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# Sensitivity of rough differential equations: an approach through the Omega lemma

Laure Coutin\*      Antoine Lejay†

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## Abstract

The Itô map assigns the solution of a Rough Differential Equation, a generalization of an Ordinary Differential Equation driven by an irregular path, when existence and uniqueness hold. By studying how a path is transformed through the vector field which is integrated, we prove that the Itô map is Hölder or Lipschitz continuous with respect to all its parameters. This result unifies and weakens the hypotheses of the regularity results already established in the literature.

**Keywords:** rough paths; rough differential equations; Itô map; Malliavin calculus; flow of diffeomorphisms.

## 1 Introduction

The theory of rough paths is now a standard tool to deal with stochastic differential equations (SDE) driven by continuous processes other than the Brownian motion such as the fractional Brownian motion. Even for standard SDE, it has been proved to be a convenient tool for dealing with large deviations or for numerical purposes. We refer the reader to [15, 20, 22, 24, 37, 38, 42, 43] for a presentation of this theory with several different points of view.

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\*Institut de Mathématiques de Toulouse, F-31062 Toulouse Cedex 9, France. E-mail: [laure.coutin@math.univ-toulouse.fr](mailto:laure.coutin@math.univ-toulouse.fr)

†Université de Lorraine, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54600, France  
CNRS, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54600, France  
Inria, Villers-lès-Nancy, F-54600, France  
E-mail: [Antoine.Lejay@univ-lorraine.fr](mailto:Antoine.Lejay@univ-lorraine.fr)

For a Banach space  $U$ , a time horizon  $T > 0$ , a regularity indice  $p \in [2, 3)$  and a control  $\omega$  (See Section 4 for a definition), we denote by  $C_p(U)$  the space of paths from  $[0, T]$  to  $U$  of finite  $p$ -variation with respect to the control  $\omega$ . By this, we mean a path  $x : [0, T] \rightarrow U$  such that with  $|x_t - x_s| \leq C\omega(s, t)^{1/p}$  for some constant  $C$ .

A *rough path*  $\mathbf{x}$  is an extension in the non-commutative tensor space  $1 \oplus U \oplus (U \otimes U)$  of a path in  $C_p(U)$ . This extension  $\mathbf{x}$  is defined through algebraic and analytic properties. There exists a natural projection from  $R_p(U)$  onto  $C_p(U)$ . Yet there exists no canonical way to lift a path from  $C_p(U)$  to  $R_p(U)$ . The space of rough paths of finite  $p$ -variations with respect to the control by  $\omega$  is denoted by  $R_p(U)$ . It is equipped with the topology induced by the  $p$ -variation distance with respect to  $\omega$ .

Given a rough path  $\mathbf{x} \in R_p(U)$  and a vector field  $f : V \rightarrow L(U, V)$  for a Banach space  $V$ , a controlled differential equation

$$y_t = a + \int_0^t f(y_s) d\mathbf{x}_s \quad (1)$$

is well defined provided that  $f$  is regular enough. This equation is called a *rough differential equation* (RDE). For a smooth path  $x$ , a rough path  $\mathbf{x}$  could be naturally constructed using the iterated integrals of  $x$ . In this case, the solution to (1) corresponds to the solution to the ordinary differential equation  $y_t = a + \int_0^t f(y_s) dx_s$ . The theory of rough paths provides us with natural extensions of controlled differential equations.

When (1) has a unique solution  $y \in C_p(V)$  for any  $\mathbf{x} \in R_p(U)$  given that the vector field  $f$  belongs to a proper subspace  $\text{Fi}$ , the map  $\mathfrak{I} : (a, \mathbf{x}, f) \mapsto y$  from  $U \times R_p(U) \times \text{Fi}$  to  $C_p(V)$  is called the *Itô map*. The Itô map is actually locally Lipschitz continuous on  $U \times R_p(U) \times \text{Fi}$  when  $\text{Fi}$  is equipped with the proper topology [22, 34, 35].

Together with the Itô map  $\mathfrak{I}$ , we could consider for each  $t \in [0, T]$   $\mathfrak{f}_t(a, \mathbf{x}, f) = \mathbf{e}_t \circ \mathfrak{I}(a, \mathbf{x}, f)$  from  $U \times R_p(U) \times \text{Fi}$  to  $V$ , where  $\mathbf{e}_t$  is the *evaluation map*  $\mathbf{e}_t(x) = x(t)$ . The family  $\{\mathfrak{f}_t(a, \mathbf{x}, f)\}_{t \in [0, T]}$  is the *flow* associated to the RDE (1). For ordinary differential equations, the flow defines a family of homeomorphisms or diffeomorphisms.

The differentiability properties of the Itô map or the flow are very important in view of applications. For SDE, Malliavin calculus opens the door to existence of a density and its regularity [47, 50], large deviation results [30], Monte Carlo methods [23, 47], ...

In this article, we then consider the differentiability properties, understood as Fréchet differentiability, of the Itô map under minimal regularity conditions on the vector field. Therefore, we extend the current result by proving Hölder

continuity of the Itô map. This generalizes to  $2 < p \leq 3$  the results of T. Lyons and X. Li [41].

Several strategies have been developed to consider the regularity of the Itô map and flows, which we review now. For this, we need to introduce some notations.

★ The space of *geometric rough paths*  $G_p(U)$  is roughly described as the limit of the natural lift of smooth rough paths through their iterated integrals.

★ For a path  $\mathbf{x} \in R_p(U)$  and a Banach space,  $P_p(\mathbf{x}, V)$  is the space of *controlled rough path* (CRP). It contains paths  $z : [0, T] \rightarrow V$  whose increments are like those of  $\mathbf{x}$ , that is  $z_{s,t} = z_s^\dagger x_{s,t} + z_{s,t}^\sharp$  for proper  $(z^\dagger, z^\sharp)$  (for a formal definition, see Section 5). A CRP  $z$  is best identified with the pair  $(z, z^\dagger)$  (See Remark 2 below).

★ For a path  $\mathbf{x} \in R_p(U)$  and  $h \in C_q(U)$  with  $1/p + 1/q > 1$  (which means that  $1 \leq q < 2$ ), there exists a natural way to construct a rough path  $\mathbf{x}(h) \in R_p(\Omega)$  such that  $\pi(\mathbf{x}(h)) = \pi(\mathbf{x}) + h$ , where  $\pi$  is the natural projection from  $R_p(U)$  onto  $C_p(U)$ , and  $\mathbf{x}(0) = \mathbf{x}$ . We write  $\mathfrak{I}_{\mathbf{x}}(h) = \mathbf{x}(h)$ . This map is  $\mathcal{C}^\infty$ -Fréchet from  $C_q(U)$  to  $R_p(U)$ .

★ A vector field  $f : V \rightarrow L(U, V)$  is said to be  $\gamma$ -Lipschitz ( $\gamma > 0$ ) if it is differentiable up to order  $[\gamma]$  with a derivative of order  $[\gamma]$  which is  $(\gamma - [\gamma])$ -Hölder continuous (See [22], p. 213).

The flow and differentiability properties have already been subject to many articles:

★ In [45], T. Lyons and Z. Qian have studied the flow property for solutions to  $y_t = a + \int_0^t f(y_s) d\mathbf{x}_s + \int_0^t g(y_s) dh_s$  for a “regular” path  $h$  subject to a perturbation.

★ In [44], T. Lyons and Z. Qian showed that the Itô map provides a flow of diffeomorphisms when the driving rough path is geometric.

★ T. Lyons and Z. Qian [46] and more recently Z. Qian and J. Tudor [52] have studied the perturbation of the Itô map when the rough path is perturbed by a regular path  $h$  and the structure of the tangent spaces. As these constructions hold in tensor spaces, quadratic terms are involved. They lead to rather complex expressions.

★ The properties of flow also arise directly from the constructions from almost flows, in the approach from I. Bailleul [3]. Firstly, only Lipschitz flow were considered, but recently in [5], it was extended to Hölder continuous flows.

★ In the case  $1 \leq p < 2$ , T. Lyons and Z. Li proved in [41] (see also [36]) that

$$x \in C_p(U) \mapsto \mathfrak{I}(a, x, f) \in C_p(V) \text{ is locally } \mathcal{C}^k\text{-Fréchet differentiable}$$

provided that  $U$  and  $V$  are finite dimensional Banach spaces and  $f$  is  $\mathcal{C}^{k+\alpha+\epsilon}$ ,  $k \geq 1$ ,  $\alpha \in (p-1, 1-\epsilon)$ ,  $\epsilon \in (0, 1)$ .

★ In the book of P. Friz and N. Victoir [22], it is proved that

$$(b, h) \in U \times C_q(U) \mapsto \mathfrak{I}(a + b, \mathbf{x}(h), f) \in C_p(U)$$

is locally  $\mathcal{C}^k$ -Fréchet at  $(a, \mathbf{x}) \in U \times G_p(U)$

provided that  $\mathbf{x} \in G_p(U)$ ,  $U$  is finite-dimensional and  $f$  is of class  $\mathcal{C}^{k-1+\gamma}(U, V)$  with  $\gamma > p$  and  $k \geq 1$  (See [22, Theorem 11.6, p. 287]). It is also proved that  $\mathfrak{f} : [0, T] \times U \mapsto V$  is a flow of  $\mathcal{C}^k$ -diffeomorphisms, and that  $\mathfrak{f}$  and its derivatives are uniformly continuous with respect to  $\mathbf{x} \in G_p(U)$  (See [22, Section 11.2, p. 289]). Transposed in the context of CRP in [20], the flow  $a \mapsto \mathfrak{f}_t(a, \mathbf{x}, f)$  is locally a diffeomorphism of class  $\mathcal{C}^{k+1}$  for a vector field  $f$  is of class  $\mathcal{C}^{k+3}$ .

★ In a series of articles [30–33] (See also [55]), Y. Inahama and H. Kawabi have studied various aspects of stochastic Taylor developments in  $\epsilon$  of solutions to

$$y_t^\epsilon = a + \int_0^t f(y_s^\epsilon) d\mathfrak{d}_\epsilon \mathbf{x}_s + \int_0^t g(y_s^\epsilon) dh_s \quad (2)$$

in term of  $\epsilon$  around the solutions to  $z_t = a + \int_0^t g(z_s) dh_s$ , when  $\mathbf{x} \in R_p(W)$ ,  $h \in C_q(U)$  with  $1/p + 1/q > 1$ ,  $f : V \rightarrow L(W, V)$ ,  $g : V \rightarrow L(U, V)$  and  $\mathfrak{d}_\epsilon : R_p(W) \rightarrow R_p(W)$  is the dilatation operator defined by  $\mathfrak{d}_\epsilon \mathbf{x} := 1 + \epsilon \mathbf{x}^1 + \epsilon^2 \mathbf{x}^2$ . Up to the natural injection of  $\mathbf{x}$  to  $R_p(W \oplus U)$ ,  $h$  to  $C_p(W \oplus U)$ ,  $f : V \rightarrow L(W \oplus U, V)$  and  $g : V \rightarrow L(W \oplus U, V)$ , (2) is recast as

$$y^\epsilon = \mathfrak{I}(a, \mathfrak{T}_{\mathfrak{d}_\epsilon \mathbf{x}}(h), f + g).$$

★ Using a Banach space version of the Implicit Functions Theorem, I. Bailleul recently proved in [4] that

$$(z, f) \in P_p(\mathbf{b}, U) \times \mathcal{C}^k(U, V) \mapsto \mathfrak{I}(a, \mathfrak{P}_\mathbf{b}(z), f) \in P_p(\mathbf{b}, V)$$

is  $\mathcal{C}^{\lfloor k \rfloor - 2}$ -Fréchet differentiable

provided that  $\mathbf{b} \in G_p(B)$  and  $k \geq 3$ , where  $\mathfrak{P}_\mathbf{b}(z)$  lifts  $z \in P_p(\mathbf{b}, U)$  to a rough path.

★ SDE driven by fractional Brownian motion and Gaussian processes attracted a lot of attention [7–12, 14, 17, 18, 26–28, 40, 48, 51]. Since integrability is the key to derive some integration by parts formula in Malliavin type calculus, several articles deal with moments estimates for solutions to linear equations driven by Gaussian rough paths [6, 13, 21, 25, 29, 53].

In this article,

- We establish that the Itô map  $\mathfrak{I}$  is locally of class  $\mathcal{C}^{\gamma-\epsilon}$  for all the types of perturbations of the driving rough path  $\mathbf{x}$  seen above. We then generalized the results in [4], [20, Sect. 8.4] or [22] by providing the Hölder regularity, and not only differentiability. At the exception of the work of T. Lyons and X.D. Li [41] for the Young case ( $1 \leq p < 2$ ), none of the works cited above deal with the Hölder regularity of the derivatives of the Itô map. They all impose stronger regularity conditions than ours on the vector field. In [4], the starting point is kept fixed. While in [20, 22], only the regularity with respect to the starting point and perturbation  $\mathbf{x}(h)$  of the driver are considered, as in [41]. Unlike [22], the chain rule may be applied as our solutions are constructed as CRP.
- By adding more flexibility in the notion of CRP, we define a bilinear continuous integral with CRP both as integrand and integrators. This simple tricks allows one to focus on the effect of a non-linear function applied to a CRP and weaken the regularity assumptions imposed on the vector fields in [4, 20].
- Following the approach of [1], we provide a version of the so-called *Omega lemma* for paths of finite  $p$ -variations and for CRP. For a function  $f$  of given regularity, this lemma states the regularity of  $\mathfrak{D} : y \mapsto f(y)$  when  $y$  is a path of finite  $p$ -variation or a CRP. When dealing with continuous paths with the sup-norms, where  $\mathfrak{D}f$  has the regularity of  $f$  (for functions of class  $\mathcal{C}^k$ , but also Hölder or Sobolev). When dealing with paths of finite  $p$ -variation, the situation is more cumbersome. This explains the losses in the regularity observed in [41].
- We provide a “genuine rough path” approach which removes the restriction implied by smooth rough paths, the restriction to geometric rough paths (which could be dealt otherwise with  $(p, p/2)$ -rough paths [39]) as well as any restriction on the dimensions of the Banach spaces  $U$  and  $V$ . Although we use a CRP, the Duhamel formula shown in [16] could serves to prove a similar result for (partial) rough paths and not only CRP.
- We exemplify the difference between the “rough situation” and the smooth one. For ordinary differential equation, the spirit of the Omega lemma together with the Implicit Functions Theorem is that the regularity of the vector field is transported into the regularity of the Itô map. Once the Omega lemma is stated for our spaces of paths, its implication on the regularity of the Itô map is immediate.

## 2 Notations

For two Banach spaces  $U$  and  $V$ , we denote by  $L(U, V)$  the set of linear *continuous* maps from  $U$  to  $V$ . If  $A \in L(U, V)$  is invertible, then its inverse  $A^{-1}$  is itself continuous and thus belongs to  $L(V, U)$ .

For a functional  $\mathfrak{F}$  from  $U$  to  $V$ , we denote by  $D\mathfrak{F}$  its Fréchet derivative, which is a map from  $V$  to  $L(V, U)$ , and by  $D_y\mathfrak{F}$  its Fréchet derivative in the direction of a variable  $y$ . of  $U$ .

For a function  $f$  and some  $\alpha \in (0, 1]$ , we denote by  $H_\alpha(f)$  its  $\alpha$ -Hölder norm  $H_\alpha(f) := \sup_{x \neq y} |f(x) - f(y)|/|x - y|^\alpha$ .

For any  $\alpha > 0$ , we denote by  $\mathcal{C}^\alpha(U, V)$  the set of continuous, bounded functions from  $U$  to  $V$  with, bounded continuous (Fréchet) derivatives up to order  $k = \lfloor \alpha \rfloor$ , and its derivative of order  $k$  is locally  $(\alpha - k)$ -Hölder continuous. A function in  $\mathcal{C}^\alpha(U, V)$  is simply said to be of class  $\mathcal{C}^\alpha$  (in this article, we restrict ourselves to non-integer values of  $\alpha$ ).

With the convention that  $\nabla^0 f := f$ , we write for  $f$  in  $\mathcal{C}^\alpha(U, V)$ ,

$$\|f\|_\alpha := \max_{j=0, \dots, k} \{ \|\nabla^j f\|_\infty, H_{\alpha-k}(\nabla^k f) \}.$$

Our result are stated for bounded functions  $f$ . For proving existence and uniqueness, these conditions could somehow be relaxed proved that only the derivatives of  $f$  are bounded (See *e.g.* [22, 34, 35], ...). Once existence is proved, there is no problem at assuming that  $f$  is bounded since we only use estimates locally around given points. This justify our choice for the sake of simplicity.

## 3 The Implicit Functions Theorem

Let us consider two Banach spaces  $P$  and  $\Lambda$  as well as a functional  $\mathfrak{F}$  from  $P \times \Lambda$  to  $P$ . Here,  $P$  plays the role of the spaces of paths, while  $\Lambda$  is the space of parameters of the equation.

We consider first solutions  $y$  to the fixed point problem

$$y = \mathfrak{F}(y, \lambda) + b, \quad (\lambda, b) \in \Lambda \times P. \quad (3)$$

To ensure uniqueness, we slightly change the problem. We assume that  $P$  contains a Banach sub-space  $P^*$ . As  $P^*$  is stable under addition, we consider the quotient space  $P^\sim := P/P^*$  defined by  $x \sim y$  when  $x - y \in P^*$ . This is nothing more than encoding the starting point.

We now consider instead of (3) the problem

$$y^* = \mathfrak{F}(y^* + z, \lambda) - z + b, \quad b \in P^*, \quad z \in P^\sim, \quad \lambda \in \Lambda. \quad (4)$$

There is clearly no problem in restricting  $b$  to  $P^*$ , since otherwise one has to change  $z$  accordingly. Solving (4) implies that (3) is solved for  $y = y^* + z$ .

For an integer  $k \geq 0$  and  $0 < \alpha \leq 1$ , if  $\mathfrak{G}(y^*, z, \lambda) := \mathfrak{F}(y^* + z, \lambda)$  and  $\mathfrak{F}$  is of class  $\mathcal{C}^{k+\alpha}$  with respect to  $(y, \lambda)$ , then  $\mathfrak{G}$  is of class  $\mathcal{C}^{k+\alpha}$  with respect to  $(y^*, z, \lambda)$

The reason for considering (4) instead of (3) is that for the cases we consider,  $\mathfrak{G}$  will be strictly contractive in  $y^*$ , ensuring the existence of the unique solution to (4).

We use the following version of the Implicit Functions Theorem (See *e.g.* [1, § 2.5.7, p. 121] for a  $\mathcal{C}^k$ -version of the Implicit Functions Theorem<sup>1</sup>).

**Theorem 1** (Implicit Functions Theorem). *Let us assume that*

*i) The map  $\mathfrak{G}(y^*, z, \lambda) := \mathfrak{F}(y^* + z, \lambda)$  is of class  $\mathcal{C}^{k+\alpha}$  from  $X := P^* \times P^\sim \times \Lambda$  to  $P^*$  for  $k \geq 1$ ,  $0 < \alpha \leq 1$  with respect to  $(y^*, z, \lambda) \in X$ .*

*ii) For some  $(\hat{y}^*, \hat{z}, \hat{\lambda}) \in X$  and any  $b \in P^*$ , there exists a unique solution  $h^*$  in  $P^*$  to*

$$h^* = D_{y^*} \mathfrak{G}(\hat{y}^*, \hat{z}, \hat{\lambda})(h^*) + b$$

*with  $\|h^*\|_P \leq C\|b\|_P$  for some constant  $C \geq 0$ . This means that  $\text{Id} - D_{\hat{y}^*} \mathfrak{G}(\cdot, \hat{z}, \hat{\lambda})$  is invertible from  $P^*$  to  $P^*$  with a bounded inverse.*

*Then there exists a neighborhood  $U$  of  $(\hat{z}, \hat{\lambda}) \in P^\sim \times \Lambda$  a neighborhood  $V$  of  $\mathfrak{G}(\hat{y}^*, \hat{z}, \hat{\lambda})$ , as well as a unique map  $\mathfrak{H}$  from  $V \times U$  to  $P^*$  which solves*

$$\mathfrak{H}(b, z, \lambda) = \mathfrak{G}(\mathfrak{H}(b, z, \lambda), z, \lambda) + b, \quad \forall (b, z, \lambda) \in V \times U.$$

*In other words,  $\mathfrak{I}(b, z, \lambda) := z + \mathfrak{H}(b, z, \lambda)$  is locally the solution in  $z + P^*$  to  $\mathfrak{I}(b, z, \lambda) = z + \mathfrak{F}(\mathfrak{I}(b, z, \lambda), \lambda) + b$ .*

*Remark 1.* Actually, we do not use this theorem in this form. We show that  $D_{y^*} \mathfrak{G}(\cdot, z, \lambda)$  is contractive when restricted to a bounded, closed, convex set  $C$  of  $P^* \times P^\sim \times \Lambda$ , and only on a time interval  $\tau$  which is small enough, in function of the radius of  $C$ . The controls we get allows us to solve iteratively the equations on abutting time intervals  $\tau_i$  and to “stack them up” on get the result on any finite time interval (and even globally for suitable functions). As for this, we have only to re-use with slight adaptations what is already largely been done, we do not treat these issues.

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<sup>1</sup>To extend it to  $\mathcal{C}^{k+\alpha}$ -Hölder continuous functions, we have just to note that the derivative of  $(b, z, \lambda) \mapsto (\mathfrak{H}(b, z, \lambda), z, \lambda)$  below, which is given by the inverse function theorem, is the composition of  $z \mapsto z^{-1}$ , which is  $\mathcal{C}^\infty$ ,  $\mathfrak{G}$ , which is  $\mathcal{C}^{k-1+\alpha}$  and  $\mathfrak{H}$  which is  $\mathcal{C}^k$ , see [49]).

## 4 The Omega lemma for paths of finite $p$ -variation

We consider a time horizon  $T > 0$ . A *control*  $\omega$  is a non-negative function defined on sub-intervals  $[s, t] \subset [0, T]$  which is super-additive and continuous close to the diagonal  $\{(t, t) \mid t \in [0, T]\}$ . This means that is  $\omega_{r,s} + \omega_{s,t} \leq \omega_{r,t}$  for  $0 \leq r \leq s \leq t \leq T$ .

For a path  $x$  from  $[0, T]$  to  $V$ , we set  $x_{s,t} := x_{[s,t]} := x_t - x_s$ . For some  $p \geq 1$ , a path of finite  $p$ -variation controlled by  $\omega$  is a path such that

$$\|x\|_p := \sup_{\substack{[s,t] \subset [0,T] \\ s \neq t}} \frac{|x_t - x_s|}{\omega_{s,t}^{1/p}}.$$

We denote by  $C_p(V)$  the space of such paths, which is a Banach space with the norm

$$\|x\|_{\bullet p} := |x_0| + \|x\|_p.$$

The space  $C_p(V)$  is continuously embedded in the space of continuous functions  $C(V)$  with the sup-norm  $\|\cdot\|_\infty$ , with

$$\|x\|_\infty \leq |x_0| + \|x\|_p \omega_{0,T}^{1/p}. \quad (5)$$

For any  $q \geq p$ ,  $C_p(V)$  is also continuously embedded in  $C_q(V)$ .

**Proposition 1** (L.C. Young [54]). *Let  $p, q \geq 1$  such that  $1/p + 1/q > 1$ . There exists a continuous, bilinear map*

$$\begin{aligned} C_p(U) \times C_q(L(V, U)) &\rightarrow C_p(V) \\ (x, y) &\mapsto \int y \, dx \end{aligned}$$

which satisfies for any  $(x, y)$  and any  $[s, t] \subset [0, T]$ ,

$$\left| \int_s^t y_r \, dx_r - y_s x_{s,t} \right| \leq K \|x\|_q \|x\|_p \omega_{s,t}^{\frac{1}{p} + \frac{1}{q}} \quad (6)$$

for some universal constant  $K$ .

For some  $\kappa \in [0, 1]$ , we set  $\bar{\kappa} := 1 - \kappa$ .

**Lemma 1.** *Let  $g \in \mathcal{C}^\gamma(V, V)$ . Then for any  $\kappa \in [0, 1]$*

$$|g(z) - g(y) - g(z') + g(y')| \leq H_\gamma(g) (|y' - y|^{\kappa\gamma} + |z' - z|^{\kappa\gamma}) (|z' - y'|^{\gamma\bar{\kappa}} + |z - y|^{\gamma\bar{\kappa}}). \quad (7)$$

*Proof.* First,

$$|g(z) - g(y) - g(z') + g(y')| \leq H_\gamma(g)(|y' - y|^\gamma + |z' - z|^\gamma).$$

By inverting the roles of  $z'$  and  $y$  in the above equation, we get a similar inequality. Choosing  $\kappa \in [0, 1]$  and raising the first inequality to power  $\kappa$  and the second one to power  $\bar{\kappa}$  leads to the result.  $\square$

We now give an alternative proof of the one in [41, Theorems 2.13 and 2.15], which mostly differs in the use of the converse of Taylor's theorem. For CRP, our proof will be modelled on this one.

For  $1 \leq p < 2$ , we consider  $f$  of class  $\mathcal{C}^{k+\gamma}$  from  $V$  to  $W := L(U, V)$  with  $k \geq 1$  and  $0 < \gamma \leq 1$ . We set  $q := p/\kappa\gamma$ . Besides, we set

$$C_p^*(V) := \{y \in C_p(V) \mid y_0 = 0\} \text{ and } C_q^*(V) := \{y \in C_q(V) \mid y_0 = 0\}.$$

**Proposition 2** (The Omega lemma for paths of finite  $p$ -variation). *With the above notations and assumptions, for any  $0 < \kappa < 1$ ,  $\mathfrak{D}f(y) := (f(y_t))_{t \in [0, T]}$  is of class  $\mathcal{C}^{k+\bar{\kappa}\gamma}$  from any ball of radius  $\rho > 0$  of  $C_p(V)$  to  $C_q(W)$ . Besides,  $D_y \mathfrak{D}f \cdot h = (Df(y_t) \cdot h_t)_{t \in [0, T]} \in C_q(W)$  for any  $y, h \in C_p(V)$ . If  $k \geq 1$ , then  $D\mathfrak{D}(y) \cdot h \in C_q^*(W)$  when  $h \in C_p^*(V)$ .*

*Proof.* I) Assume that  $f \in \mathcal{C}^\gamma$ . Thanks to the embedding from  $C_{p/\gamma}(L(U, V))$  to  $C_q(L(U, V))$ ,  $\mathfrak{D}f$  maps  $C_p^*(V)$  to  $C_q(L(U, V))$  with  $\|\mathfrak{D}f(y)\|_q \leq CH_\gamma(f)\|y\|_p$  for all  $y \in C_p^*(V)$ .

With Lemma 1, we easily obtain that

$$\|\mathfrak{D}f(y) - \mathfrak{D}f(z)\|_{\bullet, q} \leq H_\gamma(f)(1 + \omega_{0, T}^{\bar{\kappa}\gamma/p})\|y - z\|_{\bullet, p}^{\bar{\kappa}\gamma}(\|y\|_p^{\kappa\gamma} + \|z\|_p^{\kappa\gamma}).$$

II) For some Banach space  $W'$ , if  $y \in C_q(L(V \otimes W', V))$  and  $z \in C_p(V)$  (resp.  $y \in C_q(V)$ ,  $z \in C_p(U)$ ), then with (5), it is straightforward to show that  $yz \in C_q(L(W', V))$  (resp.  $y \otimes z \in C_q(V \otimes U)$ ) with (with  $y \cdot z = (y_t z_t)_{t \in [0, t]}$  or  $y \cdot z = (y_t \otimes z_t)_{t \in [0, T]}$ )

$$\|y \cdot z\|_q \leq (1 + 2\omega_{0, T}^{1/q})\|y\|_{\bullet, q}\|z\|_{\bullet, q} \text{ and } \|y \cdot z\|_{\bullet, q} \leq |y_0| \cdot |z_0| + \|y \cdot z\|_q.$$

Besides, if  $z_0 = 0$ , then  $(yz)_0 = 0$  (resp.  $(y \otimes z)_0 = 0$ ).

III) Let us assume now that  $f \in \mathcal{C}^{k+\gamma}$  for some  $k \geq 1$ . With the Taylor development of  $f$  up to order  $k$ ,

$$f(y + z) = f(y) + \sum_{i=1}^k \frac{1}{i!} D^i f(y) z^{\otimes i} + R(y, z)$$

with

$$R(y, z) = \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} (D^k f(y+sz) - D^k f(y)) z^{\otimes k} ds.$$

Since  $D^i f$  is locally of class  $\mathcal{C}^{k-i+\gamma}$  from  $V$  to  $L(V^{\otimes i}, L(U, V))$  identified with  $L(V^{\otimes i} \otimes U, V)$ , then  $y \in C_p(V) \mapsto (D^i f(y_t))_{t \in [0, T]}$  takes its values in  $C_q(L(V^{\otimes i} \otimes U, V))$ .

For  $y, z^{(1)}, \dots, z^{(i)} \in C_p(V)$ , write

$$\phi_i(y) \cdot z^{(1)} \otimes \dots \otimes z^{(i)} := (D^i f(y_t) \cdot z^{(1)} \otimes \dots \otimes z^{(i)})_{t \in [0, T]}$$

Since  $C_p(V)$  is continuously embedded in  $C_q(V)$ , II) implies that  $\phi_i(y)$  is multi-linear and continuous from  $C_p(V)^{\otimes i}$  to  $C_q(W)$ .

Similarly, for some constant  $C$  that depends only on  $\rho$  (the radius of the ball such that  $\|y\|_{\bullet p} \leq \rho$ ),  $p, q$  and  $\omega_{0, T}$ ,

$$\frac{\|R(y, z)\|_{\bullet q}}{\|z\|_{\bullet p}^k} \leq CH_\gamma(D^k f) \|y\|_{\bullet p}^{\bar{\kappa}\gamma}.$$

The converse of the Taylor's theorem [2] implies that  $\mathfrak{D}f$  is of class  $\mathcal{C}^k$  from  $C_p(V)$  to  $C_q(W)$ . In addition, it is easily shown from I) that since  $D^k f$  is of class  $\mathcal{C}^\gamma$ ,  $\mathfrak{D}f$  is of class  $\mathcal{C}^{k+\bar{\kappa}\gamma}$  from  $C_p(V)$  to  $C_q(W)$ .  $\square$

Given  $0 < \kappa < 1$  with  $1 + \kappa\gamma > p$ , we then define

$$\mathfrak{F}(y, x, f) := \int \mathfrak{D}f(y) dx \text{ for } (y, x, f) \in C_p(V) \times C_p(U) \times \mathcal{C}^{k+\gamma}(V, W),$$

as the integral is well defined as a Young integral using our constraint on  $\gamma, \kappa$  and  $p$ . The map  $\mathfrak{F}$  is linear and continuous with respect to  $(z, f)$ . It is of class  $\mathcal{C}^{k+\bar{\kappa}\gamma}$  with respect to  $y$ . When  $1 + \gamma > p$ , it is well known that  $y = \mathfrak{F}(y, z, f)$  has a unique solution (see *e.g.* [20, 22, 42]). Besides, it is evident that for any  $a \in V$ ,

$$y_t = b_t + \int_0^t f(y_s) dx_s, \quad t \in [0, T] \text{ if and only if } y_t^* = b_t^* + \int_0^t f(a + y_s^*) dx_s, \quad t \in [0, T]$$

when  $y = a + y^*$ ,  $b = a + b^*$ ,  $b^*, y^* \in C_p^*(V)$ , so that  $C_p^\sim(V) = C_p(V)/C_p^*(V)$  is identified  $V$  and  $a = y_0$ .

The Fréchet derivatives of  $\mathfrak{F}$  is the direction of  $y$  is

$$D_y \mathfrak{F}(y, z, f) \cdot h = \int (\mathfrak{D}Df(y) \cdot h) dz$$

which is also well defined as a Young integral.

From II), if  $h \in C_p^*(V)$ , then  $Df(y) \cdot h$  takes its values in  $C_p^*(V)$ . With (6), (5) and I), for  $T$  is small enough (depending only on  $\|f\|_{1+\gamma}$  and  $\|x\|_p$ ),  $h \mapsto \int Df(y) \cdot h dx$  is strictly contractive on  $C_p^*(V)$ . This proves that  $\mathfrak{G}(y^*, a, x, f) := \mathfrak{F}(a + y^*, x, f)$  satisfies the conditions of application of the Implicit Function Theorem 1 (See Remark 1).

We then recover and extend the result in [41].

**Theorem 2.** *For  $p \in [1, 2)$ ,  $x \in C_p(U)$ ,  $a \in V$  and  $f \in \mathcal{C}^{k+\gamma}(V, L(U, V))$ ,  $\kappa \in (0, 1)$ ,  $k \geq 1$ , provided that  $1 + \kappa\gamma > q$ , there exists a unique solution  $\mathfrak{J}(a, f, b)$  in  $C_p(V)$  to*

$$y_t = a + \int_0^t f(y_s) dx_s + b_{0,t}, \quad t \in [0, T].$$

Besides,  $\mathfrak{J}$  is locally of class  $\mathcal{C}^{k+\bar{\kappa}\gamma}$  from  $V \times \mathcal{C}^{k+\gamma}(V, L(U, V)) \times C_p(U) \times C_p(V)$  to  $C_p(V)$ .

For some  $t \in [0, T]$ , let  $\mathbf{e}_t : C_p(V) \rightarrow V$  be the evaluation map  $\mathbf{e}_t(y) = y_t$ . Thanks to (5),  $\mathbf{e}_t$  is continuous. We then define  $\mathfrak{f}_t(a) := \mathbf{e}_t \circ \mathfrak{J}(a, f, 0)$  for some vector field  $f$ .

**Corollary 1.** *Under the conditions of Theorem 2,  $\mathfrak{f}_t : V \rightarrow V$  is locally a diffeomorphism of class  $\mathcal{C}^{k+\bar{\kappa}\gamma}$  for any  $t \in [0, T]$ .*

*Proof.* Let us consider the solution to  $y_{t,r}(a) = a + \int_r^t f(y_{s,r}(a)) dx_s$  for  $0 \leq r \leq t$ . Using the chain rule,

$$\begin{aligned} D_a y_{t,r}(a) &= \text{Id} + \int_r^t Df(y_s(a)) D_a y_{s,r}(a) dx_s \\ &= \text{Id} + \int_r^t Df(y_{s,r}(a)) \mathbf{e}_s \circ D_a \mathfrak{J}(a, f, 0) dx_s. \end{aligned} \quad (8)$$

Owing to the controls given above, the right-hand side of (8) is bounded when  $a$  belongs to a set  $\{|a| \leq R\}$ . Besides, due to (5), it is easily seen that for  $t - r$  small enough (in function of  $f$ ,  $\gamma$ ,  $\omega_{r,t}$ ,  $p$  and  $R$  such that  $|a| \leq R$ ), one may choose a constant  $0 < k < 1$  such that

$$\|D_a y_{t,r}(a) - \text{Id}\| \leq k.$$

It follows that  $D_a y_{t-r}$  is invertible at the point  $a$ . The Inverse Function Theorem asserts that  $y_{t,r}$  is then locally a  $\mathcal{C}^{k+\bar{\kappa}\gamma}$ -diffeomorphism around any point  $a \in V$ .

Using the additive property of the integral and the uniqueness,  $y_{t,r}(a) = y_{t,s}(y_{s,r}(a))$ . Hence, the regularity of  $a \mapsto y_{t,r}(a)$  for a arbitrary times  $r, t$  is treated using the stability of  $\mathcal{C}^{k+\bar{\kappa}\gamma}$  under composition and the flow property.

To conclude, it remains to remark that  $\mathfrak{f}_t(a) = y_{t,0}(a)$ .  $\square$

## 5 The Omega lemma for controlled rough paths

Let us fix  $p, q, r$  with  $2 \leq p < 3$ ,  $q > 0$  and  $r > 0$ . We consider a rough path  $\mathbf{x} \in \mathbb{R}_p(\mathbb{U})$ .

A Controlled Rough Path (CRP) is a path  $y : [0, T] \rightarrow \mathbb{V}$  which admits the decomposition

$$y_{s,t} = y_s^\dagger \mathbf{x}_{s,t}^1 + y_{s,t}^\sharp \text{ with } y^\dagger \in C_q(L(\mathbb{U}, \mathbb{V})) \text{ and } \|y^\sharp\|_r < +\infty.$$

A CRP is identified as the pair of paths  $(y, y^\dagger)$ , as  $y^\sharp$  may be computed from  $y$  and  $y^\dagger$ . We write

$$\|y\|_{\mathbf{x}} := \|y^\dagger\|_q + \|y^\sharp\|_r \text{ and } \|y\|_{\bullet, \mathbf{x}} := |y_0| + |y_0^\dagger| + \|y\|_{\mathbf{x}}.$$

The space of CRP is denoted by  $\mathbb{P}_{p,q,r}(\mathbf{x}, \mathbb{V})$ . This is a Banach space with the norm  $\|\cdot\|_{\bullet, \mathbf{x}}$ .

*Remark 2.* This definition, which involves three indices, is more general than the one in [20] (in which  $(p, q, r) = (p, p, p/2)$ ) and the seminal article [24] in which  $(p, q, r) = (p, q, p+q)$ . The reason of this flexibility will appear with the Omega lemma.

For any  $s \in [0, T]$ , we extend  $y_s^\dagger$  as an operator in  $L(\mathbb{U} \otimes \mathbb{U}, \mathbb{U} \otimes \mathbb{V})$  by  $y_s^\dagger(a \otimes b) = a \times y_s^\dagger b$  for all  $a, b \in \mathbb{U}$ .

**Proposition 3.** *Assume that*

$$\theta := \max\left\{\frac{2}{p} + \frac{1}{q}, \frac{1}{p} + \frac{1}{r}\right\} > 1.$$

*There exists a continuous linear map  $y \mapsto y^\flat$  on  $\mathbb{P}_{p,q,r}(\mathbf{x}, \mathbb{V})$  which transforms  $y$  to  $\{y_{s,t}^\flat\}_{[s,t] \subset [0,T]}$  with values in  $\mathbb{U} \otimes \mathbb{V}$  such that*

$$y_{r,s}^\flat + y_{s,t}^\flat + \mathbf{x}_{r,s}^1 \otimes y_{s,t} = y_{r,t}^\flat, \quad \forall 0 \leq r \leq s \leq t \leq T. \quad (9)$$

*Moreover, for some universal constant  $K$ ,*

$$|y_{s,t}^\flat - y_s^\dagger \mathbf{x}_{s,t}^2| \leq K \|y\|_{\mathbf{x}} \|\mathbf{x}\|_p \omega_{s,t}^\theta \quad (10)$$

*and*

$$\|y^\flat\|_{p/2} \leq (|y_0^\dagger| + (\omega_{0,T}^{1/p} + K \omega_{0,T}^{\theta-2/p} \|y\|_{\mathbf{x}})) \|\mathbf{x}\|_p. \quad (11)$$

*Proof.* Let us introduce on the linear space  $\mathbb{W} := \mathbb{U} \oplus \mathbb{V} \oplus (\mathbb{U} \otimes \mathbb{V})$  the (non-commutative) operation

$$(a, b, c) \boxtimes (a', b', c') = (a + a', b + b', c + c' + a \otimes a')$$

and the norm  $|(a, b, c)| := \max\{|a|, |b|, |c|\}$ .

For  $Y, X, Z \in W$ , it is clear that  $|\cdot|$  is Lipschitz continuous and

$$|Z \boxtimes Y - Z \boxtimes X| \leq (1 + |Z|)|Y - X| \text{ and } |Y \boxtimes Z - X \boxtimes Y| \leq (1 + |Z|)|Y - X|.$$

With  $\boxtimes$ ,  $W$  is a monoid for which the hypotheses of the Multiplicative Sewing Lemma are fulfilled [19].

We then define for  $y \in P_{p,q,r}(\mathbf{x}, L(U, V))$  the family of operators

$$\phi_{s,t}(y) := (\mathbf{x}_{s,t}^1, y_{s,t}, y_s^\dagger \mathbf{x}_{s,t}^2), \quad [s, t] \subset [0, T].$$

Thus,

$$\phi_{r,s}(y) \boxtimes \phi_{s,t}(y) - \phi_{r,t}(y) = (0, 0, y_{r,s}^\dagger \mathbf{x}_{s,t}^2 + \mathbf{x}_{r,s}^1 y_{s,t}^\# - \mathbf{x}_{r,s}^1 \otimes y_{r,s}^\dagger \mathbf{x}_{s,t}^1). \quad (12)$$

The Multiplicative Sewing Lemma [19] on  $(\phi_{s,t}(y))_{[s,t] \subset [0,T]}$  yields the existence of a family  $\{Y_{s,t}\}_{[s,t] \subset [0,T]}$  taking its values in  $W$  with  $|Y_{s,t} - \phi_{s,t}(y)| \leq C\omega_{s,t}^\theta$  for any  $[s, t] \subset [0, T]$ . We define  $y^b$  as the part in  $U \otimes V$  in the decomposition of  $Y$  as  $Y_{s,t} = (\mathbf{x}_{s,t}^1, y_{s,t}, y_{s,t}^b)$ . Since  $Y$  satisfies  $Y_{r,s} \boxtimes Y_{s,t} = Y_{r,t}$ ,  $y^b$  satisfies (9). From (12), we easily obtain (10) and (11).

For  $y, z \in P_{p,q,r}(\mathbf{x}, V)$ ,

$$\phi_{r,s}(y + z) \boxtimes \phi_{s,t}(y + z) = \phi_{r,s}(y) \boxtimes \phi_{s,t}(y) + \phi_{r,s}(z) \boxtimes \phi_{s,t}(z).$$

From this additivity property and the construction of the multiplicative sewing lemma,  $y \mapsto y^b$  is also linear and continuous.  $\square$

When  $y$  takes its values in  $L(V, W)$ ,  $y^\dagger$  takes its values in  $L(U, L(V, W)) \simeq L(U \otimes V, W)$ .

**Proposition 4.** Fix  $(p, q, r)$  and  $(p, q', r')$  such that  $2 \leq p < 3$ . Assume that

$$\theta := \max\left\{\frac{2}{p} + \frac{1}{q}, \frac{1}{p} + \frac{1}{r}\right\} > 1 \text{ and } \theta' := \max\left\{\frac{2}{p} + \frac{1}{q'}, \frac{1}{p} + \frac{1}{r'}\right\} > 1.$$

There exists a bilinear continuous mapping

$$\begin{aligned} P_{p,q,r}(\mathbf{x}, L(W, V)) \times P_{p,q',r'}(\mathbf{x}, W) &\mapsto P_{p,p \vee q', p/2}(\mathbf{x}, V) \\ (y, z) &\mapsto \int y \, dz \end{aligned}$$

such that  $(\int y \, dz)_s^\dagger = y_s z_s^\dagger$ ,

$$\left| \int_s^t y_r \, dz_r - y_s z_{s,t} - y_s^\dagger z_{s,t}^b \right| \leq K \|y\|_{\mathbf{x}} \|z\|_{\mathbf{x}} \|\mathbf{x}\|_p \omega_{s,t}^\theta \quad (13)$$

and

$$\begin{aligned} \left\| \int_{\mathbf{x}} y \, dz \right\|_{\mathbf{x}} &\leq (|y_0| + \omega_{0,T}^{\theta-1/p} \|y\|_{\mathbf{x}}) \|z\|_{\mathbf{x}} \|\mathbf{x}\|_p \\ &+ (|y_0^\dagger| + K(\omega_{0,T}^{\theta-1/p-1/q} + \omega_{0,T}^{\theta-2/p}) \|y\|_{\mathbf{x}}) \|z\|_{\mathbf{x}} \|\mathbf{x}\|_p \end{aligned} \quad (14)$$

for some universal constant  $K$ .

*Proof.* We set  $Y_{s,t} := y_s z_{s,t} + y_s^\dagger z_{s,t}^b$ . Thus, for any  $r \leq s \leq t$ ,

$$Y_{r,s} + Y_{s,t} - Y_{r,t} = y_{r,s}^\# z_{s,t} + y_{r,s}^\dagger z_{s,t}^b.$$

The existence of the integrals follows from the Additive Sewing Lemma [19]. Therefore, the inequalities (13) and (14) are straightforward.  $\square$

From now, let us fix  $2 \leq p < 3$ ,  $q, r \geq 1$  as well as  $0 < \gamma \leq 1$  and  $0 < \kappa < 1$ . For two Banach spaces  $V$  and  $W$  and a rough path  $\mathbf{x} \in \mathbb{R}_p(U)$ , we set

$$\begin{aligned} PV &:= P_{p,q,r}(\mathbf{x}, V), \quad P^*V := \{y \in PV \mid (y_0, y_0^\dagger) = (0, 0)\} \\ QW &:= P_{p, \frac{q\vee p}{\kappa\gamma}, \frac{r\vee p}{1+\kappa\gamma}}(\mathbf{x}, W) \text{ and } Q^*W := \{y \in QW \mid (y_0, y_0^\dagger) = (0, 0)\}. \end{aligned}$$

The spaces  $P^*V$  and  $Q^*W$  are Banach sub-spaces of  $PV$  and  $QW$ .

**Proposition 5** (The Omega lemma for CRP). *Assume that  $f \in \mathcal{C}^{k+1+\gamma}(V, W)$ . Then  $\mathfrak{D}f := (f(y_t))_{t \in [0, T]}$  is of class  $\mathcal{C}^{k+\bar{\kappa}\gamma}$  from  $PV$  to  $QW$  with  $\mathfrak{D}f(y)^\dagger = \nabla f(y)y^\dagger$ . Besides,*

$$D\mathfrak{D}f(y) \cdot z = (D^i f(y)_t \cdot z_t)_{t \in [0, T]} \in QW, \quad \forall y, z \in PV.$$

In addition, if  $z \in P^*V$ , then  $D\mathfrak{D}f(y)z \in Q^*W$  for any  $y \in PV$ .

*Proof.* I) Let us prove first that  $\mathfrak{D}f$  maps  $PV$  to  $P_{p,p/\gamma, r\vee p/(1+\gamma)}(\mathbf{x}, W)$ . As the latter space is continuously embedded in  $QW$ , this proves that  $\mathfrak{D}f$  maps  $PV$  to  $QW$ .

Set for  $0 \leq s \leq t \leq T$ ,

$$\begin{aligned} Y_t &:= f \circ y_t, \quad Y_t^\dagger := \nabla f(y_t)y_t^\dagger \\ \text{and } Y_{s,t}^\# &:= \nabla f(y_s)y_{s,t}^\# + \int_0^t (\nabla f(y_s + \theta y_{s,t}) - \nabla f(y_s))y_{s,t} \, d\theta. \end{aligned}$$

Clearly,  $Y_{s,t} = Y_s^\dagger x_{s,t}^1 + Y_{s,t}^\#$ . Besides,

$$\begin{aligned} |Y_{s,t}^\#| &\leq \|\nabla f\|_\infty \|y\|_{\mathbf{x}} \omega(s, t)^{1/r} + H_\gamma(\nabla f) \|y\|_p^{1+\gamma} \omega(s, t)^{(1+\gamma)/p}, \\ |Y_{s,t}^\dagger| &\leq H_\gamma(\nabla f) |y_{s,t}|^\gamma \cdot \|y^\dagger\|_\infty + \|\nabla f\|_\infty \cdot |y_{s,t}^\dagger|. \end{aligned}$$

We deduce that  $Y \in P_{p,p/\gamma,r\wedge p/(1+\gamma)}^{0,0}(\mathbf{x}, W)$  with

$$\|Y\|_{\mathbf{x}} \leq C_T \|f\|_{1+\gamma} \max\{\|y\|_{\mathbf{x}}, \|y\|_{\mathbf{x}}^{1+\gamma}\},$$

for a constant  $C_T$  that depends only on  $\omega_{0,T}$  and  $p$ .

II) In particular, setting  $f(y, z) = yz$  (resp.  $f(y, z) = y \otimes z$ ) for  $y \in PL(W, V)$  (resp.  $y \in PV$ ) and  $z \in PW$  shows that  $yz \in PV$  (resp.  $y \otimes z \in PW \otimes V$ ). Moreover,  $(yz)_t^\dagger = y_t^\dagger z_t + y_t z_t^\dagger$  (resp.  $(y \otimes z)_t^\dagger = y_t^\dagger \otimes z_t + y_t \otimes z_t^\dagger$ ) for any  $t \in [0, T]$ .

In particular,  $z_0 = 0$  and  $z_0^\dagger = 0$  implies that  $(yz) \in P^*V$  (resp.  $y \otimes z \in P^*W \otimes V$ ) when  $z \in P^*W$ .

In addition, it is easily obtained that for the product  $y \cdot z = yz$  or  $y \cdot z = y \otimes z$

$$\|xy\|_{\bullet\mathbf{x}} \leq C \|x\|_{\bullet\mathbf{x}} \|y\|_{\bullet\mathbf{x}}$$

for some universal constant  $C$ .

III) We consider that  $f \in \mathcal{C}^{1+\gamma}(V, W)$ . Let  $y, z \in PV$  and set  $Y_t := f(y_t)$ ,  $Z_t := f(z_t)$ . Thus

$$\begin{aligned} & Z_{s,t}^\dagger - Y_{s,t}^\dagger \\ &= (\nabla f(z_t) - \nabla f(z_s) - \nabla f(y_t) + \nabla f(y_s)) z_t^\dagger + (\nabla f(z_s) - \nabla f(y_s)) z_{s,t}^\dagger + \nabla f(y_s) (z_{s,t}^\dagger - y_{s,t}^\dagger). \end{aligned}$$

Applying (7) in Lemma 1, for  $0 < \kappa < 1$ ,

$$\begin{aligned} |Z_{s,t}^\dagger - Y_{s,t}^\dagger| &\leq H_\gamma(\nabla f) \|z - y\|_\infty^{\bar{\kappa}\gamma} (\|z\|_p^{\kappa\gamma} + \|y\|_p^{\kappa\gamma}) \omega_{s,t}^{\kappa\gamma/q} \|z^\dagger\|_\infty \\ &\quad + H_\gamma(\nabla f) \|z - y\|_\infty^\gamma \|z^\dagger\|_q \omega_{s,t}^{\gamma/q} + \|\nabla f\|_\infty \|z^\dagger - y^\dagger\|_q \omega_{s,t}^{\gamma/q}. \end{aligned}$$

From this,

$$\begin{aligned} \frac{|Z_{s,t}^\dagger - Y_{s,t}^\dagger|}{\omega_{s,t}^{\kappa\gamma/q}} &\leq K_p^{\bar{\kappa}\gamma} K_q H_\gamma(\nabla f) \|z - y\|_{\mathbf{x}}^{\bar{\kappa}\gamma} (\|z\|_{\mathbf{x}}^{\kappa\gamma} + \|y\|_{\mathbf{x}}^{\kappa\gamma}) \|z\|_{\mathbf{x}} \\ &\quad + K_p^\gamma H_\gamma(\nabla f) \|z - y\|_{\mathbf{x}}^\gamma \|z\|_{\mathbf{x}} \omega_{[0,T]}^{\bar{\kappa}\gamma/q} + \|\nabla f\|_\infty \omega_{0,T}^{\bar{\kappa}\gamma/q} \|z - y\|_{\mathbf{x}}^\gamma. \quad (15) \end{aligned}$$

Besides,

$$\begin{aligned} Z_{s,t}^\# - Y_{s,t}^\# &= \nabla f(z_s) z_{s,t}^\# - \nabla f(y_s) y_{s,t}^\# + \int_0^1 (\nabla f(y_s + \tau y_{s,t}) - \nabla f(y_s)) (z_{s,t} - y_{s,t}) d\tau \\ &\quad + \int_0^t \mathcal{D}_2 \nabla f(z_s + \tau z_{s,t}, z_s, y_s + \tau y_{s,t}, z_s) y_{s,t} d\tau. \end{aligned}$$

With Lemma 1,

$$\begin{aligned}
|Z_{s,t}^\# - Y_{s,t}^\#| &\leq \|z\|_{\mathbf{x}} H_\gamma(\nabla f) \|z - y\|_\infty^\gamma \omega_{s,t}^{1/r} \\
&\quad + H_\gamma(\nabla f) \|z - y\|_{\mathbf{x}} \omega_{s,t}^{\gamma/q+1/r} + 2\omega_{s,t}^{1/p} \|\nabla f\|_\infty \|y - z\|_p \\
&\quad + 4^{\bar{\kappa}\gamma} \|y\|_q \omega_{s,t}^{(1+\bar{\kappa}\gamma)/p} \|z - y\|_\infty^{\bar{\kappa}\gamma} (\|z\|_p^{\bar{\kappa}\gamma} + \|y\|_p^{\bar{\kappa}\gamma}). \quad (16)
\end{aligned}$$

For any  $y \in \text{PV}$ ,

$$\begin{aligned}
\|y\|_\infty &\leq |y_0| + \omega_{0,T}^{1/p} \|y\|_p \|x\|_p + \omega_{0,T}^{1/r} \|y\|_{\mathbf{x}} \\
&\leq |y_0| + |y_0^\dagger| \omega_{0,T}^{1/p} \|x\|_p + \|x\|_p \|y\|_{\mathbf{x}} \omega_{[0,T]}^{1/q+1/p} + \|y\|_{\mathbf{x}} \omega_{0,T}^{1/r}.
\end{aligned}$$

Using the latter inequality, (16) and (15) could be summarized as

$$\|Z - Y\|_{\mathbf{x}} \leq K \|f\|_{1+\gamma} \|x\| (|z_0 - y_0| + |z_0^\dagger - y_0^\dagger| + L(\omega_{0,T}) \|z - y\|_{\mathbf{x}}^{\bar{\kappa}\gamma})$$

where  $K$  is a constant which depends on  $\|z\|_{\bullet\mathbf{x}}$ ,  $\|y\|_{\bullet\mathbf{x}}$ ,  $(p, q, r)$ , and  $L$  is a continuous increasing function which depends only on  $p, q, r$ .

Moreover,  $|f(y_0) - f(z_0)| \leq \|\nabla f\|_\infty |y_0 - z_0|$  and

$$|f(y_0)^\dagger - f(z_0)^\dagger| \leq H_\gamma(\nabla f) |y_0 - z_0| + \|\nabla f\|_\infty |y_0^\dagger - z_0^\dagger|.$$

Up to changing  $K$ , we get a similar inequality as  $\|Z - Y\|_{\mathbf{x}}$  is replaced by  $\|Z - Y\|_{\bullet\mathbf{x}}$ .

We have then proved that  $\mathfrak{D}f$  is locally of class  $\mathfrak{C}^{\bar{\kappa}\gamma}$  from PV to QW.

IV) For dealing with the general case  $f \in \mathfrak{C}^{k+1+\gamma}(\mathbb{V}, \mathbb{W})$ , we apply the converse of the Taylor theorem as in the proof of Proposition 2, using II), the proof being in all points similar.  $\square$

When considering a fixed point for  $y \mapsto \mathfrak{F}(y, z, f) := \int f(y) dz$ ,  $\mathfrak{F}$  should map PV to PV. Owing to Propositions 4 and 5, a suitable choice is

$$q = q' = p, \quad r = r' = p/2.$$

From now, use these values  $(q', r') = (p, p/2)$ .

We state now existence and uniqueness results on solutions of RDE, whose proof may be found, up to a straightforward modification for dealing with  $b \neq 0$ , in [20, 24].

**Proposition 6.** *For any  $f \in \mathfrak{C}^{k+1+\gamma}(\mathbb{V}, L(\mathbb{W}, \mathbb{V}))$  with  $k \geq 0$ , any  $z \in \text{PW}$  and any  $a \in \mathbb{V}$ ,  $b \in L(\mathbb{W}, \mathbb{V})$ , then there exists a unique CRP  $y \in \text{PV}$  which solves*

$$y_t = a + \int_0^t f(y_s) dz_s + bz_{0,t} \quad \text{with } y_t^\dagger = f(y_t) z_t^\dagger + bz_t^\dagger \quad (17)$$

for any  $t \in [0, T]$ . Besides, when  $|a| + |b| \leq R$ , then such that  $y$  belongs to a closed ball of PV whose radius depends only on  $\|f\|_{k+1+\gamma}$ ,  $\omega_{0,T}$ ,  $p$ ,  $\gamma$ ,  $\|z\|_{\mathbf{x}}$  and  $R$ .

Let us define

$$X_{k,\gamma} := V \times L(V, W) \times \mathcal{C}^{k+1+\gamma}(V, L(W, V)) \times \text{PW}.$$

We then define the *Itô map*  $\mathfrak{I}$  as the map sending  $(a, b, f, z) \in X_{k,\gamma}$  to  $(y, y^\dagger)$  given by (17). Besides, (17) is equivalent in finding  $y^* \in P^*V$  which solves

$$y_t^* = \int_0^t f(y_s^* + a + bz_{0,s}) dz_s$$

and then to set  $y_t := y_t^* + a + bz_{0,t}$ .

The proof of the regularity result is now in all points similar to the one for paths of finite  $p$ -variation so that we skip it.

**Theorem 3.** *Under the above conditions, the Itô map  $\mathfrak{I} : X_{k,\gamma} \rightarrow \text{PV}$  is locally of class  $\mathcal{C}^{k+\bar{\kappa}\gamma}$  for any  $\bar{\kappa} \in (0, 1)$  with  $1 + \bar{\kappa}\gamma > p$ .*

For  $t \in [0, T]$ , let  $\mathbf{e}_t$  be the evaluation map  $\mathbf{e}_t(y, y^\dagger) = y_t$  (this choice forces the value  $y_t^\dagger = f(y_t)$ ).

The proof of the next result is in all points similar to the one of Corollary 1.

**Corollary 2.** *Under the hypotheses of Theorem 3, for any  $t > 0$ ,  $\mathbf{f}_t(a) := \mathbf{e}_t \circ \mathfrak{I}(a, 0, f, 0)$  is locally a  $\mathcal{C}^{k+\bar{\kappa}\gamma}$ -diffeomorphism from  $V$  to  $V$ .*

Since  $P_{p,p,p/2}(\mathbf{x}, V)$  is continuously embedded in  $C_p(V)$ , we could consider this approach for studying the regularity of

$$y_t = a + \int_0^t f(y_s) dz_s + \int_0^t g(y_s) dh_s + b_t, \quad t \geq 0, \quad (18)$$

where for a rough path  $\mathbf{x}$ ,  $z \in P_{p,p,p/2}(\mathbf{x}, W)$ ,  $h \in C_q(W')$ ,  $f$  and  $g$  are maps respectively of class  $\mathcal{C}^{k+1+\gamma}$  from  $V$  to  $L(W, V)$  and of class  $\mathcal{C}^{k+\delta}$  from  $V$  to  $L(W', V)$ , provided that

$$\frac{1}{p} + \frac{1}{q} > 1, \quad 1 + \gamma > p \quad \text{and} \quad 1 + \delta > q \quad \text{for} \quad 0 < \gamma, \delta \leq 1.$$

The map  $\mathfrak{I}(f, g, b, y_0, y_0^\dagger)$  giving the solution to (18) is then of class  $\mathcal{C}^{k+(1-\bar{\kappa})\min\{\delta,\gamma\}}$ .

Using for  $z$  the decomposition  $z^\dagger = 1$  and  $z^\sharp = 0$ , and replacing the vector field  $f$  by  $\epsilon f$ , it is easily seen that we may consider the problem

$$y_t^\epsilon = a + \int_0^t f(y_s^\epsilon) d\mathfrak{d}_\epsilon \mathbf{x}_s + \int_0^t g(y_s^\epsilon) dh_s \quad \text{for} \quad a = (y_0, y_0^\dagger) \quad \text{given.}$$

Asymptotic expansions in  $\epsilon$  can then be performed as in [5, 30–33].

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