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► **To cite this version:**

Thomas Braibant, Jacques-Henri Jourdan, David Monniaux. Implementing and reasoning about hash-consed data structures in Coq. *Journal of Automated Reasoning*, 2014, 53 (3), pp.271-304. 10.1007/s10817-014-9306-0 . hal-00881085v3

**HAL Id: hal-00881085**

**<https://inria.hal.science/hal-00881085v3>**

Submitted on 13 Dec 2014 (v3), last revised 25 Sep 2015 (v4)

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# Implementing and reasoning about hash-consed data structures in Coq

Thomas Braibant · Jacques-Henri Jourdan ·  
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November 2013

**Abstract** We report on four different approaches to implementing hash-consing in Coq programs. The use cases include execution inside Coq, or execution of the extracted OCaml code. We explore the different trade-offs between faithful use of pristine extracted code, and code that is fine-tuned to make use of OCaml programming constructs not available in Coq. We discuss the possible consequences in terms of performances and guarantees. We use the running example of binary decision diagrams and then demonstrate the generality of our solutions by applying them to other examples of hash-consed data structures.

## 1 Introduction

*Hash-consing* is a programming technique used to share identical immutable values in memory, keeping a single copy of semantically equivalent objects. It is especially useful in order to get a compact representation of abstract syntax trees. A hash-consing library maintains a global pool of expressions and never recreates an expression equal to one already in memory, instead reusing the one already present. In typical implementations, this pool is a global hash-table. Hence, the phrase *hash-consing* denotes the technique of replacing node *construction* by a lookup in a hash table returning a preexisting object, or creation of the object followed by insertion into the table if previously nonexistent. This makes it possible to get *maximal sharing* between objects, if hash-consing is used systematically when creating objects.

Moreover, a unique identifier is given to each object, allowing fast hashing and comparisons. This makes it possible to do efficient *memoization* (also known as *dynamic programming*): the results of an operation are tabulated so as to be returned immediately when an identical sub-problem is encountered.

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Hash-consing and memoization are examples of imperative techniques that are of prime importance for performance. The typical way of implementing hash-consing (a global hash table) does not translate easily into Coq. The reason is that the Gallina programming language at the heart of the Coq proof assistant is a purely applicative language, without imperative traits such as hash tables, pointers, or pointer equality. In the following, we discuss how hash-consing can be implemented using the Coq proof assistant (without modifying it) with two possible use cases in mind:

- execution inside Coq with reasonable efficiency, e.g. for proofs by reflection;
- efficient execution when extracted to OCaml, e.g. for use in a model-checking or static analysis tool proved correct in Coq.

More precisely, we present a few *design patterns* to implement hash-consing and memoization on various examples, and we use binary decision diagrams (BDDs) as our prime example. We evaluate these design patterns based on several criterion. For instance, we discuss for each pattern how we handle the maximal sharing property and how easy it is to prove the soundness of our implementation. We also discuss whether or not it is possible to compute inside Coq with a given solution.

*Reading this article.* In the following, we will try to separate some implementation details or Coq specific comments from the main text of the article. These remarks, titled “For the practical mind”, can be safely skipped on the first reading.

*Outline.* This paper is organized as follows. Since hash-consing can be seen as a particular kind of memoization, where one memoizes constructor applications, we begin this paper with a short review of the techniques that are available in Coq in order to memoize functions (§2). Then, we give a short primer on BDDs in §3 and present the “menu” of our four implementations of BDD libraries in §4. These implementations are subsequently described in §5 and §6 and use different design patterns. We discuss the pros and cons of each design pattern in §7. We adapt these design patterns to implement strong reduction of hash-consed lambda-terms in §8. We discuss related work and conclude this paper in §9.

## 2 Memoization in Coq: state of the art

Generally speaking, there are two ways to use Coq code: either to execute it inside Coq, e.g., to define reflective decision procedures; or to extract it to OCaml (or other supported languages), then compile the extracted code. These two usages impose constraints on the possible design patterns. In the remainder of this section, we describe what the state of the art in memoization approaches is with respect to these two use cases.

### 2.1 State monad

The first idea to implement memoization in Coq is to use a dedicated data structure to tabulate function calls: that is, use a finite map of some sort to store the pairs

```

Section t.
Context {A : Type}.
CoInductive lazy : Type :=
| Thunk : A → lazy.

CoInductive lazies : Type :=
| Cons : lazy → lazies → lazies.

Fixpoint nth n cst :=
  match n, cst with
  | 0, Cons (Thunk xi) _ => xi
  | S p, Cons _ c => nth p c
  end.

CoFixpoint mk_lazy {B} f (n : B) :=
  Thunk (f n).

CoFixpoint memo' f n :=
  Cons (mk_lazy f n) (memo' f (S n)).

Definition memo f := memo' f 0.
End t.

(* A costly version of the identity *)
Fixpoint big {A} n : A → A :=
  match n with
  | 0 => fun x => x
  | S n => fun x => big n (big n x)
  end.

Definition id n := big n n.
Definition k := memo id.
Definition run x := nth x k.

```

Fig. 1: Memoization using co-inductives

(*argument, result*) for a given function. Users of a library implemented in this fashion must, then, thread this state through the program using a state monad.

While this solution always works, it has non-negligible verification cost: we must prove that the bindings in the memoization table are correct, i.e., the values held in the table correspond to the result of the would-be corresponding function calls.

Then, if we memoize a function of type  $A \rightarrow B$ , we get a function of type  $A \rightarrow M B$ , where  $M$  is the type constructor associated with our state monad. Even if the latter seems equivalent to the former, these types are different: therefore, this tiny modification (memoizing a function) requires modifying every caller of the memoized function to use a monadic style.

Since it is cumbersome to perform all computations inside a state monad, we continue this section with a review of other (partial) solutions to the memoization problem.

## 2.2 Shallow memoization

It is possible in some cases to make a *shallow embedding* of this memoization table in Coq. Melquiond [17] describes how to use co-inductive objects as a cache that stores previously computed values of a given function. In this section, we present his clever idea in Fig. 1, with a few cosmetic changes<sup>1</sup>. A *lazy* value is defined as a co-inductive, which will effectively be represented as a thunk when executed by Coq's virtual machine, and will be extracted to OCaml as a `Lazy.t` thunk. A *thunk* is a term whose evaluation is frozen until it is actually needed (lazy evaluation), then the computed value is cached in the thunk so that it is instantly returned in case the term is evaluated again (thus, performing a kind of memoization).

Then, `lazies` represents a stream (an infinite list) of thunks, that can be peeked at using `nth`. Each time the virtual machine has to destruct the `cst` argument of

<sup>1</sup> We found out that similar definitions actually made their way into Coq's standard library under the name `StreamMemo.v`, yet we believe that this idea deserves more explanation than what is available.

`nth`, it will remember the two arguments of this `cons` cell. Then, it suffices to build a stream `memo` that computes the value of a function `f` for each possible natural number: in practice, `f` will be evaluated lazily, and only once per input value.

As an example, the following snippet of code memoizes the time-consuming function `id`. (Remark that it also demonstrates that the intermediate calls of `id` are not memoized.)

```
Time Eval vm_compute in run 26. (* 6s *)      Time Eval vm_compute in id 25. (* 3s *)
Time Eval vm_compute in run 26. (* 0s *)      Time Eval vm_compute in id 25. (* 3s *)
Time Eval vm_compute in run 25. (* 3s *)
Time Eval vm_compute in run 25. (* 0s *)
```

Note that this clever trick is not limited to functions over Peano numbers, and could be adapted for branching structures, e.g. memoizing functions over binary numbers. Unfortunately, it seems hard to adapt this idea to memoize recursive calls: consider for instance the function

```
Fixpoint exp n := match n with 0 => 1 | S n => exp n + exp n end.
```

In order to compute `exp` using a linear number of recursive calls we need a memoizing fixed-point combinator rather than a way to memoize a predefined function.

(To be complete, let us add that functions `f : nat → A` that can be expressed as the iteration of a given function `g : A → A`, can actually be memoized using the aforementioned technique. While we obviously could rewrite `exp` to fit into that scheme, this is not the case of the other functions that we use through our developments.)

This problem is somewhat representative of the one that we have to face when it comes to BDDs: first, memoizing fixed-point combinators are crucial in our work; second, the function that we memoize depends on the global state of the hash-consing machinery, which evolves through times. In both cases, we fall out of the scope of the above trick: there is no predefined function to memoize.

### 2.3 Memoization in extracted code

Now, we turn to a partial solution to the memoization problem, that works when the user is solely interested in executing the code extracted from a Coq program.

Vafeiadis [21] introduced recently a restricted form of mutable state called *adjustable references*. The idea is that adjustable references store some internal value that can only be updated in innocuous ways. That is, adjustable references are equipped with an observation function; and the update function ensures that the result of the observations remain equals through updates. It is therefore possible to adjust the values held in the references in a way that changes, e.g., the costs of some computations, yet does not change the result of these computations.

Vafeiadis demonstrated how to use adjustable references to memoize a function `f` by tabulating its results: the adjustable reference internal state is a finite map, while the observation function simply returns `f`. Updating the contents of the finite map does not change the observation function; but subsequent reads may make use of the fact that a value was previously computed and stored inside the memoization cache. We refer the reader to Vafeiadis' article for more details about this idea.

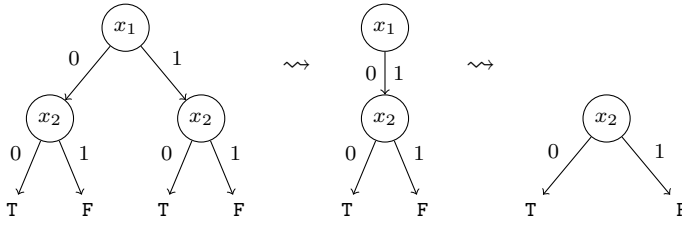


Fig. 2: The BDD for the function  $f(0,0) = \text{T}$ ,  $f(0,1) = \text{F}$ ,  $f(1,0) = \text{T}$ ,  $f(1,1) = \text{F}$

Remark that it is again the case that this technique does not scale to the definition of memoizing fixed-point combinators. Therefore, adjustable references do not quite fit our needs either.

## 2.4 Summary

Out of the three memoization techniques available in Coq that we described, only the first one suits our need: it makes it possible to define memoizing fixed-point combinators. In this paper, we use this *state monad* approach, then we will come up with a new idiom.

## 3 A primer on binary decision diagrams

In the following, we use “reduced ordered binary decision diagrams” (ROBDDs, BDDs for short) as a running example of hash-consed data structures. BDDs are representations of Boolean functions and are often used in software and hardware formal verification tools, in particular *model checkers* [16].

*The data structure.* A Boolean function  $f : \{0,1\}^n \rightarrow \{0,1\}$  can be represented as a complete binary tree with  $2^n - 1$  decision nodes, labeled by variables  $x_i$  according to the depth  $i$  from the root (thus the adjective *ordered*); edges are labeled 0 and 1; leaves are labeled T (for true) or F (for false). The semantics of such a diagram is:  $f(x_1, \dots, x_n)$  is obtained by traversing from the root and following the 0 or 1 edge of a node labeled by  $x_i$  according to the value of  $x_i$ .

Such a tree can be *reduced* by merging identical subtrees, thus becoming a connected *directed acyclic graph* (DAG; see second diagram in Fig. 2); furthermore, decision nodes with identical children are removed (see third diagram in Fig. 2). These transformations preserve semantics. The reduced representation is *canonical*: given a variable ordering  $x_1, \dots, x_n$ , a function is represented by a unique ROBDD.

*Building BDDs.* In practice, one directly constructs the reduced tree; let us see how. All BDD nodes allocated so far are stored in a global hash table, so that a new object is created only if not already in the hash table: *there are never two identical objects at two different locations in memory*, a crucial invariant that must be maintained throughout the execution.

Each BDD node is given a unique identifier: for instance, the current value of a global 64-bit counter incremented following each allocation<sup>2</sup>. Since a new object is never created if an identical object already exists, two objects are equal if and only if they have the same unique identifier. The equality test, which is usually expensive over tree structures, since it requires full traversal, is instead implemented by a very fast comparison of unique identifiers.

Unique identifiers are also used for hashing. When attempting to construct a node  $(v, b_0, b_1)$  where  $v$  is the variable,  $b_0$  is the subtree labeled 0 and  $b_1$  is the subtree labeled 1, one computes the hash value of the node as  $H(v, u_0, u_1)$  where  $u_0$  and  $u_1$  are the respective unique identifiers of the subtrees  $b_0$  and  $b_1$ . This hash value is then used to look up the node in the hash table. Again, such shallow hashing is considerably faster than having to traverse the tree structures.

*Operations on BDDs.* The principal way to build BDDs is to combine the diagrams of two functions  $f$  and  $g$  in order to obtain the BDD for other functions such as  $f \wedge g$ ,  $f \vee g$  or  $f \oplus g$ . The basic combinator of BDDs is called node *melding* [16]. Intuitively, melding corresponds to a traversal of the internal nodes of two BDDs to build a third diagram. This traversal respects the order on variables, and thus, the third diagram will naturally be ordered. Then, all binary operations on BDDs are defined as one particular instance of melding.

Suppose that we have two nodes  $\alpha = (v, l, h)$  and  $\alpha' = (v', l', h')$ . The melding of  $\alpha$  and  $\alpha'$ , denoted by  $\alpha \diamond \alpha'$  is defined as follows:

$$\alpha \diamond \alpha' = \begin{cases} (v, l \diamond l', h \diamond h') & \text{if } v = v' \\ (v, l \diamond \alpha', h \diamond \alpha') & \text{if } v < v' \\ (v', \alpha \diamond l', \alpha \diamond h') & \text{if } v > v' \end{cases}$$

Then, the binary operation  $\diamond$  is entirely defined by the results of

$$\mathbf{F} \diamond \alpha \quad \alpha \diamond \mathbf{F} \quad \mathbf{T} \diamond \alpha \quad \alpha \diamond \mathbf{T}$$

For instance, the conjunction operation  $f \wedge g$  can be defined by melding the BDDs for  $f$  and  $g$ , and using the following rewrite rules for the leaf cases:

$$\mathbf{F} \diamond \alpha \rightarrow \mathbf{F} \quad \alpha \diamond \mathbf{F} \rightarrow \mathbf{F} \quad \mathbf{T} \diamond \alpha \rightarrow \alpha \quad \alpha \diamond \mathbf{T} \rightarrow \alpha$$

For the sake of clarity, we focus on binary operations in the following: they are representative of the difficulties we had to face<sup>3</sup>.

*Memoization.* The fact that each node has a unique identifier also makes it possible to memoize the results of BDD operations. One keeps a map from sub-problems (a pair of nodes  $\alpha$  and  $\alpha'$ ) to nodes (the result of  $\alpha \diamond \alpha'$ ), so as to be returned immediately when a previously solved sub-problem is encountered.

In a BDD library, memoization is crucial to implement the or/and/xor operations with time complexity in  $O(|a| \cdot |b|)$  where  $|a|$  and  $|b|$  are the sizes of the inputs: as the function is executed, its results on the subtrees of the original problem are stored in a structure indexed by  $(u_a, u_b)$  where  $u_a$  and  $u_b$  are the unique identifiers of the  $a$  and  $b$  inputs. In contrast, the naive approach has exponential complexity, since the function may be evaluated exponentially often on the same couple of subtrees.

<sup>2</sup> In certain implementations, the unique identifier is the address of the node; this supposes that objects never change address, which is not the case in OCaml.

<sup>3</sup> Note that we only need two rewrite rules for commutative binary operations.

## 4 Implementing BDDs in Coq.

The following two sections §5 and §6 describe a total of *four* implementations of BDDs in Coq. To make things clear for the reader, we give each of these implementations a reference name, a pointer to the relevant section of the paper and a short description.

**PURE-DEEP.** See §5.1. A pure Coq implementation of BDDs that makes a deep embedding of memory as finite maps and uses indices as surrogates of pointers.

**PURE-SHALLOW.** See §5.2. A pure Coq implementation of BDDs that uses a shallow embedding of memory.

**SMART.** See §6.1. An "impure" implementation of BDDs in Coq: we implement hash-consing and memoization through the extraction mechanism of Coq.

**SMART+UID.** See §6.2. A variation on the previous approach in which we discuss how to expose and axiomatize the operations on the unique identifiers associated with BDD nodes.

## 5 Pure solutions

As the reader can gather from the summary above, we describe two pure Coq implementations of BDDs libraries in this section, that differ mainly by the way we model the memory and the allocation of nodes.

### 5.1 The PURE-DEEP approach

The idea here is to model the memory using finite maps inside Coq and use indices in the maps as surrogates for pointers, implementing BDD operations as manipulations of these persistent maps. Such an implementation<sup>4</sup> was described in [22, 23], but we propose a new one here for the sake of completeness, and because the old one did not age well w.r.t. the evolution of Coq. Our implementation is defined as follows.

First, we assign a unique identifier to each decision node. Second, we represent the directed acyclic graph underlying a BDD as a Coq finite map from identifiers to decision nodes (that is, tuples that hold the left child, the node variable and the right child). For instance, the following graph, on the left, can be represented using the map on the right.



Then, we implement the hash-consing pool using another map from decision nodes to node identifiers and a `next` counter that is used to assign a unique identifier to a fresh node. In the situation above, `next` is equal to 4 and the hash-consing map is defined as follows

<sup>4</sup> One should also mention the seminal work by E. Ledinot in 1993 on the canonicity of binary decision dags, available from the Coq contributions.



$$\begin{aligned}
(F, x_1, N\ 2) &\mapsto 1 \\
(F, x_2, N\ 3) &\mapsto 2 \\
(F, x_3, T) &\mapsto 3
\end{aligned}$$

*Notations.* Through this paper, we stay as close as possible to our Coq code. Yet we allow us some liberty when it comes to finite maps (that are pervasive in our code): the notation  $\mathfrak{t} \rightsquigarrow \mathfrak{s}$  denotes a type of efficient maps from type  $\mathfrak{t}$  to type  $\mathfrak{s}$ ; and we use indiscriminately the notations `get` and `set` that respectively access and update such finite maps. Implicitly, these two notations have the following types:

$$\begin{aligned}
\text{get} &: A \rightarrow (A \rightsquigarrow B) \rightarrow \text{option } B \\
\text{set} &: A \rightarrow B \rightarrow (A \rightsquigarrow B) \rightarrow (A \rightsquigarrow B)
\end{aligned}$$

(Note that we do use the efficient finite maps from the Coq standard library; but we chose here to abstract from the particular module definitions and names that we have to use in our code, for the sake of legibility.)

*For the practical mind.* In the following, we use Coq’s positive numbers `positive` as unique identifiers, and use Patricia trees for finite maps.  $\square$

Fig. 3 shows our inductive definitions and the code of the associated allocation function `mk_node`, knowing that `alloc n st` allocates the fresh node `n` in the hash-consing state `st` (taking care of updating both finite maps and incrementing the “next fresh” counter). Then, equality between BDDs (`eqb`) is provided by decidable equality over node identifiers.

All operations thread the current global state in a monadic fashion. The correctness of BDD operations corresponds to the facts that 1) the global state is used in a monotonic fashion (that is the structure of the resulting global state is a refinement of the input one, see Fig. 4); 2) the resulting global state is well-formed; 3) the denotation of the resulting BDD expression is correct.

*Invariants and well-formedness properties.* We present the well-formedness properties we preserve over expressions (`wf_expr`) and over hash-consing states (`wf_hashcons`) in Fig. 5. The inductive predicate `wf_expr st v e` depends on a variable level `v` that indicates a bound over the variable identifiers appearing in the BDD `e`. This ensures that variable are kept ordered. Then, a well-formed expression is either a leaf (cases `wf_T` and `wf_F`) or a pointer to a hash-consing node with a head variable `w` less than the bound `v`. (Remark that this definition of well-formed expression is not directly recursive, because the actual recursion takes place in the definition of `wf_hashcons` below; this notion of well-formed expressions coupled with well-formedness of the state suffice to prove that expressions are DAGs with a suitable ordering on the variables.)

The predicate `wf_hashcons` is the general well-formedness property over hash-consing states. The predicate `wf_bijection` ensures that the two maps `hmap` and `graph` form a bijection; `wf_expr_lt_next` ensures that the next fresh variable counter will indeed produce fresh variables; `wf_map_wf_expr_l` and `wf_map_wf_expr_r` ensure that a node stored in the hash-consing structure has well-formed children, respecting the variable order; `wf_reduced` ensures that all those nodes are reduced (ie. their left child is different from their right child).

Note that the statement that the hash-consing structure is correct corresponds to `wf_bijection` and `wf_expr_lt_next`. The other statements correspond to properties about BDDs, namely the facts that they are reduced and ordered.

```

Inductive expr := F | T | N : positive → expr.

Definition eqb a b :=
  match a,b with | T,T ⇒ true | F,F ⇒ true | N x, N y ⇒ (x =? y) | _, _ ⇒ false end.

Definition node := (expr * var * expr).

Record hashcons_state := {
  graph : positive ↔ node;
  hmap : node ↔ positive;
  next : positive
}.

Definition alloc u st :=
  (st.(next), {
    graph := set st.(next) u st.(graph);
    hmap := set u st.(next) st.(hmap);
    next := st.(next) + 1
  }).

Definition mk_node (l : expr) (v : var) (h : expr) st :=
  if eqb l h then (l,st)
  else match get (l,v,h) st.(hmap) with
    | Some x ⇒ (N x, st)
    | None ⇒ let (x, st) := alloc (l,v,h) st in (N x, st)
  end.

```

Fig. 3: Hash-consing in pure Coq, using a deep-embedding

```

Record incr (st1 st2 : hashcons_state) : Prop := {
  incr_lt : st1.(next) ≤ st2.(next);
  incr_find : ∀ p x, get p st1.(graph) = Some x → get p st2.(graph) = Some x
}.

```

Fig. 4: Monotonicity predicate over hash-consing states

*The denotation of BDD expressions.* Recall that BDD expression, as we have defined them in this section, do not have an inductive structure we can follow: this means that we cannot define the semantics of BDD expressions as a Coq fixed-point. Rather, we have to define the semantics of BDD expressions as an inductive predicate that uses the state of the hash-consing data structure to go through the graph. The inductive `value` is shown on Fig. 6. It is defined as a binary relation with two parameters (the valuation of variables `env` and the state of the hash-consing structure `st`) and two arguments: the expression and its denotation (a Boolean).

*For the practical mind.* We only use the type of environments `env` to establish a semantics, thus efficiency is not an issue. However, we prefer to use finite maps of some kinds to represent environments: using a function of type `var → bool` would force the client to reason about the fact that some BDD expressions are insensitive to the valuation of some variables. Using a finite map make this reasoning internally in our library and eases the client's life. □

```

Inductive wf_expr st : var → expr → Prop :=
| wf_e_T : ∀ v, wf_expr st v T
| wf_e_F : ∀ v, wf_expr st v F
| wf_e_N : ∀ (p : positive) (l : expr) (w : var) (h : expr),
  get p st.(graph) = Some (l, w, h) →
  ∀ (v : var), (w < v) →
  wf_expr st v (N p).

Record wf_hashcons (st : hashcons_state) : Prop := {
  wf_bijection : ∀ p n, get n st.(hmap) = Some p ↔ get p st.(graph) = Some n;
  wf_expr_lt_next : ∀ p v, wf_expr st v (N p) → p < st.(next);
  wf_map_wf_expr_l : ∀ p x v y, get p st.(graph) = Some (x, v, y) → wf_expr st v x;
  wf_map_wf_expr_h : ∀ p x v y, get p st.(graph) = Some (x, v, y) → wf_expr st v y;
  wf_reduced : ∀ p l v h, get p st.(graph) = Some (l, v, h) → l <> h
}.

```

Fig. 5: Well-formedness of hash-consing state

```

Inductive value_env st : expr → bool → Prop :=
| value_T : value_env st T true
| value_F : value_env st F false
| value_N : ∀ (p : positive) (l : expr) (v : var) (h : expr),
  get p st.(graph) = Some (l, v, h) →
  ∀ (vv : bool), get env v = Some vv →
  ∀ (vhl : bool), value_env st (if vv then h else l) vhl →
  value_env st (N p) vhl.

```

Fig. 6: The denotation of BDD expressions

*Termination of BDD operations.* The first problem that we have to solve in our Coq representation is that, as can be expected from our data structure, BDD operations cannot be defined using structural recursion (there is no inductive structure to follow). Unfortunately, we cannot easily use well-founded recursion here because the well-founded relation involves both parameters of the function and the global state.

The problem is that the termination of the BDD operations relies on the fact that the graph of nodes is acyclic; but the graph is not fixed through an execution of the melding operation! Rather, the global state is threaded in the recursive calls of  $\diamond$ . Therefore, proving the recursive calls to be well-founded requires to prove that the  $\diamond$  operation is monotonic w.r.t. graph inclusion. In order to prove termination, we would have to define the fixed-point and prove this monotonicity property *at the same time*. This would involve embedding invariants directly in the global state, using dependent types. This has two major drawbacks: first, defining terms containing complex dependent types is cumbersome and proving properties about them is often very challenging. Second, we would have to pay close attention to prevent Coq from normalizing those proof terms if we want to use this library for reflection purposes.

In the end, we resorted to define partial functions that use a *fuel* argument to ensure termination: that is, they use an explicit bound on the number of iterations to do.

```

Record memo := {
  mand : (positive * positive) ~> expr;
  mor  : (positive * positive) ~> expr;
  mxor : (positive * positive) ~> expr;
  mneg : positive ~> expr}.

Record state := {
  ... :> hashcons_state;
  ... :> memo}.

```

Fig. 7: Adding memoization tables to the global state

```

Record wf_memo2 (st : hashcons_state) (m : (positive * positive) ~> expr)
  (opb : bool → bool → bool)
:= {
  wf_memo2_find_wf_res :
    ∀ x y e, get (x, y) m = Some e → wf_expr st v (N x) → wf_expr st v (N y) →
      wf_expr st v e;
  wf_memo2_find_wf :
    ∀ x y e, get (x, y) m = Some e → ∃ v, wf_expr st v (N x) ∧ wf_expr st v (N y);
  wf_memo2_find_sem :
    ∀ x y res, get (x, y) m = Some res →
      ∀ env vx vy, value env st (N x) vx → value env st (N y) vy →
        value env st res (opb va vb)
}.

Record wf_memo_neg (st : hashcons_state) (m : positive ~> expr) := { ... }.

Record wf_memo (st : state) := {
  wf_memo_mand : wf_memo2 st st.(mand) Datatypes.andb;
  wf_memo_mor  : wf_memo2 st st.(mor)  Datatypes.orb;
  wf_memo_mxor : wf_memo2 st st.(mxor) Datatypes.xorb;
  wf_memo_mneg : wf_memo_neg st st.(mneg)
}.

Record wf_st (st : state) : Prop := { ... : wf_hashcons st; ... : wf_memo b }.

```

Fig. 8: Invariant over the memoization information

*Memoizing operations.* Finally, it is possible to enrich our hash-consing structure with memoization tables in order to tabulate the results of BDD operations.

The memoization tables (see Fig. 7) are passed around by the state monad, just as the hash-consing structure. It is then necessary to maintain invariants on the memoization information.

*For the practical mind.* In the definition of state, we use two *field coercions* (denoted  $:>$ ) to declare coercions from the type `state` to the types `hashcons_state` and `memo`. In practice, this means that one can use a `state` in any expression that would expect the result of one of these two projections.  $\square$

We present the invariants over the memoization maps for the binary operations<sup>5</sup> in Fig. 8. First, we have to prove that the nodes referenced in the domain and in the codomain of those tables are well-formed and that those tables keep the bounds over the variables correct. Then, we have to state that the memoization information is semantically correct. One downside of this data structure definition is that we are forced to define one table per operation that we want to memoize, and that

<sup>5</sup> The case of the negation operation is similar and is not detailed.

this is not modular: adding a new operation requires modifying the definition of the memoization state and add the corresponding field in the `wf_memo` record.

(Note that finding the correct pattern of memoization for a program is still an art rather than a science: using the data structure above, we keep the memoized values from one run of an operation to the other. In this section, we will settle on this conservative strategy, but other memoization strategies are possible that yield different performance and memory consumption profiles.)

*A mouthful of code.* The final version of our code is shown on Fig. 9. We use a do-notation à la Haskell to make it more palatable.

Then, we prove that under some hypotheses, this combinator is correct: that is, it produces well-formed hash-consing structures and memoization tables, and the denotation of the resulting expression is correct. For the sake of clarity, we will not expose these hypotheses nor the resulting correctness theorem in this paper, and refer the interested reader to the archive of code that accompanies this paper [5].

*Implementing the and operation.* However, we demonstrate the use of this combinator on the particular example of the `and` function; all other binary operations follow the same pattern. First, we have to define a function `upd_and` that updates the memoization state: it is simply a wrapper that add an element to the right memoization table, and leave the others untouched. Then, we define the function `mk_and` as a simple call to the binary combinator. The crux here is the choice of the functions `Fx` and `Tx` that specify the behavior of the combinator at the leaves of the DAG.

```

Definition upd_and na nb res (st : state) :=
mk_state st
{| mand := set (na,nb) res st.(mand);
  mor := st.(mor);
  mxor := st.(mxor);
  mneg := st.(mneg)
|}).

Definition mk_and :=
combinator
(fun x st => Some (F,st) ) (* Fx *)
(fun x st => Some (x,st) ) (* Tx *)
(fun a b st => get (a,b) st.(mand))
upd_and.

```

Then, the semantic correctness theorem for this operation is defined as follows.

```

Theorem mk_and_correct depth v0 (st: state) a b :
wf_st st →
wf_expr st v0 a → wf_expr st v0 b →
∀ res st', mk_and depth a b st = Some (res, st') →
wf_st st' ^ incr st st' ^
wf_expr st' v0 res ^
∀ env va vb,
value env st a va →
value env st b vb →
value env st' res (va && vb).

```

This statement proves three things under the hypotheses that the given state and BDDs are well-formed : first, `mk_and` returns a well formed state. Second, it manipulates the state in an increasing manner (and, in particular, previously well-defined BDDs will still be well-defined and keep their semantic values). Third, the returned BDD have the expected semantic interpretations.

```

Section combinator.
(* fx is the base case when one operand is F *)
Variable fx : expr → state → option (expr * state).
(* tx is the base case when one operand is T *)
Variable tx : expr → state → option (expr * state).
Variable memo_get : positive → positive → state → option expr.
Variable memo_update : positive → positive → expr → state → state.

Definition memo_node a b l v h st :=
  let (res, st) := mk_node l v h st in
  let st := memo_update a b res st in
  (res,st).

Fixpoint combinator depth : expr → expr → state → option (expr * state) :=
  fun a b st =>
  match depth with
  | 0 => None
  | S depth =>
    match a,b with
    | F, _ => fx b st
    | _, F => fx a st
    | T, _ => tx b st
    | _, T => tx a st
    | N na, N nb =>
      match memo_get na nb st with
      | Some p => Some (p,st)
      | None =>
        do nodea ← get na st.(graph);
        do nodeb ← get nb st.(graph);
        let '(l1,v1,h1) := nodea in
        let '(l2,v2,h2) := nodeb in
        match Pos.compare v1 v2 with
        | Eq =>
          do x, st ← combinator depth l1 l2 st;
          do y, st ← combinator depth h1 h2 st;
          Some (memo_node na nb x v1 y st)
        | Gt =>
          do x, st ← combinator depth l1 b st;
          do y, st ← combinator depth h1 b st;
          Some (memo_node na nb x v1 y st)
        | Lt =>
          do x, st ← combinator depth a l2 st;
          do y, st ← combinator depth a h2 st;
          Some (memo_node na nb x v2 y st)
        end
      end
    end
  end.
End combinator.

```

Fig. 9: Deep combinator for binary operations

*Canonicity.* We can prove that this representation of BDDs is canonical: that is, well-formed equivalent expressions are mapped to the same nodes.

**Definition** `equiv st e1 e2 :=`  
 $\forall \text{ env } v1 \ v2, \text{ value env st } e1 \ v1 \rightarrow \text{ value env st } e2 \ v2 \rightarrow v1 = v2.$

**Lemma** `canonicity st v e1 e2 :`  
 $\text{wf\_st st} \rightarrow \text{wf\_expr st } v \ e1 \rightarrow \text{wf\_expr st } v \ e2 \rightarrow$   
 $\text{equiv st } e1 \ e2 \rightarrow e1 = e2.$

From this result, it follows that the (non-recursive) `eqb` function from Fig. 3 is a correct and complete characterization of semantic equivalence of expressions.

**Lemma** `eqb_correct st v e1 e2 :`  
 $\text{wf\_st st} \rightarrow \text{wf\_expr st } v \ e1 \rightarrow \text{wf\_expr st } v \ e2 \rightarrow$   
 $(\text{eqb } e1 \ e2 = \text{true} \leftrightarrow \text{equiv st } e1 \ e2).$

*Garbage collection.* In the above version of BDDs, we have not implemented garbage collection. That is, allocated nodes are never destroyed, until the allocation map becomes unreachable as a whole. Garbage collection could be added, e.g., using a stop and copy operation that preserve a set of roots. This is beyond the scope of this paper.

## 5.2 The PURE-SHALLOW approach

The previous implementation uses a *deep embedding* of the representation of the BDD in memory via the `graph` map. This is a natural way to encode a directed acyclic graph, but, as we saw, makes it difficult (if not unfeasible) to deal properly with termination. Therefore, we would like to be able to reason about BDDs as if they formed an inductive type, while keeping the ability to share sub-terms at runtime.

There is actually no need to look further than inductive types to do that. The standard intuition about inductive types is that they define the smallest type closed under application of constructors: the mental image that we get from that is a *tree*.<sup>6</sup> Yet, there is nothing that prevent us to use the system to share sub-terms.<sup>6</sup>

In this section, we demonstrate that we can encode BDDs in Coq using a representation that looks like binary decision *trees*, yet has runtime performances similar to the PURE-DEEP implementation (see §5.1), using explicit sharing. We present on Fig. 10 our inductive definitions, and the associated allocation function `mk_node`. This has to be compared with Fig. 3: the difference mainly lies in the deletion of the `graph` map, and the explicit recursive structure of the `expr` type.

*For the practical mind.* The reader may wonder why we chose to define `expr` as two mutual inductive types `expr` and `hc_expr`. Indeed, we explain in §8.3 that it is possible to inline `hc_expr` inside the definition of `expr`. The mutual inductive solution is inspired by the hash-consing library in OCaml by Conchon and Filliâtre [9]. This

<sup>6</sup> Such an approach may be interesting even if each structure is a tree, but different trees can share sub-trees. For instance, the Astrée static analyzer implements maps as balanced binary trees with *opportunistic* sharing [3, §6.2]: e.g. when an idempotent operation  $f(x, x) = x$  is to be performed on the images of two maps  $g$  and  $h$ , returning the map  $x \mapsto f(g(x), h(x))$ , then if  $g$  and  $h$  are determined to be identical through pointer equality,  $g$  is directly returned.

```

Inductive expr := | F | T | N : hc_expr → var → hc_expr → expr
with hc_expr := HC : expr → positive → hc_expr.

(* Two extra definitions that are used as coercions in the following code*)
Definition unhc (e : hc_expr) := let 'HC res _ := e in res.
Definition ident (e : hc_expr) := let 'HC _ res := e in res.

Definition eqb a b :=
  match a,b with
  | T,T | F,F ⇒ true
  | N (HC _ id1) x (HC _ id2), N (HC _ id1') x' (HC _ id2') ⇒
    (id1 =? id1') && (x =? x') && (id2 =? id2')
  | _, _ ⇒ false
  end.

Record hashcons_state := {hmap : expr ~> hc_expr; next : positive}.

Definition alloc u st :=
  let r := HC u b.(next) in
  (r, { | hmap := set u r b.(hmap); next := b.(next) + 1|}).

Definition hc_node (e : expr) (st : hashcons_state) :=
  match get e st.(hmap) with
  | Some x ⇒ (x, st)
  | None ⇒ alloc e st
  end.

Definition mk_node (l : hc_expr) (v : var) (h : hc_expr) st :=
  if (ident l =? ident h) then (l,st) else hc_node (N l v h) st.

```

Fig. 10: A shallow-embedding of sharing

library makes a clear distinction between hash-consing nodes (the equivalent of our `hc_expr` inductive) and the actual values that are hash-consed. In Coq, this makes some inductive proofs a little bit more involved: we need to use mutual induction on the two data-types.  $\square$

The code of the `hc_node` function is subtle: a call to `hc_node e bdd` will perform a lookup in the hash-consing map `hmap`: if the same expression was previously allocated, then we return the old version; otherwise, we update the map `hmap` with a mapping from the expression to its hash-consed version. The lookup ensures that equal expressions (modulo the comparison function used to index `hmap`) are mapped to the same hash-consed expression. Then, `mk_node l v h bdd` will first test the identifiers of `l` and `h` for equality: if it is the case, then there is no need to introduce a new node; otherwise, we perform a call to `hc_node`.

As an example, assume that `x` and `y` are expressions with different identifiers. The following code

```

let (a,st) := mk_node x v y st in
let (b,st) := mk_node x v y st in
let (c,st) := mk_node a v' b st in ...

```

will make `a`, `b` and `c` point to the same memory location! However, if `x` and `y` were not shared maximally, then neither are `a`, `b` nor `c`.



*A word on memoization.* There is no difference at all in the way we handle memoization in this implementation w.r.t. §5.1. That is, we implement the same `state` record as before; and pass the same memoization tables around.

*A word on termination.* It is now easier to define recursive functions that operate on BDDs by taking advantage of the inductive definition of `expr`. We have to stress that *easier* is not *easy* because the Coq termination checker requires that recursive calls are made on a structurally smaller argument: there is no built-in support for recursive definitions with pairs of arguments that are decreasing w.r.t. a lexicographic order. Therefore, we have to use nested fixed-points or prove that the recursive calls are well-founded. In this section, we choose the former.

*Re-implementing the combinator.* We are now ready to describe the code of the implementation of the binary combinator presented in Fig. 11. The code is similar to the one in Fig. 9 with a few key differences: thanks to the inductive definition of expressions, we do not have to perform lookups in the `graph` map and we do not use fuel anymore. This makes the combinator function *total*; and we can get rid of the Maybe monad.

*Canonicity.* Again, we prove that this representation of BDDs is canonical: well-formed equivalent expressions are mapped to the same nodes. Again, we have the corollary that the (non-recursive) equality test from Fig. 10 that inspects the (top-level) node identifiers is a complete and correct characterization of semantic equivalence.

**Definition** `equiv e1 e2 :=`  
 $\forall \text{env } v1 \ v2, \text{value env } e1 \ v1 \rightarrow \text{value env } e2 \ v2 \rightarrow v1 = v2.$

**Lemma** `eqb_correct st e1 e2 v :`  
 $\text{wf\_st } st \rightarrow \text{wf\_expr } st \ v \ e1 \rightarrow \text{wf\_expr } st \ v \ e2 \rightarrow$   
 $(\text{eqb } e1 \ e2 = \text{true} \leftrightarrow \text{equiv } e1 \ e2).$

*Comparison with the previous approach.* The implementation presented in this section is derived from the previous one, with the following improvements: the proofs are roughly 20% shorter; the performances are slightly better when executing the code inside Coq (there is less administrative book-keeping to do in our data structures); the functions that operates on BDDs are total. Furthermore, it would probably be easier to implement garbage collection in this setting than in the previous one, thanks to the simpler definition of the global state.

The situation is almost ideal for the equality test: we prove that the (non-recursive) equality function that inspects the top-level identifiers of nodes is a correct and complete characterization of semantics equivalence of BDD expressions. However, we have no way to prove that it corresponds to physical equality. Actually, we cannot *state* within Coq that it is never the case that two identical representations of the same term coexists, even if we could argue at a meta-level that it is indeed not the case.

```

Section combinator.
Variable fx : hc_expr → state → hc_expr * state.
Variable tx : hc_expr → state → hc_expr * state.
Variable memo_get : positive → positive → state → option (hc_expr).
Variable memo_update : positive → positive → hc_expr → state → state.

Fixpoint combinator : hc_expr → hc_expr → state → hc_expr * state := fun a =>
  fix combinator_rec b st :=
  match unhc a, unhc b with
  | F, _ => fx b st
  | _, F => fx a st
  | T, _ => tx b st
  | _, T => tx a st
  | N l1 v1 h1, N l2 v2 h2 =>
    let ida := ident a in
    let idb := ident b in
    match memo_get ida idb st with
    | Some p => (p,st)
    | None =>
      let '(res, st) :=
        match Pos.compare v1 v2 with
        | Eq =>
          let '(x, st) := combinator l1 l2 st in
          let '(y, st) := combinator h1 h2 st in
          mk_node x v1 y st
        | Gt =>
          let '(x, st) := combinator l1 b st in
          let '(y, st) := combinator h1 b st in
          mk_node x v1 y st
        | Lt =>
          let '(x, st) := combinator_rec l2 st in
          let '(y, st) := combinator_rec h2 st in
          mk_node x v2 y st
        end
      in
      let '(_, st) := memo_update ida idb res st in
      (res, st)
    end
  end.
End combinator.

```

Fig. 11: Shallow combinator for binary operations

## 6 From pure data structures to persistent data structures via extraction

In the previous section, we use a state monad to store information about hash-consing and memoization. However, one can see that, even if these programming constructs use a mutable state, they behave transparently with respect to the pure Coq definitions.

We have seen earlier (see §2.3) that if we abandon (efficient) executability inside Coq, we can express new idioms. In the following, we implement the BDD library in Coq as if manipulating decision trees with neither sharing, nor hash-consing tables, nor memoization tables, then add the hash-consing and memoization code by using special features of the extraction mechanism to remap some constants arbitrarily to custom OCaml code.

```

Inductive bdd: Type :=
| T | F | N : var → bdd → bdd.
Extract Inductive bdd ⇒
"bdd" ["hT" "hF" "hN"] "bdd_match".

```

(a) BDDs in Coq as decision trees

```

type bdd =
| T of tag | F of tag | N of positive * bdd * bdd * tag

```

```

module HCbdd = Hashcons.Make(...)
let hT = HCbdd.hashcons (T Weakhtbl.dummy_tag)
let hF = HCbdd.hashcons (F Weakhtbl.dummy_tag)
let hN (v, t, f) = HCbdd.hashcons (N (v, t, f, Weakhtbl.dummy_tag))

```

```

let bdd_match fT fF fN t =
match t with
| T _ → fT ()
| F _ → fF ()
| N (v, t, f, _) → fN v t f

```

(b) Hash-consed OCaml BDD type

Fig. 12: Implementing BDDs in Coq, extracting them using smart constructors

### 6.1 The SMART approach

More precisely, we define our BDDs as binary decision trees (see Fig. 12a), and implement operations in Coq on this simple data structure. Then, we tell Coq to extract the `bdd` inductive type to a custom `bdd` OCaml type (see left of Fig. 12b) and to extract constructors into *smart constructors* that maintain the maximal sharing property. The type defined in OCaml is identical to the Coq one, except that it carries one extra field of type `tag`, morally containing the associated unique identifier. The smart constructors make use of the hash-consing library used in Why3 [20], a recent version of a library by Conchon and Filliâtre [9]. It defines the `Hashcons.Make` functor, that we instantiate. The generated module provides a `HCbdd.hashcons` function that returns a unique hash-consed representative for the parameter.

The reader may notice that we choose to name `bdd` in Coq what is clearly a representation of a binary decision tree, and which corresponds to what was previously named `expr`. We believe that this particular choice of name makes sense if we consider values of type `bdd` to represent Boolean functions.

*Discussion: the status of the equality test.* In Coq, we define the obvious recursive function `bdd_eqb` of type `bdd → bdd → bool`, that decides structural equality of BDDs. Then, we extract this function into OCaml's physical equality:

```

Fixpoint bdd_eqb (b1 b2: bdd): bool :=
  match b1, b2 with
  | T, T | F, F ⇒ true
  | N v1 b1t b1f, N v2 b2t b2f ⇒
    Pos.eqb v1 v2 && bdd_eqb b1t b2t && bdd_eqb b1f b2f
  | _, _ ⇒ false
  end.

```

```

Extract Inlined Constant bdd_eqb ⇒ "(==)".

```

From a meta-level perspective, we argue that the two are equivalent thanks to the physical unicity of hash-consed structures, provided that all values are constructed using our smart-constructors, which is the case if we create all nodes from code extracted from Coq (of course, handwritten OCaml code may break this

invariant). The key point here is that the way we build terms enforces the fact that the BDDs they build are maximally shared.

*Keeping the BDD reduced.* One could be tempted to put in the OCaml code of the smart constructor `hN` a test that would enforce that BDDs are reduced. That is, it would not build a node if its two children were identical.

```
let hN (v,t,f) =
  if t == f
  then t
  else HCbdd.hashcons (N (v, t, f, Weaktbl.dummy_tag))
```

To see the problem with this idea, consider the following Coq code.

```
match N (v,T,T) with
| T => false
| N (_,_,_) => true
end
```

In Coq, the above expression reduces to `true` while the extracted version would reduce to `false`. Therefore, this idea is wrong: avoiding the node construction makes subsequent case analysis behave inconsistently between Coq and OCaml.

In the end, the correct way to implement reduction is to use the following helper function, written in Coq, that builds a node only when necessary.

```
Definition N_check (v : var) (bt bf : bdd) : bdd :=
  if eqb bt bf then bt else N v bt bf.
```

Note that the user is forced to use this function instead of using the `N` constructor<sup>7</sup>. Failing to do so will result in code that is correct, but does not build reduced binary decision diagrams. More precisely, extracting `eqb` to `==` would make it possible to prove that BDD operations are semantically correct, but this would make diagrams non-canonical: well-formed expressions could be represented by several different nodes.

*Implementing the combinator.* The last ingredient needed to transform a decision tree library into a BDD library is memoization. We use the same kind of ideas: we define our functions as if not memoized, but we use a special well-founded fixed-point combinator, that we extract to a memoizing fixed-point combinator. The details can be seen on Fig. 13: we declare an abstract type class `memoizer` of types `A` such that we know how to memoize functions of type  $\forall x : A, P x$ . This is extracted in OCaml to the type of polymorphic fixed-point combinators, with an extra technical `int` parameter used to specify the initial size of the used hash map. In Coq, we then define a memoizing combinator `memo` and a memoizing fixed-point combinator `memo_rec` as if they were not using memoization, but we ask the extraction mechanism to map them to special OCaml functions, that make use of the type class instance given as parameter. These functions are observationally equivalent to the Coq ones, provided that the type class instance is correct, and that the memoized function is pure.

It is worth noting that directly exposing the type `'key memoizer` in Coq would be unsound, because this allows to use its instances in a non-terminating manner:

<sup>7</sup> Still, the `N` constructor that occurs in the definition `N_check` must be extracted using a smart-constructor.

```

Parameter memoizer : ∀ A : Type, Type.
Existing Class memoizer.
Extract Constant memoizer "'key" ⇒ "'key Helpers_common.memoizer".

Definition memo A {H : memoizer A} P := @id (∀ x : A, P x).
Extract Inlined Constant memo ⇒ "Helpers_common.memo".
Arguments memo [A] {H} [P] _ _ . (* Set implicit arguments for memo*)

Definition memo_rec A {H : memoizer A} := @Coq.Wf.Fix A.
Extract Inlined Constant memo_rec ⇒ "Helpers_common.memo_rec".
Arguments memo_rec [A] {H} [R] Rwf [P] F x. (* Set implicit arguments for memo_rec *)

```

(a) Coq part

```

type 'key memoizer =
  { memo : α. int → (('key → α) → ('key → α)) → ('key → α) }

let memo m f = m.memo 5 (fun _ x → f x)

let memo_rec m f = m.memo 5 (fun frec x → f x (fun y _ → frec y))

```

(b) OCaml part

Fig. 13: Memoizing combinators

```

Parameter memoizer_N : memoizer N.
Existing Instance memoizer_N.
Extract Inlined Constant memoizer_N ⇒ "Helpers_common.memoizer_N".

```

(a) Coq part

```

module NHT =
  Hashtbl.Make (struct
    type t = coq_N
    let equal = (=)
    let hash = N.to_int
  end)

let memoizer_N =
  { memo = fun n f →
    let h = NHT.create n in
    let rec aux x =
      try NHT.find h x with Not_found →
        let r = f aux x in NHT.replace h x r; r
    in aux }

```

(b) OCaml part

Fig. 14: An instance of memoizer for Coq.Numbers.BinNums.N

the `memo_rec` wrapper makes sure the recursive calls are well-founded. Moreover, it is important to understand that we have not axiomatized these combinators. Instead, we give real implementations, semantically equivalent to their OCaml counterparts. This is important, because it then becomes clear that we do not introduce any logical inconsistencies in Coq, and these terms keep a computational content which could be very useful in proofs.

This part of the code is modular, and can be used to memoize functions of any domain. It is up to the user to give instances of the `memoizer` type class: to do so, he can provide dedicated code, as shown in Fig. 14 for the type `N` of binary

```

Definition bdd_and : bdd → bdd → bdd.
refine
  (memo_rec (well_founded_ltof _ bdd_size) (fun x ⇒
    match x with
    | T ⇒ fun _ y ⇒ y
    | F ⇒ fun _ _ ⇒ F
    | N xv xt xf ⇒ fun recx ⇒
      memo_rec (well_founded_ltof _ bdd_size) (fun y ⇒
        match y with
        | T ⇒ fun _ ⇒ x
        | F ⇒ fun _ ⇒ F
        | N yv yt yf ⇒ fun recy ⇒
          match Pos.compare xv yv with
          | Eq ⇒ N_check xv (recx xt _ yt) (recx xf _ yf)
          | Lt ⇒ N_check yv (recy yt _) (recy yf _)
          | Gt ⇒ N_check xv (recx xt _ y) (recx xf _ y)
        end
      end)
    end));
  unfold ltof; simpl; clear; abstract omega.
Defined.

```

Fig. 15: The and operation on bdds

natural integers, using simple hash tables. Alternatively, for the `bdd` type, our type class instance is just an OCaml wrapper around the hash-consing library built-in memoization mechanisms.

Again, we use the example of the `bdd_and` operation, shown in Fig. 15. As in Fig. 11, we define this function using two nested fixed-points, in order to handle the special recursion scheme of this function. The definition of `bdd_and` uses `memo_rec` twice, in a nested fashion. This is needed in order to handle the special recursion scheme of this function (decreasing on one of its two parameters).

*Garbage collection.* The strategy we use in §5.2 for garbage collection is very naive: we kept everything alive, forbidding any garbage collection. Here, the hash-consing library we use allow the garbage collector to reclaim any node that is not referenced by the program, by using suitable weak hash tables. Moreover, it reclaims any memoized value that is associated with a dead key (we will not give more details here and refer the reader to [9]). While it avoids memory leaks, this strategy does not necessarily give the best performances (see §7). We believe the changes necessary to implement a new strategy are small, and do not involve rewriting the proofs: only OCaml code is involved.

It is important to note that we have to memoize the functions each argument after the other (keeping them curried). Indeed, one should be tempted to memoize a function `bdd * bdd → bdd`. This is not a good idea, since we use weak hash-tables for the memoization: in this case, the pairs containing the arguments are no longer accessible after the function call, so that the garbage collector is going to collect most of the memoized data at each cycle. Using our pattern, we make sure a memoized datum is not reclaimed as long as both parameters are still alive in memory. (Note that if we choose to use regular hash-tables instead of weak ones,

this argument does not stand anymore, but the program has a different performance and memory consumption profile.)

*Discussion: comparison with previous approaches.* In this instance of our BDD library, all Coq definitions are kept simple and proofs are straightforward. That is, we can prove semantic correctness of all operations directly using structural induction on decision trees and there is no state holding structures. There is a nice separation between the hash-consing and memoization code, that is generic, written in OCaml and not proved and the BDD code, mostly written and proved in Coq. We discuss the lack of confidence in the OCaml code and how it can be restored in §9.1.

We do not need to use monads in the Coq code, so that the interface of the BDD library remains modular and easy to use. Moreover, it is straightforward to implement garbage collection strategies in order to avoid memory leaks.

## 6.2 The SMART+UID approach

The previous SMART approach totally hides the unique identifiers from the Coq code. Yet, exposing these unique identifiers may be useful at times.

Consider the following use case: from a BDD  $B$  we would like to build an equivalent propositional logic formula of *linear size*, for instance for feeding into a satisfiability modulo theory solver. In order to avoid an exponential blow-up, each shared sub-tree should generate one single sub-formula, used in a “let” binder so that its value can be used in multiple occurrences. The obvious way to implement such a transformation is to first detect which sub-trees are shared, using a set of shared subtrees seen so far, then to perform the transformation, using a table of mappings from subtrees to bound variables.

It seems therefore highly desirable to be able to build sets and maps over our hash-consed type. Generic functional sets and maps are usually implemented using balanced trees over a totally ordered data type; for efficiency, the comparison function should be very fast. An obvious choice would be to expose the unique identifiers to the Coq code (through a function `bdd → uid`), or at least the total order that they induce (through a function `bdd → bdd → comparison`).

Unfortunately, doing so without precautions can lead to unsoundness. Consider a program where, in succession, two nodes  $A$  and  $B$  are allocated, then a node  $A'$  isomorphic to  $A$  is created; let  $u_A < u_B$ ,  $u_{A'}$  be the successive unique identifiers. If  $A$  is collected between the allocations of  $B$  and  $A'$ , then  $A'$  will be allocated, with  $u_{A'} > u_B$ . Yet,  $A'$  and  $A$  are, from the point of view of the Coq code, identical; thus  $u_{A'} = u_A$ , yielding an inconsistency.

The workaround is to use a normal hash-table, as opposed to a weak hash-table, which prevents the collection of unreachable nodes. Then, two identical nodes created within the same execution are necessarily physically equal and thus share the same identifier.

One difficulty remains. Gallina is a purely functional language; the evaluation of a given term always yields the same result, and one expects the same property to extend to the extracted OCaml program, as long as it does not interact with the external world (e.g. reading from files). Yet, this is not necessarily the case if one exposes the unique identifiers. Consider a program  $P_1$ , such that the extracted OCaml code allocates two nodes  $A$  and  $B$  in this order. If  $P_1$  is run stand-alone,

then  $u_A < u_B$ . Yet, if another program  $P_2$  allocating  $B'$  isomorphic to  $B$  is first run, and  $B'$  is not collected, then  $B'$  is the same as  $B$  and  $u_B = u_{B'} < u_A$ .

It seems questionable that the result of an evaluation should depend on whether or not some other (unrelated) evaluation has taken place, if only because it makes debugging difficult.<sup>8</sup>

### 6.2.1 Maps and sets over hash-consed types

Arguably, the final result of an algorithm using functional sets and tables should not depend on the order relation used. One could thus opt to expose neither the unique identifiers, nor the order relation they induce, but only a functional map interface (with *find*, *remove* and *update*). Then, the results cannot depend on garbage collection and other disturbances to the ordering. Yet, one could also want these sets and maps to provide a *fold* operation; this would need to be restricted to associative and commutative operations (so that their result do not depend on the ordering). Such an interface would fit the *map-reduce* approach to aggregate objects.

One additional advantage is that, with this approach, it is possible to use weak hash tables and garbage collection, as long as the sets and maps point to the nodes used as keys and thus prevent their collection: the ordering relation is then exposed to the Coq code only through equality testing, be it explicit or implicit (through the use of set or maps).<sup>9</sup>

While this line of work seems powerful, we did not pursue it: the many axioms needed do not inspire confidence; some nontrivial meta-theoretic reasoning about the sequential execution of OCaml code extracted from Coq would be needed.

### 6.2.2 Axioms

In the SMART approach, hash-consing and memoization are done after the fact, and are completely transparent for the user; yet we felt a need to break this transparency by exposing the unique identifiers. In the following, we instead make more explicit the hypotheses that we make on the representation of BDDs. That is, we make visible in the inductive type of BDDs that each BDD node has a “unique identifier” field (see Fig. 16) and we take the node construction function as an axiom, which is implemented in OCaml. Note that nothing prevents the Coq program from creating new BDD nodes without calling this function `mkN`. Yet, only objects created by it (or copies thereof) satisfy the `valid` predicate; we must declare another axiom

<sup>8</sup> One could argue that, with certified programs in the Coq fashion, there is no need to debug: each function or module comes with a proof of its correctness, which compositionally entail the correctness of the whole program. Yet, commonly one only proves the results to be *correct*, not necessarily *optimal*, and one proves very seldom that the computation has the expected complexity. Furthermore, some computations are split between an untrusted solving procedure, and a trusted checker; a failed check entails having to debug the untrusted procedure, which may be hard if behaviors are hard to reproduce independently of the rest of program.

<sup>9</sup> It is of paramount importance that the sets and maps use the actual nodes as keys (ordered by the order on unique identifiers), not just the unique identifiers: this prevents scenarios such as creating an object  $A$ , inserting a key  $u_A$  into some set  $S$ , collecting  $A$ , creating another object  $A'$  isomorphic to  $A$ , but with a different key  $u_{A'}$ , which could be distinguished from  $A$  by membership in  $S$ . In a nutshell, it is crucial that the Coq program cannot test for equality of unique identifiers separated from their associated objects.



```

Axiom var : Set.
Axiom uid : Set.
Axiom uid_eqb : uid → uid → bool.
Axiom uid_eq_correct : ∀ x y : uid,
  (uid_eqb x y = true) ↔ x = y.

Inductive bdd : Set :=
| T | F
| N : uid → var → bdd → bdd → bdd.

Axiom mkN : var → bdd → bdd → bdd.

Axiom mkN_ok :
∀ v : var, ∀ bt bf : bdd,
  ∃ id, mkN v bt bf = N id v bt bf.

Inductive valid : bdd → Prop :=
| valid_T : valid T
| valid_F : valid F
| valid_N : ∀ var bt bf,
  (valid bt) → (valid bf) →
  (valid (mkN var bt bf)).

Axiom shallow_equal_ok :
∀ id1 id2 : uid,
  ∀ var1 var2 : var,
  ∀ bt1 bf1 bt2 bf2 : bdd,
  valid (N id1 var1 bt1 bf1) →
  valid (N id2 var2 bt2 bf2) →
  id1 = id2 →
  N id1 var1 bt1 bf1 =
  N id2 var2 bt2 bf2.

```

Fig. 16: Axiomatization of equality using unique identifiers

stating that unique identifier equality is equivalent to Coq’s Leibniz equality *for valid nodes*. Then, we can use unique identifiers to check for equality.

This approach is close to the SMART approach. It has one advantage, the fact that unique identifiers are directly accessible from the Coq code. The previously mentioned limitation applies: since unique identifiers are visible, one cannot use weak hash tables and the OCaml garbage collector.

The use of axioms is debatable. On the one hand, the use of axioms somewhat lowers the confidence we can give in the proofs, and they make the code not executable within Coq. On the other hand, these axioms are actually used implicitly when extracting Coq constructors to “smart constructors”: they correspond to the meta-theoretical statement that these constructors behave as native Coq constructors. Thus, they make explicit some of the magic done during extraction.

We could gain confidence about the fact that our axiomatization is consistent, by using Coq’s module system. That is, we could parameterize our development in this section by a suitable interface specifying the various functions that we axiomatized. Then, we could instantiate this interface either with axioms, and extract them to an efficient yet unproven implementation in OCaml; or, we could instantiate them with a pure Coq version. This would prove that this set of axioms is not unsound in Coq. This general approach was used by Vafeiadis [21] to prove that his interface for adjustable references is sound, and we could do the same here. We have yet to conduct this experiment.

## 7 Discussion

In this section, we compare our design patterns on various aspects. Let’s start with the easy ones:

**Executability inside Coq.** The PURE-DEEP and PURE-SHALLOW implementations can be executed inside Coq, and have decent performances. The SMART approach can also be executed inside Coq, but has dreadful performance (when executed inside Coq, it uses binary decision trees). The SMART+UID approach cannot be executed inside Coq.

Trust in the extracted code. Unsurprisingly, the `SMART` and the `SMART+UID` approaches yield code that is harder to trust, while the `PURE-DEEP` and `PURE-SHALLOW` approaches leave the extracted code pristine.

**Proof.** From a proof-effort perspective, the `SMART` approach is by far the simplest.

The `SMART+UID` approach involves the burden of dealing with axioms. However, it makes it easier to trust that what is formally proven corresponds to the real behavior of the underlying runtime. By comparison, the `PURE-DEEP` and `PURE-SHALLOW` approaches required considerably more proof-engineering in order to check the validity of invariants on the global state. Note however that our proof arguments are much simpler in the latter one.

**Garbage collection.** Implementing (and proving correct) garbage collection for the `PURE-DEEP` or `PURE-SHALLOW` approaches would require a substantial amount of work. By contrast, the `SMART` approach make it possible to use OCaml’s garbage collector to reclaim unreachable nodes “for free”.

**Operations** As we have shown, binary operations can be handled with a single parametric combinator. All our implementations use such a combinator to implement conjunction, disjunction, exclusive-or and so on. There is little work to do to add support for any binary operation we may have overlooked. The situation is more complicated when it comes to ternary operations (such as the if-then-else Boolean operation). Implementing it using the `SMART` approach, and proving it correct requires around 80 lines of code. It would require a non-trivial amount of work to implement it using our `PURE-DEEP` or `PURE-SHALLOW` approaches, and we have not conducted this experiment yet. Other operations that are relevant in a BDD library are function composition and quantification. We have yet to implement these in any of our libraries. Quantification over one variable can be defined as a structural recursion over one BDDs. As such, we think it can be easily defined in all our frameworks. Unary composition (that is, substitution of one variable with another Boolean function) can be defined as a structural recursion over two BDDs. As a consequence, implementing it in the `SMART` framework seems simple. However, this particular recursion scheme does not fit the combinator of the `PURE-DEEP` or `PURE-SHALLOW` approaches, and some additional work would be needed in order to implement it using those approaches. Moreover, a non-negligible amount of work would be needed to support variable arity composition and quantification in all libraries.

## 7.1 Performances of the extracted code

We evaluate the performances of the OCaml code that is extracted from our “pure” (see §5) and “impure” (see §6) libraries, and we pit them against a reference library implemented in OCaml (available from J.C. Filliâtre’s web page). This reference library does not keep the memoization table alive from one execution of an operation to another: for instance, each time the `and` function is called, a new memoization hash-table is allocated. Therefore, to make up for a fair benchmark with our implementations, we modified this reference library to use a memoization strategy closer to ours. This alternative reference implementation is designed “reference (conservative)” in what follows.

Then, this evaluation, we use two standard benchmarks (see [23]). The first one is Urquhart’s formula  $U(n)$  defined by

$$U(n) \triangleq x_1 \iff (x_2 \iff \dots (x_n \iff (x_1 \iff \dots (x_{n-1} \iff x_n))))$$

The second kind of formula states the pigeonhole principle  $P(n)$  for  $n + 1$  pigeons: that is, if there is  $n + 1$  pigeons in  $n$  pigeonholes, then there is at least one hole that contains two pigeons. The plots of the execution time required to check that these formulas are tautology are given in Fig. 17.

First, we remark that the reference implementation that we use is (almost always<sup>10</sup>) 4 times faster than our best implementation SMART. There is no significant difference in execution time between the reference implementation with the conservative memoization strategy and the reference implementation. We have not conducted a detailed measurement of the memory consumption<sup>11</sup>, though, and the memory consumption profile is likely to be different between the two.

Then, our SMART implementation is roughly 4 times faster than the PURE-DEEP one: interestingly enough this value is consistent over our benchmark, while we could have expected logarithmic factors to show up. Indeed, recall that the pure implementation uses functional finite maps, while the smart one uses OCaml hash-maps.

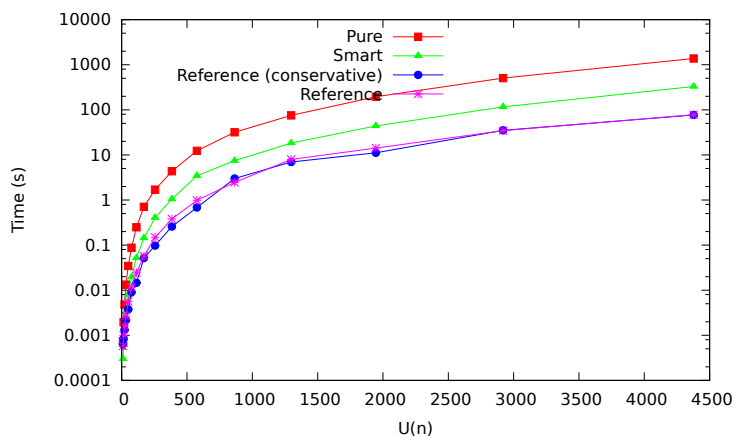
Also, we have run the same benchmark on our PURE-SHALLOW implementation, and the resulting plot is close to the PURE-DEEP plot: for small instances, the difference of execution times are of about 10%. For our largest instance of the pigeonholes problem, we observe that the PURE-SHALLOW is faster by about 30%. We do not delve too much on this result: we wonder for instance to what extent changing the implementation of finite maps that we use could result in similar differences of performances.

Primary investigation show that the main performance difference between the reference implementation and our SMART approach come from the use of the generic hash-consing library: the reference library uses a specialized hash-consing system, much faster than the generic library we use. Similarly, the difference between the SMART and the PURE-SHALLOW seems to be the use of AVL trees to represent dictionaries.

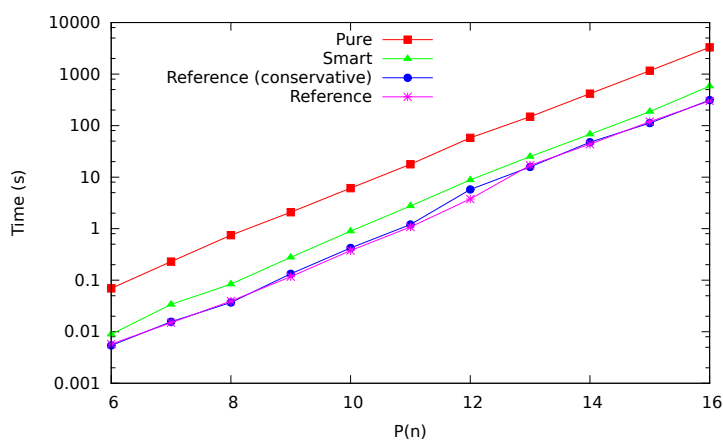
We did not investigate much the memory behavior. Our simple experiments show that it depends on the benchmark we consider. The bottom line is that all our approaches consume less than twice as much memory as the reference implementation, conservative version. Moreover, we observe that PURE-SHALLOW consumes 10%-20% less memory than PURE-DEEP. Concerning the SMART approach, it seems like our Curryfied memoization scheme creates a lot of very small hash-tables in the OCaml heap, which is not the optimized use case of OCaml hash-tables. Thus, there might be room for improvement here.

<sup>10</sup> This is not necessarily true for small formulas due to differences in the start-up costs of these libraries.

<sup>11</sup> These particular benchmarks were run on a computer with 1 TB of ram, which made it possible to avoid swapping during our experiments. Indeed, memory usage is the limiting factor on bigger instances of the tests.



(a) Urquhart's formula



(b) Pigeonhole principle

Fig. 17: Execution time for the BDD benchmarks

## 8 Implementing reduction in the $\lambda$ -calculus

In the previous sections, we have presented several design patterns about how to implement hash-consed data structures in Coq. In this section, we apply two of them to the example of reduction of  $\lambda$ -terms (the running example of Conchon and Filliâtre [9]).

### 8.1 An implementation without hash-consing

We first describe a Coq implementation, without memoization nor hash-consing, of  $\lambda$ -terms with de Bruijn indices inspired from Huet's *Constructive Computation Theory* [14]. We implement four functions on terms: `lift`, `subst`, `hnf` (that puts a term in head-normal form) and `nf` (that puts a term in normal form). This amounts

```
Inductive term : Type :=
```

```
| Var : N → term
| App : term → term → term
| Abs : term → term.
```

(a)  $\lambda$ -terms with de Bruijn indices

```
Fixpoint lifti (n : N) (t : term) (k : N) :=
```

```
  match t with
  | Var i ⇒ if N.ltb i k then Var i else Var (i + n)
  | Abs t ⇒ Abs ( lifti n t (N.succ k) )
  | App t u ⇒ App (lifti n t k) (lifti n u k)
  end.
```

```
Definition lift n t := lifti n t 0.
```

(b) A implementation of `lift` without hash-consing nor memoization

```
Definition lifti : N → term → N → term.
```

```
refine (
  memo (fun n ⇒
    memo_rec (well_founded_ltof _ size) (fun t rec k ⇒
      memo (fun k ⇒
        match t with
        | Var i ⇒ fun rec ⇒ if N.ltb i k then Var i else Var (N.add i n)
        | Abs t ⇒ fun rec ⇒ Abs (rec t _ (N.succ k) )
        | App t u ⇒ fun rec ⇒ App (rec t _ k) (rec u _ k)
        end) k rec)));
(* Proof obligations elided *)
```

```
Definition lift n t := lifti n t 0.
```

(c) Implementing `lift` the smart way

Fig. 18: Implementation of terms and `lift`

to roughly 60 lines of Coq code: we present the code for `lift` in Fig. 18b as a reference for future comparisons.

Then, we reuse Conchon and Filliâtre’s benchmark [9]: we implement a  $\lambda$ -calculus version of quicksort that operates on lists (encoded as  $\lambda$ -terms) of Church numerals; and apply this algorithm to sort the list  $L = (0 :: 3 :: 5 :: 2 :: 4 :: 1 :: nil)$ .

Using Coq’s virtual machine, sorting this list requires over 250 s on a recent desktop computer<sup>12</sup>: this involves a number of reduction that is exponential w.r.t. the size of the list and the size of the numbers. Extracting the code to OCaml makes it more efficient by a constant factor. We compile the extracted code with the native OCaml compiler, and run it to sort the list  $L$ .

time (s)	Number of allocated bytes	Max. size of the major heap (words)
30	$47 * 10^9$	$1.1 * 10^6$

The number of allocated bytes is given as reported by `Gc.allocated_bytes`: it corresponds to what is allocated during the execution of the sorting algorithm. The maximum size of the major heap is given as reported by OCaml `Gc.stat` function. In short, these numbers mean that the program allocates an awful lot of short-lived values.

<sup>12</sup> Equipped with Intel Xeon cores running at 3.60GHz.

## 8.2 Using smart constructors

The SMART approach from §6.1 involves a small amount of changes to the code, mainly to use memoizing constructs and to add `memoizer` definitions. To get a rough idea of the changes involved, we present the modified code for `lift` in Fig. 18c: it is merely a matter of replacing abstractions and fixed-point definitions with their memoizing counterparts. The overall size of the code is now roughly over 80 lines of Coq code. While they do not make it possible to compute efficiently inside Coq, these changes have dramatic consequences on the execution time of the extracted code. Again, we compile the extracted code with the native OCaml compiler, and run it to sort the list  $L$ .

time (s)	Number of allocated bytes	Max. size of the major heap (words)
0.15	$0.12 * 10^9$	$3.1 * 10^6$

Here, we have a higher top size of the major heap, which account for our use of memoization tables. This is to be weighed against the lower amount of allocated words and the 200× decrease in running time.

## 8.3 Using the shallow-embedding approach

Finally, we use the PURE-SHALLOW from §5.2. There is now roughly 300 lines of code and definitions to implement `lift`, `subst`, `hmf` and `nf`. In this section, we will make a brief review of this code, highlighting the differences with respect to what we presented in §5.2.

We give the modified definition of terms in Fig. 19a: each constructor takes an extra `positive` argument: that is, we drop the extra indirection we used in the definition of BDD expressions in §5.2. (This choice of definition yields simpler proof arguments because we do not need to perform mutual inductions on `expr` and `hc_expr`.)

We adapt the notion of global state from §5.2. Again, the `hmap` finite map is indexed using a non-recursive comparison function: it simply compare the head symbols of the terms and, if equal, compare the tags of the sub-terms. Again, we define one memoization table per function we wish to memoize.

Then, let us take the example of abstraction to describe term creation. We define the following smart constructor.

**Definition** `mk_Abs t st := mk_term (Abs t 1) st.`

The function `mk_Abs` is a simple wrapper around `mk_term`: it applies the term constructor `Abs` to the term `t` and a dummy unique identifier. Then, `mk_term` (see right of Fig. 19b) performs a lookup in the hash-consing table. If there exists a mapping from `Abs t p` (for some `p`) to `x` in the table, then we return `x`. Otherwise, the `alloc` function replaces the dummy unique identifier with the next fresh unique identifier available, and it updates the map `hmap` with a mapping from the expression `t` to its tagged version `r`.

In effect, this code maintains the invariant that all the terms in the `hmap` table are the canonical representative of their equivalence classes modulo the term comparison used to index `hmap`. That is, we ensure that the terms we build verify the following property.

**Definition** `wf_term st t := get t st.(hmap) = Some t.`

Let us jump to the definition of `lifti` (see Fig. 19c): it is defined using a well-founded fixed-point combinator `Fixm` (elided here) that ensures that the measure `msr` is decreasing through recursive calls. The bulk of this definition is `lifti_rec`: it is quite similar to the previous definitions of `lift`, except that we have to add fancy return clauses to the match definitions. They are necessary to be able to prove that the measure actually decreases through recursive calls. (Also, note that we could have inlined `lifti_rec` in `lifti`, but this two-step definitions makes it for more palatable proof goals later on).

The upside of these definition is that we are now able to compute the normal forms of our  $\lambda$ -terms inside Coq with decent efficiency: it takes roughly 4 s to sort the list  $L$  using `vm_compute`. The downside of these definitions is made apparent as soon as we start to prove that our four functions `lift`, `subst`, `hnf` and `nf` enjoy a simple correctness property: that they always return terms that are well-formed and preserve well-formedness of the global state. Proving this requires around 600 lines of code, which brings the total size of this formalization to 900 lines, before we even started to prove meta-properties like the fact that `hnf` or `nf` implement beta-reduction!

## 8.4 Discussion

In this section, we have demonstrated that our design patterns can be applied to other settings than BDDs. This exercise helped us to refine the presentation of smart constructors we gave in §6.1, and gave us the opportunity to present another flavor of the shallow-embedding technique, that does not rely on the mutually recursive inductives we used in §5.2.

In short, this is a nice *exercice de style*. Yet, we believe that this study is also significant by itself. Reduction in the  $\lambda$ -calculus is representative of the kind of computation that arises in Coq, e.g. in the conversion test. Our case study can be seen as a step stone for future works that would attempt to prove the correctness of *efficient* implementations of symbolic algorithms. On the one hand, one could use the `PURE-SHALLOW` approach to implement hash-consing and memoization, yet remain inside the bounds of what Coq’s extraction mechanism can safely handle. On the other hand, one could bite the bullet, use the `SMART` approach, and focus on more interesting details than hash-consing and memoization.

## 9 Conclusion

In this paper, we proposed two solutions to implement hash-consing in programs certified with the Coq system. The first one is to implement it using Coq data structures; the second is to use the imperative features provided by OCaml through the tuning of the extraction mechanism. The difference in flavor between the mapping of Coq constants to smart OCaml realizers or the axiomatization of there realizers in Coq is a matter of taste. In both cases, some meta-theoretical reasoning is required and requires to “sweep something under the rug”.

```

Inductive term : Type :=
| Var : N → positive → term
| App : term → term → positive → term
| Abs : term → positive → term.

Definition tag t n :=
match t with
| Var x _ ⇒ Var x n
| App t u _ ⇒ App t u n
| Abs t _ ⇒ Abs t n
end.

```

(a) Definition of terms, and the associated tagging function

```

Definition alloc t (st : state) :=
let r := tag t st.(next) in
(r, mk_state
{
  hmap := set t r st.(hmap);
  next := st.(next) + 1
}) (to_memo st)).

Definition mk_term (t : term) st :=
match get t st.(hmap) with
| Some x ⇒ (x, st)
| None ⇒ alloc t st
end.

```

(b) Updating the global state, and the term constructor

```

Let A := (term * N * state).
Let B := (term * state).
Let msr := fun (n : A) ⇒ size (fst (fst n)). (* measure function *)

Definition lifti_rec (n : N) (arg : A) (rec : ∀ x : A, msr x < msr arg → B).
refine(
match arg as arg' return arg = arg' → B with
| (t,k,st) ⇒ fun Harg ⇒
  match get (n,ident t,k) (memo_lifti st) with
  | Some t ⇒ (t,st)
  | None ⇒ let (r,st) :=
      match t as t' return t = t' → _ with
      | Var i _ ⇒ fun H ⇒ if N.ltb i k
          then mk_Var i st
          else mk_Var (N.add i n) st
      | Abs t _ ⇒ fun H ⇒ let (t,st) := rec (t,(N.succ k),st) _ in
          mk_Abs t st
      | App t u _ ⇒ fun H ⇒ let (t,st) := rec (t,k,st) _ in
          let (u,st) := rec (u,k,st) _ in
          mk_App t u st
      end eq_refl
      in (r,upd_lifti (n,ident t,k) r st)
  end
end eq_refl).
(* Proof obligations elided *)
Defined.

Definition lifti (n : N) (t : term) (k : N) (st : state) : term * state :=
Fixm msr (lifti_rec n) (t,k,st).

```

(c) Definition of `lifti`Fig. 19:  $\lambda$ -terms revisited, using the shallow-embedding approach



### 9.1 Trusting manually-tweaked extracted code

Wrapping-up this article, we have to stress the fact that using hand-written code in the extraction process yields code that is less trustworthy. Indeed, the code we use to realize axioms (e.g., inhabitants of the type `memoizer`) or to replace constructors and destructors with the suitable smart ones may introduce bugs.

For instance, we could end up producing code that fails to compile, because the code snippets used to customize the extraction are used as textual replacements. Or, we could end up producing code that is observationally not-equivalent to the pristine extracted function because of side-effects. Or we could simply ruin all our verification effort by introducing the same kind of bugs that we wanted to catch using formal proofs.

There are two simple ways to restore confidence in this code: code reviews and tests. Unfortunately, doing more than that is hard. For instance, we could imagine using the Why3 [20] framework to reason about our code. Unfortunately, Why3 is not suited for reasoning about higher-order programs, such as our memoizing fixpoint operators. More promising is Charguéraud’s CFML [7] framework, that makes it possible to reason about ML programs with effects. We wonder to what extent the verification of our combinators could be conducted in this system.

In any case, we would like to point out that using hand-written code only *increase* the trusted code base, which contains the whole extraction mechanism.

### 9.2 Trusting external imperative code

The simple act of calling external libraries that rely on side effects may introduce inconsistencies if these libraries return inconsistent values when called with identical parameters. This is because, in Coq, a given term has a unique value inside a model (or a trace of execution, in practical terms). This is the reason why exposing the internals of mutable state, such as unique identifiers, is fraught with possibilities of introducing subtle inconsistencies.

For example, the “else” branch of the following Coq program is unreachable, because `foo x` and `foo (pred (S x))` are equal, thus the theorem stating that it always returns “true”:

```
Definition essai (x : nat) : bool :=
  if eq_nat_dec (foo x) (foo (pred (S x))) then true else false.
Theorem essai_true: ∀ x: nat, (essai x) = true.
```

Yet, if `foo` is extracted to a function such as

```
let counter = ref 0 in fun x → counter := S (!counter); !counter
```

then the OCaml code for `essai 0` returns “false”.

### 9.3 Some practical details of implementation

We took care through this article *not* to use Coq’s advanced support for recursive definitions, i.e., the `PROGRAM` and `FUNCTION` commands. There are two important reasons for this. First, as we hinted at in §5.1, the termination arguments for our BDD operations are out of the scope of what both commands can automate.

Second, we could have used `PROGRAM` instead of `refine` in many of our definitions. Both can be used to build terms with holes, that are later filled using tactics. However, `PROGRAM` is based on a powerful *elaboration* [19], meaning that the final term that is built may be quite different from the one the user has written.

Generally speaking, the performances of the OCaml version of the libraries we build using the `SMART` approach are quite dependent on side-effects. For instance, keeping memoization tables from one run of a given BDD operation to the other yields different performance and memory consumption profiles from what one gets without this conservative strategy. The only change between these conservative and non-conservative strategies is when the memoization tables are allocated. Therefore, we are quite picky when it comes to program transformations that could introduce partial applications or eta-expansions. Using `refine`, we are more confident that the code that we wrote is the one that is going to be extracted, and we have a better idea of the kind of OCaml code that is produced. This makes it easier to deal with this kind of potential performance issues.

#### 9.4 Related work

*Purely functional implementations of BDDs* Christiansen and Huch [8] and Jeremy Bradley [15] implemented *lazy* BDDs in Haskell in a manner similar to our “pure-shallow” approach. In their approaches, not only are BDDs maximally shared, they are also lazily evaluated “on demand”. We did not investigate this possibility.

*Imperative features inside the specification language.* An obvious solution to implement and reason about imperative algorithms is to have these imperative features present in the modeling language of the prover. Some provers directly target high-level programming languages with data structures, references and imperatives features: an example is KeY [2], which targets a large subset of Java. While such features are not available in Coq, there are two conceptual difficulties with this approach that would have made it impractical in our case-studies. First, BDD algorithms implemented using hash-consing are functional in a high-level view: BDD operations are very clearly given functionally, by induction; but also because hash-consing is suitable only for immutable structures<sup>13</sup>. It therefore seems strange to have to program them in an imperative language, furthermore one that complicates common functional idioms (e.g. pattern-matching). Second, the meta-theory for such languages is typically huge, with intricate proof rules having to deal with mutable data, references, late binding etc. It is not obvious how much we can trust the proof system with respect to the semantics of the language.

A second option is to add to an existing functional specification language certain imperative traits as *monads*, then modify the extraction function from the specification language to the target language so as to translate monadic operations into imperative calls, as has been done for Isabelle/HOL [6]; this approach has been used to verify a BDD package [10]. Special proof idioms have to be used for monadic programs, in addition to the general difficulty of programming in

<sup>13</sup> Or at least for structures behaving as though they were immutable; for instance, we can perform hash-consing on a structure if the mutable information in the structure is just used for caching and does not affect the hashcode.

monadic style (see §2.1). To the best of our knowledge, this approach has not been investigated in Coq.

A third option is to integrate into the specification language some essentially imperative data structures (e.g. mutable arrays, from which hash tables can be implemented), but present them in a functional fashion (e.g. an update to a mutable array is treated as returning a new array, same as the previous except for the updated location). The imperative features are then implemented efficiently for evaluation of expressions inside the prover, and are mapped to native imperative features of the target language during extraction. For instance, an experimental version of Coq exists, with native integers and arrays [1]. Again, all difficulties with monads (see §2.1) apply, plus there is the problem of running a nonstandard version of Coq.

A fourth option is to *deeply embed* a subset of an imperative language into Coq: the programs of this imperative language are given a semantics inside Coq, and correctness properties are proved with respect to this semantics. This idea is discussed by Vafeiadis [21], but, as he remarks, this largely precludes the use of regular proof tactics: we have to develop proof steps specific to the language being embedded, and prove these steps correct with respect to the semantics.

*Verification of BDD algorithms in Coq.* Verma et al. [22, 23] implemented and proved correct in Coq a BDD library featuring efficient negation and disjunction; other operations like conjunction, implication and so on are implemented as derived operations. The BDDs produced are reduced and shared.

As we said, we chose to implement a fresh one because the code associated to their paper did not age well w.r.t. the evolution of Coq. Beside the fact that they investigated garbage collection, there is no conceptual difference between their library and our PURE-DEEP approach.

*Verification of BDD algorithms in other theorem provers* In Isabelle/HOL, Ortner and Schirmer [18] verified the implementation of a normalization algorithm for binary decision diagrams. That is, their algorithm takes as input a BDD, and outputs the corresponding ROBDD. Their formalization is built on the Burstall-Bornat memory model: they build one heap of type `ref`  $\rightsquigarrow$  `value` for each component of a BDD node, with `ref` the abstract type of memory addresses. Using a split-heap model makes it easier to reason about heap-allocated data structures in tools such as Why3. We believe however, that our formalization is more direct than theirs, and more suitable for efficient implementations in Coq.

Boyer and Hunt [4] developed an extension of ACL2 that uses hash-consing to give canonical representatives to ACL2 objects. This makes it possible to memoize some ACL2 user defined functions. As a case study, they implemented a BDD library in ACL2. Remark that this implementation is based on the fact that ACL2 exposes hash-consing primitives and the associated reasoning principles to the user. It is unclear to what extent these features could be added to the Coq proof assistant.

*Hash-consing in the execution language* Jean Goubault-Larrecq’s HimML programming language [11–13] is an extension of core Standard ML with primitive finite set and map data types with a run-time system designed around the concept of maximal sharing, or systematic hash-consing.

## 9.5 Future work

We conclude with directions for future work. First, we believe that the smart constructors approach is generalizable to a huge variety of inductive types. One can imagine it could be the job of Coq's extraction mechanism to implement on-demand such smart constructors and memoizers as it was the case for other imperative constructs [1]. Where to stop? Should the extraction mechanism also provide built-in functional maps and sets upon hash-consed types?

Second, in §5.2, we hinted at the fact that using two mutually inductive data-types for expressions and hash-consed expressions yields more complicated proofs than the one we had to do in §8.3. However, these two mutually inductive data-types are at the heart of Conchon and Filliâtre hash-consing *library*. We believe that following their approach closely is a good starting point for investigating to what extent one could provide a certified version of this hash-consing library.

Third, we confess that our implementations of BDDs are a bit rough. We would like to polish the implementations of our PURE-SHALLOW and SMART approaches, and release them as user-friendly libraries. In particular, we need to add some operations like quantification or functional composition.

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