ &\quad - \int_{\Omega_1} \pi_{1,h}^n \nabla \cdot \mathbf{v}_{1,h} + \int_{\Omega_1} q_{1,h} \nabla \cdot \mathbf{u}_{1,h}^n \\ &\quad + \rho \int_{\Omega_1} (\mathbf{u}_{1,h}^n \nabla \mathbf{u}_{1,h}^{n-1} \cdot \mathbf{v}_{1,h} - \mathbf{v}_{1,h} \nabla \mathbf{u}_{1,h}^{n-1} \cdot \mathbf{u}_{1,h}^n), \end{aligned} \quad (32)$$

$$\begin{aligned} A_{2,\delta t}^\pi[\mathbf{u}_{2,h}^{n-1}; (\mathbf{u}_{2,h}^n, \pi_{2,h}^n), (\mathbf{v}_{2,h}, q_{2,h})] &:= \rho \int_{\Omega_2} \partial_{\delta t} \mathbf{u}_{2,h}^n \cdot \mathbf{v}_{2,h} + 2\mu \int_{\Omega_2} \boldsymbol{\epsilon}(\mathbf{u}_{2,h}^n) : \boldsymbol{\epsilon}(\mathbf{v}_{2,h}) \\ &\quad - \int_{\Omega_2} \pi_{2,h}^n \nabla \cdot \mathbf{v}_{2,h} + \int_{\Omega_2} q_{2,h} \nabla \cdot \mathbf{u}_{2,h}^n \\ &\quad + \rho \int_{\Omega_2} (\mathbf{u}_{2,h}^n \nabla \mathbf{u}_{2,h}^{n-1} \cdot \mathbf{v}_{2,h} - \mathbf{v}_{2,h} \nabla \mathbf{u}_{2,h}^{n-1} \cdot \mathbf{u}_{2,h}^n). \end{aligned} \quad (33)$$

Note that with this new formulation, the last integral in (6) and (7) (Temam's trick) is no longer necessary. The new staggered scheme is presented in Algorithm 4.2.

-
1. Find $(\mathbf{u}_{1,h}^n, \pi_{1,h}^n) \in W_{1,h} \times Q_{1,h}$ satisfying the essential boundary conditions and such that

$$\begin{aligned} & A_{1,\delta t}^\pi[\mathbf{u}_{1,h}^{n-1}; (\mathbf{u}_{1,h}^n, \pi_{1,h}^n), (\mathbf{v}_{1,h}, q_{1,h})] + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{1,h}^n - \mathbf{u}_{2,h}^{n-1}) \cdot \mathbf{v}_{1,h} \\ & + \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}^{n-1}, \pi_{2,h}^{n-1}) \cdot \mathbf{n}_2 \mathbf{v}_{1,h} = 0, \end{aligned} \quad (34)$$

for all $(\mathbf{v}_{1,h}, \pi_{1,h}) \in V_{1,h} \times Q_{1,h}$.

2. Find $(\mathbf{u}_{2,h}^n, \pi_{2,h}^n) \in W_{2,h} \times Q_{2,h}$ satisfying the essential boundary conditions and such that

$$\begin{aligned} & A_{2,\delta t}^\pi[\mathbf{u}_{2,h}^{n-1}; (\mathbf{u}_{2,h}^n, \pi_{2,h}^n), (\mathbf{v}_{2,h}, q_{2,h})] + S(\pi_{2,h}^n, q_{2,h}) \\ & + \frac{\gamma\mu}{h} \int_{\Sigma} (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n) \cdot \mathbf{v}_{2,h} - \int_{\Sigma} \boldsymbol{\sigma}(\mathbf{u}_{2,h}^{n-1}, \pi_{2,h}^{n-1}) \cdot \mathbf{n}_2 \mathbf{v}_{2,h} \\ & - \int_{\Sigma} (\mathbf{u}_{2,h}^n - \mathbf{u}_{1,h}^n) \cdot \mathbf{n}_2 q_{2,h} = 0, \end{aligned} \quad (35)$$

for all $(\mathbf{v}_{2,h}, \pi_{2,h}) \in V_{2,h} \times Q_{2,h}$.

3. Go to next time-step.
-

As with the previous formulation, the pressure fluctuation at the interface are controlled by the term $S(\pi_{2,h}^n, q_{2,h})$ in sub-domain 2. Again, the two sub-problems can be seen as coupled through the Robin-Robin transmission conditions:

$$\begin{cases} \boldsymbol{\sigma}(\mathbf{u}_1^n, \pi_1^n) \cdot \mathbf{n}_1 + \frac{\gamma\mu}{h} \mathbf{u}_1^n = \frac{\gamma\mu}{h} \mathbf{u}_2^{n-1} - \boldsymbol{\sigma}(\mathbf{u}_2^{n-1}, \pi_2^{n-1}) \cdot \mathbf{n}_2, & \text{on } \Sigma, \\ \boldsymbol{\sigma}(\mathbf{u}_2^n, \pi_2^n) \cdot \mathbf{n}_2 + \frac{\gamma\mu}{h} \mathbf{u}_2^n = \frac{\gamma\mu}{h} \mathbf{u}_1^n + \boldsymbol{\sigma}(\mathbf{u}_2^{n-1}, \pi_2^{n-1}) \cdot \mathbf{n}_2, & \text{on } \Sigma. \end{cases} \quad (36)$$

This is similar to (23), but the static pressure has been replaced by the total pressure.

Remark 2 The total pressure formulation can be obviously used in the implicit case. The proof of the stability analysis is performed in the same way as in Proposition 2, with the total pressure variable $\pi_{i,h}$ instead of $p_{i,h}$, $i = 1, 2$ but without the interface stabilization terms (8) and (9).

The next proposition shows that Algorithm 4.2 is conditionally stable in the energy norm.

Proposition 4 *Let $(\mathbf{u}_{1,h}^n, \pi_{1,h}^n, \mathbf{u}_{2,h}^n, \pi_{2,h}^n)$ be the solution of (34)-(35). Under the conditions (24), there holds:*

$$\begin{aligned} E^n + \frac{\gamma\mu}{2h} \delta t \|\mathbf{u}_{2,h}^n\|_{\Sigma}^2 + \frac{1}{2} \frac{\gamma_0 h}{\gamma\mu} \delta t \|\pi_{2,h}^n\|_{\Omega_2}^2 & \leq 3E^0 + \frac{3}{2} C_{\Sigma\mu} \|\mathbf{u}_{2,h}^0\|_{\Sigma}^2 \\ & + \frac{3}{2} \frac{\gamma_0 h}{\gamma\mu} \delta t \|\pi_{2,h}^0\|_{\Sigma}^2 + 24\delta t \|\boldsymbol{\epsilon}_{2,h}^0\|_{\Omega_2}^2 \end{aligned}$$

where E^n , $n \geq 1$, and E^0 are defined in (19) and (20).

Proof. Taking $(\mathbf{v}_{1,h}, q_{1,h}, \mathbf{v}_{2,h}, q_{2,h}) = (\mathbf{u}_{1,h}^n, \pi_{1,h}^n, \mathbf{u}_{2,h}^n, \pi_{2,h}^n)$, the terms related to advection in (32) and (33) cancel each others. Therefore, multiplying for δt and summing over $1 \leq m \leq n$,

the system (34)-(35) can be written as

$$\begin{aligned}
& \frac{\rho}{2} \sum_{m=1}^n \left(\|\mathbf{u}_{1,h}^m\|_{\Omega_1}^2 - \|\mathbf{u}_{1,h}^{m-1}\|_{\Omega_1}^2 \right) + \frac{\rho}{2} \sum_{m=1}^n \|\mathbf{u}_{1,h}^m - \mathbf{u}_{1,h}^{m-1}\|_{\Omega_1}^2 + 2\mu\delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{1,h}^m\|_{\Omega_1}^2 \\
& + \frac{\rho}{2} \sum_{m=1}^n \left(\|\mathbf{u}_{2,h}^m\|_{\Omega_2}^2 - \|\mathbf{u}_{2,h}^{m-1}\|_{\Omega_2}^2 \right) + \frac{\rho}{2} \sum_{m=1}^n \|\mathbf{u}_{2,h}^m - \mathbf{u}_{2,h}^{m-1}\|_{\Omega_2}^2 + 2\mu\delta t \sum_{m=1}^n \|\boldsymbol{\epsilon}_{2,h}^m\|_{\Omega_2}^2 \\
= & \underbrace{\delta t \sum_{m=1}^n \int_{\Sigma} (\pi_{2,h}^m - \pi_{2,h}^{m-1}) \mathbf{n}_2 \cdot (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m) - \frac{\gamma_0 h}{\gamma \mu} \delta t \sum_{m=1}^n \int_{\Sigma} (\pi_{2,h}^m - \pi_{2,h}^{m-1}) \pi_{2,h}^m}_{T_1} \\
& - \underbrace{\frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m) \cdot \mathbf{u}_{2,h}^m + \frac{\gamma \mu}{h} \delta t \sum_{m=1}^n \int_{\Sigma} (\mathbf{u}_{2,h}^{m-1} - \mathbf{u}_{1,h}^m) \cdot \mathbf{u}_{1,h}^m}_{T_2} \\
& + \underbrace{2\mu\delta t \sum_{m=1}^n \int_{\Sigma} \boldsymbol{\epsilon}_{2,h}^{m-1} \cdot \mathbf{n}_2 (\mathbf{u}_{2,h}^m - \mathbf{u}_{1,h}^m)}_{T_3}.
\end{aligned}$$

The terms in the right-hand side are bounded *mutatis mutandis* as in the proof of Proposition 3. This completes the proof. \square

5 Numerical Examples

In this section, we first discuss some tests to illustrate the influence of the stabilization parameters γ and γ_0 , then we concentrate on hemodynamics applications. For comparison purposes, the following error indicators will be used:

$$\varepsilon_{x_i} := \frac{\|x_i - x_{\text{ref}}\|_{L^2(\Gamma)}}{\|x_{\text{ref}}\|_{L^2(\Gamma)}}, \quad \varepsilon_x := \frac{\|x_1 - x_2\|_{L^2(\Sigma)}}{\|x_{\text{ref}}\|_{L^2(\Sigma)}}.$$

The first indicator measures the relative error on a part of the boundary $\Gamma \subset \partial\Omega_i$ for the physical quantity x_i (defined in Ω_i), $i = 1, 2$. The second indicator gives the relative interface drop of the variable x across the interface Σ . The subscript $_{\text{ref}}$ indicates a quantity from the reference solution, obtained by solving problems (1) or (31) in the whole domain Ω .

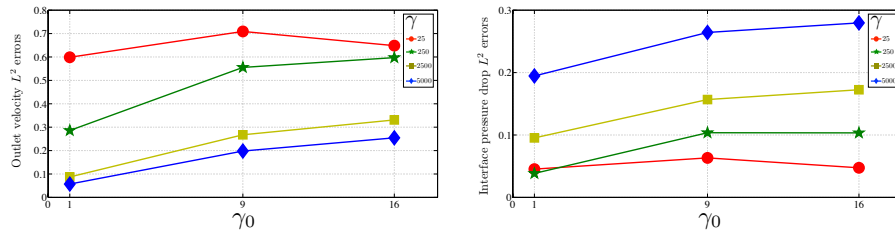
5.1 Impact of the stabilization parameters

In this paragraph we investigate the impact of the stabilization parameters γ and γ_0 on the accuracy of Algorithm 4.2. We consider a two coupled Stokes problems in a rectangular domain $[0, 1] \times [0, 1] \cup [1, 2] \times [0, 1]$. Note that in this case, only the interface pressure stabilization (22) is considered, since the convective non-linear terms are neglected. A constant velocity $(400, 0)^T$ cm/s is imposed on the left boundary and zero traction on the right boundary. A no-slip condition is enforced on the upper and lower sides. The density $\rho = 1.06$ g/cm³ and the dynamic viscosity $\mu = 0.04$ poise are those typically encountered in blood flow simulations. The spatial discretization is based on P_1/P_1 stabilized finite elements. The time-step is $\tau = 10^{-4}$ s and the space step is $h = 0.05$ cm. The tests are run with $\gamma_0 = 0, 1, 9, 16$ and $\gamma = 25, 250, 2500, 5000$. All the numerical computations have been performed in FreeFem++ (see [17]).

Table 1: Windkessel parameters.

	Γ_1	Γ_2	Γ_3	Γ_{out}
R_p ($dyn \cdot s \cdot cm^{-5}$)	0.005×10^4	0.19×10^4	0.075×10^4	0.015×10^4
R_d ($dyn \cdot s \cdot cm^{-5}$)	0.85×10^4	3.22×10^4	1.25×10^4	0.25×10^4
C ($cm^5 \cdot dyn^{-1}$)	0.95×10^{-4}	0.25×10^{-4}	0.64×10^{-4}	3.17×10^{-4}

In Figure 1, the relative error on the outlet velocity and on the pressure drop at the interface are plotted. These results suggest that the optimal choice of the parameters is $\gamma = 2500$ and $\gamma_0 = 1$, which gives a good compromise between the velocity and pressure errors. The values for $\gamma_0 = 0$ are not reported in Figure 1 since the corresponding numerical solution is unstable. This highlights the importance of the interface pressure stabilization (22).

Figure 1: Outlet velocity errors $\varepsilon_{\mathbf{u}_2}$ (left) and interface pressure drop errors ε_p (right).

5.2 Aortic blood flow simulation

The computational domain is the idealized geometry reported in Figure 2 (left), including the aortic root, with the valve, and the aortic arch. The two-domain partitioning is shown in Figure 2 (middle). This geometrical splitting is motivated by the fact that, in order to describe the blood dynamics through the valve, different modeling options can be incorporated within the aortic root Ω_1 (see, e.g., [1, 11]). Here, for simplicity, the aortic valve is frozen in its open configuration. No-slip boundary conditions are imposed on the three leaflets of the valve.

A sinusoidal waveform, see Figure 2 (right), is imposed on the inlet boundary Γ_{in} . The systolic phase, corresponding to the first half of the cardiac cycle, delivers a maximum flux of about $235 \text{ cm}^3/\text{s}$. In the second half (diastolic phase), the inlet flow is set to zero. The resulting cardiac output is approximately $4.5 \text{ dm}^3/\text{min}$. The outlets Γ_1 , Γ_2 and Γ_3 correspond to the brachiocephalic artery, left common carotid and subclavian artery, respectively. The outlet Γ_{out} is located in the descending aorta. The neglected portions of the circulation are taken into account by imposing the natural boundary condition $\boldsymbol{\sigma} \cdot \mathbf{n} = -P_p \mathbf{n}$ on each outlet. The proximal pressure P_p is described by a (0D) three-element Windkessel model (see, e.g., [3, 21]) given by the ODE:

$$CR_d \frac{dP_p}{dt} + P_p = Q(R_d + R_p) + CR_p R_d \frac{dQ}{dt},$$

where Q stands for the outlet flow rate, R_p and R_d denote the proximal and distal resistances and C is the capacitance representing the compliance of the blood vessels. The Windkessel parameters are reported in Table 1. For simplicity, the left and right coronaries, located in the aortic sinus, are closed. The physical parameters for blood are chosen as $\rho = 1.06 \text{ g/cm}^3$ and $\mu = 0.04 \text{ poise}$. The space discretization is based on P_1/P_1 finite elements, stabilized with the SUPG method. The stabilization parameters are $\gamma = 2500$ and $\gamma_0 = 1$. For comparison purposes,

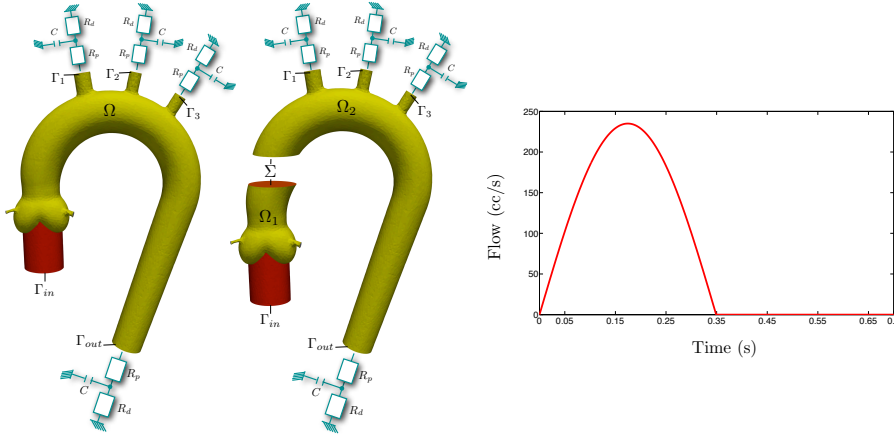


Figure 2: Reference domain (left), sub-domains definition (middle) and inflow waveform (right).

the reference solution has been generated by solving the Navier-Stokes equations associated in the whole domain Ω . The meshes of Ω , Ω_1 and Ω_2 are, respectively, made of 177 651, 53 960 and 126 200 tetrahedra. Six cardiac cycles have been simulated, using a time-step length of $\tau = 10^{-3} s$.

5.2.1 Static pressure formulations

In this paragraph the results obtained with Algorithms 4.2 and 4.3 are compared with the reference solution (1). Figure 3 displays the snapshots of the velocity magnitude at two time instants of the third cardiac cycle: $t = 1.575 s$ middle of systole (top), and $t = 1.750 s$ beginning of diastole (bottom). The solution obtained with Algorithm 4.2 is close to the reference solution on the interface and on the outlets. Note that even if we were not able to prove its energy stability (Proposition 3), the simulation is stable in this test case. Algorithm 4.3 yields a mass leak on the interface and has very poor results at the outlets. This is due to the non-consistent terms (27) and (28). This behavior is more visible at high velocity, i.e., in the systole phase.

Table 2: Interface relative errors: flow ε_{f_1} , ε_{f_2} and pressure ε_{p_1} , ε_{p_2} . Interface drop errors: flow ε_f and pressure ε_p .

	Algorithm 4.2		Algorithm 4.3				
ε_{f_1}	0.0048	ε_{p_1}	0.0135	ε_{f_1}	0.0048	ε_{p_1}	0.1864
ε_{f_2}	0.0123	ε_{p_2}	0.0140	ε_{f_2}	0.1058	ε_{p_2}	0.1866
ε_f	0.0153	ε_p	0.0013	ε_f	0.1054	ε_p	0.0012

Figures 4 and 5 show the interface flow rate and pressure in Ω_1 and Ω_2 , obtained with Algorithms 4.2 and 4.3, respectively. The results provided by Algorithm 4.2 are clearly more accurate than those obtained with Algorithm 4.3. This can also be inferred from the error indicators reported in Table 2.

The same behavior is observed for the flow rate and the pressure at the outlets. Figures 6 and 7 and Figures 8 and 9 compare the reference solution with the results obtained with the staggered algorithms in the three top arteries and the descending aorta. As shown by Table 3, the flow rate and pressure errors obtained with Algorithm 4.2 are between 1% and 5%, whereas with Algorithm 4.3 we get a 20% error.

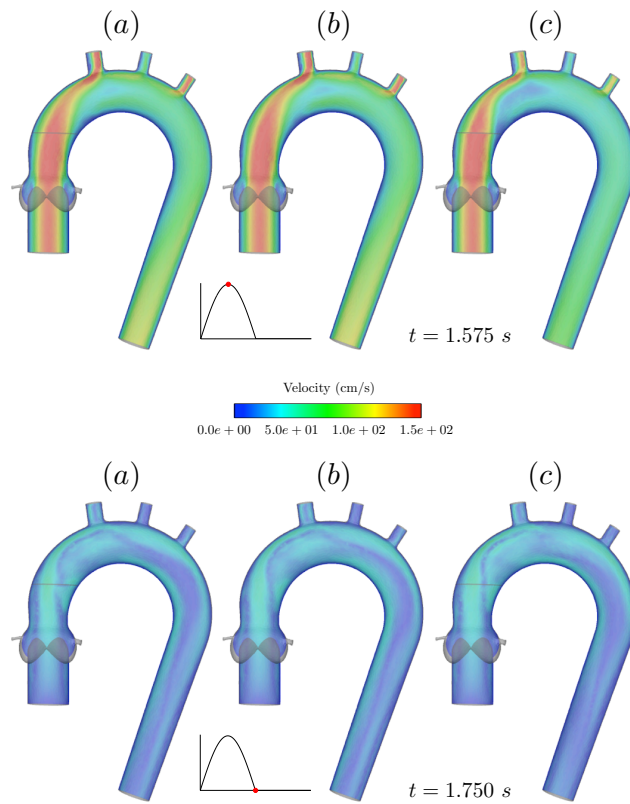


Figure 3: Static pressure formulation. Snapshot of the velocity magnitude at two time instants obtained with: (a) Algorithm 4.2; (b) Reference solution; (c) Algorithm 4.3.

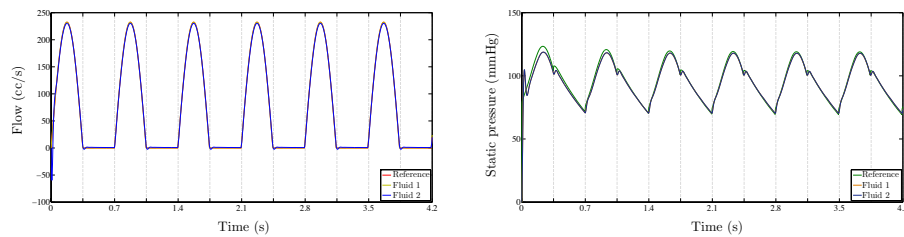


Figure 4: Reference vs Algorithm 4.2: interface flows (left) and static pressures (right).

5.2.2 Total pressure formulation

In this paragraph the results obtained with Algorithm 4.2 are presented. Figure 10 shows snapshots taken at two time instants of the third cardiac cycle, in the maximum of the systole $t = 1.575$ s and at the beginning of the diastole $t = 1.750$ s, for the explicit algorithm with the total pressure and the reference solution obtained by the discretization of (31). Note the good agreement of the results on the interface and at the outlets.

Figure 11 depicts the flow rate, on the left, and the pressure, on the right, for Fluid 1 and the Fluid 2, compared with the reference solution on the interface. Except very slight differences

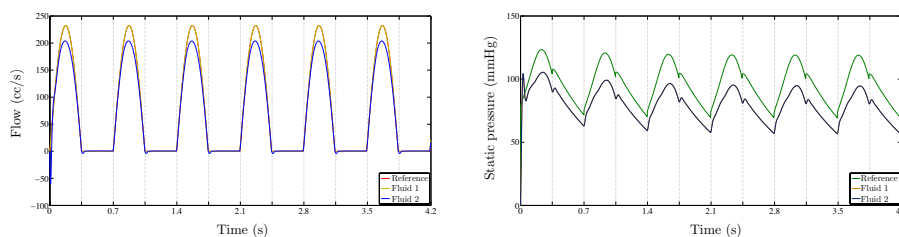


Figure 5: Reference vs Algorithm 4.3: interface flows (left) and static pressures (right).

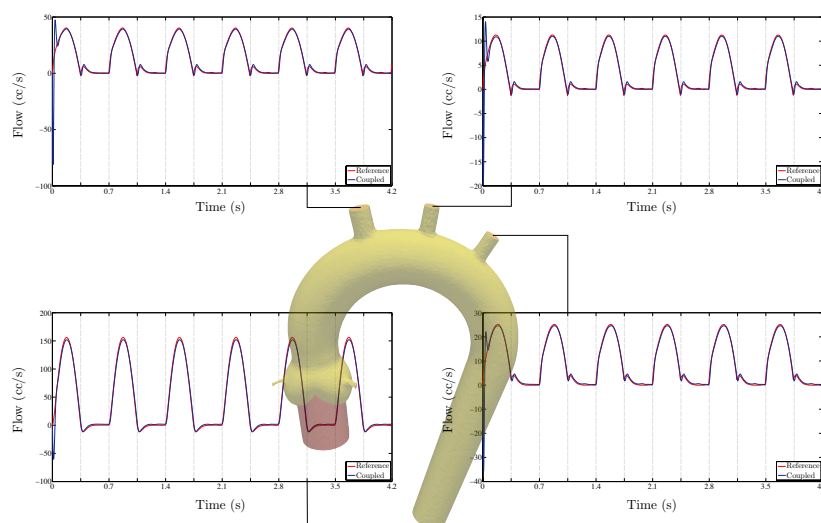


Figure 6: Reference vs Algorithm 4.2: outlet flows.

for Fluid 2 in the peak of the systole and at the beginning of the diastole, the flow rate and the pressure drop between the two fluids are around 2.86% and 1.13% respectively (Table 4).

The same behavior can be observed for the flow rate and the pressure at the outlets (Figure 12 and 13). Discrepancies still arise at the maximum of the systoles and at the beginning of the diastoles. The errors in these cases remain between 6% and 9% for the flow rate and around 3% for the pressure (Table 5).

6 Conclusions

In this work we have discussed several loosely coupled strategies for fluid-fluid interaction problems coupling the incompressible Navier-Stokes equations through standard kinematic/kinetic

Table 3: Outlet flow ε_{f_2} and static pressure ε_{p_2} errors.

Algorithm 4.2					Algorithm 4.3				
	Γ_1	Γ_2	Γ_3	Γ_{out}	Γ_1	Γ_2	Γ_3	Γ_{out}	
ε_{f_2}	0.0589	0.0596	0.0488	0.0411	0.2171	0.2237	0.2003	0.2185	
ε_{p_2}	0.0115	0.0121	0.0111	0.0122	0.1822	0.1843	0.1797	0.1828	

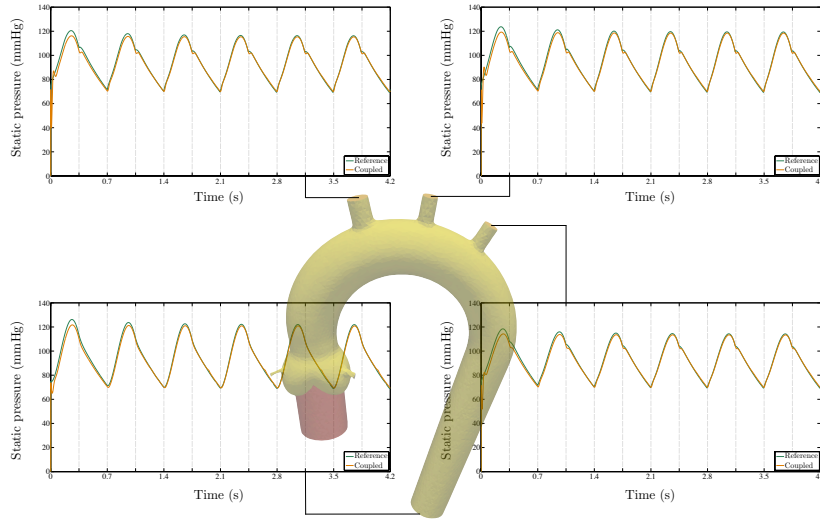


Figure 7: Reference vs Algorithm 4.2: outlet static pressures.

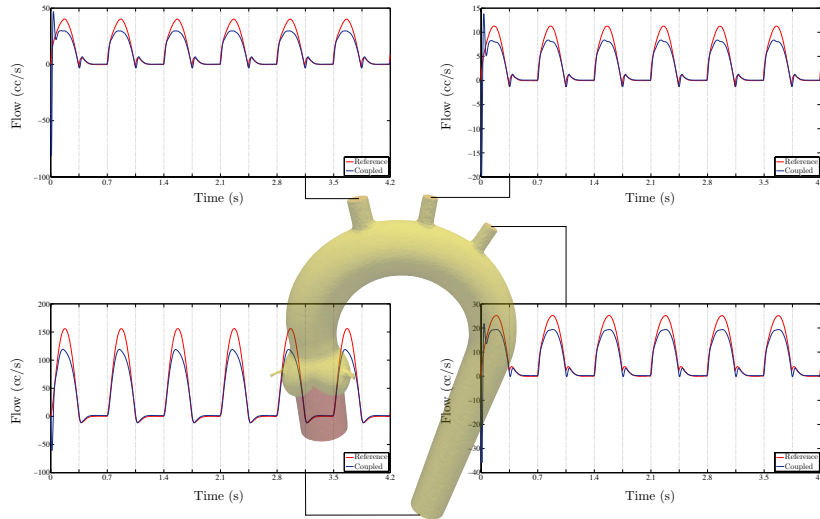


Figure 8: Reference vs Algorithm 4.3: outlet flows.

Table 4: Interface relative errors: flow ε_{f_1} , ε_{f_2} and total pressure ε_{π_1} , ε_{π_2} . Interface drop errors: flow ε_f and total pressure ε_π .

ε_{f_1}	0.0053	ε_{π_1}	0.0321
ε_{f_2}	0.0266	ε_{π_2}	0.0415
ε_f	0.0286	ε_π	0.0113

interface conditions. The main ingredients of the methods considered are:

1. an explicit Robin-Robin treatment of the interface coupling and a suitable weakly consistent artificial compressibility on the interface;

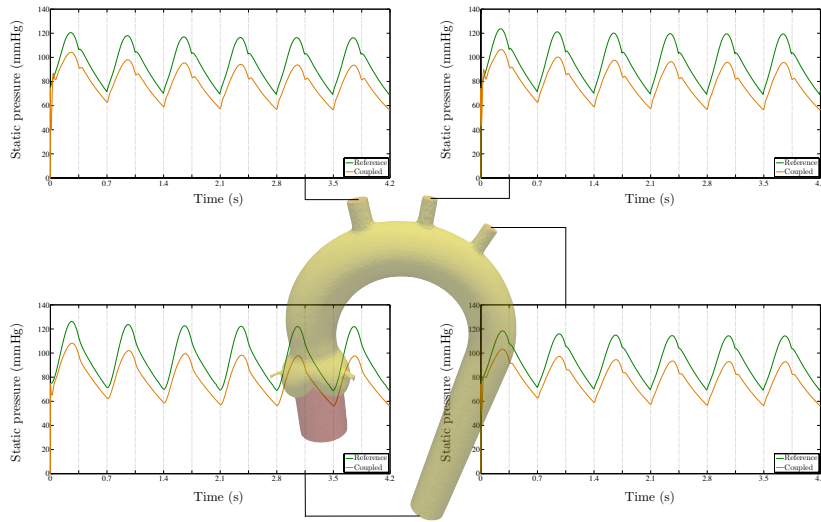


Figure 9: Reference vs Algorithm 4.3: outlet static pressures.

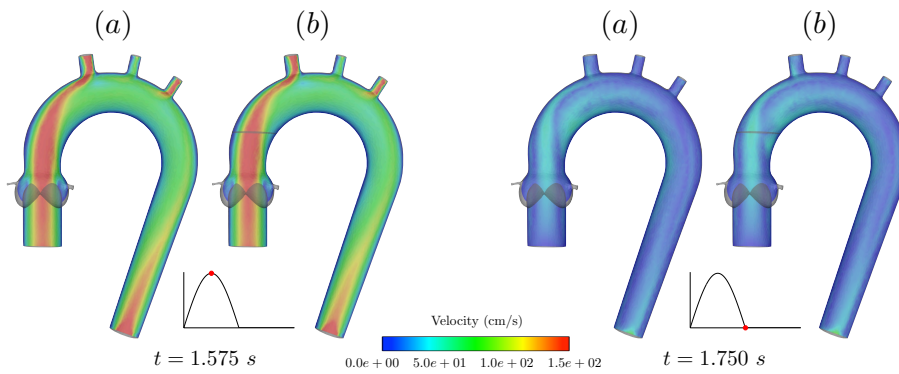


Figure 10: Total pressure formulation. Snapshot of the velocity magnitude at two time instants obtained with: Reference solution (a) and Algorithm 4.2 (b).

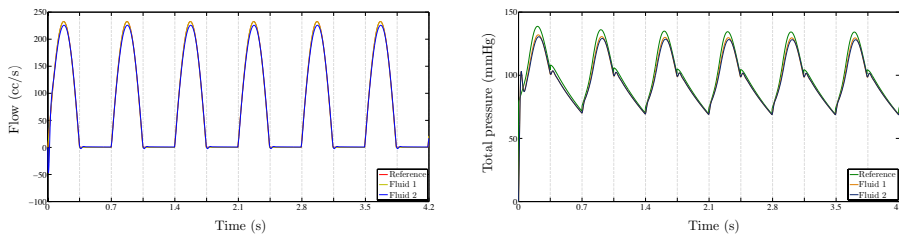


Figure 11: Reference vs Algorithm 4.2: interface flows (left) and total pressures (right).

2. a specific formulation of the convective terms: static pressure (standard and skew-symmetric) and total pressure formulations.

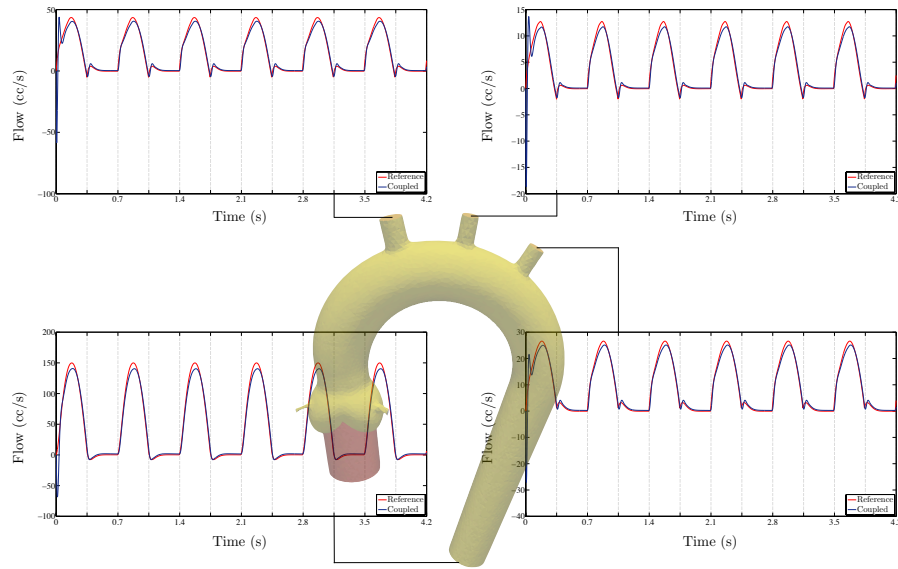


Figure 12: Reference vs Algorithm 4.2: outlet flows.

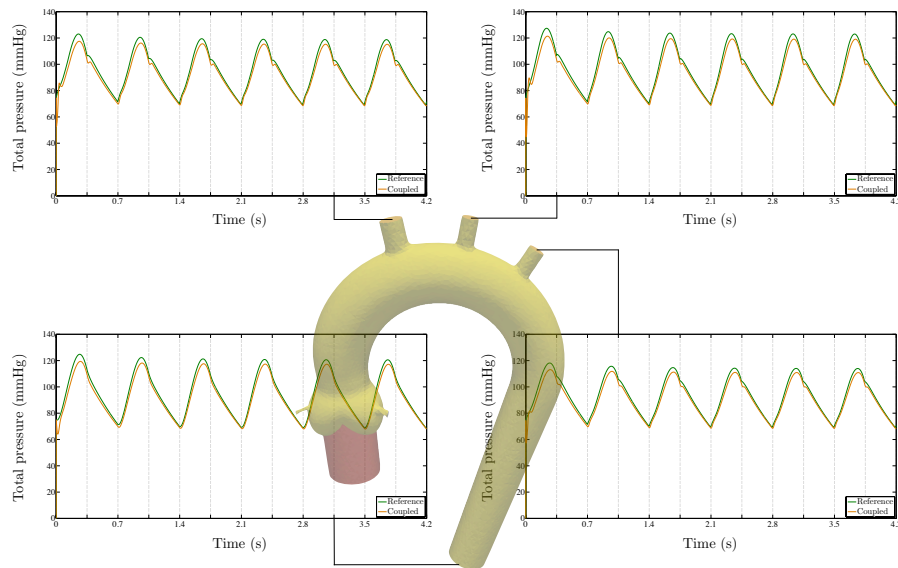


Figure 13: Reference vs Algorithm 4.2: outlet total pressures.

Table 5: Algorithm 4.2: Outlet flow ε_{f_2} and total pressure ε_{π_2} errors.

	Γ_1	Γ_2	Γ_3	Γ_{out}
ε_{f_2}	0.0853	0.0926	0.0714	0.0670
ε_{π_2}	0.0295	0.0311	0.0276	0.0284

The first controls the artificial power generated by the kinematic/kinetic splitting on the interface, while the second governs the unbalanced dynamic pressure power across the interface.

Energy stability cannot a priori be guaranteed for the standard static pressure formulation Algorithm 4.2. Yet, extensive numerical evidence has shown that the scheme is robust and accurate (with respect to a fully implicit method). This suggests that the artificial dynamic pressure power generated by the splitting has a lower impact on the overall stability of the scheme. Energy stability can be recovered with a static pressure formulation and a skew-symmetric treatment of the convection (Algorithm 4.3). Unfortunately, this formulation is not consistent with the original coupled problem due to its non-confirming character across the interface. The numerical results have confirmed its poor accuracy. For the total pressure formulation, an a priori energy estimate guaranteeing the stability of the splitting (Algorithm 4.2) has been derived. The comparison with fully coupled solutions have shown that the method gives satisfactory results.

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