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Sensitivity of rough differential equations*

Laure Coutin[†] Antoine Lejay[‡]

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Abstract

In the context of rough paths theory, we study the regularity of the Itô map with respect to the starting point, the coefficients and perturbation of the driving rough paths. In particular, we show that the Itô map is differentiable with a Hölder or Lipschitz continuous derivatives. With respect to the current literature on the subject, our proof rely on perturbation of linear Rough Differential Equations and our approach under the most general hypotheses on the driving path and on the vector field.

Keywords: rough paths, rough differential equation; Itô map; Malliavin calculus; flow of diffeomorphisms.

1 Introduction

The theory of rough paths has now become a standard tool to deal with stochastic differential equations (SDE) driven by continuous processes other than the Brownian motion such as the fractional Brownian motion. Even for standard SDE, it has been proved to be a convenient tool for dealing with the large deviations or for numerical purposes. We refer the reader to

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[17, 19, 33, 34, 38, 39] for a presentation of this theory with several different points of view.

A *rough path* \mathbf{x} is an extension in a non-commutative tensor space of a path of finite p -variation controlled by ω , that is a continuous path x from $[0, T]$ with values in a Banach space U with $|x_t - x_s| \leq C\omega(s, t)^{1/p}$ for a function $\omega : \{0 \leq s < t \leq T\} \rightarrow \mathbb{R}_+$ which is continuous close to its diagonal and such that $\omega(s, r) + \omega(r, t) \leq \omega(s, t)$. This extension \mathbf{x} is defined through algebraic and analytical properties but is not uniquely associated to x . Here, we consider only values of p in $[2, 3)$. For $p < 2$, then the theory is simpler and relies on Young integrals [32].

The idea behind this theory is that when using \mathbf{x} and given a vector field $f : V \rightarrow L(U, V)$ for a Banach space V , then

$$y_t = a + \int_0^t f(y_s) d\mathbf{x}_s \quad (1)$$

is well defined provided that f is regular enough. This equation is called a *rough differential equation* (RDE). The success of the rough paths theory is that for a smooth path x , then there is a natural way to construct \mathbf{x} as the iterated integrals of x . In this case, the solution to (1) corresponds to the solution to the ordinary differential equation $y_t = a + \int_0^t f(y_s) dx_s$. The theory of rough paths provides us with natural extensions of controlled differential equations.

The map $\mathfrak{I} : x \mapsto y$, called the *Itô map*, is continuous when one uses the topology induced by the p -variation distance with respect to ω (one may also use Hölder norms when $\omega(s, t) = t - s$). In particular, it may be shown that \mathfrak{I} is locally Lipschitz continuous with respect to the path, the vector field and the starting point [30, 31]. It seems that the Hölder continuity of the derivative of the Itô map has never been established.

However, one may be willing to study deeper the differentiability properties in the Itô map. This point is very important toward applications. For SDE, Malliavin calculus opens the door to existence of a density and its regularity [43, 45], large deviation results [23], Monte Carlo methods [18, 43], ...

There are several ways to consider a perturbation of the Itô map:

- ★ Perturbation of the starting point.
- ★ Perturbation of the rough path \mathbf{x} by a path of finite q -variation h with $1/p + 1/q > 1$.
- ★ Perturbation of the parameter when we consider a family of vector field $f(\cdot, \lambda)$ with a regular variation.

The flow properties have already been subject to many articles:

- ★ In [41], T. Lyons and Z. Qian have studied the flow property for solutions to $y_t = a + \int_0^t f(y_s) d\mathbf{x}_s + \int_0^t g(y_s) dh_s$ for a “regular” path h subject to a perturbation.
- ★ In [40], T. Lyons and Z. Qian showed that the Itô map provides a flow of diffeomorphisms when the driving rough path is geometric.
- ★ In [2], I. Bailleul give an approach which is a non-linear interpretation of [15], where the RDE is constructed through its flow.
- ★ In the case $p < 2$, the Itô map has a Hölder continuous derivative [32, 37].
- ★ T. Lyons and Z. Qian [42], S. Aida [1] and more recently Z. Qian and J. Tudor [47] have studied the perturbation of the Itô map when the rough path is perturbed by a regular path h and the structure of the resulting tangent space.
- ★ The book of P. Friz and N. Victoir [17] also presents flow properties for geometric rough paths.
- ★ Y. Inahama and H. Kawabi have studied various aspects of stochastic Taylor developments of solutions to $y_t^\epsilon = a + \int_0^t \epsilon f(y_s^\epsilon) d\mathbf{x}_s + \int_0^t g(y_s^\epsilon) dh_s$ in term of ϵ around the solutions to $z_t = a + \int_0^t g(z_s) dh_s$ [23, 26–28]. See also [50] for some bounds and applications to the fractional Brownian motion.
- ★ The special case of SDE driven by fractional Brownian motion has been subject to a lot of attentions [4–8, 10, 13, 14, 22, 24, 36, 44, 46]. Since integrability is the key to derive some integration by parts formula in Malliavin type calculus, several articles deal with moments estimates for solutions to linear equations driven by Gaussian rough paths [3, 9, 16, 21, 25, 48].

The goal of this article is to present the regularity properties of the Itô map. In particular, our approach is “rough” in the sense that we do not rely on approximations by smooth paths as in [17] or [47]. Our approach may then be applied for any non-geometric rough paths as well as in infinite dimension.

In particular, we show that the Itô map \mathfrak{I} which transform a rough path \mathbf{x} of finite p -variation with $p \in [2, 3)$ into the solution to $y_t = a + \int_0^t f(y_s, \lambda) d\mathbf{x}_s$ is Fréchet differentiable with a locally Hölder or Lipschitz continuous derivative with respect to a and λ , according to the regularity of f . Besides, we also show that the Itô map is Fréchet differentiable with a locally Hölder or Lipschitz continuous derivative when the rough path \mathbf{x} is perturbed by a path h of finite q -variation with $1/p + 1/q > 1$.

The key tool is the study of the perturbed linear RDE which we develop in the companion paper [11]. By this, we mean equations of type

$$y_t = y_0 + \int_0^t d\mathbf{A}_s y_s + z_t \quad (2)$$

for an operator valued rough path \mathbf{A} . We endow that to construct the solution to (2), it is necessary to know not only z but also some equivalent in some sense of the cross-iterated integrals between z and \mathbf{A} . However, it is not necessary to know a rough path \mathbf{z} lying above z . This consideration simplifies the analysis.

Outline. In Section 2, we introduce some notations. In Section 3, we introduce the notion of linear RDE and in Section 4, we show their relation with equations encountered while one derives the Itô map. Our main results on the derivation of the Itô map with respect to the starting point and the vector field are contained in Section 5 and 6. In Section 7, we show similar results when the driving rough path of finite p -variation is perturbed by a path of finite q -variation with $1/p + 1/q > 1$.

2 Notations

Let U and V be Banach spaces. The norms on such spaces are denoted by $|\cdot|$. For two Banach spaces U and V , we use a norm $|\cdot|$ on the tensor product $U \otimes V$ which is such that $|u \otimes v| \leq |u| \cdot |v|$ for $(u, v) \in U \times V$.

Let us fix a time horizon $T > 0$.

A path $x : [0, T] \rightarrow U$ is said to be of *finite p -variation controlled by ω* if $x_{s,t} := x_t - x_s$ satisfies $|x_{s,t}| \leq C\omega(s, t)^{1/p}$ for all $0 \leq s \leq t \leq T$. The best constant such that this inequality holds is denoted by $\|x\|_p$.

If $\{x_{s,t}\}_{0 \leq s \leq t \leq T}$ is a family indexed by (s, t) and there exists a constant $C < +\infty$ such that $|x_{s,t}| \leq C\omega(s, t)^{1/p}$, then the best constant such that this inequality holds is still denoted by $\|x\|_p$.

Let \mathbf{x} be a rough path of finite p -variation that lies above a path x with values in a Banach space U with $p \in [2, 3)$. The component of \mathbf{x} in U is denoted by x^1 , while its component in $U \otimes U$ is denoted by x^2 . We set

$$\|\mathbf{x}\| := \sup_{0 \leq s < t \leq T} \max \left\{ \frac{|x_{s,t}^1|}{\omega(s, t)^{1/p}}, \frac{|x_{s,t}^2|}{\omega(s, t)^{2/p}} \right\} = \max\{\|x^1\|_p, \|x^2\|_{p/2}\}.$$

Provided that a map $f : V \mapsto L(U, V)$ satisfies appropriate conditions, there exists a unique solution to the rough differential equation (RDE) $y_t = y_0 + \int_0^t f(y_s) d\mathbf{x}_s$. For this, we assume either

- (A) The vector field f is continuous and bounded with bounded first and second derivatives, and $\nabla^2 f$ is γ -Hölder continuous with $2 + \gamma > p$;

or

(B) The space V is finite-dimensional, the vector field f is continuous with bounded first and second derivative, and $f\nabla f$ has a bounded derivative which is γ -Hölder continuous with $2 + \gamma > p$.

Note that these two sets of hypotheses leads to two different notions of solutions [30, 31].

Let us consider linear subspaces X and Y of $T_2(V)$ such that $T_2(V) = X \oplus Y$ with $T_2(V) = \mathbb{R} \oplus V \oplus (V \otimes V)$ and $\mathbb{R} \subset X$. This also means that X is the space $T_2(V)$ quotiented by the relation $x \equiv y$ if $x - y \in Y$. The projection from V onto X along Y is denoted by π_X .

Let us consider a family $(\mathbf{P}_{s,t})_{0 \leq s \leq t \leq T}$ with values in X such that either (a) $X = V$ or (b) $\mathbb{R} \subset X$ and $\pi_{\mathbb{R}}(\mathbf{P}_{s,t}) = 1$. Then \mathbf{P} is an *almost rough path* if for some constant C ,

$$\|\mathbf{P}\| < +\infty \text{ and } |\mathbf{P}_{s,r,t}| \leq C\omega(s,t)^\theta, \quad \forall 0 \leq s \leq r \leq t \leq T \text{ with } \theta > 1,$$

where $\mathbf{P}_{s,r,t} := \pi_X(\mathbf{P}_{s,t} - \mathbf{P}_{s,r} \otimes \mathbf{P}_{r,t})$. To such an almost rough path is associated a unique *rough path* $(\mathbf{Q}_t)_{t \in [0,T]}$ which is a path with values in X such that $\mathbf{Q}_{s,t} := \pi_X(\mathbf{Q}_s^{-1} \otimes \mathbf{Q}_t^{-1})$ satisfies $|\mathbf{Q}_{s,t} - \mathbf{P}_{s,t}| \leq CK\omega(s,t)^\theta$ for some universal constant K that depends only on $\omega(0,T)$ and θ [15, 19, 33, 34, 38]. This is the main technical point of the theory of rough paths and is sometimes called the *sewing lemma*. We denote by \mathfrak{S}_X the map giving a rough path from an almost rough path.

Remark 1. When $X = V$, then $\mathfrak{S}_V : \mathbf{P} \mapsto \mathbf{Q}$ is linear.

Given two paths y and x of finite p -variation controlled by ω respectively with values in V and U , a *cross-iterated integral* $y \ltimes x$ between y and x is a family $(y \ltimes x_{s,t})_{0 \leq s \leq t \leq T}$ with values in $U \otimes V$ such that

$$y \ltimes x_{s,t} = y \ltimes x_{s,r} + y \ltimes x_{r,t} + y_{s,r} \otimes y_{r,t} \quad (3)$$

$$|y \ltimes x_{s,t}| \leq C\omega(s,t)^{2/p}. \quad (4)$$

For a rough path \mathbf{x} and a solution y to $y_t = y_0 + \int_0^t f(y_s) d\mathbf{x}_s$ with values in a Banach space V , a natural choice of $y \ltimes x$ is given by

$$y \ltimes x_{s,t} := \pi_{V \otimes U}(\mathfrak{S}_X[\{1 + x_{s,t} + y_{s,t} + f(y_s) \otimes 1 \cdot x_{s,t}^2\}_{0 \leq s \leq t \leq T}]) \quad (5)$$

with $X = \mathbb{R} \oplus U \oplus V \oplus V \otimes U$. There are indeed an infinite number of choices of $y \ltimes x$ satisfying (3) and (4). However, our choice is natural in the sense that if \mathbf{x} is a smooth rough path lying above x , then y is solution to the ordinary differential equation $y_t = y_0 + \int_0^t f(y_s) dx_s$ and $y \ltimes x_{s,t} = \int_s^t (y_r - y_s) dx_r$.

From now, when y is constructed from an RDE driven by x , then we choose for $y \ltimes x$ this natural choice.

Let x and y be two paths of finite p -variation respectively with values in U and V such that $y \ltimes x$ exists. Let $g : V \mapsto L(V, W)$ be smooth enough. By definition, $\int g(y_s) d\mathbf{x}_s$ is the rough path with values in W associated to the almost rough path

$$R_{s,t} = g(y_s)x_{s,t}^1 + \nabla g(y_s)y \ltimes x_{s,t}.$$

For a rough path \mathbf{x} and y solution to the RDE $y_t = y_0 + \int_0^t f(y_s) d\mathbf{x}_s$, we denote by $\int_0^t g(y_s) d\mathbf{x}_s$ the rough path associated to R when $y \ltimes x$ has been defined through (5).

It is immediate that for g and h smooth enough, $\int g(y_s) d\mathbf{x}_s + \int h(y_s) d\mathbf{x}_s = \int (g+h)(y_s) d\mathbf{x}_s$.

3 Perturbed linear RDE

3.1 Motivations

Let us first recall some classical results about the derivative of a ordinary differential equations [20, § I-14, p. 94].

Let us consider the ordinary differential equation driven by a smooth path x : $y_t^a = a + \int_0^t f(y_s^a) dx_s$ for a smooth vector field. It is well known that $a \mapsto y_t^a$ defines a family of diffeomorphisms. Differentiating formally with respect to a , we obtain that for $Y_t(h) := \lim_{\epsilon \rightarrow 0} \epsilon^{-1}(y_t^{a+\epsilon h} - y_t^a)$, then the linear operator Y_t is solution to the linear equation

$$Y_t = \text{Id} + \int_0^t \nabla f(y_s^a) Y_s dx_s,$$

Let us consider now another smooth path h , as well as the solution to

$$y_t(\epsilon) = a + \int_0^t f(y_s(\epsilon)) dx_s + \epsilon \int_0^t f(y_s(\epsilon)) dh_s.$$

Again, a formal computation shows that $\mathfrak{D}y_t := \lim_{\epsilon \rightarrow 0} \epsilon^{-1}(y_t(\epsilon) - y_t(0))$ is solution to

$$\mathfrak{D}y_t = \int_0^t \nabla f(y_s(0)) \mathfrak{D}y_s dx_s + \int_0^t f(y_s(0)) dh_s.$$

The usual way to consider the solution to $\mathfrak{D}y_t$ is to note first that for

$$Z_t = \text{Id} - \int_0^t Z_s \nabla f(y_s^a) dx_s,$$

then $Y_t Z_t = \text{Id}$ for any $t \in [0, T]$. A variation of constant formula implies that

$$\mathfrak{D}y_t = Y_t \int_0^t Z_s f(y_s) dh_s. \quad (6)$$

This approach could be carried as well when x is a Brownian motion and the integrals are understood as Stratonovich ones [29, 45].

On the case of rough paths, the existence of a solution of a linear RDE is known for a long time [15, 17, 40]. However, one may wonder if (6) still holds in general, excepted in the Young case, that is when $p < 2$ [32, 37], or for geometric rough paths. Yet this impose to restrict ourselves to rough paths that could be naturally approximated using smooth paths [17, 40, 47]. By using an appropriate definition of the integral, we have shown in [11] that (6) holds true for non-geometric rough paths as well by using a notion of perturbed linear RDE.

3.2 Linear RDE

Linear RDE have been considered by several works in the theory of rough paths [15, 17, 40]. The theory itself took its inspiration in some algebraic approach for linear differential equations. Yet, excepted in [15], linear RDE are seen as a tool for studying flow properties, not as a mathematical object by themselves for which one may be willing to extend the results of the theory of linear ODE. Here, we consider linear RDE as in the spirit of the approach [15] and presents quickly some definition and results whose proofs may be found in the companion paper [11].

Here, there is no restriction on the range of $p \in (0, 1]$.

Definition 1 (Rough resolvent). Let V be a Banach space. We say that a family $A = (A_{s,t})_{0 \leq s \leq t \leq T}$ is a *p-rough resolvent* controlled by ω if

- (i) For any $0 \leq s \leq t \leq T$, $A_{s,t}$ belongs to $L(V, V)$ and $\|A_{s,t} - \text{Id}\| \leq C\omega(s, t)^{1/p}$ for $0 \leq s \leq t \leq T$.
- (ii) For any $0 \leq s \leq r \leq t \leq T$,

$$A_{s,t} = A_{r,t} A_{s,r}.$$

The algebraic and analytical conditions (i) and (ii) are similar to the one encountered for rough paths, excepted that the tensor product is replaced by the multiplication between operators.

Definition 2 (Almost rough semi-group). An *almost p -rough resolvent* is a family of operators $\mathbf{B} = (\mathbf{B}_{s,t})_{0 \leq s \leq t \leq T}$ with $\mathbf{B}_{s,t} \in L(V, V)$ such that for some constants L_1, L_2 and $\theta > 1$,

$$\|\mathbf{B}_{s,t} - \mathbf{B}_{r,t}\mathbf{B}_{s,r}\| \leq C\omega(s,t)^\theta \text{ and } \|\mathbf{B}_{s,t} - \text{Id}\| \leq C\omega(s,t)^{1/p}$$

for any $0 \leq s \leq r \leq t \leq T$.

The next proposition is an adaptation of the sewing lemma. In the Hölder case, that is when $\omega(s,t) = t - s$, a simple proof may be found in [15, Theorem 15].

Proposition 1. *To any almost p -rough resolvent \mathbf{B} , there exists a unique p -rough resolvent \mathbf{A} . By uniqueness, we mean that for any p -rough resolvent \mathbf{C} such that $\|\mathbf{B}_{s,t} - \mathbf{C}_{s,t}\| \leq C\omega(s,t)^\theta$ for $C > 0$, $\theta > 1$ and all $0 \leq s \leq t \leq T$, then $\mathbf{C} = \mathbf{A}$.*

Definition 3 (Linear RDE). A solution to a *linear RDE* is a continuous path $y : [0, T] \rightarrow V$ of finite p -variation controlled by ω satisfying $y_t = \mathbf{A}_{s,t}y_s$ for all $0 \leq s \leq t \leq T$ for a p -rough resolvent $\mathbf{A} = \text{Id} + \mathbf{A}^\sharp$. We also write that y is the solution to $y_t = y_s + \int_s^t d\mathbf{A}_r^\sharp y_r$.

Indeed, the solution of a linear RDE may also be understood as a solution in the sense given by A.M. Davie [12] or by M. Gubinelli [19].

Proposition 2. *Let \mathbf{B} be an almost p -rough resolvent whose associated rough resolvent is \mathbf{A} . A path y of finite p -variation controlled by ω is solution to the linear RDE $y_t = y_s + \int_s^t d\mathbf{A}_r^\sharp y_r$ if and only if there exists a family $(y_{s,t}^\dagger)_{0 \leq s \leq t \leq T}$ of elements of V such that*

$$y_t = \mathbf{B}_{s,t}y_s + y_{s,t}^\dagger \text{ with } |y_{s,t}^\dagger| \leq C\omega(s,t)^\theta, \theta > 1$$

for any $0 \leq s \leq t \leq T$.

3.3 Perturbed linear RDE

Our main theorem concerning the perturbation of linear RDE is a generalization of the variation of constant method, also called the Duhamel principle. We consider that $p \in [2, 3)$ and that \mathbf{A} is a p -rough resolvent controlled by ω decomposed as $\mathbf{A}_{s,t} = \text{Id} + \mathbf{A}_{s,t}^{(1)} + \mathbf{A}_{s,t}^{(2)}$ with $\|\mathbf{A}^{(1)}\|_p < +\infty$ and $\|\mathbf{A}^{(2)}\|_{p/2} < +\infty$.

Theorem 1. (I) *Let z be a p -rough path controlled by ω with values in V such that $\mathbf{A} \cdot \bowtie z$ exists, that is a family $\{\mathbf{A} \cdot \bowtie z_{s,t}\}_{0 \leq s \leq t \leq T}$ with values in V such that*

$$\mathbf{A} \cdot \bowtie z_{s,t} = \mathbf{A} \cdot \bowtie z_{s,r} + \mathbf{A} \cdot \bowtie z_{r,t} + \mathbf{A}_{s,r}^{(1)} z_{r,t}, \quad \forall 0 \leq s \leq r \leq t \leq T. \quad (7)$$

Then there exists a unique solution taking its values in V to the perturbed linear RDE

$$y_t = a + \int_0^t dA_s^\# y_s + z_{0,t}.$$

By this, we mean a path $y : [0, T] \rightarrow V$ of finite p -variation controlled by ω such that $y_0 = a$ and for some constant $L \geq 0$ and $\theta > 1$,

$$|y_t - A_{s,t}y_s - z_{s,t} - A \cdot \ltimes z_{s,t}| \leq L\omega(s, t)^\theta, \quad \forall 0 \leq s \leq t \leq T. \quad (8)$$

Besides, there exists a constant C depending only on $\|A\|_p$, $\omega(0, T)$ and p such that

$$\|y\|_p \leq C \max \{|a|, \|z\|_p, \|A \cdot \ltimes z\|_{p/2}\}$$

for all $0 \leq s \leq t \leq T$. In addition, set $y_t := \phi_{s,t}(a)$, where y is the solution to $y_r = a + \int_s^r dA_u^\# y_u + z_{s,r}$, $r \in [s, t]$. Then $\{\phi_{s,t}\}_{0 \leq s \leq t \leq T}$ defines a flow of homeomorphisms.

(II) Let z be a path of finite q -variation with $1/p + 1/q > 1$. Then there exists a unique path $y : [0, T] \rightarrow V$ of finite p -variation controlled by ω to the perturbed linear RDE (8) in the sense that

$$|y_t - A_{s,t}y_s - z_{s,t}| \leq C\omega(s, t)^\theta, \quad \forall 0 \leq s \leq t \leq T.$$

Besides, there exists a constant C depending only on $\|A\|_p$, $\omega(0, T)$ and p such that

$$\|y\|_p \leq C \max \{|a|, \|h\|_q\}$$

for all $0 \leq s \leq t \leq T$. In addition, for $\phi_{s,t}(a) = y_t$ with $y_r = a + \int_s^r dA_u^\# y_u + z_{s,r}$, $r \in [s, t]$, the family of maps $\{\phi_{s,t}\}_{0 \leq s \leq t \leq T}$ defines a flow of homeomorphisms.

4 Linear RDE arising in flow equation

Let $f : V \rightarrow L(U, V)$ be a vector field with bounded first and second derivatives and such that $\nabla^2 f$ is γ -Hölder continuous.

For two Banach spaces X and Y and g in $L(X, Y)$, we may also denote for convenience the value $g(x)$ as $\langle g, x \rangle$.

Let \mathbf{x} be a rough path with values in the second order tensor space $\mathbb{R} \oplus U \oplus U \otimes U$.

Let us note that ∇f maps V into $L(V, L(U, V))$ which is equivalent to $L(V \otimes U, V)$. Similarly, $\nabla^2 f$ maps V into $L(V \otimes V, L(U, V))$ which is equivalent to $L(V \otimes V \otimes U, V)$. To simplify the expressions, for (x, x') in $U \otimes U$

and y, z in V , we denote by $\nabla f(y)x$, $\nabla^2 f(y)z \otimes x$ and by $\nabla f(y)\nabla f(y)x \otimes x'$ the linear maps from V to V defined by

$$\langle \nabla f(y)x, h \rangle = \nabla f(y)h \otimes x, \quad \langle \nabla^2 f(y)z \otimes x, h \rangle = \nabla f(y)h \otimes z \otimes x,$$

and

$$\langle \nabla f(y)\nabla f(y)x \otimes x', h \rangle = \langle \nabla f(y), \langle \nabla f(y)h \otimes x \rangle \otimes x' \rangle$$

for h in V .

Let y be the unique solution to the RDE $y_t = y_0 + \int_0^t f(y_s) d\mathbf{x}_s$ and let $y \ltimes x$ the iterated integral between y and x , which takes its values in $V \otimes U$. For $0 \leq s \leq t \leq T$, set

$$B_{s,t} = \text{Id} + \nabla f(y_s)x_{s,t}^1 + \nabla^2 f(y_s)y \ltimes x_{s,t} + \nabla f(y_s)\nabla f(y_s)x_{s,t}^2$$

Then $B_{s,t}$ takes its values in $L(V, V)$.

As for constructing rough integrals, it is now sufficient to remark that for some constant C depending on $\omega(0, T)$, p , $\|\mathbf{x}\|$ and $\|\nabla f\|_\infty$, $\|\nabla^2 f\|_\infty$ and $H_\gamma(\nabla^2 f)$,

$$|B_{s,t} - B_{r,t}B_{s,r}| \leq C\omega(s, t)^{(2+\gamma)/p} \text{ for any } 0 \leq s \leq r \leq t \leq T.$$

Hence B is an almost rough resolvent which may be uniquely associated to a rough resolvent A .

In order to keep natural notations, we denote $\int_0^t dA_s u_s$ by $\int_0^t \nabla f(y_s)u_s d\mathbf{x}_s$.

Proposition 3. *Let $F(y, u)$ be the vector field from $V \otimes V$ to $L(U, V)$ defined by $F(y, u) := \nabla f(y)u$. The solution to the linear RDE $u_t = u_0 + \int_0^t dA_s u_s$ if and only if it is also solution to the RDE $u_t = u_0 + \int_0^t F(y_s, u_s) d\mathbf{x}_s$, in the sense that there exists a cross-iterated integrals $u \ltimes x$ such that $\int_0^t F(y_s, u_s) d\mathbf{x}_s$ exists as a rough integral and is equal to $u_t - u_0$.*

4.1 Case of non-geometric rough paths

Let us consider a p -rough path \mathbf{x} controlled by ω which is not geometric. One knows from [35] that \mathbf{x} may be decomposed as $\widehat{\mathbf{x}} + \phi$, where $\widehat{\mathbf{x}}$ is a geometric p -rough path controlled by ω , and ϕ is a path of $p/2$ -finite variation with values in $\text{Sym}(U \otimes U)$, the space of elements in $U \otimes U$ which are symmetric, that is such that $\tau(a) = a$ when τ is the linear map from $U^{\otimes 2}$ to $U^{\otimes 2}$ defined by $\tau(x \otimes x') = x' \otimes x$ for $x \otimes x' \in U^{\otimes 2}$.

Note that $\widehat{x}^1 = x$ but $\widehat{\mathbf{x}}$ and \mathbf{x} differ by their part in $U^{\otimes 2}$. It is easily seen that

$$y \ltimes x_{s,t} = y \ltimes \widehat{x}_{s,t} + \int_s^t (f \otimes 1)(y_r) d\phi_r,$$

where $(f \otimes 1)(y)(x \otimes x') = f(y)x \otimes x'$ for $(y, x, x') \in V \otimes U \otimes U$.

The integral $\int (f \otimes 1)(y_r) d\phi_r$ is a path of finite $p/2$ -variation controlled by ω with values in $V \otimes U$.

Set

$$\widehat{B}_{s,t} = \text{Id} + \nabla f(y_s)x_{s,t}^1 + \nabla^2 f(y_s)y \bowtie \widehat{x}_{s,t} + \nabla f(y_s)\nabla f(y_s)\widehat{x}_{s,t}^2$$

and let \widehat{A} be its associated rough resolvent. Then

$$A_{s,t} = \widehat{A}_{s,t} + \int_s^t \nabla^2 f(y_r)f(y_r) d\phi_r + \int_s^t \nabla f(y_r)\nabla f(y_r) d\phi_r,$$

where $\int_s^t \nabla^2 f(y_r)f(y_r) d\phi_r$ denotes the linear map in $L(V, V)$ defined by

$$\left\langle \int_s^t \nabla^2 f(y_r)f(y_r) d\phi_r, h \right\rangle = \int_s^t \langle \nabla^2 f(y_r), h \otimes f(y_r) \rangle d\phi_r,$$

where $\langle \nabla^2 f(y, r), \cdot \rangle$ is seen as a linear map from $V^{\otimes 2} \otimes U^{\otimes 2}$ to V .

5 Derivations of RDE with respect to the starting point and the coefficients

5.1 Results on the differentiability of the Itô map

Let $f(\cdot, \lambda)$ be a family of vector fields that depends on a parameter λ living in a convex set Λ_0 of a Banach space. This means that $f : V \times \Lambda_0 \mapsto L(U, V)$.

We assume that each $f(\cdot, \lambda)$ satisfies either **(A)** or **(B)** to ensure the existence of a unique solution $y^{\lambda,a}$ to $y_t^{\lambda,a} = a + \int_0^t f(y^{\lambda,a}, \lambda) d\mathbf{x}_s$, where a belongs to a convex set Λ_1 of V .

From now on, we assume that

(H) There exists a constant $\rho > 0$ such that $\|y^{\lambda,a}\| \leq \rho$ and $\|y^{\lambda,a}\|_\infty \leq \rho$ for $(\lambda, a) \in \Lambda_0 \times \Lambda_1$.

For a function g on $V \times \Lambda_0$, we denote by $\|g\|_\infty$ and $H_\gamma(g)$ the norm and semi-norm

$$\|g\|_\infty := \sup_{|y| \leq \rho, \lambda \in \Lambda_0} |g(y, \lambda)| \text{ and } H_\gamma(g) := \sup_{|x, y| \leq \rho, \lambda \in \Lambda_0} \frac{|g(x, \lambda) - g(y, \lambda)|}{|x - y|^\gamma}.$$

For a function g on $V \times \Lambda_0$, we set

$$H_\beta^{(\lambda)}(g) = \sup_{\mu, \lambda \in \Lambda_0} \sup_{y \in V} \frac{|g(y, \mu) - g(y, \lambda)|}{|\mu - \lambda|^\beta}.$$

We also assume

(**H**_①) The function f is of class $\mathcal{C}^{3,2}(\mathbf{V} \times \Lambda_0, L(\mathbf{U}, \mathbf{V}))$ and for some $\gamma \in (0, 1]$ with $2 + \gamma > p$, $H_\gamma(\nabla_{yyy}^3 f)$, $\|f\|_\infty$, $\|\nabla_y f\|_\infty$, $\|\nabla_{yy}^2 f\|_\infty$, $\|\nabla_{yyy}^3 f\|_\infty$, $\|\nabla_\lambda f\|_\infty$, $\|\nabla_{\lambda y}^2 f\|_\infty$, $\|\nabla_{\lambda yy}^3 f\|_\infty$, $\|\nabla_{\lambda \lambda y}^3 f\|_\infty$ and $\|\nabla_{\lambda \lambda yy}^4 f\|_\infty$ are finite. We denote by $N(f)$ the maximum of these quantities.

or

(**H**_②) The function f is of class $\mathcal{C}^{2,1}(\mathbf{V} \times \Lambda_0, L(\mathbf{U}, \mathbf{V}))$ and for some $\gamma \in (0, 1]$ with $2 + \gamma > p$ and for some $\beta \in (0, 1]$, $H_\gamma(\nabla_{yy}^2 f)$, $\|f\|_\infty$, $\|\nabla_y f\|_\infty$, $\|\nabla_{yy}^2 f\|_\infty$, $\|\nabla_\lambda f\|_\infty$, $H_\beta^{(\lambda)}(\nabla_\lambda f)$, $H_\beta^{(\lambda)}(\nabla_{\lambda y}^2 f)$ and $H_\beta^{(\lambda)}(\nabla_{\lambda yy}^3 f)$ are finite. We denote by $N(f)$ the maximum of these quantities.

A statement starting by ① means that we work under (**H**_①), while a standard starting by ② means that we work under (**H**_②).

Remark 2. If (**H**_①) or (**H**_②) hold with $\rho = +\infty$, then (**H**) holds for a finite ρ . Yet there is no need to assume that f is globally bounded. With appropriate conditions on f and its derivatives (which need to be bounded), (**H**) holds for a finite ρ and then (**H**_①) or (**H**_②) hold if Λ_0 is chosen to be compact. See [30, 31] or [17] for discussion about existence of solution for non bounded vector fields.

Definition 4. We say that a constant C depends only on the structure if it depends only on $N(f)$, ρ , $\omega(0, T)$, p , γ , β and $\|\mathbf{x}\|$.

For $\lambda \in \Lambda_0$, let $\mathfrak{P}(\lambda, a; h, k)$ be the solution to the perturbed RDE defined by

$$\mathfrak{P}(\lambda, a; h, k)_t = k + \int_0^t \nabla_y f(y_s^{\lambda, a}, \lambda) \mathfrak{P}(\lambda, a; h, k)_s d\mathbf{x}_s + \int_0^t \nabla_\lambda f(y_s^{\lambda, a}, \lambda) h d\mathbf{x}_s.$$

Note that $\mathfrak{P}(\lambda, a; \cdot, \cdot)$ is bilinear.

Theorem 2. Under Hypotheses (**H**) and (**H**_①) or (**H**_②), there exists a constant C that depends only on the structure such that

$$\|y^{\lambda+h, a+\epsilon} - y^{\lambda, a} - \mathfrak{P}(\lambda, a; h, \epsilon)\|_p \leq C(|h|^\delta + |\epsilon|^\delta)$$

with

$$\delta := \begin{cases} 2 & \text{under } (\mathbf{H}_\text{①}), \\ 1 + (1 - \kappa)(\beta \wedge \gamma) \text{ when } \kappa \in (0, 1), 2 + \kappa\gamma > p & \text{under } (\mathbf{H}_\text{②}). \end{cases}$$

This implies the existence of a derivative to the Itô map.

Corollary 1. *Under Hypotheses (\mathbf{H}) and $(\mathbf{H}_{\textcircled{1}})$ or $(\mathbf{H}_{\textcircled{2}})$, the Itô map $\mathfrak{I} : (\lambda, a) \mapsto y^{\lambda, a}$ is Fréchet differentiable at any point $(\lambda, a) \in \Lambda_0 \times \Lambda_1$ and its derivative is $\mathfrak{P}(\lambda, a; \cdot, \cdot)$.*

The derivative of the Itô map is itself regular, which gives a stronger statement than Theorem 2.

Theorem 3. *For fixed $h \in \Lambda_0$ and $k \in \Lambda_1$, then*

$$\|\mathfrak{P}(\lambda, a; h, \epsilon) - \mathfrak{P}(\mu, b; h, \epsilon)\|_p \leq C(|\epsilon| + |h|)(|\mu - \lambda|^\zeta + |b - a|^\xi)$$

with

$$\zeta = (1 - \kappa)\beta, \quad \xi = (1 - \kappa)\gamma \text{ for } 2 + \kappa\gamma > p, \quad \kappa \in (0, 1)$$

and a constant C that depends only on the structure.

Theorem 3 is proved in Section 6.

Remark 3. Of course, with more regularity on the vector field f , it is then possible to derivate the map $(\lambda, a) \mapsto (y^{\lambda, a}, \mathfrak{P}(\lambda, a; h, \epsilon))$ which is itself defined as the solution of some RDE and then to obtain higher derivatives of \mathfrak{I} .

5.2 Some preliminary estimates

The proof of Theorem 2 relies on the construction of a suitable vector field $F : V \times V \mapsto L(U, V)$ and to study the norm of $\int F(y_s, z_s) d\mathbf{x}_s$ for two solutions $y_t = y_0 + \int_0^t f(y_s, \lambda) d\mathbf{x}_s$ and $z_t = z_0 + \int_0^t f(z_s, \mu) d\mathbf{x}_s$ with $(y_0, z_0) \in \Lambda_1^2$, $(\lambda, \mu) \in \Lambda_0^2$.

The norm of $\int F(y_s, z_s) d\mathbf{x}_s$ is then related to $|y_0 - z_0|$ and to $|\mu - \lambda|$. Afterwards, it is seen that $\Delta_t := z_t - y_t$ is solution to a linear perturbed equation.

Lemma 1. ① *Let f be a function of class $\mathcal{C}^{1+\gamma}$ on V . Then for any y, y', z, y' in V ,*

$$\begin{aligned} & |f(z) - f(y) - f(z') + f(y')| \\ & \leq \|\nabla_y f\|_\infty |z' - z - y' + y| + H_\gamma(\nabla_y f)(|y' - y|^\gamma + |z' - z|^\gamma) |z' - y'|. \end{aligned}$$

② *Let f be a function of class \mathcal{C}^γ . Then for $\kappa \in [0, 1]$ and $\bar{\kappa} := 1 - \kappa$,*

$$\begin{aligned} & |f(z') - f(y') - f(z) + f(y)| \\ & \leq H_\gamma(f)(|y' - y|^{\kappa\gamma} + |z' - z|^{\kappa\gamma})(|z' - y'|^{\gamma\bar{\kappa}} + |z - y|^{\gamma\bar{\kappa}}). \quad (9) \end{aligned}$$

Proof. ① This follows from

$$\begin{aligned}
& |f(z) - f(y) - f(z') + f(y')| \\
&= \left| \int_0^1 \nabla_y f(y + \tau(z - y))(z - y) d\tau - \int_0^1 \nabla_y f(y' + \tau(z' - y'))(z' - y') d\tau \right| \\
&\leq \left| \int_0^1 \nabla_y f(y + \tau(z - y))(z - y - z' + y') d\tau \right| \\
&\quad + \left| \int_0^1 (\nabla_y f(y' + \tau(z' - y')) - \nabla_y f(y + \tau(z - y)))(z' - y') d\tau \right|.
\end{aligned}$$

The result stems from this inequality and

$$\begin{aligned}
|\nabla_y f(y' + \tau(z' - y')) - \nabla_y f(y + \tau(z - y))| &\leq H_\gamma(\nabla_y f) |(1 - \tau)(y' - y) + \tau(z' - z)|^\gamma \\
&\leq H_\gamma(\nabla_y f) (|y' - y|^\gamma + |z' - z|^\gamma)
\end{aligned}$$

for $\tau \in [0, 1]$.

② Inequality (9) follows from a combination using κ of the Hölder inequality applied to $f(y) - f(y')$ (resp. $f(z) - f(z')$) and $f(y) - f(z)$ (resp. $f(y') - f(z')$) and from the inequality $(a + b)^\delta \leq a^\delta + b^\delta$ for $a, b \geq 0$ and $\delta \in [0, 1]$. \square

The next result is an immediate consequence of this lemma.

Corollary 2. *Let y and z be two paths of finite p -variation controlled by ω such that $\max\{\|y\|_\infty, \|z\|_\infty, \|y\|_p, \|z\|_p\} \leq \rho$. Set $\Delta_t := z_t - y_t$.*

① *For $f \in \mathcal{C}^{1+\gamma}$ and $0 \leq s \leq t \leq T$,*

$$|f(z_t) - f(z_s) - f(y_t) + f(y_s)| \leq C_1(\|\Delta\|_p + |\Delta_0|)\omega(s, t)^{\gamma/p},$$

where C_1 depends only on ρ , $N(f)$, $\omega(0, T)$, p and γ .

② *For $f \in \mathcal{C}^\gamma$, $\kappa \in [0, 1]$ and $0 \leq s \leq t \leq T$*

$$|f(z_t) - f(z_s) - f(y_t) + f(y_s)| \leq H_\gamma(f)C_2(\|\Delta\|_p + |\Delta_0|)^{\bar{\kappa}\gamma}\omega(s, t)^{\gamma\kappa/p},$$

where C_2 depends only on ρ , $\omega(0, T)$, p , κ and γ .

Proof. This follows from Lemma 1 and the facts that $\|\Delta\|_p \leq \|y\|_p + \|z\|_p$ and $\|\Delta\|_\infty \leq |\Delta_0| + \omega(0, T)^{1/p}\|\Delta\|_p$. \square

For a function f in \mathcal{C}^1 , set

$$\mathfrak{D}[f](y, z) := f(z) - f(y) - \nabla_y f(y)(z - y).$$

Lemma 2. ① Let f be of class $\mathcal{C}^{2+\gamma}$ on V with $\|\nabla^2 f\|_\infty < +\infty$ and $H_\gamma(\nabla^2 f) < +\infty$. Then

$$|\mathfrak{D}[f](y, z)| \leq \|\nabla^2 f\|_\infty |z - y|^2 \quad (10)$$

and

$$\begin{aligned} |\mathfrak{D}[f](y, z) - \mathfrak{D}[f](y', z')| &\leq \|\nabla^2 f\|_\infty |z' - y' - z + y| \cdot (|z - y| + |z' - y'|) \\ &\quad + H_\gamma(\nabla^2 f)(|y' - y|^\gamma + |z' - z|^\gamma) |z' - y'| \cdot |z - y|. \end{aligned}$$

② Let f be of class $\mathcal{C}^{1+\gamma}$ on V with $\|\nabla_y f\|_\infty < +\infty$ and $H_\gamma(\nabla_y f) < +\infty$. Then

$$|\mathfrak{D}[f](y, z)| \leq H_\gamma(\nabla_y f) |z - y|^{1+\gamma} \quad (11)$$

and

$$\begin{aligned} |\mathfrak{D}[f](y, z) - \mathfrak{D}[f](y', z')| &\leq H_\gamma(\nabla_y f) |z' - y' - z + y| \cdot |z' - y'|^\gamma \\ &\quad + H_\gamma(\nabla_y f)(|z' - z|^{\kappa\gamma} + |y' - y|^{\kappa\gamma})(|z' - y'|^{\bar{\kappa}\gamma} + |z - y|^{\bar{\kappa}\gamma}) |z - y|. \end{aligned}$$

Proof. Inequalities (10) and (11) follow immediately from

$$\mathfrak{D}[f](y, z) = \int_0^1 (\nabla f(y + \tau(z - y)) - \nabla f(y))(z - y) \, d\tau.$$

Besides,

$$\begin{aligned} &\mathfrak{D}[f](y, z) - \mathfrak{D}[f](y', z') \\ &= \int_0^1 (\nabla f(y + \tau(z - y)) - \nabla f(y) - \nabla f(y' + \tau(z' - y')) + \nabla f(y'))(z - y) \, d\tau \\ &\quad + \int_0^1 (\nabla f(y' + \tau(z' - y')) - \nabla f(y'))(z - y - z' + y') \, d\tau. \end{aligned}$$

We conclude with Lemma 1 applied to ∇f . \square

For f satisfying $(\mathbf{H}_\textcircled{1})$ or $(\mathbf{H}_\textcircled{2})$, let us consider the vector field

$$F(y, z) := f(z, \mu) - f(y, \lambda) - \nabla_y f(y, \lambda)(z - y) - \nabla_\lambda f(z, \lambda)(\mu - \lambda).$$

Lemma 3. ① It holds that

$$|F(y, z)| \leq \|\nabla_{yy}^2 f\|_\infty |z - y|^2 + \|\nabla_{\lambda\lambda}^2 f\|_\infty |\lambda - \mu|^2. \quad (12)$$

② It holds that

$$|F(y, z)| \leq \|\nabla_{yy}^2 f\|_\infty |z - y|^2 + H_\beta^{(\lambda)}(\nabla_\lambda f) |\lambda - \mu|^{1+\beta}. \quad (13)$$

Proof. Transforming $F(y, z)$ into

$$\begin{aligned} F(y, z) &= \int_0^1 (\nabla_\lambda f(z, \lambda + \tau(\mu - \lambda)) - \nabla_\lambda f(z, \lambda))(\mu - \lambda) d\tau \\ &\quad + \int_0^1 (\nabla_y f(y + \tau(z - y), \lambda) - \nabla_y f(y, \lambda))(z - y) d\tau, \end{aligned}$$

leads to inequalities (12) and (13). \square

Lemma 4. ① For $y, z \in V$,

$$|\nabla_y F(y, z)| \leq \|\nabla_{yy}^2 f\|_\infty |z - y| \quad (14)$$

and

$$\begin{aligned} |\nabla_y F(y, z) - \nabla_y F(y', z')| &\leq \|\nabla_{yy}^2 f\|_\infty |y - y' - z + z'| \\ &\quad + |y' - y| \cdot |y' - z'| \|\nabla_{yyy}^3 f\|_\infty. \end{aligned} \quad (15)$$

② For $y, z \in V$, (14) holds and

$$\begin{aligned} |\nabla_y F(y, z) - \nabla_y F(y', z')| &\leq \|\nabla_{yy}^2 f\|_\infty |y - y' - z + z'| \\ &\quad + |y' - y|^\gamma \cdot |y' - z'| H_\gamma(\nabla_{yy}^2 f). \end{aligned} \quad (16)$$

Proof. The gradient of F with respect to the y -variables is

$$\nabla_y F(y, z) = -\nabla_{yy}^2 f(y, \lambda)(z - y).$$

Inequalities (14), (15) and (16) are immediate. \square

Lemma 5. Set

$$\mathfrak{F}[F](y, z) := \nabla_y F(y, z) + \nabla_z F(y, z).$$

① It holds that

$$|\mathfrak{F}[F](y, z)| \leq \|\nabla_{yyy}^3 f\|_\infty |z - y|^2 + \|\nabla_{\lambda\lambda y}^3 f\|_\infty |\mu - \lambda|^2 \quad (17)$$

and

$$\begin{aligned} |\mathfrak{F}[F](y, z) - \mathfrak{F}[F](y', z')| &\leq \|\nabla_{\lambda\lambda yy}^4 f\|_\infty |\mu - \lambda|^2 |z - z'| \\ &\quad + H_\gamma(\nabla_{yyy}^3 f)(|y' - y|^\gamma + |z' - z|^\gamma) |z' - y'| \cdot |z - y| \\ &\quad + \|\nabla_{yyy}^3 f\|_\infty |z' - y' - z + y| \cdot (|z - y| + |z' - y'|). \end{aligned} \quad (18)$$

② It holds that

$$|\mathfrak{I}[F](y, z)| \leq H_\gamma(\nabla_{yy}^2 f)|z - y|^{1+\gamma} + H_\beta^{(\lambda)}(\nabla_{\lambda y}^2 f)|\mu - \lambda|^{1+\beta} \quad (19)$$

and

$$\begin{aligned} |\mathfrak{I}[F](y, z) - \mathfrak{I}[F](y', z')| &\leq H_\gamma(\nabla_{yy}^2 f)|z' - y' - z + y| \cdot |z' - y'|^\gamma \\ &\quad + H_\gamma(\nabla_{yy}^2 f)(|z' - y'| + |z - y|)^{\kappa\gamma}(|z' - z| + |y' - y|)^{\bar{\kappa}\gamma}|z - y| \\ &\quad + 2H_\gamma(\nabla_{\lambda y}^2 f)^\kappa H_\beta^{(\lambda)}(\nabla_{\lambda y}^2 f)^{\bar{\kappa}}|z' - z|^{\gamma\kappa}|\mu - \lambda|^{1+\bar{\kappa}\beta}. \end{aligned} \quad (20)$$

Proof. The gradient of F with respect to the z -variables is

$$\begin{aligned} \nabla_z F(y, z) &= \nabla_y f(z, \mu) - \nabla_y f(y, \lambda) - \nabla_{\lambda y}^2 f(z, \lambda)(\mu - \lambda) \\ &= \int_0^1 (\nabla_{\lambda y}^2 f(z, \lambda + \tau(\mu - \lambda)) - \nabla_{\lambda y}^2 f(z, \lambda))(\mu - \lambda) d\tau \\ &\quad + \nabla_y f(z, \lambda) - \nabla_y f(y, \lambda). \end{aligned}$$

Since $\nabla_y F(y, z) = -\nabla_{yy}^2 f(y, \lambda)(z - y)$,

$$\mathfrak{I}[F](y, z) = A_1(z) + \mathfrak{D}[\nabla_y f](y, z)$$

with

$$A_1(z) := \int_0^1 (\nabla_{\lambda y}^2 f(z, \lambda + \tau(\mu - \lambda)) - \nabla_{\lambda y}^2 f(z, \lambda))(\mu - \lambda) d\tau.$$

Under $(\mathbf{H}_{\textcircled{1}})$,

$$A_1(z) = \int_0^1 \int_0^1 \tau \nabla_{\lambda \lambda y}^3 f(z, \lambda + \rho\tau(\mu - \lambda))\tau(\mu - \lambda, \mu - \lambda) d\rho d\tau.$$

Besides, $|A_1(z)| \leq \|\nabla_{\lambda \lambda y}^3 f\|_\infty |\mu - \lambda|^2$ and

$$|A_1(z) - A_1(z')| \leq \|\nabla_{\lambda \lambda y y}^4 f\|_\infty |z - z'| |\mu - \lambda|^2.$$

Under $(\mathbf{H}_{\textcircled{2}})$, $|A_1(z)| \leq H_\beta^{(\lambda)}(\nabla_{\lambda y}^2 f)|\mu - \lambda|^{1+\beta}$ and

$$|A_1(z) - A_1(z')| \leq \begin{cases} H_\gamma(\nabla_{\lambda y}^2 f)|z' - z|^\gamma |\mu - \lambda|, \\ 2H_\beta^{(\lambda)}(\nabla_{\lambda y}^2 f)|\mu - \lambda|^{1+\beta}. \end{cases}$$

We combine these two estimates by raising one to the power $\kappa \in [0, 1]$, and the second to the power $(1 - \kappa)$ and multiplying them.

Inequalities (17), (18), (19) and (20) stem from these computations and Lemma 2 applied to $\nabla_y f(\cdot, \lambda)$. \square

Corollary 3. *Let y and z be two paths of finite p -variation controlled by ω for which $\max\{\|y\|_\infty, \|z\|_\infty, \|y\|_p, \|z\|_p\} \leq \rho$. Set $\Delta_t := z_t - y_t$.*

① *For any $0 \leq s \leq t \leq T$,*

$$|\mathfrak{T}[F](y_s, z_s) - \mathfrak{T}[F](y_s, z_s)| \leq C_3 \omega(s, t)^{\gamma/p} (|\Delta_0|^2 + \|\Delta\|_p^2 + |\mu - \lambda|^2), \quad (21)$$

and

$$\begin{aligned} |\mathfrak{T}[F](y_s + \tau y_{s,t}, z_s + \tau z_{s,t}) - \mathfrak{T}[F](y_s, z_s)| \\ \leq C_3 \omega(s, t)^{\gamma/p} (|\Delta_0|^2 + \|\Delta\|_p^2 + |\mu - \lambda|^2) \end{aligned} \quad (22)$$

for some constant C_3 that depends only on the structure.

② *For $\kappa \in [0, 1]$ and for any $0 \leq s \leq t \leq T$,*

$$\begin{aligned} |\mathfrak{T}[F](y_s, z_s) - \mathfrak{T}[F](y_s, z_s)| \\ \leq C_4 \omega(s, t)^{\kappa\gamma/p} (|\Delta_0|^{1+\bar{\kappa}\gamma} + \|\Delta\|_p^{1+\bar{\kappa}\gamma} + |\mu - \lambda|^{1+\beta}), \end{aligned} \quad (23)$$

and

$$\begin{aligned} |\mathfrak{T}[F](y_s + \tau y_{s,t}, z_s + \tau z_{s,t}) - \mathfrak{T}[F](y_s, z_s)| \\ \leq C_4 \omega(s, t)^{\kappa\gamma/p} (|\Delta_0|^{1+\bar{\kappa}\gamma} + \|\Delta\|_p^{1+\bar{\kappa}\gamma} + |\mu - \lambda|^{1+\beta}) \end{aligned} \quad (24)$$

for some constant C_4 that depends only on the structure.

Proof. With Lemma 5 under $(\mathbf{H}_{\textcircled{1}})$,

$$\begin{aligned} |\mathfrak{T}[F](y_r, z_r) - \mathfrak{T}[F](y_s, z_s)| \leq C_5 |\mu - \lambda|^2 \omega(s, t)^{1/p} + C_6 \omega(s, t)^{\gamma/p} \|\Delta\|_\infty^2 \\ + C_7 \omega(s, t)^{1/p} \|\Delta\|_\infty \|\Delta\|_p \end{aligned}$$

with $C_5 := \|\nabla_{\lambda\lambda yy}^4 f\|_\infty \|z\|_p$, $C_6 := H_\gamma(\nabla_{yyy}^3 f)(\|y\|_p^\gamma + \|z\|_p^\gamma)$ and $C_7 := \|\nabla_{yyy}^3 f\|_\infty$. Inequality (21) follows easily from $\|\Delta\|_p \leq \|z\| + \|y\|_p$, $\|\Delta\|_\infty \leq |\Delta_0| + \omega(0, T)\|\Delta\|_p$ and $|\Delta_0| \cdot \|\Delta\|_p \leq \frac{1}{2}|\Delta_0|^2 + \frac{1}{2}\|\Delta\|_p^2$. Inequality (22) follows from the same kind of computations.

Under $(\mathbf{H}_{\textcircled{2}})$, the same computations leads to (23) and (24). \square

Lemma 6. *Let y and z be two paths of finite p -variation controlled by ω for which $\max\{\|y\|_\infty, \|z\|_\infty, \|y\|_p, \|z\|_p\} \leq \rho$. Set $\Delta_t = z_t - y_t$ and*

$$A_3(s, r) := F(y_r, z_r) - F(y_s, z_s) - \nabla_y F(y_s, z_s) y_{s,r} - \nabla_z F(y_s, z_s) z_{s,r}.$$

Then

$$|A_3(s, r)| \leq C_8 \begin{cases} \omega(s, r)^{(1+\gamma)/p} (\|\Delta\|_p^2 + |\Delta_0|^2 + |\mu - \lambda|^2) & \text{under } (\mathbf{H}_{\textcircled{1}}), \\ \omega(s, r)^{(1+\kappa\gamma)/p} (\|\Delta\|_p^{1+\bar{\kappa}\gamma} + |\Delta_0|^{1+\bar{\kappa}\gamma} + |\mu - \lambda|^{1+\beta}) & \text{under } (\mathbf{H}_{\textcircled{2}}), \end{cases}$$

where C_8 depends only on $N(f)$, ρ , $\omega(0, T)$, γ , κ and p .

Proof. Let us note that

$$\begin{aligned}
A_3(s, r) &= \int_0^1 (\nabla_y F(y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) - \nabla_y F(y_s, z_s)) y_{s,r} d\tau \\
&\quad + \int_0^1 (\nabla_z F(y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) - \nabla_z F(y_s, z_s)) z_{s,r} d\tau \\
&= \int_0^1 (\nabla_y F(y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) - \nabla_y F(y_s, z_s)) (y_{s,r} - z_{s,r}) d\tau \\
&\quad + \int_0^1 \mathfrak{T}[F](y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) - \mathfrak{T}[F](y_s, z_s) z_{s,r} d\tau.
\end{aligned}$$

Under $(\mathbf{H}_{\textcircled{1}})$ (resp. $(\mathbf{H}_{\textcircled{2}})$), Inequality (22) (resp. (24)) in Corollary 3 and Inequality (15) (resp. (16)) in Lemma 4 allow one to conclude. \square

5.3 An estimate on the integral $\int F(y_s, z_s) d\mathbf{x}_s$

For $\lambda, \mu \in \Lambda_0$ and $y_0, z_0 \in \Lambda_1$, let y and z be the solutions to

$$y_t = y_0 + \int_0^t f(y_s, \lambda) d\mathbf{x}_s \text{ and } z_t = z_0 + \int_0^t f(z_s, \mu) d\mathbf{x}_s,$$

and set $\Delta_t := z_t - y_t$.

In the next theorem, we combine both results on solutions in the sense of Lyons and solutions in the sense of Davie [30], which are equivalent here thanks to Hypotheses $(\mathbf{H}_{\textcircled{1}})$ or $(\mathbf{H}_{\textcircled{2}})$.

Theorem 4 (Theorem 4 in [31], Theorem 1 in [30]). *It holds that*

$$\|\Delta\|_p \leq C_9(|\Delta_0| + C_{10}|\mu - \lambda|) \quad (25)$$

$$C_{10} := \|\nabla_\lambda f\|_\infty \|\nabla_{yy}^2 f\|_\infty + \|f\|_\infty \|\nabla_{\lambda yy}^3 f\|_\infty + 2\|\nabla_\lambda f\|_\infty \|\nabla f\|_\infty,$$

and

$$\begin{aligned}
& |(y_{s,t} - f(y_s, \lambda)x_{s,t}^1 - (f\nabla_y f)(y_s, \lambda)x_{s,t}^2) - (z_{s,t} - f(z_s, \mu)x_{s,t}^1 - (f\nabla_y f)(z_s, \mu)x_{s,t}^2)| \\
& \leq C_{11}(\|\Delta\|_p + |\Delta_0| + C_{10}|\mu - \lambda|) \begin{cases} \omega(s, t)^{3/p} & \text{under } (\mathbf{H}_{\textcircled{1}}), \\ \omega(s, t)^{(2+\gamma)/p} & \text{under } (\mathbf{H}_{\textcircled{2}}), \end{cases} \quad (26)
\end{aligned}$$

where C_9 and C_{11} depend only on the structure.

Let us denote by $y \bowtie x$ (resp. $z \bowtie x$) the cross-iterated integrals between y (resp. z) and x given by 5. The cross-iterated integral $(z - y) \bowtie x$ between $z - y$ and x also exists and $(z - y) \bowtie x = z \bowtie x - y \bowtie x$.

Corollary 4. *There exists a constant C_{12} depending only on the structure such that*

$$|(z - y) \ltimes x_{s,t}| \leq \omega(s, t)^{2/p} C_{12} (|\mu - \lambda| + |\Delta_0| + \|\Delta\|_p)$$

for any $0 \leq s < t \leq T$.

Proof. Set $X := \mathbb{R} \oplus U \oplus V \oplus V \otimes U$,

$$\mathbf{P}_{s,t} := 1 + x_{s,t} + z_{s,t} - y_{s,t} + (f(z_s, \mu) - f(y_s, \lambda)) \otimes 1 \cdot x_{s,t}^2$$

and let $\mathbf{P}_{s,r,t} := \pi_X(\mathbf{P}_{s,t} - \mathbf{P}_{s,r} \otimes \mathbf{P}_{r,t})$. Note that in the expression of \mathbf{P} , $z - y$ lives in V . Hence,

$$\begin{aligned} \mathbf{P}_{s,r,t} &= (f(z_r, \mu) - f(y_r, \lambda) - f(z_s, \mu) + f(y_s, \lambda)) x_{r,t}^2 \\ &\quad + (f(z_s, \mu) x_{s,r}^1 - z_{s,r}^1 - f(y_s, \lambda) x_{s,r}^1 + y_{s,r}^1) \otimes x_{r,t}^1. \end{aligned}$$

Then

$$\begin{aligned} &|f(z_r, \mu) - f(y_r, \lambda) - f(z_s, \mu) + f(y_s, \lambda)| \\ &\leq \left| \int_0^t (\nabla_\lambda f(z_r, \lambda + \tau(\mu - \lambda)) - \nabla_\lambda f(z_s, \lambda + \tau(\mu - \lambda))) \cdot (\mu - \lambda) d\tau \right| \\ &\leq |\lambda - \mu| \|\nabla_{\lambda y}^2 f\|_\infty \|z\| \omega(s, t)^{1/p}. \end{aligned}$$

With this and Corollary 2 applied to $f(\cdot, \lambda)$,

$$\begin{aligned} &|(f(z_r, \mu) - f(y_r, \lambda) - f(z_s, \mu) + f(y_s, \lambda)) x_{r,t}^2| \\ &\leq C_{13} (|\mu - \lambda| + \|\Delta\|_p + |\Delta_0|) \omega(s, t)^{3/p} \quad (27) \end{aligned}$$

for some constant C_{13} that depends only on the structure.

For $g(z, \lambda) = (f \nabla_y f)(z, \lambda)$, we have

$$|g(z_s, \mu) - g(y_s, \lambda)| \leq |\mu - \lambda| \cdot \|\nabla_\lambda g\|_\infty + \|\Delta\|_\infty \|\nabla_y g\|_\infty.$$

With (26) in Theorem 4 and (27),

$$|\mathbf{P}_{s,r,t}| \leq C_{14} (|\mu - \lambda| + \|\Delta\|_p + |\Delta_0|) \begin{cases} \omega(s, t)^{3/p} & \text{under } (\mathbf{H}_{\textcircled{1}}), \\ \omega(s, t)^{(2+\gamma)/p} & \text{under } (\mathbf{H}_{\textcircled{2}}), \end{cases}$$

for some constant C_{14} depending only on the structure

Since $2 + \gamma > p$, \mathbf{P} is an almost rough path. Its associated rough path $\mathfrak{G}_X[\mathbf{P}]$ satisfies for some constant K that depends only on $\omega(0, T)$, γ and p ,

$$|\mathfrak{G}_X[\mathbf{P}]_{s,t} - \mathbf{P}_{s,t}| \leq K \omega(s, t)^\theta \sup_{0 \leq s \leq r \leq t \leq T} \frac{|\mathbf{P}_{s,r,t}|}{\omega(s, t)^\theta}, \quad (28)$$

with $\theta = 3/p$ under $(\mathbf{H}_{\textcircled{1}})$ and $\theta = (2 + \gamma)/p$ under $(\mathbf{H}_{\textcircled{2}})$. This rough path $\mathfrak{S}_X[\mathbf{P}]$ is by definition $x + z - y + (z - y) \times x$ (See (5)). A control on $\|\mathfrak{S}_X[\mathbf{P}]\|$ is then easily deduced from (28) and

$$|f(y_s, \lambda) - f(z_s, \mu)| \leq C_{15}(\|\Delta\|_\infty + |\lambda - \mu|)$$

for some constant C_{15} that depends only on $N(f)$. \square

Lemma 7. For $0 \leq s \leq r \leq T$,

$$|F(y_r, z_r) - F(y_s, z_s)| \leq C_{16} \begin{cases} |\Delta_0|^2 + |\mu - \lambda|^2 & \text{under } (\mathbf{H}_{\textcircled{1}}), \\ |\Delta_0|^{1+\gamma} + |\mu - \lambda|^{1+\beta} & \text{under } (\mathbf{H}_{\textcircled{2}}), \end{cases}$$

for some constant C_{16} that depends only on the structure.

Proof. Let us note that

$$\begin{aligned} & F(y_r, z_r) - F(y_s, z_s) \\ &= \int_0^1 \nabla_y F(y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) y_{s,r} \, d\tau + \int_0^1 \nabla_z F(y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) z_{s,r} \, d\tau \\ &= \int_0^1 \nabla_y F(y_s + \tau y_{s,r}, z_s + \tau z_{s,r})(y_{s,r} - z_{s,r}) \, d\tau + \int_0^1 \mathfrak{T}[F](y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) z_{s,r} \, d\tau. \end{aligned}$$

With Lemma 4, for $\tau \in [0, 1]$,

$$|\nabla_y F(y_s + \tau y_{s,r}, z_s + \tau z_{s,r})| \leq \|\nabla_{yy}^2 f\|_\infty (|y_s - z_s| + |z_{s,r} - y_{s,r}|).$$

On the other hand, Lemma 5 and the convexity of $x \mapsto x^\nu$ for $\nu \geq 1$, for some constant C_{17} that depends only on the structure,

$$\begin{aligned} & |\mathfrak{T}[F](y_s + \tau y_{s,r}, z_s + \tau z_{s,r})| \\ & \leq C_{17} \begin{cases} |z_s - y_s|^2 + |z_r - y_r|^2 + |\mu - \lambda|^2 & \text{under } (\mathbf{H}_{\textcircled{1}}), \\ |z_s - y_s|^{1+\gamma} + |z_r - y_r|^{1+\gamma} + |\mu - \lambda|^{1+\beta} & \text{under } (\mathbf{H}_{\textcircled{2}}). \end{cases} \end{aligned}$$

Together with Theorem 4, this leads to the result. \square

5.4 Constructing integrals

We may now construct several integrals from \mathbf{x} and the solutions y and z .

The first one will be the linear semi-group

$$A_{s,t} := \int_0^t \nabla_y f(y_s, \lambda) \, d\mathbf{x}_s$$

acting on V and defined as being associated to

$$z \mapsto \nabla_y f(y_s, \lambda) z \cdot x_{s,t}^1 + \nabla_y^2 f(y_s, \lambda) z \cdot y \times x_{s,t} + \nabla_y f(y_s, \lambda) \nabla_y f(y_s, \lambda) z \cdot x_{s,t}^2.$$

Lemma 8. *There exists a family $\{x \ltimes \mathbf{A}_{s,t}\}_{0 \leq s \leq t \leq T}$ with values in $\mathbf{U} \otimes L(\mathbf{V}, \mathbf{V})$ such that*

$$x \ltimes \mathbf{A}_{s,t} = x \ltimes \mathbf{A}_{s,r} + x \ltimes \mathbf{A}_{r,t} + x_{s,r}^1 \otimes \mathbf{A}_{r,t}^{(1)}$$

for all $0 \leq s \leq r \leq t \leq T$. Besides, $\|x \ltimes \mathbf{A}\|_{p/2} \leq C_{18}$ for a constant C_{18} that depends only on the structure.

Proof. Let us consider $\mathbf{X} := \mathbb{R} \oplus \mathbf{U} \oplus L(\mathbf{V}, \mathbf{V}) \oplus \mathbf{U} \otimes L(\mathbf{V}, \mathbf{V})$,

$$\mathbf{P}_{s,t} := 1 + x_{s,t}^1 + \mathbf{A}_{s,t} + 1 \otimes \nabla f(y_s) \cdot x_{s,t}^2$$

and $\mathbf{P}_{s,r,t} := \pi_{\mathbf{X}}(\mathbf{P}_{s,t} - \mathbf{P}_{s,r} \otimes \mathbf{P}_{r,t})$. Then

$$\begin{aligned} \mathbf{P}_{s,r,t} &= 1 \otimes (\nabla f(y_r) - \nabla f(y_s)) \cdot x_{r,t}^2 + 1 \otimes (\nabla f(y_r) - \nabla f(y_s)) \cdot x_{s,r}^1 \otimes x_{r,t}^1 \\ &\quad + 1 \otimes \nabla f(y_r) x_{s,r}^1 \otimes x_{r,t}^1 - x_{s,r}^1 \otimes \mathbf{A}_{r,t}. \end{aligned}$$

But from the definition of \mathbf{A}

$$|\mathbf{A}_{r,t} - \text{Id} - \nabla_y f(y_s, \lambda) x_{s,t}^1| \leq C_{19} \omega(s, t)^{(2+\gamma)/p}$$

for some constant C_{19} that depends only on the structure. Hence, it is easily shown that \mathbf{P} is an almost rough path so that $x \ltimes \mathbf{A}$ is constructed as $\pi_{\mathbf{U} \otimes L(\mathbf{V}, \mathbf{V})}(\mathfrak{S}_{\mathbf{X}}(\mathbf{P}))$ and satisfies the required estimate. \square

Proposition 4. *For some constants C_{20} and C_{21} that depend only on the structure, the rough integral $R := \int F(y_s, z_s) d\mathbf{x}_s$ with values in \mathbf{V} exists and satisfies*

$$\|R\|_p \leq C_{20}(|\mu - \lambda|^\delta + |\Delta_0|^\delta) \text{ with } \delta := \begin{cases} 2 & \text{under } (\mathbf{H}_{\textcircled{1}}), \\ 1 + \bar{\kappa}(\beta \wedge \gamma) & \text{under } (\mathbf{H}_{\textcircled{2}}), \end{cases} \quad (29)$$

and for $0 \leq s \leq t \leq T$,

$$|R_{s,t} - F(y_s, z_s) x_{s,t}^1| \leq C_{21} \omega(s, t)^{2/p} (|\mu - \lambda|^\delta + |\Delta_0|^\delta). \quad (30)$$

Proof. Let us set

$$H_{s,t} := F(y_s, z_s) x_{s,t}^1 + \nabla_y F(y_s, z_s) y \ltimes x_{s,t} + \nabla_z F(y_s, z_s) z \ltimes x_{s,t}.$$

For $H_{s,r,t} = H_{s,t} - H_{s,r} - H_{r,t}$, it follows from standard computations that

$$\begin{aligned} H_{s,r,t} &= -(F(y_r, z_r) - F(y_s, z_s) - \nabla_y F(y_s, z_s) y_{s,r} - \nabla_z F(y_s, z_s) z_{s,r}) x_{r,t}^1 \\ &\quad + (\nabla_y F(y_s, z_s) - \nabla_y F(y_r, z_r)) y \ltimes x_{r,t} + (\nabla_z F(y_s, z_s) - \nabla_z F(y_r, z_r)) z \ltimes x_{r,t} \\ &= -A_3(s, r) x_{r,t}^1 + (\nabla_y F(y_s, z_s) - \nabla_y F(y_r, z_r)) (y - z) \ltimes x_{r,t} \\ &\quad + (\mathfrak{T}[F](y_s, z_s) - \mathfrak{T}[F](y_r, z_r)) z \ltimes x_{r,t}. \end{aligned}$$

Using Lemma 6, Lemma 4 and Corollary 4 as well as Corollary 3, there exists some constant C_{22} depending only on the structure such that

$$\begin{aligned} |H_{s,r,t}| &\leq C_{22} \\ \times \begin{cases} \omega(s,t)^{(2+\gamma)/p} (|\Delta_0|^2 + \|\Delta\|_p^2 + |\mu - \lambda|^2) & \text{under } (\mathbf{H}_1), \\ \omega(s,t)^{(2+\kappa\gamma)/p} (|\Delta_0|^{1+\bar{\kappa}(\gamma\wedge\beta)} + \|\Delta\|_p^{1+\bar{\kappa}(\gamma\wedge\beta)} + |\mu - \lambda|^{1+\bar{\kappa}\beta}) & \text{under } (\mathbf{H}_2). \end{cases} \end{aligned} \quad (31)$$

Under (\mathbf{H}_2) , when one combines Lemma 4 with Corollary 4, one gets a products of type

$$C_{23}(\|\Delta\|_p + |\Delta_0|)(\omega(s,t)^{1/p} + \omega(s,t)^{\gamma/p})(|\mu - \lambda|^{\bar{\kappa}\beta} + \|\Delta\|^{\bar{\kappa}\gamma}).$$

Hence, with the Young inequality,

$$\|\Delta\|_p \times |\mu - \lambda|^{\bar{\kappa}\beta} \leq \frac{1}{1 + \bar{\kappa}\beta} \|\Delta\|_p^{1+\bar{\kappa}\beta} + \frac{\bar{\kappa}\beta}{1 + \bar{\kappa}\beta} |\mu - \lambda|^{1+\bar{\kappa}\beta}.$$

As $\|\Delta\|_p$ may be controlled by a constant that depends only on the structure, this explains the appearance of $\|\Delta\|_p$ and $|\Delta_0|$ to the power $1+\bar{\kappa}(\beta\wedge\gamma)$ in (31).

Then H is an almost rough path whose associated rough path $\mathfrak{S}_V(H)$ is by definition $\int F(y_s, z_s) d\mathbf{x}_s$. Besides, for some constant K depending only on $\omega(0, T)$ and p ,

$$\left| \int_s^t F(y_r, z_r) d\mathbf{x}_r - H_{s,t} \right| \leq K\omega(s,t)^\theta \sup_{0 \leq s \leq r \leq t \leq T} \frac{|H_{s,r,t}|}{\omega(s,t)^\theta} \quad (32)$$

with $\theta = (2 + \gamma)/p$ under (\mathbf{H}_1) and $\theta = (2 + \kappa\gamma)/p$ under (\mathbf{H}_2) . On the other hand,

$$\begin{aligned} \nabla_y F(y_s, z_s) y \times x_{s,t} + \nabla_z F(y_s, z_s) z \times x_{s,t} \\ = \nabla_y F(y_s, z_s) (y - z) \times x_{s,t} + \mathfrak{T}[F](y_s, y_r) z \times x_{s,t}, \end{aligned}$$

which implies with Lemma 4, Lemma 5 and Corollary 4 that

$$|H_{s,t}| \leq C_{24}\omega(s,t)^{1/p} \begin{cases} (\|\Delta\|_p^2 + |\Delta_0|^2 + |\mu - \lambda|^2) & \text{under } (\mathbf{H}_1), \\ (\|\Delta\|_p^{1+\bar{\kappa}(\gamma\wedge\beta)} + |\Delta_0|^{1+\bar{\kappa}(\gamma\wedge\beta)} + |\mu - \lambda|^{1+\bar{\kappa}\beta}) & \text{under } (\mathbf{H}_2) \end{cases} \quad (33)$$

for some constant C_{24} that depends only on the structure. Inequalities (29) and (30) follow from (33) and (32) together with Lemma 4 for (30) and from (25), since $\|\Delta\|_p$ is bounded in a linear way by $|\Delta_0| + |\mu - \lambda|$. \square

Indeed, we have applied there a general procedure which we describe in the following lemma, but with some adaptations.

Lemma 9. *Let U be a path of finite p -variation with values in a Banach space W and let F be a function of class $\mathcal{C}^{1+\gamma}$, $2+\gamma > p$ from W to $L(U, V)$.*

(I) *Let us assume that $U \bowtie x$ exists*

$$\begin{aligned} |\nabla F(U_s + \tau U_{s,t}) - \nabla F(U_s)| &\leq L_1 \omega(s, t)^{\gamma/p}, \quad \forall \tau \in [0, 1], \\ |\nabla F(U_s)| &\leq L_2, \\ |F(U_s)| &\leq L_3. \end{aligned}$$

Then $\int F(U_s) d\mathbf{x}_s$ exists as a path with values in V and

$$\left\| \int F(U_s) d\mathbf{x}_s \right\| \leq C_{25} \max\{L_1, L_2, L_3\}$$

for a constant C_{25} that depends only on $\omega(0, T)$, \mathbf{x} , $\|U\|$, $\|U \bowtie x\|$, γ and p .

(II) *Let us assume that $\int F(U_s) d\mathbf{x}_s$ exists and*

$$|F(U_s) - F(U_r)| \leq L_4 \omega(s, t)^{\gamma/p} \quad (34)$$

$$\text{and } \left| \int_s^t F(U_s) d\mathbf{x}_s - F(U_s) x_{s,t}^1 \right| \leq L_5 \omega(s, t)^{2/p}, \quad (35)$$

then a family $\left\{ \mathbf{A} \cdot \bowtie \left(\int F(U) d\mathbf{x} \right)_{s,t} \right\}_{0 \leq s \leq t \leq T}$ taking its values in V and satisfying (7) exists. Besides,

$$\left| \mathbf{A} \cdot \bowtie \left(\int F(U) d\mathbf{x} \right)_{s,t} \right| \leq C_{26} \max\{L_4, L_5\}$$

for a constant C_{26} that depends only on $\omega(0, T)$, \mathbf{x} , $x \bowtie \mathbf{A}$, γ and p .

Proof. Set

$$P_{s,t} := F(U_s) x_{s,t}^1 + \nabla F(U_s) U \bowtie x_{s,t} \text{ and } P_{s,r,t} := P_{s,t} - P_{s,r} - P_{r,t}.$$

Since $F(U_r) - F(U_s) = \int_0^1 \nabla F(U_s + \tau U_{s,t}) U_{s,t} d\tau$ and $U \bowtie x_{s,t} = U \bowtie x_{s,r} + U \bowtie x_{r,t} + U_{s,r} \otimes x_{r,t}$, we have

$$\begin{aligned} P_{s,r,t} = & - \left(\int_0^1 (\nabla F(U_s + \tau U_{s,t}) - \nabla F(U_s)) U_{s,t} d\tau \right) x_{r,t}^1 \\ & - (\nabla F(U_r) - \nabla F(U_s)) U \bowtie x_{r,t}. \end{aligned}$$

Then P is an almost rough path with values in V . In addition, For a constant K that depends only on $\omega(0, T)$, γ and p ,

$$\sup_{0 \leq s \leq t \leq T} |\mathfrak{S}_V[P]_{s,t} - P_{s,t}| \leq KL_1(\|U \ltimes x\|_{p/2} + \|U\|_p \|\mathbf{x}\|) \omega(s, t)^{(2+\gamma)/p}.$$

By definition, $\int F(U_s) d\mathbf{x}_s := \mathfrak{S}_V[P]$. Hence, we deduce that

$$\begin{aligned} |\mathfrak{S}_V[P]_{s,t}| &\leq L_1(\|U \ltimes x\|_{p/2} + \|U\|_p \|\mathbf{x}\|) \omega(s, t)^{(2+\gamma)/p} \\ &\quad + L_2 \|U \ltimes x\|_{p/2} \omega(s, t)^{2/p} + L_3 \|\mathbf{x}\| \omega(s, t)^{1/p}. \end{aligned}$$

For the second part of the lemma, let us consider

$$\mathbf{P}_{s,t} := 1 + \mathbf{A}_{s,t} + \mathfrak{S}_V[P]_{s,t} + 1 \otimes F(U_s) \cdot x \ltimes \mathbf{A}_{s,t}$$

with values in $Y := \mathbb{R} \oplus L(V, V) \oplus V \oplus V \otimes L(V, V)$ and set $\mathbf{P}_{s,r,t} := \pi_Y(\mathbf{P}_{s,t} - \mathbf{P}_{s,r} \otimes \mathbf{P}_{r,t})$ for $0 \leq s \leq r \leq t \leq T$. Then

$$\mathbf{P}_{s,r,t} = (F(U_s) - F(U_r)) \otimes 1 \cdot x \ltimes \mathbf{A}_{r,t} - \left(\int_s^t F(U_v) d\mathbf{x}_v - F(U_r) x_{s,r}^1 \right) \otimes \mathbf{A}_{r,t}.$$

With (34) and (35), \mathbf{P} is an almost rough path with values in Y . To conclude, we have just to apply the linear map $\Phi : Y \rightarrow V$ such that $\Phi(a) = 0$ for $a \in \mathbb{R}$, $a \in V$ or $a \in L(V, V)$, and $\Phi(a \otimes b) = ba$ for $a \in V$ and $b \in L(V, V)$. Then $\mathbf{A} \cdot \ltimes \int F(U_s) d\mathbf{x}_s$ is defined as $\Phi(\mathfrak{S}_Y[\mathbf{P}])$. \square

Corollary 5. *A family $\{\mathbf{A} \cdot \ltimes R_{s,t}\}_{0 \leq s \leq t \leq T}$ satisfying (7) exists and*

$$\|\mathbf{A} \cdot \ltimes R\|_{p/2} \leq C_{27}(|\Delta_0|^\delta + |\mu - \lambda|^\delta)$$

for some constant C_{27} that depends only on the structure and δ defined in (29).

Proof. The result is an immediate consequence of Lemma 9(II) with Lemma 7 and Inequality (30) of Proposition 6. \square

5.5 The path Δ as solution to a perturbed linear RDE

We now study the process $\Delta = z - y$ and show that it is solution to a perturbed linear RDE.

Proposition 5. *The process $\Delta = z - y$ is solution to the perturbed linear RDE*

$$\Delta_t = \Delta_0 + \int_0^t \nabla_y f(y_s, \lambda) \Delta_s d\mathbf{x}_s + R_{0,t} + (\mu - \lambda) \int_0^t \nabla_\lambda f(z_s, \lambda) d\mathbf{x}_s, \quad (36)$$

where R has been introduced in Proposition 4.

Proof. Set

$$G(y, z) = \nabla_y f(y, \lambda)(z - y) + \nabla_\lambda f(z, \lambda)(\mu - \lambda)$$

so that

$$F(y, z) = f(z, \mu) - f(y, \lambda) - G(y, z).$$

This way,

$$\begin{aligned} & F(y_s, z_s)x_{s,t}^1 + \nabla_y F(y_s, z_s)y \times x_{s,t} + \nabla_z F(y_s, z_s)z \times x_{s,t} \\ &= f(z_s, \mu)x_{s,t}^1 + \nabla_y f(z_s, \mu)z \times x_{s,t} - f(y_s, \lambda)x_{s,t}^1 - \nabla_y f(y_s, \lambda)y \times x_{s,t} \\ & \quad + G(y_s, z_s)x_{s,t}^1 + \nabla_y G(y_s, z_s)y \times x_{s,t} + \nabla_z G(y_s, z_s)z \times x_{s,t}. \end{aligned}$$

Using the linearity of \mathfrak{S}_V (See Remark 1),

$$R_{s,t} = \int_s^t f(z_s, \mu) d\mathbf{x}_s - \int_s^t f(y_s, \lambda) d\mathbf{x}_s - \int_s^t G(y_s, z_s) d\mathbf{x}_s. \quad (37)$$

On the other hand, both $\nabla_\lambda f(z_s, \lambda)x_{s,t}^1 + \nabla_{\lambda y}^2 f(z_s, \lambda)z \times x_{s,t}$ and

$$\begin{aligned} A_4(s, t) &:= \nabla_y f(y_s, \lambda)(z_s - y_s)x_{s,t}^1 + \nabla_{yy}^2 f(y_s, \lambda)(z_s - y_s)y \times x_{s,t} \\ & \quad - \nabla_y f(y_s, \lambda)y \times x_{s,t} + \nabla_y f(y_s, \lambda)z \times x_{s,t} \end{aligned}$$

are almost rough paths associated respectively to the rough integrals $\int \nabla_\lambda f(z_s, \lambda) d\mathbf{x}_s$ and $\int \nabla_y f(y_s, \lambda)(z_s - y_s) d\mathbf{x}_s$ with values in V . Yet

$$A_4(s, t) = \nabla_y f(y_s, \lambda)\Delta_s x_{s,t}^1 + \nabla_{yy}^2 f(y_s, \lambda)\Delta_s y \times x_{s,t} + \nabla_y f(y_s, \lambda)\Delta_s \times x_{s,t}.$$

Again with the linearity of \mathfrak{S}_V ,

$$\int_s^t G(y_s, z_s) d\mathbf{x}_s = \int_0^t \nabla_y f(y_s, \lambda)\Delta_s d\mathbf{x}_s + (\mu - \lambda) \int_0^t \nabla_\lambda f(z_s, \lambda) d\mathbf{x}_s.$$

Relation (36) follows from (37). \square

Proposition 6. *Let $H(y, z) := \nabla_\lambda f(z, \lambda) - \nabla_\lambda f(y, \lambda)$. Then $Q = \int H(y_r, z_r) d\mathbf{x}_r$ is well defined as an integral with values in $L(\Lambda_0, V)$. Besides, a family $\{A \cdot \times Q_{s,t}\}_{s \leq t \leq T}$ exists with values in $L(\Lambda_0, V)$ exists. In addition,*

$$\|Q\|_p \leq C_{28}(|\Delta_0| + |\mu - \lambda|) \text{ and } \|A \cdot \times Q\|_{p/2} \leq C_{29}(|\Delta_0| + |\mu - \lambda|)$$

for constants C_{28} and C_{29} that depend only on the structure.

Proof. Set

$$\widehat{Q}_{s,t} := H(y_s, z_s)x_{s,t}^1 + \nabla_y H(y_s, z_s)y \ltimes x_{s,t} + \nabla_z H(y_s, z_s)z \ltimes x_{s,t}$$

and $\widehat{Q}_{s,r,t} := \widehat{Q}_{s,t} - \widehat{Q}_{s,u} - \widehat{Q}_{u,t}$. Then

$$\begin{aligned} \widehat{Q}_{s,r,t} &= (H(y_r, z_r)x_{s,r}^1 - H(y_s, z_s)x_{s,r}^1 - \nabla_y H(y_s, z_s)y_{s,r} - \nabla_z H(y_s, z_s)z_{s,r})x_{r,t}^1 \\ &\quad - (\nabla_y H(y_r, z_r) - \nabla_y H(y_s, z_s))y \ltimes x_{s,r} - (\nabla_z H(y_r, z_r) - \nabla_z H(y_s, z_s))z \ltimes x_{s,r}. \end{aligned}$$

Note that

$$\begin{aligned} \nabla_y H(y_r, z_r) - \nabla_y H(y_s, z_s) \\ = \nabla_{\lambda y}^2 f(y_r, \lambda) - \nabla_{\lambda y}^2 f(z_r, \lambda) - \nabla_{\lambda y}^2 f(y_s, \lambda) + \nabla_{\lambda y}^2 f(z_s, \lambda) \end{aligned}$$

and $\nabla_{\lambda y}^2 f(\cdot, \lambda)$ is of class $\mathcal{C}^{1+\gamma}$ under (\mathbf{H}_1) and (\mathbf{H}_2) , $2 + \gamma > p$. A similar equality holds for $\nabla_z H$. Besides,

$$\begin{aligned} &(H(y_r, z_r)x_{s,r}^1 - H(y_s, z_s)x_{s,r}^1 - \nabla_y H(y_s, z_s)y_{s,r} - \nabla_z H(y_s, z_s)z_{s,r}) \\ &= \int_0^1 (\nabla_y H(y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) - \nabla_y H(y_s, z_s))y_{s,r} d\tau \\ &\quad + \int_0^t (\nabla_z H(y_s + \tau y_{s,r}, z_s + \tau z_{s,r}) - \nabla_z H(y_s, z_s))z_{s,r} d\tau \end{aligned}$$

With Lemma 1 applied to $\nabla_{\lambda y}^2 f(\cdot, \lambda)$, we obtain that

$$|\widehat{Q}_{s,r,t}| \leq C_{30}(\|\Delta\|_p + |\Delta_0|)\omega(s, t)^{(2+\gamma)/p}$$

for some constant C_{30} that depends only on the structure. Hence, \widehat{Q} is an almost rough path with values in $L(\Lambda_0, V)$ and its is easily proved using Theorem 4 that $\|Q\|_p \leq C_{28}(|\Delta_0| + |\lambda - \mu|)$ with $Q = \mathfrak{S}_{L(\Lambda_0, V)}[\widehat{Q}]$ and

$$|Q_{s,t} - H(y_s, z_s)x_{s,t}^1| \leq C_{31}(|\Delta_0| + \|\Delta\|_p)\omega(s, t)^{2/p}$$

and some constants C_{28} and C_{31} that depend only on the structure.

Lemma 1 applied on $H(z_t, y_t) - H(z_s, y_s)$ and Lemma 9(II) allow one to conclude regarding the construction of $A \ltimes Q$ and the given estimate. \square

Theorem 5 (Theorem 2 rephrased). *Let N be the solution to the perturbed linear RDE*

$$N_t = \Delta_0 + \int_0^t \nabla_y f(y_s, \lambda)N_s d\mathbf{x}_s + (\mu - \lambda) \int_0^t \nabla_\lambda f(y_s, \lambda) d\mathbf{x}_s.$$

Then

$$\|\Delta - N\|_p \leq C_{32}(|\mu - \lambda|^\delta + |\Delta_0|^\delta)$$

for some constant C_{32} that depends only on the structure and δ is defined in (29).

Proof. Using Proposition 5, by linearity of linear RDE, $\Delta - N$ is solution to

$$(\Delta - N)_t = \int_0^t \nabla f(y_s, \lambda)(\Delta - N) d\mathbf{x}_s + R_{0,t} + Q_{0,t} \cdot (\mu - \lambda).$$

With Theorem 1,

$$\begin{aligned} \|\Delta - N\|_p &\leq C_{33}(\|R\|_p + \|\mathbf{A} \bowtie R\|_{p/2} + |\mu - \lambda| \cdot \|Q\|_p + |\mu - \lambda| \cdot \|\mathbf{A} \bowtie Q\|_{p/2}) \\ &\leq C_{34}(|\Delta_0|^\delta + |\mu - \lambda|^\delta). \end{aligned}$$

The result follows from Proposition 4, Corollary 5, Proposition 6 combined with Theorem 1. \square

6 Regularity of the derivative of the Itô map

Let us consider now

$$y_t = a + \int_0^t f(y_s, \lambda) d\mathbf{x}_s \text{ and } z_t = b + \int_0^t f(z_s, \mu) d\mathbf{x}_s,$$

as well as the two perturbed linear RDE for $\epsilon \in V$ and h in Λ_0 such that $a + \epsilon, b + \epsilon \in \Lambda_1$ and $\lambda + h, \mu + h \in \Lambda_0$. According to Corollary 1, the derivatives of the Itô map around (λ, a) and (μ, b) in the direction (ϵ, h) are solutions to

$$\begin{aligned} Y_t &= \epsilon + \int_0^t \nabla_y f(y_s, \lambda) Y_s d\mathbf{x}_s + \int_0^t \nabla_\lambda f(y_s, \lambda) h d\mathbf{x}_s \\ \text{and } Z_t &= \epsilon + \int_0^t \nabla_y f(z_s, \mu) Z_s d\mathbf{x}_s + \int_0^t \nabla_\lambda f(z_s, \mu) h d\mathbf{x}_s. \end{aligned}$$

Hence, $N := Z - Y$ is solution to

$$\begin{aligned} N_t &= \int_0^t \nabla_y f(y_s) N_s d\mathbf{x}_s + \int_0^t F(y_s, z_s, Z_s) d\mathbf{x}_s \\ &\quad + \int_0^t G(z_s, Z_s) d\mathbf{x}_s + \int_0^t (\nabla_\lambda f(z_s, \mu) - \nabla_\lambda f(y_s, \lambda)) h d\mathbf{x}_s \end{aligned}$$

with

$$F(y, z, u) = (\nabla_y f(z, \lambda) - \nabla_y f(y, \lambda))u \text{ and } G(z, u) = (\nabla_y f(z, \mu) - \nabla_y f(z, \lambda))u.$$

Lemma 10. *The cross-iterated integral $Z \bowtie x$ exists and*

$$\|Z\|_p + \|Z \bowtie x\|_{p/2} \leq C_{35}(|\epsilon| + |h|)$$

for some constant C_{35} that depends only on the structure.

Proof. The control on $\|Z\|_p$ follows from Theorem 1. The cross-iterated integral $Z \bowtie x$ is constructed from the almost rough path

$$\mathbf{P}_{s,t} := 1 + Z_{s,t} + x_{s,t}^1 + (\nabla_y f(z_s)Z_s + \nabla_\lambda f(z_s)h)x_{s,t}^2$$

and the result follows from standard computations. \square

The proof of Theorem 3 follows from Theorem 1 and some controls on the involved integrals.

Let us start with a control on $\int F(y_s, z_s, Z_s) d\mathbf{x}_s$ which follows from Lemma 12 and 13 below, which are close to Lemma 9.

Lemma 11. *The integral $P := \int F(y_s, z_s, Z_s) d\mathbf{x}_s$ exists and*

$$\|P\|_p \leq C_{36}(|\epsilon| + |h|)(\|\Delta\|_p + |\Delta_0|)^\xi$$

for some constant C_{37} that depends only on the structure, where

$$\xi := \begin{cases} 1 & \text{under } (\mathbf{H}_1), \\ (1 - \kappa)\gamma \text{ with } \kappa \in (0, 1], 2 + \kappa\gamma > p & \text{under } (\mathbf{H}_2). \end{cases}$$

Let us consider

$$\begin{aligned} \widehat{P}_{s,t} &= F(y_s, z_s, Z_s)x_{s,t}^1 + \nabla_y F(y_s, z_s, Z_s)y \bowtie x_{s,t} \\ &\quad + \nabla_z F(y_s, z_s, Z_s)z \bowtie x_{s,t} + \nabla_u F(y_s, z_s, Z_s)Z \bowtie x_{s,t}. \end{aligned}$$

Lemma 12. *With $\widehat{P}_{s,r,t} := \widehat{P}_{s,t} - \widehat{P}_{s,r} - \widehat{P}_{r,t}$ for $0 \leq s \leq r \leq t \leq T$,*

$$|\widehat{P}_{s,r,t}| \leq C_{37}(|\epsilon| + |h|) \begin{cases} \omega(s, t)^{(2+\gamma)/p} (|\Delta_0| + \|\Delta\|_p) & \text{under } (\mathbf{H}_1), \\ \omega(s, t)^{(2+\kappa\gamma)/p} (|\Delta_0| + \|\Delta\|_p)^\xi & \text{under } (\mathbf{H}_2), \end{cases} \quad (38)$$

for some constant C_{37} that depends only on the structure.

Proof. In order to show that \widehat{P} is an almost rough path, we start first by some estimations. First, for $\tau \in [0, 1]$, set $g_{s,t}(\tau) = g_s + \tau(g_t - g_s)$ for a continuous function g and

$$\begin{aligned} E_1(s, t; \tau) &:= \nabla_y F(y_{s,t}(\tau), z_s + \tau z_{s,r}, Z_{s,t}(\tau)) - \nabla_y F(y_s, z_s, Z_s) \\ &\quad + \nabla_z F(y_{s,t}(\tau), z_{s,t}(\tau), Z_{s,t}(\tau)) - \nabla_z F(y_s, z_s, Z_s), \end{aligned}$$

we have

$$\begin{aligned}
& |E_1(s, t; \tau)| \\
&= |\nabla_{yy}^2 f(y_{s,t}(\tau)) Z_{s,t}(\tau) - \nabla_{yy}^2 f(y_s) Z_s - \nabla_{yy}^2 f(z_{s,t}(\tau)) Z_{s,t}(\tau) - \nabla_{yy}^2 f(z_s) Z_s| \\
&\leq |\nabla_{yy}^2 f(y_{s,t}(\tau)) - \nabla_{yy}^2 f(y_s) - \nabla_{yy}^2 f(z_{s,t}(\tau)) - \nabla_{yy}^2 f(z_s)| \cdot \|Z\|_\infty \\
&\quad + |Z_t - Z_s| \cdot |\nabla_{yy}^2 f(y_t) - \nabla_{yy}^2 f(z_t)| \\
&\leq C_{38}(|\epsilon| + |h|) \begin{cases} (|\Delta_0| + \|\Delta\|_p) \omega(s, t)^{1/p} & \text{under } (\mathbf{H}_\textcircled{1}), \\ (|\Delta_0| + \|\Delta\|_p)^{\bar{\kappa}\gamma} \omega(s, t)^{\bar{\kappa}\gamma/p} & \text{under } (\mathbf{H}_\textcircled{2}) \end{cases}
\end{aligned}$$

for some constant C_{38} that depends only on the structure.

For $0 \leq s \leq r \leq t \leq T$ and $\tau \in [0, 1]$,

$$\begin{aligned}
E_2(s, t) &:= (\nabla_y F(y_t, z_t, Z_t) - \nabla_y F(y_s, z_s, Z_s)) y \bowtie x_{r,t} \\
&\quad + (\nabla_z F(y_t, z_t, Z_t) - \nabla_z F(y_s, z_s, Z_s)) z \bowtie x_{r,t} \\
&= E_1(s, t; 1) y \bowtie x_{r,t} + (\nabla_z F(y_t, z_t, Z_t) - \nabla_z F(y_s, z_s, Z_s)) \Delta \bowtie x_{r,t}.
\end{aligned}$$

With Corollary 4, we obtain easily that

$$|E_2(s, t; 1)| \leq C_{39}(|\epsilon| + |h|) \begin{cases} \omega(s, t)^{(2+\gamma)/p} (|\Delta_0| + \|\Delta\|_p) & \text{under } (\mathbf{H}_\textcircled{1}), \\ \omega(s, t)^{(2+\kappa\gamma)/p} (|\Delta_0| + \|\Delta\|_p)^{\bar{\kappa}\gamma} & \text{under } (\mathbf{H}_\textcircled{2}). \end{cases} \quad (39)$$

Similarly, for $\tau \in [0, 1]$,

$$\begin{aligned}
E_3(s, t; \tau) &:= (\nabla_y F(y_{s,t}(\tau), z_{s,t}(\tau), Z_{s,t}(\tau)) - \nabla_y F(y_s, z_s, Z_s)) y_{s,t} \\
&\quad + (\nabla_z F(y_{s,t}(\tau), z_{s,t}(\tau), Z_{s,t}(\tau)) - \nabla_z F(y_s, z_s, Z_s)) z_{s,t} \\
&= E_1(s, t; \tau) y_{s,t} + (\nabla_z F(y_{s,t}(\tau), z_{s,t}(\tau), Z_{s,t}(\tau)) - \nabla_z F(y_s, z_s, Z_s)) \Delta_{s,t}
\end{aligned}$$

and then

$$|E_3(s, t; \tau)| \leq C_{40}(|\epsilon| + |h|) \begin{cases} \omega(s, t)^{(1+\gamma)/p} (|\Delta_0| + \|\Delta\|_p) & \text{under } (\mathbf{H}_\textcircled{1}), \\ \omega(s, t)^{(1+\kappa\gamma)/p} (|\Delta_0| + \|\Delta\|_p)^{\bar{\kappa}\gamma} & \text{under } (\mathbf{H}_\textcircled{2}) \end{cases} \quad (40)$$

for some constant C_{40} that depends only on the structure.

On the other hand,

$$\begin{aligned}
E_4(s, t) &:= (\nabla_u F(y_t, z_t, Z_t) - \nabla_u F(y_s, z_s, Z_s)) \\
&= (\nabla f(y_t) - \nabla f(y_s) + \nabla f(z_t) - \nabla f(z_s))
\end{aligned}$$

so that with Lemma 1,

$$|E_4(s, t)| \leq C_{41}(|\Delta_0| + \|\Delta\|_p) \begin{cases} \omega(s, t)^{1/p} & \text{under } (\mathbf{H}_{\textcircled{1}}), \\ \omega(s, t)^{\gamma/p} & \text{under } (\mathbf{H}_{\textcircled{2}}), \end{cases} \quad (41)$$

for some constant C_{39} that depends only on the structure.

At least, standard computations show that $\widehat{P}_{s,r,t}$ is equal to

$$\widehat{P}_{s,r,t} = \left(\int_0^1 E_3(s, t; \tau) d\tau \right) x_{r,t}^1 + E_2(s, t) + E_4(s, t) Z \ltimes x_{r,t}.$$

With (39), (40) and (41), this proves (38). \square

Lemma 13. *For a constant C_{42} that depends only on the structure,*

$$\|\widehat{P}\|_p \leq C_{42}(|\epsilon| + |h|)(|\Delta_0| + \|\Delta\|_p)^\xi. \quad (42)$$

Proof. It holds that

$$\begin{aligned} & |\nabla_y F(y_s, z_s, Z_s) + \nabla_z F(y_s, z_s, Z_s)| \\ &= |\nabla_{yy}^2 f(y_s) - \nabla_{yy}^2 f(z_s)| \cdot |Z_s| \leq C_{43}(|\epsilon| + |h|)(|\Delta_0| + \|\Delta\|_p)^\nu \end{aligned}$$

for some constant C_{37} that depends only on the structure, with $\nu = 1$ under $(\mathbf{H}_{\textcircled{1}})$ and $\nu = \gamma$ under $(\mathbf{H}_{\textcircled{2}})$. As above for providing a bound in $E_2(s, t; \tau)$,

$$\begin{aligned} & |\nabla_y F(y_s, z_s, Z_s) y \ltimes x_{s,t} + \nabla_z F(y_s, z_s, Z_s) z \ltimes x_{s,t}| \\ & \leq C_{44}(|\epsilon| + |h|) \omega(s, t)^{2/p} (|\Delta_0| + \|\Delta\|_p)^\xi \end{aligned}$$

for some constant C_{37} that depends only on the structure. Besides,

$$|F(y_s, z_s, Z_s)| \leq \|\nabla_y f\|_\infty \|\Delta\|_\infty (|\epsilon| + |h|) \text{ and } |\nabla_u F(y_s, z_s, Z_s)| \leq \|\nabla^2 f\|_\infty \|\Delta\|_\infty.$$

This proves (42). \square

Lemma 14. *There exists a family $\{\mathbf{A} \cdot \ltimes P_{s,t}\}_{0 \leq s \leq t \leq T}$ satisfying (7) and*

$$\|\mathbf{A} \cdot \ltimes P\|_{p/2} \leq C_{45}(|\epsilon| + |h|)(|\Delta_0| + \|\Delta\|_p)^\xi$$

for some constant C_{45} that depends only on the structure.

Proof. This follows from Lemma 9(II) and from the computations of Lemma 12 and Lemma 12, as well as an estimate on $|F(y_t, z_t, Z_t) - F(y_s, z_s, Z_s)|$ which follows immediately from Lemma 1. \square

Lemma 15. *Let us set $R := \int G(z_s, Z_s) d\mathbf{x}_s$. Then $\|R\|_p \leq C_{46}(|\epsilon| + |h|) \cdot |\mu - \lambda|$. Besides, a family $\{\mathbf{A} \ltimes R_{s,t}\}_{0 \leq s \leq t \leq T}$ satisfying (7) exists and*

$$\|\mathbf{A} \ltimes R_{s,t}\|_{p/2} \leq C_{47}(|\epsilon| + |h|) \cdot |\mu - \lambda|. \quad (43)$$

Proof. The integral R is the rough path associated to

$$\mathbf{P}_{s,t} := G(z_s, Z_s)x_{s,t}^1 + \nabla_y G(z_s, Z_s)z \ltimes x_{s,t} + (\nabla_y f(z_s, \mu) - \nabla_y f(z_s, \lambda))Z \ltimes x_{s,t}.$$

For $\tau \in [0, 1]$,

$$\begin{aligned} & |\nabla_y G(z_s + \tau z_{s,t}, Z_s + \tau Z_s) - \nabla_y G(z_s, Z_s)| \\ & \leq |\nabla_{yy}^2 f(z_s + \tau z_{s,t}, \mu) - \nabla_{yy}^2 f(z_s, \mu) - \nabla_{yy}^2 f(z_s + \tau z_{s,t}, \lambda) + \nabla_{yy}^2 f(z_s, \lambda)| \cdot |Z_s| \\ & \quad + \|\nabla_{y\lambda}^2 f\|_\infty |\mu - \lambda| \|Z\|_p \omega(s, t)^{1/p} \\ & \leq \int_0^1 \|Z\|_\infty |\nabla_{yy\lambda}^3 f(z_s + \tau z_{s,r}, \lambda + \sigma(\mu - \lambda)) - \nabla_{yy\lambda}^3 f(z_s, \lambda + \sigma(\mu - \lambda))| \cdot |\mu - \lambda| d\sigma \\ & \quad + \|\nabla_{y\lambda}^2 f\|_\infty |\mu - \lambda| \|Z\|_p \omega(s, t)^{1/p} \\ & \leq |\mu - \lambda| (\|Z\|_p + \|Z\|_\infty) \begin{cases} \|z\|_p \|\nabla_{yy\lambda}^3 f\|_\infty \omega(s, t)^{1/p} & \text{under } (\mathbf{H}_\textcircled{1}), \\ \|z\|_p^\gamma H_\gamma(\nabla_{yy\lambda}^3 f) \omega(s, t)^{\gamma/p} & \text{under } (\mathbf{H}_\textcircled{2}). \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} |G(z_s, Z_s)| & \leq \|z\|_\infty \|Z\|_\infty |\mu - \lambda| \|\nabla_{\lambda y}^2 f\|_\infty, \\ |\nabla_u G(z_s, Z_s)| & = |\nabla_y f(z_s, \mu) - \nabla_y f(z_s, \lambda)| \leq \|z\|_\infty |\mu - \lambda| \|\nabla_{\lambda y}^2 f\|_\infty \\ \text{and } |\nabla_y G(z_s, Z_s)| & \leq \|Z\|_\infty \|z\|_\infty |\mu - \lambda| \|\nabla_{\lambda yy}^3 f\|_\infty. \end{aligned}$$

With Lemma 9(I), this is sufficient to prove the existence of R and the given estimate. In addition,

$$\begin{aligned} |G(z_t, Z_t) - G(z_s, Z_s)| & \leq |\mu - \lambda| \cdot \|Z\|_p \omega(s, t)^{1/p} \|\nabla_{\lambda y}^2 f\|_\infty \\ & \quad + |\mu - \lambda| \cdot \|Z\|_\infty \begin{cases} \|z\|_p \|\nabla_{\lambda yy}^4 f\|_\infty \omega(s, t)^{1/p} & \text{under } (\mathbf{H}_\textcircled{1}), \\ \|z\|_p^\gamma H_\gamma(\nabla_{\lambda yy}^3 f) \omega(s, t)^{\gamma/p} & \text{under } (\mathbf{H}_\textcircled{2}), \end{cases} \end{aligned}$$

With Lemma 9(II) and the above inequalities, this is sufficient to show that $\mathbf{A} \ltimes R$ exists and satisfies (43). \square

Let us note that

$$\int_0^t (\nabla_y f(z_s, \mu) - \nabla_y f(z_s, \lambda)) d\mathbf{x}_s = \int_0^t (\nabla_y f(z_s, \mu) - \nabla_y f(z_s, \lambda)) d\mathbf{x}_s + Q_{0,t}$$

and $Q_{0,t}$ has already been studied in Proposition 6.

Lemma 16. *The integral*

$$U_{s,t} := \int_0^t (\nabla_\lambda f(z_s, \mu) - \nabla_\lambda f(z_s, \lambda)) \, d\mathbf{x}_s$$

exists, as well as family $\{\mathbf{A} \cdot \ltimes U_{s,t}\}_{0 \leq s \leq t \leq T}$ satisfying (7). In addition,

$$\|U\|_p \leq C_{48} |\mu - \lambda|^\zeta \text{ and } \|\mathbf{A} \cdot \ltimes U\|_{p/2} \leq C_{49} |\mu - \lambda|^\zeta$$

for some constants C_{48} and C_{49} that depend only on the structure and

$$\zeta := \begin{cases} 1 & \text{under } (\mathbf{H}_\textcircled{1}), \\ (1 - \kappa)\beta \text{ with } 2 + \kappa\gamma > 1 & \text{under } (\mathbf{H}_\textcircled{2}). \end{cases}$$

Proof. We have

$$\begin{aligned} |\nabla_\lambda f(z_s, \mu) - \nabla_\lambda f(z_s, \lambda)| &\leq \begin{cases} \|\nabla_{\lambda\lambda}^2 f\|_\infty \|z\|_\infty |\mu - \lambda| & \text{under } (\mathbf{H}_\textcircled{1}), \\ H_\beta^{(\lambda)}(\nabla_\lambda f) \|z\|_\infty |\mu - \lambda|^\beta & \text{under } (\mathbf{H}_\textcircled{2}), \end{cases} \\ |\nabla_{y\lambda}^2 f(z_s, \mu) - \nabla_{y\lambda}^2 f(z_s, \lambda)| &\leq \begin{cases} \|\nabla_{\lambda\lambda y}^3 f\|_\infty \|z\|_\infty |\mu - \lambda| & \text{under } (\mathbf{H}_\textcircled{1}), \\ H_\beta^{(\lambda)}(\nabla_{y\lambda}^2 f) \|z\|_\infty |\mu - \lambda|^\beta & \text{under } (\mathbf{H}_\textcircled{2}), \end{cases} \end{aligned}$$

and for $\kappa \in [0, 1]$ and $\bar{\kappa} := 1 - \kappa$,

$$\begin{aligned} &|\nabla_{y\lambda}^2 f(z_s + \tau z_{s,r}, \mu) - \nabla_{y\lambda}^2 f(z_s, \mu) + \nabla_{y\lambda}^2 f(z_s + \tau z_{s,r}, \lambda) - \nabla_{y\lambda}^2 f(z_s, \lambda)| \\ &\leq \begin{cases} \|\nabla_{\lambda\lambda y}^3 f\|_\infty |\mu - \lambda| \|z\|_p \omega(s, t)^{1/p} & \text{under } (\mathbf{H}_\textcircled{1}), \\ 2H_\beta^{(\lambda)}(\nabla_{y\lambda}^2 f)^{\bar{\kappa}} H_\gamma(\nabla_{yy}^2 f)^\kappa \|z\|_p^{\kappa\gamma} \omega(s, t)^{\kappa\gamma/p} |\mu - \lambda|^{\bar{\kappa}\beta} & \text{under } (\mathbf{H}_\textcircled{2}). \end{cases} \end{aligned}$$

In addition

$$\begin{aligned} &|\nabla_\lambda f(z_t, \mu) - \nabla_\lambda f(z_s, \mu) + \nabla_\lambda f(z_t, \lambda) - \nabla_\lambda f(z_s, \lambda)| \\ &\leq \begin{cases} \|\nabla_{\lambda y}^2 f\|_\infty |\mu - \lambda| \|z\|_p \omega(s, t)^{1/p} & \text{under } (\mathbf{H}_\textcircled{1}), \\ 2H_\beta^{(\lambda)}(\nabla_\lambda f)^{\bar{\kappa}} \|\nabla_{\lambda y}^2 f\|_\infty^\kappa \|z\|_p^{\kappa/p} \omega(s, t)^{\kappa/p} |\mu - \lambda|^{\bar{\kappa}\beta} & \text{under } (\mathbf{H}_\textcircled{2}). \end{cases} \end{aligned}$$

The result follows from Lemma 9. \square

We may end with the proof of Theorem 3.

Proof of Theorem 3. With Theorem 4, $\|\Delta\|_p \leq C_{50}(|\mu - \lambda| + |b - a|)$ for a constant C_{50} that depends only on the structure. The proof is now an immediate consequence of Lemma 12, Lemma 15, Proposition 6 and Lemma 16. \square

7 Perturbation of the driving path

We may also consider the derivative of the Itô map when the rough path \mathbf{x} is subject to a “regular” perturbation.

To be more precise, let us consider a rough path \mathbf{x} of finite p -variation, $p \in [2, 3)$ and let h be path with values in U of finite q -variation with $1/p + 1/q > 1$. Note that with this condition, $q < 2$.

We denote by $\|h\|_q$ the q -variation semi-norm of h .

With this condition on p and q , it is possible to construct a rough path \mathbf{x}^h that lies above $x^1 + h$ [35]. Indeed, $x \bowtie h_{s,t} := \int_s^t (x_r - x_s) dh_r$, $h \bowtie x_{s,t} := \int_s^t (h_r - h_s) dx_r$ and $h \bowtie h_{s,t} := \int_s^t (h_r - h_s) dh_r$ are well defined as Young integrals [49] and then

$$\mathbf{x}_{s,t}^h = 1 + x_{s,t}^1 + h_{s,t} + x_{s,t}^2 + x \bowtie h_{s,t} + h \bowtie x_{s,t} + h \bowtie h_{s,t}.$$

However, there will be no need to consider these integrals.

We assume hypotheses similar to **(H)** and **(H_①)** or **(H_②)** by assuming that we only consider perturbations h in a set that ensures there exists global bounds on the solutions y and z as well as on f and its derivatives.

Let z and y be the solutions to the RDE

$$z_t = a + \int_0^t f(z_s) d\mathbf{x}^h \text{ and } y_t = a + \int_0^t f(y_s) d\mathbf{x}.$$

Indeed, z may be interpreted as a solution to the linear RDE

$$z_t = a + \int_0^t f(z_s) d\mathbf{x}_s + \int_0^t f(z_r) dh_r$$

driven by the (p, q) -rough path (\mathbf{x}, h) , where $\int f(z_r) dh_r$ is interpreted as a Young integral [35]. Using the result of [35] (See Remark at the end of Theorem 3 in [35]) or using computations similar to the ones used in this article, one gets easily the following result.

Proposition 7. *It holds that*

$$\|z\|_p \leq C_{51} \|h\|_q \text{ and } \|(z - y) \bowtie x\|_{p/2} \leq C_{52} \|h\|_q$$

for some constants C_{51} and C_{52} that depend only on the structure.

Remark 4. Here, the interpretation in term of a driving rough paths allows one to get a control on $\|z - y\|_p$ and $\|(z - y) \bowtie x\|_{p/2}$ linearly with $\|h\|_q$. Without this and if we apply only Theorem 4 in [31], we get a control in term of $\|h\|_q \vee \|h\|_q^2$ because the p -variation distance between \mathbf{x}^h and \mathbf{x} involves the integral $\int_0^\cdot h_{0,s} dh_s$.

Theorem 6. Let $\mathfrak{Q}(\mathbf{x}; h)$ be the solution to the perturbed linear RDE

$$\mathfrak{Q}(\mathbf{x}; h)_t = \int_0^t \nabla f(y_s) \mathfrak{Q}(\mathbf{x}; h)_s d\mathbf{x}_s + \int_0^t f(y_s) dh_s.$$

Then

$$\|\Delta - \mathfrak{Q}(\mathbf{x}; h)\|_p \leq C_{53} \|h\|_q^\delta$$

for some constant C_{53} that depends only on the structure and

$$\delta := \begin{cases} 2 & \text{if } f \in \mathcal{C}^{3+\gamma} \text{ with } 2 + \gamma > p, \\ 1 + (1 - \kappa)\gamma & \text{if } f \in \mathcal{C}^{2+\gamma} \text{ with } \kappa \in (0, 1], \ 2 + \kappa\gamma > p. \end{cases}$$

The Itô map $\mathfrak{J} : \mathbf{x} \mapsto y$ is Fréchet differentiable under a perturbation of \mathbf{x} by h of finite q -variation.

Theorem 7. For h and k of finite q -variation, there exists a constant C_{54} that depends only on the structure such that

$$\|\mathfrak{Q}(\mathbf{x}^k; h) - \mathfrak{Q}(\mathbf{x}; h)\|_p \leq C_{54} \|h\|_q \|k\|_q^\xi$$

with ξ defined by

$$\xi := \begin{cases} 1 & \text{under } (\mathbf{H}_1), \\ (1 - \kappa)\gamma \text{ with } \kappa \in (0, 1], \ 2 + \kappa\gamma > p & \text{under } (\mathbf{H}_2). \end{cases}$$

The proof relies on the following lemma.

Lemma 17. Let P be the Young integral $P_t := \int_0^t (f(z_r) - f(y_r)) dh_r$. Then

$$\|P\|_q \leq C_{55} \|\Delta\|_p \cdot \|h\|_q,$$

for some constant C_{55} that depend only on the structure.

Proof. Since $f(z) - f(y)$ has finite p -variation controlled by ω , the estimate

$$\|P\|_q \leq C_{56} \|\Delta\|_p \cdot \|h\|_q$$

for some constant C_{56} that depends only on the structure follows immediately from Lemma 1 and the fact that for Young integrals,

$$|P_{s,t} - (f(z_s) - f(y_s))h_{s,t}| \leq |f(z) - f(y)| \cdot \|h\|_q \omega(s, t)^{1/p+1/q}$$

and that

$$|f(z_s) - f(y_s)| \leq \|\nabla f\|_\infty \|\Delta\|_p \omega(0, T)^{1/p} \text{ for } 0 \leq s \leq T,$$

since $y_0 = z_0 = a$. □

Proof of Theorem 6. As previously, one may write

$$\Delta_t = \int_0^t F(y_s, z_s) d\mathbf{x}_s + \int_0^t \nabla f(y_s) \Delta_s d\mathbf{x}_s + \int_0^t f(z_s) dh_s$$

with

$$F(y, z) = f(z) - f(y) - \nabla f(y)(z - y).$$

Hence $\Delta - \mathfrak{Q}(h)$ is solution to

$$\begin{aligned} (\Delta - \mathfrak{Q}(h))_t &= \int_0^t \nabla f(y_s) (\Delta - \mathfrak{Q}(h))_s d\mathbf{x}_s \\ &\quad + \int_0^t F(y_s, z_s) d\mathbf{x}_s + \int_0^t (f(z_s) - f(y_s)) dh_s. \end{aligned}$$

The result follows from Theorem 1 with the controls on $R := \int F(y_s, z_s) d\mathbf{x}_s$ and on $R \rtimes A$ given by Proposition 4 and Corollary 5 (in the statements of these results, we have replaced $\|\Delta\|_p$ by its bound in term of $|\Delta_0|$ and $|\mu - \lambda|$. Here, Proposition 7 shall be used instead), and Lemma 17. \square

Proof of Theorem 7. Let us consider now

$$y_t = a + \int_0^t f(y_s) d\mathbf{x}_s \text{ and } z_t = a + \int_0^t f(z_s) d\mathbf{x}_s^k,$$

as well as

$$\begin{aligned} Y_t &= \int_0^t \nabla f(y_s) Y_s d\mathbf{x}_s + \int_0^t f(y_s) dh_s \\ \text{and } Z_t &= \int_0^t \nabla f(z_s) Z_s d\mathbf{x}_s^k + \int_0^t f(z_s) dh_s \end{aligned}$$

so that $Y_t = \mathfrak{Q}(\mathbf{x}; h)$ and $Z_t = \mathfrak{Q}(\mathbf{x}^k; h)$. For $N_t := Z_t - Y_t$,

$$\begin{aligned} N_t &:= \int_0^t \nabla f(y_s) N_s d\mathbf{x}_s - \int_0^t (\nabla f(y_s) - \nabla f(z_s)) Z_s d\mathbf{x}_s \\ &\quad + \int_0^t (f(z_s) - f(y_s)) dh_s + \int_0^t \nabla f(z_s) Z_s dk_s. \end{aligned}$$

As for the proof of Lemma 17, we have $\|Z\|_p \leq C_{57} \|h\|_q$ for a constant C_{57} that depends only on the structure. Hence

$$|\nabla f(z_t) Z_t - \nabla f(z_s) Z_s| \leq \omega(s, t)^{1/p} C_{57} \|h\|_q (\|\nabla f\|_\infty + \|\nabla^2 f\|_\infty \|z\|_p \omega(0, T)^{1/p})$$

and then for $R_t := \int_0^t f(z_s) Z_s dk_s$ we have as in proof of Lemma 17,

$$\|R\|_p \leq C_{58} \|h\|_q \|k\|_q$$

for some constants C_{58} that depends only on the structure.

The result follows from Theorem 1(II) and Lemma 17, as well as Lemma 11 and Lemma 14 in which $|\epsilon| + |h|$ is replaced by $\|h\|_q$ (since this term depends on the bounds of $\|Z\|_p$ and $\|Z \ltimes x\|_{p/2}$ computed in Lemma 10 which has to be adapted here). \square

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