Sensitivity of rough differential equations

Laure Coutin, Antoine Lejay

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Laure Coutin† Antoine Lejay‡

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Abstract

In the context of rough paths theory, we study the regularity of the Itô map with respect to the starting point, the coefficients and perturbation of the driving rough paths. In particular, we show that the Itô map is differentiable with a Hölder or Lipschitz continuous derivatives under general hypotheses. With respect to the current literature on the subject, our proof relies on perturbation of linear Rough Differential Equations, and the Hölder regularity of the Itô map is established.

Keywords: rough paths, rough differential equation; Itô map; Malliavin calculus; flow of diffeomorphisms.

1 Introduction

The theory of rough paths is now a standard tool to deal with stochastic differential equations (SDE) driven by continuous processes other than the Brownian motion such as the fractional Brownian motion. Even for standard SDE, it has been proved to be a convenient tool for dealing with large deviations or for numerical purposes. We refer the reader to [12, 18, 20, 22, 35, 36, 39, 40] for a presentation of this theory with several different points of view.

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†Institut de Mathématiques de Toulouse, F-31062 Toulouse Cedex 9, France. E-mail: `laure.coutin@math.univ-toulouse.fr`

‡Université de Lorraine, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54600, France
CNRS, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54600, France
Inria, Villers-lès-Nancy, F-54600, France
E-mail: `Antoine.Lejay@univ-lorraine.fr`

A *rough path* $\mathbf{x}$ is an extension in a non-commutative tensor space of a path of finite p-variation controlled by ω, that is a continuous path x from $[0,T]$ with values in a Banach space U with $|x_t - x_s| \leq C\omega(s,t)^{1/p}$ for a control ω (See Section 2 for a definition). This extension $\mathbf{x}$ is defined through algebraic and analytic properties but is not uniquely associated to x. Here, we consider only values of p in $[2,3)$. For $p<2$, then the theory is simpler and relies on Young integrals [34].

The idea is that given a rough path $\mathbf{x}$ and a vector field $f : \mathrm{V} \to L(\mathrm{U},\mathrm{V})$ for a Banach space V, a controlled differential equation

$$y_t = a + \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s \tag{1}$$

is well defined provided that f is regular enough. This equation is called a *rough differential equation* (RDE). For a smooth path x, a rough path $\mathbf{x}$ could be naturally constructed using the iterated integrals of x. In this case, the solution to (1) corresponds to the solution to the ordinary differential equation $y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s$. The theory of rough paths provides us with natural extensions of controlled differential equations.

The map $\mathfrak{I} : x \mapsto y$, called the *Itô map*, is continuous when one uses the topology induced by the p-variation distance with respect to ω (one may also use $1/p$-Hölder norms when $\omega(s,t) = t-s$). In particular, it may be shown that $\mathfrak{I}$ is locally Lipschitz continuous with respect to the path, the vector field and the starting point [*]lejay1,lejay2.

However, one may be willing to study deeper the differentiability properties in the Itô map. This point is very important toward applications. For SDE, Malliavin calculus opens the door to existence of a density and its regularity [44,46], large deviation results [28], Monte Carlo methods [21,44], ...

There are several ways to consider a perturbation of the Itô map:

- ⋆ Perturbation of the starting point.
- ⋆ Perturbation of the rough path $\mathbf{x}$ by a path h of finite q-variation with $1/p + 1/q > 1$, or of finite p-variation, provided that a cross-iterated integral exists between h and $\mathbf{x}$.
- ⋆ Perturbation of the parameter λ when we consider a family of vector fields $f(\cdot,\lambda)$ with a regular variation.

The flow properties have already been subject to many articles:

- ⋆ In [42], T. Lyons and Z. Qian have studied the flow property for solutions to $y_t = a + \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s + \int_0^t g(y_s)\,\mathrm{d}h_s$ for a "regular" path h subject to a perturbation.
- ⋆ In [41], T. Lyons and Z. Qian showed that the Itô map provides a flow of diffeomorphisms when the driving rough path is geometric.

- ⋆ In [2], I. Bailleul give an approach which is a non-linear interpretation of [17], where the RDE is constructed through its flow.
- ⋆ In the case $p < 2$, the Itô map has a Hölder continuous derivative [34,38].
- ⋆ T. Lyons and Z. Qian [43], S. Aida [1] and more recently Z. Qian and J. Tudor [48] have studied the perturbation of the Itô map when the rough path is perturbed by a regular path h and the structure of the resulting tangent space.
- ⋆ The book of P. Friz and N. Victoir [20] also presents flow properties for geometric rough paths.
- ⋆ Y. Inahama and H. Kawabi have studied various aspects of stochastic Taylor developments of solutions to $y^\epsilon_t = a + \int_0^t \epsilon f(y^\epsilon_s)\,\mathrm{d}\mathbf{x}_s + \int_0^t g(y^\epsilon_s)\,\mathrm{d}h_s$ in term of ϵ around the solutions to $z_t = a + \int_0^t g(z_s)\,\mathrm{d}h_s$ [28–31]. See also [50] for some bounds and applications to the fractional Brownian motion.
- ⋆ The special case of SDE driven by fractional Brownian motion and Gaussian processes has been subject to a lot of attentions [4–9,11,15,16,24–26,37,45,47]. Since integrability is the key to derive some integration by parts formula in Malliavin type calculus, several articles deal with moments estimates for solutions to linear equations driven by Gaussian rough paths [3,10,19,23,27,49].

In this article, we establish regularity properties of the Itô map in a general setting. In particular, our approach is "rough" in the sense that we do not rely on approximations by smooth paths as in [20] or [48]. Our approach may then be applied for any for non-geometric rough paths as well as in infinite dimension. Besides, it provides some general regularity results in the case where $p > 2$ on the derivative of the Itô map.

In particular, we show that the Itô map $\mathfrak{I}$ which transforms a rough path $\mathbf{x}$ of finite p-variation with $p \in [2, 3)$ into the solution to $y^{\lambda,a}_t = a + \int_0^t f(y^{\lambda,a}_s, \lambda)\,\mathrm{d}\mathbf{x}_s$ is Fréchet differentiable with a locally Hölder or Lipschitz continuous derivative with respect to a and λ, according to the regularity of f. Although already known for $p \in [1, 2)$, the Hölder regularity of the derivative of the Itô map for $p \in [2, 3)$ is new up to our knowledge.

The key idea is the show that the difference between two solution is itself solution to perturbed linear RDE and the to use the theory developed in the companion paper [13]. By this, we mean that

$$y^{\mu,b}_t - y^{\lambda,a}_t = (b-a) + \int_0^t \mathrm{d}\mathsf{A}^{\lambda,a}_s(y^{\mu,b}_s - y^{\lambda,a}_s) + P(a,b,\lambda,\mu)_t + R(a,b,\lambda,\mu)_t \quad (2)$$

for an operator valued rough path $\mathsf{A}^{\lambda,a}$ and paths $P(a, b, \lambda, \mu)$ and $R(a, b, \lambda, \mu)$, with, roughly speaking, the p-variation norm of $R(a, b, \lambda, \mu)$ being of order $|b - a|^\delta + |\mu - \lambda|^\delta$ for some $\delta > 1$ that depends on the regularity of $\mathbf{x}$ and f.

We endow that to construct the solution to (2), it is necessary to know not only $P(a, b, \lambda, \mu)$ but also some equivalent in some sense of the cross-iterated integrals between $P(a, b, \lambda, \mu)$ and $\mathsf{A}^{\lambda,a}$.

Hence, (2) is then identified with a perturbed linear RDE

$$\Delta_t = b - a + \int_0^t \mathrm{d}\mathsf{A}_s^{\lambda,a} \Delta_s + P(a, b, \lambda, \mu)_t + R(a, b, \lambda, \mu)_t$$

to which we subtract the solution to

$$N(a, b, \lambda, \mu)_t = b - a + \int_0^t \mathrm{d}\mathsf{A}_s^{\lambda,a} N(a, b, \lambda, \mu)_s + P(a, b, \lambda, \mu)_t.$$

Using the Duhamel principle constructed in [13], the control over the norm of $R(a, b, \lambda, \mu)$ is transferred as a control over the norm of $\Delta_t - N(a, b, \lambda, \mu)$ with $\Delta := y_t^{\mu,b} - y^{\lambda,a}$.

The main points of this approach is then it identify and control the terms in (2). To do, we actually mix several of the approaches (T. Lyons, A.M. Davie and M. Gubinelli) to take benefits of the best of each approach in terms of control. We also complete the results in [13] by stating the continuity result on $N(a, b, \lambda, \mu)$.

Outline. In Section 2, we introduce some notations. Our main results are presented in Section 3. Section 4, we present the sketch of the proof and the intermediate results, which are proved in Section 5. Finally, Theorems 2 and 1 are proved in Sections 6 and 7. In Appendix A, we summarized some results about controlled rough path which we use through all this article.

2 Notations

Let U and V be Banach spaces. The norms on such spaces are denoted by $|\cdot|$. On the tensor products of type $\mathrm{U} \otimes \mathrm{V}$, we use a norm $|\cdot|$ which is such that $|u \otimes v| \leq |u| \cdot |v|$ for $(u, v) \in \mathrm{U} \times \mathrm{V}$.

Let us fix a time horizon $T > 0$ and set $\mathbb{T} := \{(s, t) \,|\, 0 \leq s \leq t \leq T\}$.

A *control* is a function ω from $\mathbb{T}$ to $\mathbb{R}_+$ which is continuous close to the diagonal $\{(s, s) \,|\, s \in [0, T]\}$ and super-additive, that is $\omega(r, s) + \omega(s, t) \leq \omega(r, t)$ for any $0 \leq r \leq s \leq t \leq T$.

A path $x : [0, T] \to \mathrm{U}$ is said to be of *finite p-variation controlled by* ω if $x_{s,t} := x_t - x_s$ satisfies $|x_{s,t}| \leq C\omega(s, t)^{1/p}$ for all $(s, t) \in \mathbb{T}$. The best constant such that this inequality holds is denoted by $\|x\|_p$.

If $(x_{s,t})_{(s,t)\in\mathbb{T}}$ satisfies $|x_{s,t}| \leq C\omega(s, t)^{1/p}$ for some constant $C < +\infty$, then the best constant such that this inequality holds is still denoted by $\|x\|_p$.

Let $\mathbf{x}$ be a rough path of finite p-variation that lies above a path x with values in a Banach space U with $p \in [2, 3)$. It is a path with values in the truncated tensor algebra $T_2(\mathrm{U}) := \mathbb{R} \oplus \mathrm{U} \oplus (\mathrm{U} \otimes \mathrm{U})$. The component of $\mathbf{x}$ in U is denoted by x^1, while its component in $\mathrm{U} \otimes \mathrm{U}$ is denoted by x^2. We set

$$\|\mathbf{x}\|_p := \sup_{(s,t)\in\mathbb{T},\ s\neq t} \max\left\{\frac{|x^1_{s,t}|}{\omega(s,t)^{1/p}}, \frac{|x^2_{s,t}|}{\omega(s,t)^{2/p}}\right\} = \max\{\|x^1\|_p, \|x^2\|_{p/2}\}.$$

Given a path $y : [0, T] \to \mathrm{V}$ and a rough path $\mathbf{x} : [0, T] \to T_2(\mathrm{U})$, we denote by $y\ltimes\mathbf{x} : \mathbb{T} \to \mathrm{V} \otimes \mathrm{U}$ the *cross-iterated integral* between y and $\mathbf{x}$, when it exists. By definition, it is a family which satisfies $\|y\ltimes\mathbf{x}\|_{p/2} < +\infty$ and

$$y\ltimes\mathbf{x}_{r,t} = y\ltimes\mathbf{x}_{r,s} + y\ltimes\mathbf{x}_{s,t} + y_{r,s} \otimes \mathbf{x}_{s,t} \text{ for all } 0 \leq r \leq s \leq t \leq T.$$

Using the framework of controlled rough paths [18, 22], $y\ltimes\mathbf{x}$ exists when y is a rough path controlled by $\mathbf{x}$.

Using the viewpoint of *controlled rough paths* introduced first by M. Gubinelli [22] (See also [18]), we have mainly to focus on the “first level” and took benefits of a canonical way to construct cross-iterated integrals and rough lifts. The basic idea behind a controlled rough path y is that its increments “looks like” the ones of a p-rough path $\mathbf{x}$. More precisely, a *controlled rough path* is a path such that

$$y_{s,t} = y^\dagger_s x^1_{s,t} + y^\sharp_{s,t} \tag{3}$$

$$\text{with } \|y\|_{\mathbf{x}} := |y_0| + |y^\dagger_0| + \sup_{\substack{(s,t)\in\mathbb{T}\\ s\neq t}} \left\{\frac{|y^\dagger_{s,t}|}{\omega(s,t)^{1/p}} + \frac{|y^\sharp_{s,t}|}{\omega(s,t)^{2/p}}\right\} < +\infty. \tag{4}$$

Let us denote by $\mathrm{P}(T, \mathbf{x}, \mathrm{V})$ the space of paths $y : [0, T] \to \mathrm{V}$ satisfying (3) and (4). With $\|\cdot\|_{\mathbf{x}}$, this is a Banach space. The main properties of controlled rough paths for our purpose are presented in Appendix A.

The solution to $y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s$ we consider is a path from $[0, T]$ to V, obtained for example from the construction of A.M. Davie [14] which coincide when T is finite with the original one of T. Lyons in which a solution is defined as a full rough path lying above $(\mathbf{x}, y)$ [39, 40]. When T is finite, the two notions coincide [32, 33] It is then a controlled rough path with $y^\dagger = f(y_s)$, and the cross-iterated integrals between y and x can then be canonically constructed.

3 Main result: properties of the derivative of the Itô map

Provided that a map $f : \mathrm{V} \to L(\mathrm{U}, \mathrm{V})$ satisfies appropriate conditions, there exists a unique solution to the rough differential equation (RDE) $y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}\mathbf{x}_s$ up to any time T. For this, we assume either

> (**A**) The vector field f is continuous and bounded with bounded first and second derivatives, and $\nabla^2 f$ is γ-Hölder continuous with $2+\gamma > p$;

or

> (**B**) The vector field f is continuous with bounded first and second derivative, and $f\nabla f$ has a bounded derivative which is γ-Hölder continuous with $2 + \gamma > p$.

Although these two sets of hypotheses lead to two different notions of solutions, they coincide when T is finite [32, 33].

Let us consider now that $f = f(\cdot, \lambda)$ depends itself of a parameter λ taken in a convex subset Λ_{p} of a Banach space Λ.

We assume that each $f(\cdot, \lambda)$ satisfies either (**A**) or (**B**) to ensure the existence of a unique solution $y^{\lambda,a}$ to $y_t^{\lambda,a} = a + \int_0^t f(y^{\lambda,a}, \lambda)\,\mathrm{d}\mathbf{x}_s$, where a belongs to a convex set Λ_{s} of V, and $\lambda \in \Lambda_{\mathrm{p}}$.

Provided that the controls on $f(\cdot, \lambda)$ are uniform in λ, in view of the controls on the Itô map, for a solution on the time interval $[0, T]$, we assume without loss of generalities that

> (**H**) There exists a constant $\rho > 0$ such that $\|y^{\lambda,a}\|_p \leq \rho$ and $\|y^{\lambda,a}\|_\infty \leq \rho$ for $(\lambda, a) \in \Lambda_{\mathrm{p}} \times \Lambda_{\mathrm{s}}$.

For a function g on $\mathrm{V} \times \Lambda_{\mathrm{p}}$, we write

$$\|g\|_\infty := \sup_{|y|\leq\rho, \lambda\in\Lambda_{\mathrm{p}}} |g(y, \lambda)| \text{ and } H_\gamma(g) := \sup_{|x|,|y|\leq\rho} \sup_{\lambda\in\Lambda_{\mathrm{p}}} \frac{|g(x, \lambda) - g(y, \lambda)|}{|x - y|^\gamma}$$

$$\text{and } H_\beta^{(\lambda)}(g) = \sup_{\mu,\lambda\in\Lambda_{\mathrm{p}}} \sup_{|y|\leq\rho} \frac{|g(y, \mu) - g(y, \lambda)|}{|\mu - \lambda|^\beta}.$$

Our main hypotheses on the regularity of f will be either

> ($\mathbf{H}_{①}$) The function f is of class $\mathscr{C}^{3,2}(\mathrm{V} \times \Lambda_{\mathrm{p}}, L(\mathrm{U}, \mathrm{V}))$ and for some $\gamma \in (0, 1]$ with $2 + \gamma > p$, $H_\gamma(\nabla^3_{yyy} f)$, $H_\gamma(\nabla^3_{\lambda yy} f)$, $\|f\|_\infty$, $\|\nabla_y f\|_\infty$, $\|\nabla^2_{yy} f\|_\infty$, $\|\nabla^3_{yyy} f\|_\infty$, $\|\nabla_\lambda f\|_\infty$, $\|\nabla^2_{\lambda y} f\|_\infty$, $\|\nabla^3_{\lambda yy} f\|_\infty$, $\|\nabla^3_{\lambda\lambda y} f\|_\infty$ and $\|\nabla^4_{\lambda\lambda yy} f\|_\infty$ are finite.

or

($\mathbf{H}_{②}$) The function f is of class $\mathscr{C}^{2,1}(\mathrm{V}\times\Lambda_{\mathrm{p}}, L(\mathrm{U},\mathrm{V}))$ and for some $\beta\in(0,1]$ and $\gamma\in(0,1]$ with $2+\gamma>p$, $H_\gamma(\nabla^2_{yy}f)$, $H_\gamma(\nabla^2_{y\lambda}f)$, $\|f\|_\infty$, $\|\nabla_y f\|_\infty$, $\|\nabla^2_{yy}f\|_\infty$, $\|\nabla_\lambda f\|_\infty$, $H^{(\lambda)}_\beta(\nabla_\lambda f)$, $H^{(\lambda)}_\beta(\nabla^2_{\lambda y}f)$, $\|\nabla^2_{y\lambda}f\|_\infty$, $H_\gamma(\nabla^2_{y\lambda}f)$ and $H^{(\lambda)}_\beta(\nabla^3_{\lambda yy}f)$ are finite.

A statement starting by ① means that we work under ($\mathbf{H}_{①}$), while a statement starting by ② means that we work under ($\mathbf{H}_{②}$).

Let us fix $\kappa=0$ under ($\mathbf{H}_{①}$) and $\kappa\in(0,1)$ with $2+\kappa\gamma>p$ under ($\mathbf{H}_{②}$). We write $\overline{\kappa}:=1-\kappa$.

Definition 1. We say that a constant depends only on the structure when it depends on ρ, the norms of f and its derivatives (uniformly over $(a,\lambda)\in\Lambda_{\mathrm{s}}\times\Lambda_{\mathrm{p}}$), κ, T, p, γ and ω. We write $a\preccurlyeq b$ to say that $a\leq Cb$ where C depends only on the structure.

The main results of this article are presented in Theorem 1 and Theorem 2 below. Actually, the proof of Theorem 1 relies partially on Theorem 2.

Theorem 1. *For any $(a,b,\lambda,\mu)\in\Lambda^2_{\mathrm{s}}\times\Lambda^2_{\mathrm{p}}$, the solution to the perturbed linear RDE*

$$N(a,b,\lambda,\mu)_t = b-a+\int_0^t \nabla_y f(y^{\lambda,a}_s,\lambda)N(a,b,\lambda,\mu)_s\,\mathrm{d}\mathbf{x}_s + \int_0^t \nabla_\lambda f(y^{\lambda,a}_s,\lambda)(\mu-\lambda)\,\mathrm{d}\mathbf{x}_s \quad (5)$$

satisfies for any $\kappa\in(0,1)$ with $2+\kappa\gamma>p$,

$$\|y^{\mu,b}-y^{\lambda,a}-N(a,b,\lambda,\mu)\|_p \preccurlyeq |b-a|^\delta+|\mu-\lambda|^\delta$$

with

$$\delta:=\begin{cases}2 & \text{under } (\mathbf{H}_{①}),\\ 1+\overline{\kappa}(\beta\wedge\gamma) & \text{under } (\mathbf{H}_{②}).\end{cases} \quad (6)$$

Using the framework developed in [13], let us consider the family $(\mathsf{A}^{\lambda,a}_{t,s})_{(s,t)\in\mathbb{T}}$ of linear operators from V to V defined by

$$\mathsf{A}^{\lambda,a}_{t,s}[z] := z+\nabla_y f(y^{\lambda,a}_s,\lambda)[z]\mathbf{x}^1_{s,t} + \nabla^2_{yy}f(y^{\lambda,a}_s,\lambda)[z]y^{\lambda,a}\ltimes\mathbf{x}_{s,t}+\nabla_y f(y^{\lambda,a}_s,\lambda)\nabla^2_{yy}f(y^{\lambda,a}_s,\lambda)[z]\mathbf{x}^2_{s,t} \quad (7)$$

for $z\in\mathrm{V}$. This linear operator is an almost rough resolvent, from which we construct the solution to the operator-valued linear RDE $\mathsf{L}^{\lambda,a}_t = \mathsf{Id}+$

$\int_0^t \nabla f(y_s^{\lambda,a})\mathsf{L}^{\lambda,a}\,\mathrm{d}\mathbf{x}_s$. In particular, $\mathsf{A}_{t,s}^{\lambda,a}$ is an approximation of $\mathsf{L}_t^{\lambda,a}(\mathsf{L}^{\lambda,a})_s^{-1}$ in short time and carry enough information to reconstruct $(\mathsf{L}_t^{\lambda,a})_{t\in[0,T]}$.

Let also $(\mathsf{B}_{t,s}^{\lambda,a})_{(s,t)\in\mathbb{T}}$ be a family of linear operators on Λ_{p} defined by

$$\mathsf{B}_{t,s}^{\lambda,a}\nu := \nabla_\lambda f(y_s^{\lambda,a},\lambda)\nu\mathbf{x}_{s,t}^1,\ \ \nu\in\Lambda_{\mathrm{p}}. \tag{8}$$

This term will be an approximation of the perturbation of the linear equation, and we will show that it could be employed in the Duhamel principle.

Theorem 2. *Let $\mathsf{A}^{\lambda,a}$ and $\mathsf{B}^{\lambda,a}$ be defined by (7) and (8) above.*

1) *The family $(\mathsf{A}_{t,s}^{\lambda,a})_{(s,t)\in\mathbb{T}}$ is an almost rough resolvent with a left rough resolvent $(\mathsf{L}_{t,s}^{\lambda,a})_{(s,t)\in\mathbb{T}}$. Besides, $(\mathsf{L}_{t,s}^{\lambda,a})_{(s,t)\in\mathbb{T}}$ is a controlled rough path with $\|\mathsf{L}^{\lambda,a}\|_{\mathbf{x}}\preccurlyeq 1$.*
2) *There exists a left p-rough lift $(\underleftarrow{\mathsf{B}}_{t,s}^{\lambda,a})_{(s,t)\in\mathbb{T}}$ with respect to $(\mathsf{L}_{t,s}^{\lambda,a})_{(s,t)\in\mathbb{T}}$ of $(\mathsf{B}_{t,s}^{\lambda,a})_{(s,t)\in\mathbb{T}}$,*
3) *The maps $\mathsf{L}_{t,s}^{\lambda,a}$ and $\underleftarrow{\mathsf{B}}_{t,s}^{\lambda,a}$ are $(\delta-1)$-Hölder continuous (by convention, 1-Hölder continuous means Lipschitz continuous) with respect to λ and a in the space of operators valued paths with the norm $\|\cdot\|_p$.*

We have seen in [13] that the existence of a p-rough lift allows one to prove a perturbation formula. We refer to the article for the definition of the "backward" integral denoted by d^{b}.

Corollary 1. *For any $(\lambda,\mu,a,b)\in\Lambda_{\mathrm{p}}^2\times\Lambda_{\mathrm{s}}^2$,*

$$N(a,b,\lambda,\mu)_{0,t} = \int_0^t \mathsf{L}_{t,r}^{\lambda,a}\mathrm{d}^{\mathrm{b}}\underleftarrow{\mathsf{B}}_r^{\lambda,a}\cdot(\mu-\lambda) + \mathsf{L}_{t,s}^{\lambda,a}(b-a),$$

and $(a,\lambda)\mapsto N(a,a+\epsilon,\lambda,\lambda+\nu)$, when $(a+\epsilon,\lambda+\nu)\in\Lambda_{\mathrm{s}}\times\Lambda_{\mathrm{p}}$ is locally $(\delta-1)$-Hölder continuous in the space of V*-valued paths with the norm $\|\cdot\|_p$.*

We could specialize our results to a variation of the parameters.

- ⋆ **Regularity with respect to the starting point:** We assume that f is constant with respect to the parameter λ. Then $\mathsf{L}_{t,0}^{\lambda,a}$ gives the derivative of the Itô map with respect to the starting point.
- ⋆ **Derivation of the underlying path:** Let U, V and Y be Banach spaces, and $\mathbf{X}$ be a rough path with values in $T_2(\mathrm{W})$ with $\mathrm{W}=\mathrm{U}\otimes\mathrm{V}$ such that the projection of X onto $T_2(\mathrm{U})$ is $\mathbf{x}$ and onto $T_2(\mathrm{V})$ is $\mathbf{y}$. For $\epsilon\in[0,1)$, we consider $F:\mathrm{Y}\times[0,1)\to L(\mathrm{W},\mathrm{Y})$ be defined by

$$F(z,\epsilon)x\otimes y = f(z)x + \epsilon f(z)y.$$

We then consider

$$z_t^\epsilon = a + \int_0^t f(z_s^\epsilon, \epsilon)\,\mathrm{d}\mathbf{X}_s.$$

We are interested in the derivation of the RDE driven by $\mathbf{x}$ which is perturbed in the direction of the rough path $\mathbf{y}$, which involves knowing cross-iterated integrals between the paths x and y the rough paths $\mathbf{x}$ and $\mathbf{y}$ lie above.
Let us consider $\mathsf{B}_{t,s} = f(z_s^0)y_{s,t}$ and $\underleftarrow{\mathsf{B}}_{t,s} = f(z_s^0)y_{s,t} + f\nabla f(z_s^0)x\ltimes\mathbf{y}_{s,t}$. It is easily seen that $(\underleftarrow{\mathsf{B}}_{t,s})_{(s,t)\in\mathbb{T}}$ is an almost left p-rough lift of $(\mathsf{B}_{t,s})_{(s,t)\in\mathbb{T}}$ with respect to the rough resolvent associated to $\int \nabla f(z_s^0)[\cdot]\,\mathrm{d}\mathbf{x}_s$ defined in Theorem 2.
Hence, $N(a,a,0,\epsilon) = K_t\epsilon$ for K be defined by $K_t := \int_0^t \mathsf{L}_{t,r}^{0,a}\,\mathrm{d}^{\mathrm{b}}\underleftarrow{\mathsf{B}}_r$. This means that the derivative ∇z^ϵ of z^ϵ with respect to ϵ at $\epsilon = 0$ is the solution to

$$\nabla z_t^0 = \int_0^t \nabla f(z_s^0)\nabla z_s^0\,\mathrm{d}\mathbf{x}_s + \int_0^t f(z_s^0)\,\mathrm{d}\mathbf{y}_s.$$

⋆ **Derivation of the underlying path by a path with an "adjoint" regularity:** When h is of finite q-variation with $1/q + q/p > 1$, then $h\ltimes\mathbf{x}$, $x\ltimes h$ and $h\ltimes h$ exist as a Young integrals, Hence, the above result could be used. In the Gaussian situation, h could be though as a Cameron-Martin direction [20]. Alternatively, one could perturb a path of finite p-variation with a full rough path, as in the stochastic Taylor developments [28–31].

Through the remainder of this paper, we consider that $(a,b,\lambda,\mu) \in \Lambda_{\mathrm{s}}^2 \times \Lambda_{\mathrm{p}}^2$ with $|b-a| \leq 1$, $|\lambda - \mu| \leq 1$ and we set

$$y_t = a + \int_0^t f(y_s,\lambda)\,\mathrm{d}\mathbf{x}_s \text{ and } z_t = b + \int_0^t f(z_s,\mu)\,\mathrm{d}\mathbf{x}_s,$$

where y and z are continuous paths of finite p-variation controlled by ω living in V.

4 The structure of the proof of Theorem 1

Notation 1. For $\theta > 0$, we write $\Omega_{s,t}(\theta)$ for a function on $\mathbb{T}$ satisfying $\Omega_{s,t}(\theta) \leq C\omega(s,t)^{\theta/p}$ where C is a constant that depends only on the structure.

In the spirit of the application of the Gronwall lemma, the idea is the show that $z - y$ is solution to a perturbed linear equation, and to get the right control over the perturbation. The main theorem is obtained by a the following procedure:

1. Existence and control over some "remainder" term.
2. Construction of $z-y$ as a solution to a perturbed linear equation.
3. Application of the results developed in [13] to get a control over the difference $z-y$.

Proposition 1. *Set*

$$F(y,z) := f(z,\mu) - f(y,\lambda) - \nabla_y f(y,\lambda)(z-y) - \nabla_\lambda f(z,\lambda)(\mu-\lambda). \quad (9)$$

Then $\int F(y_s,z_s)\,\mathrm{d}\mathbf{x}_s$ *is a controlled rough path with*

$$\left\| \int F(y_z,z_s)\,\mathrm{d}\mathbf{x}_s \right\|_{\mathbf{x}} \preccurlyeq |\mu-\lambda|^\delta + |\Delta_0|^\delta,$$

where δ *is defined by* (6).

Proposition 2. *Set*

$$H(y,z) := \nabla_\lambda f(z,\lambda) - \nabla_\lambda f(y,\lambda). \quad (10)$$

Then $\int H(y_r,z_r)\,\mathrm{d}\mathbf{x}_r$ *is a controlled rough path with values in* $L(\Lambda,\mathrm{V})$ *satisfying*

$$\left\| \int H(y_s,z_s)\,\mathrm{d}\mathbf{x}_s \right\|_{\mathbf{x}} \preccurlyeq |\Delta_0|^{\delta-1} + |\lambda-\mu|^{\delta-1}$$

with δ *given by* (6).

Remark 1. Since from the Young inequality, $a^{\delta-1}b \leq a^\delta(\delta-1)/\delta + b^\delta\delta$, $\left\|\int H(y_s,z_s)(\mu-\lambda)\,\mathrm{d}\mathbf{x}_s\right\|_{\mathbf{x}} \preccurlyeq |\Delta_0|^\delta + |\mu-\lambda|^\delta$.

Since $\int F(y_s,z_s)\,\mathrm{d}\mathbf{x}_s$ and $\int H(y_s,z_s)\,\mathrm{d}\mathbf{x}_s$ exist as rough integral, the following proposition allows one to decompose the equation satisfied by Δ.

Proposition 3. *The difference* $\Delta := z-y$ *is solution to*

$$\Delta_t = \Delta_0 + \int_0^t \nabla_y f(y_s,\lambda)\Delta_s\,\mathrm{d}\mathbf{x}_s + \int_0^t \nabla_\lambda f(y_s,\lambda)(\mu-\lambda)\,\mathrm{d}\mathbf{x}_s \\ + \int_0^t F(y_s,z_s)\,\mathrm{d}\mathbf{x}_s + \int_0^t H(y_s,z_s)(\mu-\lambda)\,\mathrm{d}\mathbf{x}_s. \quad (11)$$

Once this result has been proved, the framework developed in [13] could be used and Theorem 2 enforces the desired estimates by considering $\Delta - N(a,b,\lambda,\mu)$ as a solution of a perturbed linear RDE.

The next result will be useful to establish the regularity of $\mathsf{L}^{a,\lambda}$.

Proposition 4. *For* $L \in \mathrm{P}(T,\mathbf{x},L(\mathrm{U},\mathrm{V}))$*, set* $K(L)_t := \int_0^t [\nabla_y f(z_s,\mu) - \nabla_y f(y_s,\lambda)]\cdot L_s\,\mathrm{d}\mathbf{x}_s$. *Then* $(K(L)_t)_{t\in[0,T]}$ *is a controlled rough path in* $\mathrm{P}(T,\mathbf{x},\mathrm{V})$ *satisfying*

$$\|K(L)\|_{\mathbf{x}} \preccurlyeq \|L\|_{\mathbf{x}}(|\Delta_0|^{\delta-1} + |\lambda-\mu|^{\delta-1})$$

for δ *defined by* (6).

5 Proofs of the intermediate propositions

5.1 Some useful lemmas

We use frequently the following notations and lemmas.

For a function g, let us define

$$\mathscr{T}_0 g(z,y) := g(z) - g(y) \text{ and } \mathscr{T}_0^2 g(z,y,z',y') := g(z) - g(y) - g(z') + g(y').$$

When $g \in \mathscr{C}^1(\mathrm{V},\mathrm{V})$, set

$$\mathscr{T}_1 g(y,z) := g(z) - g(y) - \nabla_y g(y)(z-y).$$

Lemma 1. ① *Let $g \in \mathscr{C}^{1+\gamma}(\mathrm{V},\mathrm{V})$. Then for any y, y', z, y' in* V,

$$\begin{aligned}&|\mathscr{T}_0^2 g(z,y,z',y')| \\ &\qquad \leq \|\nabla_y g\|_\infty |z'-z-y'+y| + H_\gamma(\nabla_y g)(|y'-y|^\gamma + |z'-z|^\gamma)|z'-y'|.\end{aligned}$$

② *Let $g \in \mathscr{C}^\gamma(\mathrm{V},\mathrm{V})$. Then for $\kappa \in [0,1]$ and $\overline{\kappa} := 1-\kappa$,*

$$|\mathscr{T}_0^2 g(z,y,z',y')| \leq H_\gamma(g)(|y'-y|^{\kappa\gamma} + |z'-z|^{\kappa\gamma})(|z'-y'|^{\gamma\overline{\kappa}} + |z-y|^{\gamma\overline{\kappa}}). \quad (12)$$

Proof. ① This follows from

$$\begin{aligned}&|g(z) - g(y) - g(z') + g(y')| \\ &= \Big| \int_0^1 \nabla_y g(y + \tau(z-y))(z-y)\,\mathrm{d}\tau - \int_0^1 \nabla_y g(y' + \tau(z'-y'))(z'-y')\,\mathrm{d}\tau \Big| \\ &\qquad\qquad \leq \Big| \int_0^1 \nabla_y g(y+\tau(z-y))(z-y-z'+y')\,\mathrm{d}\tau \Big| \\ &\qquad\quad + \Big| \int_0^1 (\nabla_y g(y' + \tau(z'-y')) - \nabla_y g(y + \tau(z-y)))(z'-y')\,\mathrm{d}\tau \Big|.\end{aligned}$$

The result stems from this inequality and

$$\begin{aligned}|\nabla_y g(y'+\tau(z'-y')) - g(y+\tau(z-y))| &\leq H_\gamma(\nabla_y g)|(1-\tau)(y'-y) + \tau(z'-z)|^\gamma \\ &\leq H_\gamma(\nabla_y g)(|y'-y|^\gamma + |z'-z|^\gamma)\end{aligned}$$

for $\tau \in [0,1]$.

② Inequality (12) follows from a combination using κ of $|\mathscr{T}_0^2 g(z,y,z',y')| \leq |\mathscr{T}_0 g(z,y)| + |\mathscr{T}_0 g(z',y')|$ and $|\mathscr{T}_0^2 g(z,y,z',y')| \leq |\mathscr{T}_0 g(y,y')| + |\mathscr{T}_0 g(z,z')|$ and the inequality $(a+b)^\delta \leq a^\delta + b^\delta$ for $a, b \geq 0$ and $\delta \in [0,1]$. □

Lemma 2. ① *Let $g \in \mathscr{C}^{2+\gamma}(\mathrm{V}, \mathrm{V})$ with $\|\nabla^2 g\|_\infty < +\infty$ and $H_\gamma(\nabla^2 g) < +\infty$. Then*

$$|\mathcal{T}_1[g](y,z)| \le \|\nabla^2 g\|_\infty |z-y|^2 \tag{13}$$

and

$$\begin{aligned}|\mathcal{T}_1[g](y,z) - \mathcal{T}_1[g](y',z')| \le \|\nabla^2 g\|_\infty |z'-y'-z+y| \cdot (|z-y| + |z'-y'|) \\ + H_\gamma(\nabla^2 g)(|y'-y|^\gamma + |z'-z|^\gamma)|z'-y'| \cdot |z-y|.\end{aligned}$$

② *Let $g \in \mathscr{C}^{1+\gamma}(\mathrm{V}, \mathrm{V})$ with $\|\nabla_y g\|_\infty < +\infty$ and $H_\gamma(\nabla_y g) < +\infty$. Then*

$$|\mathcal{T}_1[g](y,z)| \le H_\gamma(\nabla_y g)|z-y|^{1+\gamma} \tag{14}$$

and

$$\begin{aligned}|\mathcal{T}_1[g](y,z) - \mathcal{T}_1[g](y',z')| \le H_\gamma(\nabla_y g)|z'-y'-z+y| \cdot |z'-y'|^\gamma \\ + 2H_\gamma(\nabla_y g)(|z'-z|^{\kappa\gamma} + |y'-y|^{\kappa\gamma})(|z'-y'|^{\overline{\kappa}\gamma} + |z-y|^{\overline{\kappa}\gamma})|z-y|.\end{aligned}$$

Proof. Inequalities (13) and (14) follow immediately from

$$\mathcal{T}_1[g](y,z) = \int_0^1 (\nabla g(y+\tau(z-y)) - \nabla g(y))(z-y)\,\mathrm{d}\tau.$$

Besides,

$$\begin{aligned}&\mathcal{T}_1[g](y,z) - \mathcal{T}_1[g](y',z') \\ &= \int_0^1 (\nabla g(y+\tau(z-y)) - \nabla g(y) - \nabla g(y'+\tau(z'-y')) + \nabla g(y'))(z-y)\,\mathrm{d}\tau \\ &\qquad + \int_0^1 (\nabla g(y'+\tau(z'-y')) - \nabla g(y'))(z-y-z'+y')\,\mathrm{d}\tau.\end{aligned}$$

We conclude with Lemma 1 applied to ∇g. □

5.2 Some controls on the path Δ

We know that the Itô map is locally Lipschitz continuous. From this follows a very useful control over $\|\Delta\|_p$.

Theorem 3. *The path Δ is a controlled rough path with*

$$\|\Delta\|_p \preccurlyeq \|\Delta\|_{\mathbf{x}} \preccurlyeq |\Delta_0| + |\mu - \lambda| \tag{15}$$

Remark 2. From (15) and since $\|\Delta\|_p \preccurlyeq \|\Delta\|_{\mathbf{x}}$, we deduce the inequality

$$\|\Delta\|_\infty \leq |\Delta_0| + \|\Delta\|_p \omega(0,T)^{1/p} \preccurlyeq |\Delta_0| + |\mu - \lambda| \tag{16}$$

which is frequently used.

Proof. We write $E_{s,t} = E_s^\dagger x_{s,t}^1 + E_{s,t}^\sharp$ with

$$E_t^\dagger := f(z_t, \mu) - f(y_t, \lambda),$$
$$E_{s,t}^\sharp := E_s' x_{s,t}^2 \text{ with } E_t' = (f\nabla f)(z_t, \mu) - (f\nabla f)(y_t, \lambda).$$

Since $f\nabla f(z,\lambda)$ is itself locally Lipschitz in z and λ,

$$\|E'\|_\infty \preccurlyeq |\Delta_0| + |\lambda - \mu|.$$

By Lemma 1 below, since $f(\cdot,\lambda)$ and $f(\cdot,\mu)$ belong to $\mathscr{C}^2(\mathrm{V}, L(\mathrm{U},\mathrm{V}))$ under both $(\mathbf{H}_{①})$ and $(\mathbf{H}_{②})$,

$$|E_{s,t}^\dagger| \preccurlyeq \Omega_{s,t}(1)(|\lambda - \mu| + |\Delta_0|),\ \forall (s,t) \in \mathbb{T}.$$

We know from Theorem 4 in [33], Theorem 1 in [32] that

$$|(y_{s,t} - f(y_s,\lambda)x_{s,t}^1 - (f\nabla_y f)(y_s,\lambda)x_{s,t}^2) - (z_{s,t} - f(z_s,\mu)x_{s,t}^1 - (f\nabla_y f)(z_s,\mu)x_{s,t}^2)|$$
$$\leq (|\Delta_0| + |\mu - \lambda|) \begin{cases} \Omega_{s,t}(3) & \text{under } (\mathbf{H}_{①}), \\ \Omega_{s,t}(2+\gamma) & \text{under } (\mathbf{H}_{②}). \end{cases}$$

This proves that $\Delta_{s,t} = \Delta_s^\dagger x_{s,t}^1 + \Delta_{s,t}^\sharp$ for $(s,t) \in \mathbb{T}$ with

$$\Delta^\dagger = E^\dagger \text{ and } |\Delta_{s,t}^\sharp| \preccurlyeq \Omega(2)\big(|E_s| + (|\Delta_0| + |\mu - \lambda|)\big) \begin{cases} \Omega_{s,t}(3) & \text{under } (\mathbf{H}_{①}), \\ \Omega_{s,t}(2+\gamma) & \text{under } (\mathbf{H}_{②}). \end{cases}$$

Then Δ is a controlled rough path and (15) holds. □

The control below on the cross-iterated integral between Δ and $\mathbf{x}$ follows then directly from Theorem 3 and Corollary 4 in Appendix A.

Corollary 2. *The cross-iterated integral* $\Delta \ltimes \mathbf{x}$ *exists and*

$$\|\Delta \ltimes \mathbf{x}_{s,t}\| \leq \Omega_{s,t}(2)(|\Delta_0| + |\lambda - \mu|),\ \forall (s,t) \in \mathbb{T}.$$

5.3 Proof of Proposition 1: Control over the remainder $\int F(y_s, z_s)\,\mathrm{d}\mathbf{x}_s$

The proof of our main theorem relies on the construction of a suitable vector field $F : \mathrm{V}\times\mathrm{V} \to L(\mathrm{U},\mathrm{V})$ and to study the norm of $\int F(y_s, z_s)\,\mathrm{d}\mathbf{x}_s$ for two solutions $y_t = y_0 + \int_0^t f(y_s,\lambda)\,\mathrm{d}\mathbf{x}_s$ and $z_t = z_0 + \int_0^t f(z_s,\mu)\,\mathrm{d}\mathbf{x}_s$ with $(y_0, z_0) \in \Lambda_{\mathrm{s}}^2$, $(\lambda,\mu) \in \Lambda_{\mathrm{p}}^2$.

For F defined by (9), $\int F(y_s, z_s)\,\mathrm{d}\mathbf{x}_s$ is defined from

$$\begin{aligned}\widehat{F}_{s,t} &:= F(y_s,z_s)x^1_{s,t} + \nabla_y F(y_s,z_s) y \ltimes \mathbf{x}_{s,t} + \nabla_z F(y_s,z_s) z \ltimes \mathbf{x}_{s,t} \\ &:= \widehat{F}^\dagger_s x^1_{s,t} + \widehat{F}^\sharp_{s,t}.\end{aligned} \tag{17}$$

In this section, our main concern is then to show that $\widehat{F}$ is an almost additive functional and to relate its norm to $|\Delta_0|$ and $|\mu-\lambda|$. Up to a multiplicative constant, the control over the norm is then transferred as a control over the integral thanks to the sewing lemma.

Let us set

$$\mathscr{D}[F](y,z) := \nabla_y F(y,z) + \nabla_z F(y,z) \tag{18}$$

and

$$\begin{aligned}A_1(s,r) &:= \mathscr{T}_1[F]((y_r,z_r),(y_s,z_s)) \\ &= F(y_r,z_r) - F(y_s,z_s) - \nabla_y F(y_s,z_s) y_{s,r} - \nabla_z F(y_s,z_s) z_{s,r}.\end{aligned} \tag{19}$$

Our main remark is that

$$\widehat{F}^\dagger_{s,t} = A_1(s,t) + \mathscr{D}[F](y_s,z_s) z_{s,t} - \nabla_y F(y_s,z_s)\Delta_{s,t}, \tag{20}$$

$$\widehat{F}^\sharp_{s,t} = -\nabla_y F(y_s,z_s)\Delta \ltimes \mathbf{x}_{s,t} + \mathscr{D}[F](y_s,z_s) z \ltimes \mathbf{x}_{s,t}. \tag{21}$$

Lemma 3. ① *It holds that*

$$|F(y,z)| \leq \|\nabla^2_{yy} f\|_\infty |z-y|^2 + \|\nabla^2_{\lambda\lambda} f\|_\infty |\lambda-\mu|^2. \tag{22}$$

② *It holds that*

$$|F(y,z)| \leq \|\nabla^2_{yy} f\|_\infty |z-y|^2 + H^{(\lambda)}_\beta(\nabla_\lambda f)|\lambda-\mu|^{1+\beta}. \tag{23}$$

Proof. Transforming $F(y,z)$ into

$$\begin{aligned}F(y,z) = &\int_0^1 (\nabla_\lambda f(z, \lambda + \tau(\mu-\lambda)) - \nabla_\lambda f(z,\lambda))(\mu-\lambda)\,\mathrm{d}\tau \\ &+ \int_0^1 (\nabla_y f(y+\tau(z-y),\lambda) - \nabla_y f(y,\lambda))(z-y)\,\mathrm{d}\tau\end{aligned}$$

leads to inequalities (22) and (23). □

The gradient of F with respect to the y-variables is

$$\nabla_y F(y,z) = -\nabla^2_{yy} f(y,\lambda)(z-y).$$

The proof of the following lemma is then immediate.

Lemma 4. *Under* $(\mathbf{H}_{①})$, $(\mathbf{H}_{②})$, *for* $y, z, y', z' \in \mathrm{V}$,

$$|\nabla_y F(y,z)| \leq \|\nabla^2_{yy} f\|_\infty |z-y|$$

and

$$\begin{aligned} |\mathscr{T}_0 \nabla_y F(y,z,y',z')| \leq \|\nabla^2_{yy} f\|_\infty |y-y'-z+z'| \\ + \begin{cases} |y'-y| \cdot |y'-z'| \|\nabla^3_{yyy} f\|_\infty & \textit{under } (\mathbf{H}_{①}), \\ |y'-y|^\gamma \cdot |y'-z'| H_\gamma(\nabla^2_{yy} f) & \textit{under } (\mathbf{H}_{②}). \end{cases} \end{aligned} \tag{24}$$

Lemma 5. ① *It holds that*

$$|\mathscr{D}[F](y,z)| \leq \|\nabla^3_{yyy} f\|_\infty |z-y|^2 + \|\nabla^3_{\lambda\lambda y} f\|_\infty |\mu-\lambda|^2 \tag{25}$$

and

$$\begin{aligned} |\mathscr{T}_0 \mathscr{D}[F](y,z,y',z')| \leq \|\nabla^4_{\lambda\lambda yy} f\|_\infty |\mu-\lambda|^2 |z-z'| \\ + H_\gamma(\nabla^3_{yyy} f)(|y'-y|^\gamma + |z'-z|^\gamma)|z'-y'| \cdot |z-y| \\ + \|\nabla^3_{yyy} f\|_\infty |z'-y'-z+y| \cdot (|z-y| + |z'-y'|). \end{aligned} \tag{26}$$

② *It holds that*

$$|\mathscr{D}[F](y,z)| \leq H_\gamma(\nabla^2_{yy} f)|z-y|^{1+\gamma} + H^{(\lambda)}_\beta(\nabla^2_{\lambda y} f)|\mu-\lambda|^{1+\beta} \tag{27}$$

and

$$\begin{aligned} |\mathscr{T}_0 \mathscr{D}[F](y,z,y',z')| \leq H_\gamma(\nabla^2_{yy} f)|z'-y'-z+y| \cdot |z'-y'|^\gamma \\ + H_\gamma(\nabla^2_{yy} f)(|z'-y'|^{\overline{\kappa}\gamma} + |z-y|^{\overline{\kappa}\gamma})(|z'-z|^{\kappa\gamma} + |y'-y|^{\kappa\gamma})|z-y| \\ + 2H_\gamma(\nabla^2_{\lambda y} f)^\kappa H^{(\lambda)}_\beta(\nabla^2_{\lambda y} f)^{\overline{\kappa}} |z'-z|^{\gamma\kappa} |\mu-\lambda|^{1+\overline{\kappa}\beta}. \end{aligned} \tag{28}$$

Proof. The gradient of F with respect to the z-variables is

$$\begin{aligned} \nabla_z F(y,z) &= \nabla_y f(z,\mu) - \nabla_y f(y,\lambda) - \nabla^2_{\lambda y} f(z,\lambda)(\mu-\lambda) \\ &= A_2(z) + \nabla_y f(z,\lambda) - \nabla_y f(y,\lambda) \end{aligned}$$

with

$$A_2(z) := \int_0^1 (\nabla^2_{\lambda y} f(z, \lambda + \tau(\mu - \lambda)) - \nabla^2_{\lambda y} f(z, \lambda))(\mu - \lambda)\,\mathrm{d}\tau.$$

Since $\nabla_y F(y, z) = -\nabla^2_{yy} f(y, \lambda)(z - y)$, it holds that $\mathscr{D}[F](y, z) = A_2(z) + \mathscr{T}_1[\nabla_y f](y, z)$.

Under $(\mathbf{H}_{①})$,

$$A_2(z) = \int_0^1 \int_0^1 \nabla^3_{\lambda\lambda y} f(z, \lambda + \rho\tau(\mu - \lambda))\tau(\mu - \lambda, \mu - \lambda)\,\mathrm{d}\rho\,\mathrm{d}\tau.$$

Besides, $|A_2(z)| \leq \|\nabla^3_{\lambda\lambda y} f\|_\infty |\mu - \lambda|^2$ and

$$|A_2(z) - A_2(z')| \leq \frac{1}{2}\|\nabla^4_{\lambda\lambda yy} f\|_\infty |z - z'| \times |\mu - \lambda|^2.$$

Under $(\mathbf{H}_{②})$, $|A_2(z)| \leq H^{(\lambda)}_\beta(\nabla^2_{\lambda y} f)|\mu - \lambda|^{1+\beta}$ and

$$|A_2(z) - A_2(z')| \leq \begin{cases} H_\gamma(\nabla^2_{\lambda y} f)|z' - z|^\gamma |\mu - \lambda|, \\ 2H^{(\lambda)}_\beta(\nabla^2_{\lambda y} f)|\mu - \lambda|^{1+\beta}. \end{cases}$$

We combine these two estimates by raising the first line to the power $\kappa \in [0, 1]$, the second line to the power $(1 - \kappa)$ and multiplying them.

Inequalities (25), (26), (27) and (28) stem from these computations and Lemma 2 applied to $\nabla_y f(\cdot, \lambda)$. □

Corollary 3. ① *For any $(s, t) \in \mathbb{T}$ and $\kappa, \tau \in [0, 1]$,*

$$\begin{aligned} &|\mathscr{T}_0\mathscr{D}[F](y_s + \tau y_{s,t}, z_s + \tau z_{s,t}, y_s, z_s)| \\ &\qquad \leq \begin{cases} \Omega_{s,t}(\gamma)(|\Delta_0|^2 + |\mu - \lambda|^2) & \text{under } (\mathbf{H}_{①}), \\ \Omega_{s,t}(\kappa\gamma)(|\Delta_0|^\delta + |\mu - \lambda|^\delta) & \text{under } (\mathbf{H}_{②}). \end{cases} \end{aligned} \tag{29}$$

Proof. With Lemma 5 under $(\mathbf{H}_{①})$,

$$\begin{aligned} &|\mathscr{T}_0\mathscr{D}[F](y_s + \tau y_{s,t}, z_s + \tau z_{s,t}, y_s, z_s)| \\ &\qquad \leq C_1|\mu - \lambda|^2\Omega_{s,t}(1) + C_2\Omega_{s,t}(\gamma)\|\Delta\|^2_\infty + C_3\Omega_{s,t}(1)\|\Delta\|_\infty\|\Delta\|_p \end{aligned}$$

with $C_1 := \|\nabla^4_{\lambda\lambda yy} f\|_\infty \|z\|_p$, $C_2 := H_\gamma(\nabla^3_{yyy} f)(\|y\|^\gamma_p + \|z\|^\gamma_p)$ and $C_3 := 2\|\nabla^3_{yyy} f\|_\infty$. Inequality (29) follows easily from $\|\Delta\|_p \leq \|z\| + \|y\|_p$, $\|\Delta\|_\infty \leq |\Delta_0| + \omega(0, T)^{1/p}\|\Delta\|_p$ and $|\Delta_0| \cdot \|\Delta\|_p \leq \frac{1}{2}|\Delta_0|^2 + \frac{1}{2}\|\Delta\|^2_p$.

Under $(\mathbf{H}_{②})$, Lemma 5 implies that

$$\begin{aligned} &|\mathscr{T}_0\mathscr{D}[F](y_s + \tau y_{s,t}, z_s + \tau z_{s,t}, y_s, z_s)| \\ &\qquad \leq \Omega(1)\|\Delta\|_p\|\Delta\|^\gamma_\infty + \|\Delta\|^{\overline{\kappa}\gamma}_p\|\Delta\|_\infty\Omega(\kappa\gamma) + \Omega(\kappa\gamma)|\mu - \lambda|^{1+\overline{\kappa}\beta}. \end{aligned}$$

Theorem 3 allows one to conclude. □

Lemma 6. *For A_1 defined by* (19),

$$|A_1(s,r)| \le C_4 \begin{cases} \Omega_{s,r}(1+\gamma)(|\Delta_0|^2 + |\mu-\lambda|^2) & \textit{under } (\mathbf{H}_{①}), \\ \Omega_{s,r}(1+\kappa\gamma)(|\Delta_0|^\delta + |\mu-\lambda|^\delta) & \textit{under } (\mathbf{H}_{②}). \end{cases}$$

Proof. Let us note that

$$\begin{aligned} A_1(s,r) &= \int_0^1 (\nabla_y F(y_s+\tau y_{s,r}, z_s+\tau z_{s,r}) - \nabla_y F(y_s,z_s)) y_{s,r}\,\mathrm{d}\tau \\ &\quad + \int_0^1 (\nabla_z F(y_s+\tau y_{s,r}, z_s+\tau z_{s,r}) - \nabla_z F(z_s)) z_{s,r}\,\mathrm{d}\tau \\ &= \int_0^1 (\nabla_y F(y_s+\tau y_{s,r}, z_s+\tau z_{s,r}) - \nabla_y F(y_s,z_s))(y_{s,r}-z_{s,r})\,\mathrm{d}\tau \\ &\quad + \int_0^1 \big(\mathscr{D}[F](y_s+\tau y_{s,r}, z_s+\tau z_{s,r}) - \mathscr{D}[F](y_s,z_s)\big) z_{s,r}\,\mathrm{d}\tau. \end{aligned}$$

Under $(\mathbf{H}_{①})$ (resp. $(\mathbf{H}_{②})$), Inequality (29) in Corollary 3 and Inequality (24) in Lemma 4 allow one to conclude. □

Lemma 7. *For $(r,s) \in \mathbb{T}$,*

$$|\mathscr{T}_0 F(y_r, z_r, y_s, z_s)| \preccurlyeq \Omega_{s,r}(1) \begin{cases} |\Delta_0|^2 + |\mu-\lambda|^2 & \textit{under } (\mathbf{H}_{①}), \\ |\Delta_0|^{1+\gamma} + |\mu-\lambda|^{1+\beta} & \textit{under } (\mathbf{H}_{②}). \end{cases}$$

Proof. Let us note that

$$\begin{aligned} &F(y_r,z_r) - F(y_s,z_s) \\ &= \int_0^1 \nabla_y F(y_s+\tau y_{s,r}, z_s+\tau z_{s,r}) y_{s,r}\,\mathrm{d}\tau + \int_0^1 \nabla_z F(y_s+\tau y_{s,r}, z_s+\tau z_{s,r}) z_{s,r}\,\mathrm{d}\tau \\ &= \int_0^1 \nabla_y F(y_s+\tau y_{s,r}, z_s+\tau z_{s,r})(y_{s,r}-z_{s,r})\,\mathrm{d}\tau + \int_0^1 \mathscr{D}[F](y_s+\tau y_{s,r}, z_s+\tau z_{s,r}) z_{s,r}\,\mathrm{d}\tau. \end{aligned}$$

With Lemma 4, for $\tau \in [0,1]$,

$$|\nabla_y F(y_s+\tau y_{s,r}, z_s+\tau z_{s,r})| \le \|\nabla^2_{yy} f\|_\infty (|y_s - z_s| + |z_{s,r} - y_{s,r}|).$$

On the other hand, Lemma 5 and the convexity of $x \mapsto x^\nu$ for $\nu \ge 1$,

$$\begin{aligned} &|\mathscr{D}[F](y_s+\tau y_{s,r}, z_z+\tau z_{s,r})| \\ &\qquad \preccurlyeq \begin{cases} |z_s - y_s|^2 + |z_r - y_r|^2 + |\mu-\lambda|^2 & \text{under } (\mathbf{H}_{①}), \\ |z_s - y_s|^{1+\gamma} + |z_r - y_r|^{1+\gamma} + |\mu-\lambda|^{1+\beta} & \text{under } (\mathbf{H}_{②}). \end{cases} \end{aligned}$$

Together with Theorem 3, this leads to the result. □

We may now combine these results.

Proof of Proposition 1. For $\widehat{F}$ defined by (17), we have seen in (9) that $\widehat{F}_{s,t} = \widehat{F}^\dagger_s x^1_{s,t} + \widehat{F}^\sharp_{s,t}$ with $\widehat{F}^\dagger_s := F(y_s, z_s)$.

We now prove that $\widehat{F}$ is an almost controlled rough path with a good control on $\|\widehat{F}\|_{\star\mathbf{x}}$, (defined in Proposition 5), from which we deduce through Proposition 5 a control over $\|\int F(y_s, z_s)\,\mathrm{d}\mathbf{x}_s\|_{\mathbf{x}}$.

With Lemma 3,

$$|\widehat{F}^\dagger_0| \leq |\Delta_0|^2 + \begin{cases} |\mu-\lambda|^2 & \text{under } (\mathbf{H}_{①}), \\ |\mu-\lambda|^{1+\beta} & \text{under } (\mathbf{H}_{②}). \end{cases}$$

Combining Lemmas 4, 5 and 6 applied to the decomposition (20),

$$|\widehat{F}^\dagger_{s,t}| \leq \Omega_{s,t}(1) \begin{cases} |\Delta_0|^2 + |\mu-\lambda|^2 & \text{under } (\mathbf{H}_{①}), \\ |\Delta_0|^\delta + |\mu-\lambda|^\delta & \text{under } (\mathbf{H}_{②}). \end{cases}$$

With Lemma 4, Corollary 2 and Lemma 5 applied to the decomposition (21),

$$\|\widehat{F}^\sharp\|_{p/2} \preccurlyeq \begin{cases} |\Delta_0|^2 + |\mu-\lambda|^2 & \text{under } (\mathbf{H}_{①}), \\ |\Delta_0|^{1+\gamma} + |\mu-\lambda|^{1+\beta} & \text{under } (\mathbf{H}_{②}). \end{cases}$$

Let us show that $\widehat{F}$ defined by (17) is an almost additive functional. It holds that

$$\begin{aligned} \widehat{F}_{s,r,t} &= -(F(y_r, z_r) - F(y_s, z_s) - \nabla_y F(y_s, z_s) y_{s,r} - \nabla_z F(y_s, z_s) z_{s,r}) x^1_{r,t} \\ &\qquad + \mathcal{T}_0 \nabla_y F(y_s, z_s, y_r, z_r) y \ltimes \mathbf{x}_{r,t} + \mathcal{T}_0 \nabla_z F(y_s, z_s, y_r, z_r) z \ltimes \mathbf{x}_{r,t} \\ &= -A_1(s,r) x^1_{r,t} + \mathcal{T}_0 \nabla_y F(y_s, z_s, y_r, z_r) \Delta \ltimes \mathbf{x}_{r,t} + \mathcal{T}_0 \mathcal{D}[F](y_s, z_s, y_r, z_r) z \ltimes \mathbf{x}_{r,t}. \end{aligned}$$

Using Lemmas 4 and 6 and Corollary 2 as well as Corollary 3,

$$|\widehat{F}_{s,r,t}| \preccurlyeq \begin{cases} \Omega_{s,t}(2+\gamma)(|\Delta_0|^2 + |\mu-\lambda|^2) & \text{under } (\mathbf{H}_{①}), \\ \Omega_{s,t}(2+\kappa\gamma)(|\Delta_0|^\delta + |\mu-\lambda|^\delta) & \text{under } (\mathbf{H}_{②}). \end{cases}$$

Hence, $(\widehat{F}_{s,t})_{(s,t)\in\mathbb{T}}$ is an almost controlled rough path, and an almost rough path. Using the additive sewing lemma, the corresponding rough path is by definition $\int F(y_s, z_s)\,\mathrm{d}\mathbf{x}_s$, which is then a controlled rough path thanks to Proposition 5 which gives the desired control over $\left\|\int F(y_s, z_s)\,\mathrm{d}\mathbf{x}_s\right\|_{\mathbf{x}}$. $\square$

5.4 Proof of Proposition 2: Control over the remainder $\int H(y_s, z_s)\,\mathrm{d}\mathbf{x}_s$

Proof of Proposition 2. For H defined by (10), set $\widehat{H}_{s,t} := \widehat{H}^\dagger_s x^1_{s,t} + \widehat{H}^\sharp_{s,t}$ with

$$\widehat{H}^\dagger_s := H(y_s, z_s) \text{ and } \widehat{H}^\sharp_{s,t} := \nabla_y H(y_s, z_s) y \ltimes \mathbf{x}_{s,t} + \nabla_z H(y_s, z_s) z \ltimes \mathbf{x}_{s,t}.$$

We note first that

$$|\widehat{H}^\dagger_0| \leq \begin{cases} \|\nabla^2_{\lambda y} f\|_\infty |\Delta_0| & \text{under } (\mathbf{H}_{①}), \\ H_\gamma(\nabla_\lambda f)|\Delta_0|^\gamma & \text{under } (\mathbf{H}_{②}). \end{cases} \tag{30}$$

With Lemma 1, since $\nabla_\lambda f(\cdot, \lambda) \in \mathscr{C}^2(\mathrm{V}, \mathrm{V})$ under $(\mathbf{H}_{①})$ and $\nabla f(\cdot, \lambda) \in \mathscr{C}^{1+\gamma}(\mathrm{V}, \mathrm{V})$ under $(\mathbf{H}_{②})$,

$$\|\widehat{H}^\dagger\|_p \leq \begin{cases} |\Delta_0| + |\mu - \lambda| & \text{under } (\mathbf{H}_{①}), \\ |\Delta_0|^\gamma + |\mu - \lambda|^\gamma & \text{under } (\mathbf{H}_{②}). \end{cases} \tag{31}$$

Since

$$\begin{aligned} &\nabla_y H(y_s, z_s) y \ltimes \mathbf{x}_{s,t} + \nabla_z H(y_s, z_s) z \ltimes \mathbf{x}_{s,t} \\ &\qquad = (\nabla^2_{\lambda y} f(z_s, \lambda) - \nabla^2_{\lambda y} f(y_s, \lambda)) y \ltimes \mathbf{x}_{s,t} - \nabla^2_{\lambda y} f(z_s, \lambda) \Delta \ltimes \mathbf{x}_{s,t}, \end{aligned}$$

Corollary 2 and (16) implies that

$$\|\widehat{H}^\sharp\|_{p/2} \preccurlyeq \begin{cases} |\mu - \lambda| + |\Delta_0| & \text{under } (\mathbf{H}_{①}), \\ |\Delta_0| + |\mu - \lambda| + \|\Delta\|^\gamma_\infty \preccurlyeq |\Delta_0|^\gamma + |\mu - \lambda|^\gamma & \text{under } (\mathbf{H}_{②}). \end{cases} \tag{32}$$

To apply Proposition 5 in Appendix A, let us consider $\widehat{H}_{r,s,t} := \widehat{H}_{r,t} - \widehat{H}_{r,s} - \widehat{H}_{s,t}$. Then

$$\widehat{H}_{r,s,t} = \mathrm{I}_{r,s} \cdot x^1_{s,t} - \mathrm{II}_{r,s,t}$$

with

$$\begin{aligned} \mathrm{I}_{r,s} &:= H(y_r, z_r) - H(y_s, z_s) + \nabla_y H(y_r, z_r) y_{r,s} + \nabla_z H(y_r, z_r) z_{r,s}, \\ \mathrm{II}_{r,s,t} &:= \mathscr{T}^2_0[\nabla^2_{\lambda y} f](y_s, y_r, z_s, z_r) y \ltimes \mathbf{x}_{s,t} + (\nabla_z H(y_s, z_s) - \nabla_z H(y_r, z_r)) \Delta \ltimes \mathbf{x}_{s,t}, \end{aligned}$$

since

$$\nabla_y H(y_s, z_s) - \nabla_y H(y_r, z_r) = \nabla^2_{\lambda y} f(y_s, \lambda) - \nabla^2_{\lambda y} f(y_r, \lambda).$$

Since $z \mapsto \nabla^2_{\lambda y} f(z,\lambda)$ is of class $\mathscr{C}^{1+\gamma}$ under $(\mathbf{H}_{①})$ and $\mathscr{C}^{\gamma}$ under $(\mathbf{H}_{②})$ with $2+\gamma > p$ so that Lemma 1 could be applied to $\nabla^2_{\lambda y} f$. In addition,

$$\begin{aligned}
-\mathrm{I}_{r,s} = \int_0^1 &(\nabla_y H(y_r + \tau y_{r,s}, z_r + \tau z_{r,s}) - \nabla_y H(y_r, z_r)) y_{r,s} \,\mathrm{d}\tau \\
&+ \int_0^t (\nabla_z H(y_r + \tau y_{r,s}, z_r + \tau z_{r,s}) - \nabla_z H(y_r, z_r)) z_{r,s} \,\mathrm{d}\tau \\
= \int_0^1 &\mathscr{T}_0^2[\nabla^2_{\lambda y} f](y_r + \tau y_{r,s}, z_r + \tau z_{r,s}, y_r, z_r)] y_{r,s} \,\mathrm{d}\tau + \int_0^t \nabla^2_{y\lambda} f(z_r + \tau z_{r,s}, z_r) \Delta_{r,s} \,\mathrm{d}\tau,
\end{aligned}$$

so that again Lemma 1 could be applied to $\nabla^2_{y\lambda} f$. Thus, combining Lemma 1 and Theorem 3, we then obtain a control over $\widehat{H}_{r,s,t}$

$$|\widehat{H}_{s,r,t}| \preccurlyeq \begin{cases} \Omega_{s,t}(2+\gamma)(|\Delta_0| + |\mu - \lambda|) & \text{under } (\mathbf{H}_{①}), \\ \Omega_{s,t}(2+\kappa\gamma)(|\Delta_0|^{\delta-1} + |\mu - \lambda|^{\delta-1}) & \text{under } (\mathbf{H}_{②}). \end{cases} \tag{33}$$

and then on $\int H(y_r, z_r) \,\mathrm{d}\mathbf{x}_r$ with Proposition 5. □

5.5 Proof of Proposition 3: Construction of a perturbed RDE

Proof of Proposition 3. Let us set

$$G(y,z) := \nabla_y f(y,\lambda)(z-y) + \nabla_\lambda f(z,\lambda)(\mu - \lambda).$$

The function G belongs to $\mathscr{C}^2(\mathrm{V}\times\mathrm{V}, L(\mathrm{U},\mathrm{V}))$ under $(\mathbf{H}_{①})$ and to $\mathscr{C}^{1+\gamma}(\mathrm{V}\times\mathrm{V}, L(\mathrm{U},\mathrm{V}))$ under $(\mathbf{H}_{②})$.

The integrals $\int f(y_s,\lambda)\,\mathrm{d}\mathbf{x}_s$, $\int f(z_s,\lambda)\,\mathrm{d}\mathbf{x}_s$ and $\int G(y_s,z_s)\,\mathrm{d}\mathbf{x}_s$ are all well defined.

We have seen that $\int F(y_s,z_s)\,\mathrm{d}\mathbf{x}_s$ is the rough integral associated to the almost additive functional $(\widehat{F}_{s,t})_{(s,t)\in\mathbb{T}}$ defined by

$$\begin{aligned}
\widehat{F}_{s,t} &= F(y_s,z_s)x^1_{s,t} + \nabla_y F(y_s,z_s) y \ltimes \mathbf{x}_{s,t} + \nabla_z F(y_s,z_s) z \ltimes \mathbf{x}_{s,t} \\
&= f(z_s,\mu)x^1_{s,t} + \nabla_y f(z_s,\mu) z \ltimes \mathbf{x}_{s,t} - f(y_s,\lambda)x^1_{s,t} - \nabla_y f(y_s,\lambda) y \ltimes \mathbf{x}_{s,t} \\
&\qquad + G(y_s,z_s)x^1_{s,t} + \nabla_y G(y_s,z_s) y \ltimes \mathbf{x}_{s,t} + \nabla_z G(y_s,z_s) z \ltimes \mathbf{x}_{s,t}.
\end{aligned}$$

Hence, using the linearity of the sewing map,

$$\int_r^t F(y_s,z_s)\,\mathrm{d}\mathbf{x}_s = \int_r^t f(z_s,\mu)\,\mathrm{d}\mathbf{x}_s - \int_r^t f(y_s,\lambda)\,\mathrm{d}\mathbf{x}_s - \int_r^t G(y_s,z_s)\,\mathrm{d}\mathbf{x}_s. \tag{34}$$

On the other hand, both $\nabla_\lambda f(z_s,\lambda)x^1_{s,t}\cdot(\mu-\lambda)x^1_{s,t}+\nabla^2_{\lambda y}f(z_s,\lambda)\cdot(\mu-\lambda)z\ltimes\mathbf{x}_{s,t}$ and

$$\begin{aligned}A_3(s,t) := \nabla_y f(y_s,\lambda)(z_s-y_s)x^1_{s,t} + \nabla^2_{yy}f(y_s,\lambda)(z_s-y_s)y\ltimes\mathbf{x}_{s,t}\\ - \nabla_y f(y_s,\lambda)y\ltimes\mathbf{x}_{s,t} + \nabla_y f(y_s,\lambda)z\ltimes\mathbf{x}_{s,t}\end{aligned}$$

are almost rough paths associated respectively to the rough integrals $\int \nabla_\lambda f(z_s,\lambda)\cdot(\mu-\lambda)\,\mathrm{d}\mathbf{x}_s$ and $\int \nabla_y f(y_s,\lambda)(z_s-y_s)\,\mathrm{d}\mathbf{x}_s$ with values in V. Yet

$$A_3(s,t) = \nabla_y f(y_s,\lambda)\Delta_s x^1_{s,t} + \nabla^2_{yy}f(y_s,\lambda)\Delta_s y\ltimes\mathbf{x}_{s,t} + \nabla_y f(y_s,\lambda)\Delta\ltimes\mathbf{x}_{s,t}.$$

Again with the linearity of the sewing map,

$$\int_r^t G(y_s,z_s)\,\mathrm{d}\mathbf{x}_s = \int_r^t \nabla_y f(y_s,\lambda)\Delta_s\,\mathrm{d}\mathbf{x}_s + \int_r^t \nabla_\lambda f(z_s,\lambda)(\mu-\lambda)\,\mathrm{d}\mathbf{x}_s.$$

Relation (11) follows from (34). □

5.6 Proof of Proposition 4

Proof of Proposition 4. For $L\in \mathrm{P}(T,\mathbf{x},\mathrm{V})$, let us write

$$(\nabla_y f(y_s,\lambda)-\nabla f(z_s,\mu))\cdot L_s = k(y_s,z_s)\cdot L_s + \ell(z_s,L_s)$$
$$\text{with } \begin{cases} k(y_z,z_s) := (\nabla_y f(y_s,\lambda)-\nabla f(z_s,\lambda)),\\ \ell(z_s,L_s) := (\nabla_y f(z_s,\lambda)-\nabla f(z_s,\mu))\cdot L_s.\end{cases}$$

Using Remark 3, Proposition 5 as well as the linearity of ℓ with respect to L,

$$\begin{aligned}\left\|\int \ell(z_s,L_s)\,\mathrm{d}\mathbf{x}_s\right\|_{\mathbf{x}} \preccurlyeq \big(|\nabla_y f(z_0,\lambda)-\nabla_y f(z_0,\mu)| + \|\nabla_y f(\cdot,\lambda)-\nabla_y f(\cdot,\mu)\|_\infty\\ + \|\nabla^2_{yy}f(\cdot,\lambda)-\nabla^2_{yy}f(\cdot,\mu)\|_\infty + H_\gamma(\nabla^2_{yy}f(\cdot,\lambda)-\nabla^2_{yy}f(\cdot,\mu))\big)\|L\|_{\mathbf{x}}.\end{aligned}$$

Since for $k=1,2$,

$$\nabla^k_{y^k}f(\cdot,\lambda)-\nabla^k_{y^k}f(\cdot,\mu) = \int_0^1 \nabla^{k+1}_{y^k\lambda}f(\cdot,\lambda+\tau\mu)(\lambda-\mu)\,\mathrm{d}\tau$$

owing to our regularity hypotheses on f,

$$\left\|\int \ell(z_s,L_s)\,\mathrm{d}\mathbf{x}_s\right\|_{\mathbf{x}} \leq |\mu-\lambda|\|L\|_{\mathbf{x}}.$$

Regarding the construction of $\int k(y_z, z_s) \cdot L_s \, \mathrm{d}x_s$, let us consider

$$K_{s,t} := k(y_s, z_s) \cdot L_s x^1_{s,t} + k^\sharp_{s,t} L_s + k(y_z, z_s) L \ltimes \mathbf{x}_{s,t},$$

where $L \ltimes \mathbf{x}$ is constructed from Proposition 6 in Appendix and

$$k^\sharp_{s,t}\nu := \nabla_y k(y_s, z_s) \cdot \nu y \ltimes \mathbf{x}_{s,t} + \nabla_z k(y_s, z_s) \cdot \nu z \ltimes \mathbf{x}_{s,t}, \ \nu \in \mathrm{V}.$$

It then follows that

$$K_{r,s,t} = (k(y_r, z_r) - k(y_s, z_s)) \cdot L_r x^1_{s,t} + k^\sharp_{r,s,t} \cdot L_r - k^\sharp_{s,t} \cdot L_{r,s}$$

Actually, the term $(k(y_r, z_r) - k(y_s, z_s)) x^1_{r,s} + k^\sharp_{r,s}$ is the same as $\widehat{H}_{r,s}$ in the proof of Proposition 2 with $\nabla_\lambda f$ replaced by $\nabla_y f$. The same computations then arise to get that

$$\begin{aligned} \|k(y_\cdot, z_\cdot)\|_p &\preccurlyeq |\Delta_0|^{\delta-1} + |\mu - \lambda|^{\delta-1}, \\ \|k^\sharp\|_{p/2} &\preccurlyeq |\Delta_0|^{\delta-1} + |\mu - \lambda|^{\delta-1} \\ \|(k(y_r, z_r) - k(y_s, z_s)) x^1_{s,t} + k^\sharp_{r,s,t}\| &\preccurlyeq |\Delta_0|^{\delta-1} + |\mu - \lambda|^{\delta-1}, \end{aligned}$$

with δ defined by (6). One get easily that

$$\left\| \int k(y_s, z_s) \cdot L_s \, \mathrm{d}\mathbf{x}_s \right\|_{\mathbf{x}} \preccurlyeq \|K\|_{\star \mathbf{x}} \leq (|\Delta_0|^{\delta-1} + |\mu - \lambda|^{\delta-1}) \|L\|_{\mathbf{x}}.$$

This proves Proposition 4. □

6 Proof of Theorem 2

Proof of Theorem 2.1). That $(\mathsf{A}^{\lambda,a}_{t,s})_{(s,t)\in\mathbb{T}}$ is an almost left p-rough resolvent under $(\mathbf{H}_{①})$ or $(\mathbf{H}_{②})$ is actually proved in Proposition 6.5 of [13].

It gives then rise to a left p-rough lift $(\mathsf{L}^{\lambda,a}_{t,s})_{(s,t)\in\mathbb{T}}$, and it is clear from the proof of Proposition 6.5 in [13] that $\mathsf{L}^{\lambda,a}$ is a controlled rough path with $\|\mathsf{L}^{\lambda,a}\|_{\mathbf{x}} \preccurlyeq 1$. □

Proof of Theorem 2.2). Let us set

$$\mathsf{B}^{\lambda,a}_{t,s} = \mathsf{B}^\dagger_s x^1_{s,t} + \mathsf{B}^\sharp_{s,t} \text{ with } \mathsf{B}^\dagger_s = \nabla_\lambda f(y_s, \lambda) \text{ and } \mathsf{B}^\sharp_{s,t} := \nabla^2_{\lambda y} f(y_s, \lambda) y \ltimes \mathbf{x}_{s,t}.$$

It follows that $\|\mathsf{B}^\dagger\|_p \preccurlyeq 1$ and $\|\mathsf{B}^\sharp\|_{p/2} \preccurlyeq 1$.

In addition,

$$\begin{aligned} \mathsf{B}^{\lambda,a}_{t,s,r} := (\nabla_\lambda f(y_s, \lambda) - \nabla_\lambda f(y_r, \lambda) - \nabla^2_{\lambda y} f(y_r, \lambda) y_{r,s}) x^1_{s,t} \\ + (\nabla^2_{\lambda y} f(y_s, \lambda) - \nabla^2_{\lambda y} f(y_r, \lambda)) y \ltimes \mathbf{x}_{s,t}. \end{aligned}$$

Hence, $\mathsf{B}^{\lambda,a}$ is an almost controlled rough path. Its associated controlled rough path $\mathsf{B}^{\lambda,a}$, which from its very construction $\int \nabla_\lambda f(y_s,\lambda)\,\mathrm{d}\mathbf{x}_s$. From Proposition 5, $\left\|\mathsf{B}^{\lambda,a}\right\|_{\mathbf{x}} \preccurlyeq 1$. Proposition 7 of Appendix A allows one to conclude. □

Proof of Theorem 2.3). Using Lemma 1 applied to $B^{\lambda,a}_{t,s,r} - B^{\mu,a}_{t,s,r}$, the Lipschitz continuity of the Itô map, the linearity of the sewing map and Proposition 5, we obtain easily that

$$\left\|\mathsf{B}^{\lambda,a} - \mathsf{B}^{\mu,b}\right\|_p \preccurlyeq \left\|\mathsf{B}^{\lambda,a} - \mathsf{B}^{\mu,b}\right\|_{\mathbf{x}} \preccurlyeq \begin{cases} |\Delta_0| + |\mu-\lambda| & \text{under } (\mathbf{H}_{①}),\\ |\Delta_0|^{\delta-1} + |\mu-\lambda|^{\delta-1} & \text{under } (\mathbf{H}_{②}) \end{cases} \tag{35}$$

when $2+\kappa\gamma > p$. Similarly, $\mathsf{L}^{\lambda,a}$ is a controlled rough path. With $\mathsf{L}^{\lambda,a})^\dagger_t = \nabla_y f(y^{\lambda,a}_t)\mathsf{L}^{\lambda_a}_t$, It satisfies from its very definition

$$|\mathsf{L}^{\lambda,a} \ltimes \mathbf{x}_{s,t} - (\mathsf{L}^{\lambda,a})^\dagger x^2_{s,t}| \preccurlyeq \omega(s,t)^\theta)$$

and, by plugging the above control in the expression of $\mathsf{A}^{\lambda,a}_{t,s}$ in (7),

$$\left|\mathsf{L}^{\lambda,a}_{s,t} - \mathsf{Id} - \nabla_y f(y_s,\lambda)\mathsf{L}^{\lambda_a} x^1_{s,t} - \nabla_y f(y_s,\lambda)(\mathsf{L}^{\lambda,a})^\dagger x^2_{s,t} - \nabla^2_{yy} f(y_s,\lambda)\mathsf{L}^{\lambda,a} y \ltimes x_{s,t}\right| \preccurlyeq \omega(s,t)^\theta$$

for some $\theta > 1$.

From this and the construction of the integral $K(L)$ given in the proof of Proposition 4, $Z_t := \mathsf{L}^{\mu,b} - \mathsf{L}^{\lambda,a}$ is solution to the perturbed linear RDE

$$Z_t = \int_0^t \nabla_y f(y^{a,\lambda}_s,\lambda) Z_s\,\mathrm{d}\mathbf{x}_s - K(\mathsf{L}^{\mu,b})_t$$

where

$$K(\mathsf{L}^{\mu,b})_t = \int_0^t [\nabla_y f(y^{b,\mu}_s,\mu) - \nabla_y f(y^{a,\lambda}_s,\lambda)] \cdot \mathsf{L}^{\mu,b}_s\,\mathrm{d}\mathbf{x}_s.$$

Then from Proposition 6.4 of [13] and Proposition 4,

$$\left\|\mathsf{L}^{\lambda,a} - \mathsf{L}^{\mu,b}\right\|_p \preccurlyeq \left\|\mathsf{L}^{\lambda,a} - \mathsf{L}^{\mu,b}\right\|_{\mathbf{x}} \preccurlyeq \left\|K(\mathsf{L}^{\mu,b})\right\|_{\mathbf{x}} \preccurlyeq \begin{cases} |\Delta_0| + |\mu-\lambda| & \text{under } (\mathbf{H}_{①}),\\ |\Delta_0|^{\delta-1} + |\mu-\lambda|^{\delta-1} & \text{under } (\mathbf{H}_{②}). \end{cases}$$

This proves the regularity of $(\lambda,a) \mapsto \mathsf{L}^{\lambda,a}$. □

7 Proof of Theorem 1 and Corollary 1

Proof of Theorem 1. Let us rewrite (11) of Proposition 3 as

$$\Delta_t = b - a + \int_0^t \nabla_y f(y_s, \lambda)\Delta_s \,\mathrm{d}\mathbf{x}_s + \mathsf{B}_{0,t}(\mu - \lambda) + \mathsf{P}_{0,t} \tag{36}$$

with

$$\mathsf{B}_{s,t} := \int_s^t \nabla_y f(y_r, \lambda)\,\mathrm{d}\mathbf{x}_r \text{ and } \mathsf{P}_{s,t} := \int_s^t F(y_r, z_r)\,\mathrm{d}\mathbf{x}_r + \int_s^t H(y_r, z_r)(\mu-\lambda)\,\mathrm{d}\mathbf{x}_r.$$

By linearity and subtracting (5) to (36), $D_t := \Delta_t - N(a, b, \lambda, \mu)_t$ is solution to

$$D_t = \int_0^t \nabla_y f(y_s, \lambda) D_s \,\mathrm{d}\mathbf{x}_s + \mathsf{P}_{0,t}.$$

Using Theorem 2 and Proposition 7 in Appendix A, there exists left a p-rough lift $\underleftarrow{\mathsf{P}}$ of P with $\|\underleftarrow{\mathsf{P}}\|_p \preccurlyeq \|\mathsf{P}\|_{\mathbf{x}}$. In addition, using the notations of [13],

$$D_t = \int_0^t \mathsf{L}^{\lambda,a}_{t,r}\,\mathrm{d}^{\mathrm{b}}\mathsf{P}_r \text{ and } \|D\|_p \preccurlyeq \|\underleftarrow{\mathsf{P}}\|_p \preccurlyeq \|\mathsf{P}\|_{\mathbf{x}}.$$

This proves the result with the help of Propositions 1 and 2. □

Proof of Corollary 1. To simplify the notations, let us set $N^0 := N(a, a + \epsilon, \lambda, \lambda + \nu)$ and $N^1 := N(a, b + \epsilon, \mu, \mu + \nu)$ when $a, a + \epsilon, b, b + \epsilon \in \Lambda_{\mathrm{s}}$ and $\lambda, \lambda + \nu, \mu, \mu + \nu \in \Lambda_{\mathrm{p}}$. Let us consider

$$N^{1/2}_t = \epsilon + \int_0^t \nabla_y f(y^{\lambda,a}_s) N^{1/2}_s \,\mathrm{d}\mathbf{x}_s + \mathsf{B}^{b,\mu}_{t,0}\nu.$$

Thanks to Proposition 7, a left p-rough lift of $\mathsf{B}^{b,\mu}$ with respect to $\mathsf{L}^{a,\lambda}$, so that $N^{1/2}$ exists. In addition, from the linearity with respect to the perturbation, $D^{0,1/2}_t := (N^{1/2} - N^0)_t$ is solution to

$$D^{0,1/2}_t = \int_0^t \nabla_y f(y^{\lambda,a}_s) D^{0,1/2}_t + (\mathsf{B}^{b,\mu}_{t,0} - \mathsf{B}^{a,\lambda}_{t,0})\nu.$$

With (35), Proposition 7 and the results in [13], it holds that

$$\|D^{0,1/2}\|_p \preccurlyeq (|b - a|^{\delta-1} + |\mu - \lambda|^{\delta-1})|\nu|.$$

It remains now to study $D^{1/2,1}_t = N^1_t - N^{1/2}_t$ which is defined as

$$\begin{aligned} D^{1/2,1}_t &= \int_0^t \nabla_y f(y^{\mu,b}_s, \mu) N^1_s \,\mathrm{d}\mathbf{x}_s - \int_0^t \nabla_y f(y^{\lambda,a}_s, \lambda) N^{1/2}_s \,\mathrm{d}\mathbf{x}_s \\ &= (\mathsf{L}^{b,\mu} - \mathsf{L}^{a,\lambda})\epsilon. \end{aligned}$$

The results follows from Theorem 2.3). □

A Controlled rough paths and continuity of a perturbed rough differential equation

For a rough path $\mathbf{x}$, we have introduced the notion of controlled rough path y as a path whose increments are decomposed as $y_{s,t} = y^\dagger_s \mathbf{x}^1_{s,t} + y^\sharp_{s,t}$ with $\|y\|_{\mathbf{x}} < +\infty$, where $\|\cdot\|_{\mathbf{x}}$ has been defined by (4). Besides, $\|y\|_p \preccurlyeq \|y\|_{\mathbf{x}}$.

The norm $\|\cdot\|_{\mathbf{x}}$ is trivially extended to family $(z_{s,t})_{(s,t)\in\mathbb{T}}$ with decomposition $z_{s,t} = z^\dagger_s \mathbf{x}^1_{s,t} + z^\sharp_{s,t}$.

Proposition 5. *Let $(z_{s,t})_{(s,t)\in\mathbb{T}}$ be a family, called an* almost controlled rough path*, such that*

$$z_{s,t} = z^\dagger_s \mathbf{x}^1_{s,t} + z^\sharp_{s,t} \text{ with } \|z\|_{\star\mathbf{x}} := \|z\|_{\mathbf{x}} + \sup_{0\leq r<s<t\leq T} \frac{|z_{r,s,t}|}{\omega(r,t)^\theta} < +\infty \text{ for } \theta > 1.$$

Then $(z_{s,t})_{(s,t)\in\mathbb{T}}$ is a almost additive functional. Its associated path $(y_s)_{s\in[0,T]}$ through the additive sewing lemma (with $y_0 = 0$) is a controlled rough path with $y^\dagger = z^\dagger$ and

$$\|y\|_{\mathbf{x}} \leq K\|z\|_{\star\mathbf{x}}$$

for a universal constant K depending only on ω, T and p.

Proof. From the additive sewing lemma, $|y_{s,t} - z_{s,t}| \leq KL\omega(s,t)^\theta$ for some universal constant K depending only on ω, T and p. The control over $\|y\|_{\mathbf{x}}$ follows easily with $y^\dagger = z^\dagger$ and $y^\sharp_{s,t} = y_{s,t} - z^\dagger_{s,t}\mathbf{x}^1_{s,t}$. □

A controlled rough path y could be integrated against a rough path $\mathbf{x}$.

Proposition 6 ([22], [18, Theorem 4.9])**.** *There exists a linear map $y \mapsto \int_0^\cdot y_s \,\mathrm{d}\mathbf{x}_s$ from $\mathrm{P}(T, \mathbf{x}, L(\mathrm{U}, \mathrm{V}))$ to $\mathrm{P}(T, \mathbf{x}, \mathrm{V})$ such that*

$$\left\| \int_0^\cdot y \,\mathrm{d}\mathbf{x} \right\|_{\mathbf{x}} \leq C(1 + \|\mathbf{x}\|_p)\|y\|_{\mathbf{x}}$$
$$\text{and } \left| \int_s^t y_u \,\mathrm{d}\mathbf{x}_u - y_s\mathbf{x}^1_{s,t} - y^\dagger_s \mathbf{x}^2_{s,t} \right| \leq C\omega(s,t)^{3/p}\|\mathbf{x}\|_p\|y\|_{\mathbf{x}},$$

for all $(s,t) \in \mathbb{T}$ and some universal constant C.

By definition, a cross-iterated integral between two paths y and x is $\int_s^t y_{s,r} \,\mathrm{d}x_r$. This notion is easily extended to controlled rough paths.

Corollary 4. *For $y \in \mathrm{P}(T, \mathbf{x}, \mathrm{V})$, then there exists a cross-iterated integral $y\ltimes\mathbf{x} : \mathbb{T} \to \mathrm{V} \otimes \mathrm{U}$ with*

$$|y\ltimes\mathbf{x}_{s,t}| \leq C\|y\|_{\mathbf{x}}\|\mathbf{x}\|_p\omega(s,t)^{2/p}, \ \forall (s,t) \in \mathbb{T}. \tag{37}$$

Proof. Let us set $y \ltimes \mathbf{x}_{r,t} := \int_r^t y_{r,s}\,\mathrm{d}\mathbf{x}_s$. Since for a constant path z, $\int_r^t z\,\mathrm{d}\mathbf{x}_s = zx^1_{r,t}$, it is easily seen that $y \ltimes \mathbf{x}$ is a cross-iterated integral, and (37) follows easily. □

Remark 3. In practice, Proposition 5 is used the following way: For $y \in \mathrm{P}(T, \mathbf{x}, \mathrm{V})$ and $f : \mathrm{V} \mapsto L(\mathrm{U}, \mathrm{V})$ be continuously differentiable with a γ-Hölder derivative, $1 + \gamma > p$,, let us set

$$\widehat{f}_{s,t} := \widehat{f}^\dagger_s x_{s,t} + \widehat{f}^\sharp_{s,t} \text{ with } \widehat{f}^\dagger_s := f(y_s) \text{ and } \widehat{f}^\sharp_{s,t} := \nabla f(y_s) y \ltimes \mathbf{x}_{s,t}.$$

Then $\widehat{f}$ is an almost controlled rough path and

$$\|\widehat{f}\|_{\star\mathbf{x}} \leq |f(y_0)| + (\|\nabla f\|_\infty + H_\gamma(\nabla f))\,\|y\|_{\mathbf{x}}.$$

The associated controlled rough path is denoted $\int f(y_s)\,\mathrm{d}\mathbf{x}_s$.

The theory of linear rough differential equation developed in [13] also fits very well in this framework.

For example, let us consider a p-rough resolvent $(\mathsf{A}_{t,s})_{(s,t)\in\mathbb{T}}$ on V which is controlled rough path with decomposition $\mathsf{A}_{t,s} = \mathsf{Id} + \mathsf{A}^\dagger_s x^1_{s,t} + \mathsf{A}^\sharp_{t,s}$.

We set $\mathsf{A}^1_{t,s} := \mathsf{A}^\dagger_s x^1_{s,t}$, which is the "main part" of the operator. We have seen in [13, Sect. 6.2] how to construct such an operator with $\mathsf{A}^\dagger_s = \nabla f(y_s)$.

A path $B : [0, T] \to \mathrm{V}$ is said to admit a *left p-rough lift with respect to* A if there exists a family $(\underleftarrow{B}_{t,s})_{(s,t)\in\mathbb{T}}$ such that $\underleftarrow{B}_{t,s} = B_{t,s} + B^{(2)}_{t,s}$ with $B_{t,s} = B_t - B_s$ and

$$\|\underleftarrow{B}\|_p := \|B\|_p + \|B^{(2)}\|_{p/2} + \sup_{\substack{0\leq r\leq s\leq t\leq T\\ r<t}} \frac{|\underleftarrow{B}_{t,r} - \underleftarrow{B}_{t,s} - \underleftarrow{B}_{r,s} - \mathsf{A}_{t,s}B_{s,r}|}{\omega(r,t)^\theta} < +\infty$$

for some $\theta > 1$. As shown in [13], the existence of a left p-rough lift allows on to construct the solution of a perturbed linear RDE $Y_t = \int_0^t \mathrm{d}\mathsf{A}_s Y_s + B_{t,0}$.

Proposition 7. *Let* $(\mathsf{A}_{t,s})_{(s,t)\in\mathbb{T}}$ *be as above. Let* $B : [0, T] \to \mathrm{V}$ *be a rough path controlled by* $\mathbf{x}$ *with decomposition* $B_{t,s} = B^\dagger_s x^1_{s,t} + B^\sharp_{t,s}$ *(for consistency with* [13]*, we have inverted the order of the indices of* B*). Then there exists a p-rough lift* $\underleftarrow{B} := B + B^2$ *with respect to* A *and*

$$\|\underleftarrow{B}\|_p \leq C\|\mathsf{A}\|_{\mathbf{x}}\|B\|_{\mathbf{x}}, \tag{38}$$

where C depends only on $\mathbf{x}$*,* ω*,* T *and* p*.*

Proof. Set $\underleftarrow{B}_{t,s} := B_{t,s} + \mathsf{A}^\dagger_s B^\dagger_s x^2_{s,t}$. Then

$$\begin{aligned}\underleftarrow{B}_{t,r} - \underleftarrow{B}_{s,r} - \underleftarrow{B}_{t,s} - \mathsf{A}^1_{t,s} B_{s,r}\\ = -\mathsf{A}^\sharp_{t,s} B_{s,r} + \mathsf{A}^\dagger_r B^\sharp_{t,s} x^1_{t,s} - A^\dagger_{s,r} B_{s,r} x^1_{s,t} + (\mathsf{A}^\dagger_r B^\dagger_r - \mathsf{A}^\dagger_s B^\dagger_s) x^2_{s,t}.\end{aligned}$$

Since for any path z on $[0,T]$ of finite p-variation with respect to ω, $\|z\|_\infty \leq |z_0| + \|z\|_p \omega(0,T)^{1/p}$, $\underleftarrow{B}$ is a p-rough lift with respect to A and (38) holds. □

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