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Existence results for Hughes' model for pedestrian flows

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Abstract

In this paper we prove two global existence results for Hughes' model of pedestrian flows under assumptions that ensures that the traces of the solutions along the turning curve are zero for any positive times.

Key words: crowd dynamics, conservation laws, eikonal equation, Hughes' model for pedestrian flows

2010 MSC: 35L65, 35F21, 90B20

1. Introduction

In this paper we study the one dimensional version of Hughes' model [16, 17] for pedestrian flows

$$\partial_t \rho - \partial_x \left[\rho v(\rho) \frac{\partial_x \varphi}{|\partial_x \varphi|} \right] = 0, \quad |\partial_x \varphi| = c(\rho), \quad (1)$$

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in the spatial domain $\Omega =]-1, 1[$, together with homogeneous Dirichlet boundary conditions

$$\rho(t, -1) = \rho(t, 1) = 0, \quad \varphi(t, -1) = \varphi(t, 1) = 0, \quad t > 0, \quad (2a)$$

and initial datum

$$\rho(0, x) = \bar{\rho}(x), \quad x \in]-1, 1[. \quad (2b)$$

Here $x \in \Omega$ is the space variable, $t \geq 0$ is the time, $\rho = \rho(t, x) \in [0, 1]$ is the (normalized) crowd density,

$$v(\rho) = (1 - \rho) \quad \text{and} \quad c(\rho) = 1/v(\rho)$$

are respectively the mean velocity and the running cost. We denote $f(\rho) = \rho v(\rho) = \rho(1 - \rho)$.

The initial datum $\bar{\rho}$ is assumed to be in $\mathbf{L}^\infty(\Omega; \mathbb{R})$ with $\|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} < 1$. This assumption, together with the maximum principle proved in [13], will ensure that the cost computed along any solution of (1) is well defined.

As already observed in [13, 2], the system (1) can be rewritten as

$$\partial_t \rho + \partial_x F(t, x, \rho) = 0, \quad (3a)$$

$$\int_{-1}^{\xi(t)} c(\rho(t, x)) dx = \int_{\xi(t)}^1 c(\rho(t, x)) dx, \quad (3b)$$

where $F(t, x, \rho) = \text{sgn}(x - \xi(t)) f(\rho)$ is discontinuous along the so called turning curve, $x = \xi(t)$, implicitly defined by (3b).

Definition 1. [13] *A map $(t, x) \mapsto \rho(t, x)$ is an entropy weak solution of the initial-boundary value problem (2), (3) if it is in $\mathbf{C}^0([0, +\infty[; \mathbf{L}^\infty(\Omega; [0, 1]))$ and for any $k \in [0, 1]$ and any test function $\psi \in \mathbf{C}_c^\infty([0, +\infty[\times \Omega; [0, +\infty[)$ it satisfies*

$$\int_0^{+\infty} \int_{-1}^1 [|\rho - k| \partial_t \psi + \mathcal{F}(t, x, \rho, k) \partial_x \psi] dx dt + \int_{-1}^1 |\bar{\rho}(x) - k| \psi(0, x) dx \quad (4a)$$

$$+ \int_0^{+\infty} [f(\rho(t, -1+)) - f(k)] \psi(t, -1) dt + \int_0^{+\infty} [f(\rho(t, 1-)) - f(k)] \psi(t, 1) dt \quad (4b)$$

$$+ 2 \int_0^{+\infty} f(k) \psi(t, \xi(t)) dt \geq 0. \quad (4c)$$

Above, we have used the notation

$$\mathcal{F}(t, x, \rho, k) = \operatorname{sgn}(\rho - k) [F(t, x, \rho) - F(t, x, k)].$$

The first line (4a) originates from the Kruřkov definition of entropy weak solution in the case of a Cauchy problem, [18]. Line (4b) comes from the boundary condition introduced by Bardos et al. in [6], see also [4, 8, 9, 19]. The latter line (4c) accounts for the discontinuity of the flux along the turning curve, see [1, 3, 5, 14, 21].

Observe that the strong traces of the solution at the boundary points exist due to the genuine non-linearity of the flux (see [20, 22]) and must satisfy

$$\begin{aligned} f(\rho(t, -1+)) &\geq f(k) && \text{for all } k \in [0, \rho(t, -1+)], \\ f(\rho(t, 1-)) &\geq f(k) && \text{for all } k \in [0, \rho(t, 1-)]. \end{aligned}$$

This in particular implies that

$$\rho(t, -1+) \leq 1/2 \quad \text{and} \quad \rho(t, 1-) \leq 1/2. \quad (5)$$

Since the cost function c is convex in $[0, 1[$, we can consider it as an entropy for (3) with entropy flux q on $x > \xi(t)$ (respectively, $-q$ on $x < \xi(t)$), where

$$q(\rho) = -c(\rho) + 2 \log c(\rho) \quad (6)$$

satisfies $q' = c' f'$. Observe that $q(0) = -1$ and $(1 - 2\rho) q'(\rho) > 0$ for $\rho \neq 1/2$.

In the perspective of studying the well-posedness for problem (1), (2), a first result was proved in [12], where the eikonal equation in (1) is replaced by an elliptic approximation. However, the techniques used in [12] cannot be applied to the original system (1), which displays a moving spatial discontinuity in the flux depending in a nonlocal way on the solution itself.

In the present article, motivated by the availability of a local Riemann solver for (1) provided in [2], we follow the wave-front tracking approach [10]: we construct a sequence of piecewise constant approximate solutions by solving locally the Riemann problems arising at each jump discontinuities, and prove their convergence by providing the uniform boundedness of total variation. Unlike the case of classical scalar conservation laws, we have to face two major problems when applying the wave-front tracking method to (1): in fact, when two wave-fronts interact or a wave-front interacts with the turning curve, several new fronts may arise from the turning curve, and the total variation of the solution may generically increase.

In this paper we show that suitable hypotheses on the initial datum prevent these situations from occurring, guaranteeing the existence of a sequence of approximate solutions with uniformly bounded total variation. Even in this somewhat simplified situation, the wave-front tracking approach is very useful when studying the variation in time of the turning curve $\xi(t)$, implicitly defined by (3b). A convenient choice of the speed of approximate rarefactions is used, see (12c).

For numerical purposes, the wave-front tracking algorithm for (1) was analyzed in [15].

As a first existence result, we give the following theorem for the “symmetric” case. Let us denote by \mathcal{S} the space of functions ρ in $\mathbf{L}^\infty(\Omega; [0, 1[)$ that are even, namely $\rho(x) = \rho(-x)$ for a.e. $x \in \Omega$.

Theorem 2. *For any initial datum $\bar{\rho}$ in \mathcal{S} there exists a unique entropy weak solution ρ of (2), (3) such that $\rho(t, \cdot) \in \mathcal{S}$ for all $t > 0$.*

PROOF. If ρ is an entropy weak solution of (2), (3) and $\rho(t, \cdot) \in \mathcal{S}$ for all $t > 0$, then necessarily $\xi \equiv 0$. Indeed, by (3b) we have that for any $t > 0$

$$\int_{\xi(t)}^1 c(\rho(t, x)) dx = \int_{-1}^{\xi(t)} c(\rho(t, x)) dx = \int_{-\xi(t)}^1 c(\rho(t, -x)) dx = \int_{-\xi(t)}^1 c(\rho(t, x)) dx$$

and therefore $\xi(t) = 0$ because by definition $c(\rho) \geq 1$. As a consequence of the Rankine-Hugoniot condition along the turning curve, we have that $f(\rho(t, 0+)) + f(\rho(t, 0-)) = 0$, namely $\rho(t, 0\pm) = 0$. Thus, for any fixed initial datum $\bar{\rho}$ in \mathcal{S} the unique candidate in \mathcal{S} to be the entropy weak solution of (2), (3) is the function $\rho : [0, +\infty[\times \Omega \rightarrow [0, \|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}]$ defined for $x \in [0, 1]$ as the entropy weak solution to the initial-boundary value problem

$$\begin{aligned} \partial_t \rho + \partial_x f(\rho) &= 0 & t > 0, x \in]0, 1[, \\ \rho(t, 0) = \rho(t, 1) &= 0 & t > 0, \\ \rho(0, x) &= \bar{\rho}(x) & x \in]0, 1[, \end{aligned}$$

namely, for any $k \in [0, 1]$ and any test function $\psi \in \mathbf{C}_c^\infty(\mathbb{R} \times [0, 1]; [0, +\infty[)$

$$\begin{aligned} & \int_0^{+\infty} \int_0^1 \{|\rho - k| \partial_t \psi + \operatorname{sgn}(\rho - k) [f(\rho) - f(k)] \partial_x \psi\} \, dx \, dt \\ & + \int_0^1 |\bar{\rho}(x) - k| \psi(0, x) \, dx + \int_0^{+\infty} [f(\rho(t, 1-)) - f(k)] \psi(t, 1) \, dt \\ & - \int_0^{+\infty} [f(\rho(t, 0+)) - f(k)] \psi(t, 0) \, dt \geq 0. \end{aligned}$$

Since for the maximum principle $\|\rho(t, \cdot)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} < 1$ for any $t > 0$, we have that $\rho \equiv 0$ in a region that contains $\{(t, x) : t > 0, x \in \Omega, |x| < v(\|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}) t\}$ and it is immediate to prove that ρ is an entropy weak solution of (2), (3) in the sense of Definition 1.

In the following theorem we treat a more general case.

Theorem 3. *If the initial datum $\bar{\rho} \in \mathbf{BV}(\Omega)$ satisfies*

$$3\|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} + \operatorname{TV}(c(\bar{\rho})) + [c(\bar{\rho}(-1+)) - c(1/2)]_+ + [c(\bar{\rho}(1-)) - c(1/2)]_+ < 2, \quad (7)$$

then there exists an entropy weak solution of (2), (3) defined globally in time.

Above $[x]_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. The proof is based on the wave-front tracking algorithm [10]. We use the results achieved in [2, 13], where the solutions of the Riemann problems associated to (2), (3) are constructed and studied. We underline that (7) implies that $\bar{\rho}(\pm 1 \mp) < (7 - \sqrt{13})/6 \sim 0.565741$.

The paper is organized as follows. In Section 2 we describe the wave-front tracking algorithm used to construct the approximate solution to (2), (3). In Section 3, provided that (7) hold on the initial data, we prove that the approximate solutions exist globally and finally prove Theorem 3. Some technical lemmas are deferred to the last section.

2. The approximate solution

For any fixed $n \in \mathbb{N}$, consider the approximation parameter $\varepsilon = 2^{-n} > 0$ and the ε -grid $\mathcal{G}^\varepsilon = \{i\varepsilon : i = 0, \dots, \varepsilon^{-1}\}$. We aim to construct ρ^ε , a piecewise constant ε -approximate solution of (2), (3) taking values in \mathcal{G}^ε and obtained by applying the wave-front tracking algorithm described below. As a first

step introduce a piecewise constant function $\bar{\rho}^\varepsilon$ in $\mathbf{BV} \cap \mathbf{L}^1(\Omega; \mathcal{G}^\varepsilon)$, with a finite number of discontinuities and such that

$$\bar{\rho}^\varepsilon(\pm 1 \mp) \leq \bar{\rho}(\pm 1 \mp), \quad \|\bar{\rho}^\varepsilon\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}, \quad (8a)$$

$$\text{TV}(c(\bar{\rho}^\varepsilon)) \leq \text{TV}(c(\bar{\rho})), \quad \|\bar{\rho} - \bar{\rho}^\varepsilon\|_{\mathbf{L}^1(\Omega; \mathbb{R})} \leq \varepsilon. \quad (8b)$$

Let $\bar{\xi}^\varepsilon$ be the solution of the equation

$$\int_{-1}^{\bar{\xi}^\varepsilon} c(\bar{\rho}^\varepsilon(x)) dx = \int_{\bar{\xi}^\varepsilon}^1 c(\bar{\rho}^\varepsilon(x)) dx. \quad (9)$$

Consider the piecewise linear function f^ε that coincides with f on \mathcal{G}^ε and whose derivative exists in $[0, 1] \setminus \mathcal{G}^\varepsilon$, namely

$$f^\varepsilon(\rho) = \sum_{i=0}^{\varepsilon^{-1}-1} [i(1+i)\varepsilon^2 + (1 - (1+2i)\varepsilon)\rho] \chi_{[i\varepsilon, (i+1)\varepsilon]}(\rho).$$

We call shock waves (respectively, rarefaction waves) the decreasing discontinuities on the left of the turning curve and the increasing discontinuities on the right of the turning curve (respectively, the increasing discontinuities on the left of the turning curve and the decreasing discontinuities on the right of the turning curve) of the solution to (2), (3) with $\bar{\rho}^\varepsilon$ instead of $\bar{\rho}$, f^ε instead of f and constructed with the classical Riemann solver \mathcal{R}_c . Introduce the simplified Riemann solver \mathcal{R}_s , that replaces any rarefaction wave given by \mathcal{R}_c with a rarefaction front (we shall state more precise assumptions below, see (12c)). Apply then \mathcal{R}_s to solve each Riemann problem associated to the boundary $\{-1, 1\}$ and to the jumps of discontinuity of $\bar{\rho}^\varepsilon$ away from $x = \bar{\xi}^\varepsilon$. Denote by ρ_L^ε , respectively ρ_R^ε , the juxtaposition of the piecewise constant functions obtained by solving with \mathcal{R}_s the Riemann problems on the left of $x = \bar{\xi}^\varepsilon$, respectively on the right of $x = \bar{\xi}^\varepsilon$. Observe that ρ_L^ε and ρ_R^ε are well defined for times sufficiently small and that in general they can not be juxtaposed (indeed, in general, $\bar{\rho}^\varepsilon(\bar{\xi}^\varepsilon -) \neq \bar{\rho}^\varepsilon(\bar{\xi}^\varepsilon +)$). By applying Theorem 6 given below, we can construct a piecewise constant function ρ_ξ^ε such that if ρ^ε is the juxtaposition of ρ_L^ε , ρ_ξ^ε and ρ_R^ε , then the corresponding turning curve $x_0^\varepsilon \equiv \xi^\varepsilon$ defined by

$$\int_{-1}^{\xi^\varepsilon(t)} c(\rho^\varepsilon(t, x)) dx = \int_{\xi^\varepsilon(t)}^1 c(\rho^\varepsilon(t, x)) dx \quad (10)$$

satisfies the Rankine-Hugoniot condition (12a) below.

Let x_i^ε , $i = -h, \dots, -1, 1, \dots, k$, with $-1 < x_i^\varepsilon(t) < x_{i+1}^\varepsilon(t) < 1$, be the curves of discontinuity of ρ^ε away from $x = \xi^\varepsilon$, which we call *fronts*. By construction $x_i^\varepsilon(0)$, $i \neq 0$, are the discontinuities of $\bar{\rho}^\varepsilon$ away from $x = \bar{\xi}^\varepsilon$, each $t \mapsto x_i^\varepsilon(t)$ is linear and ρ^ε is constant between two distinct fronts, namely for any $t > 0$ sufficiently small

$$\begin{aligned} \rho^\varepsilon(t, x) = & \rho_{-h-\frac{1}{2}}^\varepsilon \chi_{[-1, x_{-h}^\varepsilon(t)]}(x) + \sum_{i=-h}^{k-1} \rho_{i+\frac{1}{2}}^\varepsilon \chi_{[x_i^\varepsilon(t), x_{i+1}^\varepsilon(t)]}(x) \\ & + \rho_{k+\frac{1}{2}}^\varepsilon \chi_{[x_k^\varepsilon(t), 1]}(x), \end{aligned} \quad (11a)$$

where

$$\rho_{i-\frac{1}{2}}^\varepsilon \neq \rho_{i+\frac{1}{2}}^\varepsilon \text{ if } i \neq 0, \quad \rho_{-\frac{1}{2}}^\varepsilon = \rho_{\frac{1}{2}}^\varepsilon \text{ if and only if } \rho_{\pm\frac{1}{2}}^\varepsilon = 0, \quad (11b)$$

$$\rho_{-h-\frac{1}{2}}^\varepsilon \leq \frac{1}{2}, \quad \rho_{k+\frac{1}{2}}^\varepsilon \leq \frac{1}{2}. \quad (11c)$$

The ε -approximate solution ρ^ε is prolonged beyond any interaction time (namely, when one front reaches the boundary, or two (or more) fronts approach, or one front reaches the turning curve) by applying \mathcal{R}_s where the interaction takes place (if away from $x = \xi^\varepsilon$) and by applying then Theorem 6. Observe that as a result of an interaction, fronts away from $x = \xi^\varepsilon$ may disappear while new fronts may originate from the turning curve. However, the resulting ε -approximate solution ρ^ε has the same structure described above. Therefore, after each interaction time, we can keep the same notation introduced before by rearranging the indices and by considering h and k as piecewise constant functions of time. Finally, the turning curve is prolonged by applying (10) as long as ρ^ε is well defined.

In the sequel we refer to upward jumps on the left of $x = \xi^\varepsilon$ and to downward jumps on the right of $x = \xi^\varepsilon$ as *rarefaction fronts*, while the remaining jumps away from $x = \xi^\varepsilon$ are called *shock fronts*. We introduce the quantities

$$\sigma_i(t) = \operatorname{sgn}(i) \left[\rho_{i-\frac{1}{2}}^\varepsilon - \rho_{i+\frac{1}{2}}^\varepsilon \right] \quad \text{and} \quad \sigma(t) = \max_{i \neq 0} \sigma_i(t)$$

to measure the size of the jumps at the rarefaction fronts.

By construction ρ^ε satisfies the Rankine-Hugoniot jump conditions along the turning curve and the shock fronts, respectively

$$\left[\rho_{\frac{1}{2}}^\varepsilon - \rho_{-\frac{1}{2}}^\varepsilon\right] \dot{\xi}^\varepsilon = f\left(\rho_{\frac{1}{2}}^\varepsilon\right) + f\left(\rho_{-\frac{1}{2}}^\varepsilon\right), \quad (12a)$$

$$\dot{x}_i^\varepsilon = \operatorname{sgn}(i) \frac{f\left(\rho_{i+\frac{1}{2}}^\varepsilon\right) - f\left(\rho_{i-\frac{1}{2}}^\varepsilon\right)}{\rho_{i+\frac{1}{2}}^\varepsilon - \rho_{i-\frac{1}{2}}^\varepsilon} \quad \text{if } i \neq 0 \text{ and } \sigma_i < 0. \quad (12b)$$

On the other hand, we impose that any rarefaction front x_i^ε travels with speed

$$\dot{x}_i^\varepsilon = \operatorname{sgn}(i) \frac{q\left(\rho_{i+\frac{1}{2}}^\varepsilon\right) - q\left(\rho_{i-\frac{1}{2}}^\varepsilon\right)}{c\left(\rho_{i+\frac{1}{2}}^\varepsilon\right) - c\left(\rho_{i-\frac{1}{2}}^\varepsilon\right)} \quad \text{if } i \neq 0 \text{ and } \sigma_i > 0. \quad (12c)$$

Remark that in both (12b) and (12c) the wave speed are bounded by 1: $|\dot{x}_i^\varepsilon| < 1$ for $i \neq 0$. The above choice for the speed of propagation of the rarefaction fronts greatly simplifies the proof of Theorem 6 and is motivated as follows. As well known, see [7, 11], the entropy conditions hold with an equality along any classical rarefaction. Therefore (12c) results directly by a discrete representation of the classical rarefaction, and allows us to simplify the terms appearing in the representation for $\dot{\xi}$, see next formula (13).

Equation (10) has a key role in the construction of ρ_ξ^ε . In the next proposition we write (10) in an equivalent form more suitable to our purposes.

Proposition 4. *If ρ^ε satisfies (11), then condition (10) is equivalent to*

$$\left[c\left(\rho_{-\frac{1}{2}}^\varepsilon\right) + c\left(\rho_{\frac{1}{2}}^\varepsilon\right)\right] \dot{\xi}^\varepsilon = \sum_{i \neq 0} \operatorname{sgn}(i) \left[c\left(\rho_{i-\frac{1}{2}}^\varepsilon\right) - c\left(\rho_{i+\frac{1}{2}}^\varepsilon\right)\right] \dot{x}_i^\varepsilon. \quad (13)$$

PROOF. In order to simplify the notations we omit the dependence on ε and write $c(\rho_i^\varepsilon) = c_i$. By (11) we have

$$\int_{-1}^{\xi(t)} c(\rho(t, x)) \, dx = [x_{-h}(t) + 1] c_{-h-\frac{1}{2}} + \sum_{i=1}^h [x_{1-i}(t) - x_{-i}(t)] c_{\frac{1}{2}-i},$$

$$\int_{\xi(t)}^1 c(\rho(t, x)) \, dx = [1 - x_k(t)] c_{k+\frac{1}{2}} + \sum_{i=1}^k [x_i(t) - x_{i-1}(t)] c_{i-\frac{1}{2}},$$

and by rearranging the indexes we have

$$\int_{-1}^{\xi(t)} c(\rho(t, x)) \, dx = c_{-h-\frac{1}{2}} + c_{-\frac{1}{2}} \xi(t) + \sum_{i=1}^h [c_{-i-\frac{1}{2}} - c_{\frac{1}{2}-i}] x_{-i}(t),$$

$$\int_{\xi(t)}^1 c(\rho(t, x)) \, dx = c_{k+\frac{1}{2}} - c_{\frac{1}{2}} \xi(t) + \sum_{i=1}^k [c_{i-\frac{1}{2}} - c_{i+\frac{1}{2}}] x_i(t).$$

By taking the derivative with respect to t we obtain

$$\frac{d}{dt} \left[\int_{-1}^{\xi(t)} c(\rho(t, x)) \, dx \right] = c_{-\frac{1}{2}} \dot{\xi}(t) + \sum_{i=1}^h [c_{-i-\frac{1}{2}} - c_{\frac{1}{2}-i}] \dot{x}_{-i}(t), \quad (14a)$$

$$\frac{d}{dt} \left[\int_{\xi(t)}^1 c(\rho(t, x)) \, dx \right] = -c_{\frac{1}{2}} \dot{\xi}(t) + \sum_{i=1}^k [c_{i-\frac{1}{2}} - c_{i+\frac{1}{2}}] \dot{x}_i(t). \quad (14b)$$

By (10) the above quantities are equal and therefore we deduce (13). Finally, since the map $t \mapsto \xi(t)$ is piecewise linear and $\xi(0) = \bar{\xi}$, with $\bar{\xi}$ given by (9), we have indeed that (13) is equivalent to (10).

In the following definition we specify the properties of the ε -approximate solution.

Definition 5. *The piecewise constant function $(t, x) \mapsto \rho^\varepsilon(t, x)$ is an ε -admissible approximate solution of (3) if it consists of a finite number of fronts traveling according to (12), satisfies (13) and $\sigma(t) \leq \varepsilon$.*

For convenience we introduce the following notation,

$$\Psi[\rho^\varepsilon] = \sum_{i \neq 0} \Phi_i[\rho^\varepsilon], \quad \Phi_i[\rho^\varepsilon] = \begin{cases} [c(\rho_{-\frac{1}{2}}^\varepsilon) + c(\rho_{\frac{1}{2}}^\varepsilon)] \dot{\xi}^\varepsilon & \text{if } i=0, \\ \text{sgn}(i) [c(\rho_{i-\frac{1}{2}}^\varepsilon) - c(\rho_{i+\frac{1}{2}}^\varepsilon)] \dot{x}_i^\varepsilon & \text{if } i \neq 0, \end{cases} \quad (15)$$

so that (13) rewrites as

$$\Phi_0[\rho^\varepsilon] = \Psi[\rho^\varepsilon].$$

For later use, for any $\alpha, \beta \in [0, 1[$ with $\alpha \neq \beta$, introduce the quantities

$$\lambda_\xi(\alpha, \beta) = \frac{f(\beta) + f(\alpha)}{\beta - \alpha}, \quad \Phi_\xi(\alpha, \beta) = [c(\alpha) + c(\beta)] \lambda_\xi(\alpha, \beta), \quad (16a)$$

$$\lambda_s(\alpha, \beta) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}, \quad \Phi_s(\alpha, \beta) = [c(\alpha) - c(\beta)] \lambda_s(\alpha, \beta), \quad (16b)$$

$$\lambda_r(\alpha, \beta) = \frac{q(\beta) - q(\alpha)}{c(\beta) - c(\alpha)}, \quad \Phi_r(\alpha, \beta) = q(\alpha) - q(\beta). \quad (16c)$$

Clearly, if x_i^ε is a shock front, respectively a rarefaction front, then $\Phi_i[\rho^\varepsilon] = \Phi_s(\rho_{i-\frac{1}{2}}^\varepsilon, \rho_{i+\frac{1}{2}}^\varepsilon)$, respectively $\Phi_i[\rho^\varepsilon] = \Phi_r(\rho_{i-\frac{1}{2}}^\varepsilon, \rho_{i+\frac{1}{2}}^\varepsilon)$. Moreover, if $\rho_{-\frac{1}{2}}^\varepsilon \neq \rho_{\frac{1}{2}}^\varepsilon$, then $\Phi_0[\rho^\varepsilon] = \Phi_\xi(\rho_{-\frac{1}{2}}^\varepsilon, \rho_{\frac{1}{2}}^\varepsilon)$, otherwise $\rho_{\pm\frac{1}{2}}^\varepsilon = 0$, $\Phi_\xi(\rho_{-\frac{1}{2}}^\varepsilon, \rho_{\frac{1}{2}}^\varepsilon)$ is not well defined and $2\dot{\xi}^\varepsilon = \Psi[\rho^\varepsilon]$.

Some of the introduced quantities have a clear geometric interpretation in the (ρ, f) -plane. Indeed $\lambda_\xi(\alpha, \beta)$ represents the slope of the segment between $(\alpha, -f(\alpha))$ and $(\beta, f(\beta))$, $\lambda_s(\alpha, \beta)$ is the slope of the segment between $(\alpha, f(\alpha))$ and $(\beta, f(\beta))$, $v(\alpha)$ is the slope of the segment between $(0, 0)$ and $(\alpha, f(\alpha))$, and $\lambda_r(\alpha, \beta)$ represents the slope of the tangent to $\rho \mapsto f(\rho)$ in $(\gamma, f(\gamma))$ for a γ between α and β , see Figure 1, left. As a consequence, see

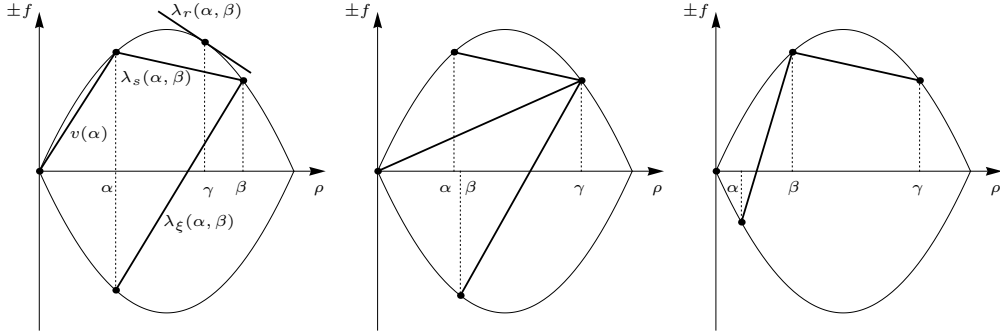


Figure 1: Left: Geometrical interpretation of the quantities introduced in (16). Center and right: Geometrical interpretation of the estimates, respectively, (17) and (18).

Figure 1, center and right,

$$\lambda_s(\alpha, \gamma) < \lambda_s(0, \gamma) = v(\gamma) = \lambda_\xi(0, \gamma) < \lambda_\xi(\beta, \gamma) \quad \text{for all } \alpha, \beta < \gamma, \quad (17)$$

$$\lambda_\xi(\alpha, \beta) > \lambda_s(\beta, \gamma) \quad \text{for all } \alpha < \beta < \gamma \quad (18)$$

and, since f' is decreasing, we have

$$f'(\beta) < \lambda_r(\beta, \alpha) < f'(\alpha) \quad \text{for all } \alpha < \beta. \quad (19)$$

By definition $\Phi_\xi(\alpha, \beta) \leq -v(\alpha) - v(\beta)$ for all $\alpha > \beta$, with the equality that holds if and only if $\beta = 0$, and $\Phi_\xi(\alpha, \beta) \geq v(\alpha) + v(\beta)$ for all $\alpha < \beta$, with the equality that holds if and only if $\alpha = 0$. Further properties of the just introduced functions are collected in Section 4.

For any interaction time $t = t_I$ or for $t = 0$, introduce also the quantity

$$\Psi[\rho_*^\varepsilon](t_I+) = \sum_{\substack{i \neq 0 \\ x_i^\varepsilon(t_I) \neq x_0^\varepsilon(t_I)}} \Phi_i[\rho^\varepsilon](t_I+), \quad (20)$$

where the sum counts the fronts of $\rho^\varepsilon(t_I+)$ that do not start from the turning curve at time $t = t_I$, namely, the fronts of ρ_L^ε and ρ_R^ε .

The next theorem, which upgrades [2, Theorem 1], shows how to construct ρ_ξ^ε once we have ρ_L^ε and ρ_R^ε .

Theorem 6. *Let $\varepsilon > 0$ be sufficiently small, and $\rho_L^\varepsilon, \rho_R^\varepsilon$ be constructed as explained above. If ρ^ε is the juxtaposition of $\rho_L^\varepsilon, \rho_\xi^\varepsilon, \rho_R^\varepsilon$ and is ε -admissible, then*

1. ρ_ξ^ε consists of rarefaction fronts on the right of the turning curve if and only if

$$0 \leq \rho_{\frac{1}{2}}^\varepsilon(t_I) < \rho_{-\frac{1}{2}}^\varepsilon(t_I) \quad \text{and} \quad \Psi[\rho_*^\varepsilon](t_I+) < \Phi_\xi\left(\rho_{-\frac{1}{2}}^\varepsilon(t_I), \rho_{\frac{1}{2}}^\varepsilon(t_I)\right). \quad (21)$$

In this case $\rho_{\frac{1}{2}}^\varepsilon(t_I) < \rho_{\frac{1}{2}}^\varepsilon(t_I+) < \rho_{-\frac{1}{2}}^\varepsilon(t_I+) = \rho_{-\frac{1}{2}}^\varepsilon(t_I)$.

2. ρ_ξ^ε consists of a shock front on the right of the turning curve if and only if either

$$0 = \rho_{-\frac{1}{2}}^\varepsilon(t_I) < \rho_{\frac{1}{2}}^\varepsilon(t_I) \quad \text{and} \quad \Psi[\rho_*^\varepsilon](t_I+) < v\left(\rho_{-\frac{1}{2}}^\varepsilon(t_I)\right) + v\left(\rho_{\frac{1}{2}}^\varepsilon(t_I)\right), \quad (22)$$

or

$$0 < \rho_{-\frac{1}{2}}^\varepsilon(t_I) < \rho_{\frac{1}{2}}^\varepsilon(t_I) \quad \text{and} \quad \Psi[\rho_*^\varepsilon](t_I+) \leq -v\left(\rho_{-\frac{1}{2}}^\varepsilon(t_I)\right) - v\left(\rho_{\frac{1}{2}}^\varepsilon(t_I)\right), \quad (23)$$

or

$$0 < \rho_{\frac{1}{2}}^\varepsilon(t_I) < \rho_{-\frac{1}{2}}^\varepsilon(t_I) \quad \text{and} \\ \Phi_\xi\left(\rho_{-\frac{1}{2}}^\varepsilon(t_I), \rho_{\frac{1}{2}}^\varepsilon(t_I)\right) < \Psi[\rho_*^\varepsilon](t_I+) \leq -v\left(\rho_{-\frac{1}{2}}^\varepsilon(t_I)\right) - v\left(\rho_{\frac{1}{2}}^\varepsilon(t_I)\right). \quad (24)$$

In the case (22) we have $\rho_{\pm\frac{1}{2}}^\varepsilon(t_I+) = 0 < \rho_{\frac{3}{2}}^\varepsilon(t_I+) = \rho_{\frac{1}{2}}^\varepsilon(t_I)$. In the cases (23), (24) we have $\rho_{\frac{1}{2}}^\varepsilon(t_I+) < \min \left\{ \rho_{-\frac{1}{2}}^\varepsilon(t_I+) = \rho_{-\frac{1}{2}}^\varepsilon(t_I), \rho_{\frac{3}{2}}^\varepsilon(t_I+) = \rho_{\frac{1}{2}}^\varepsilon(t_I) \right\}$ and $\rho_{\frac{1}{2}}^\varepsilon(t_I+) = 0$ if and only if $\Psi[\rho_*^\varepsilon](t_I+) = -v \left(\rho_{-\frac{1}{2}}^\varepsilon(t_I) \right) - v \left(\rho_{\frac{1}{2}}^\varepsilon(t_I) \right)$.

3. ρ_ξ^ε consists of two shock fronts, one on each side of the turning curve, if and only if

$$\rho_{\pm\frac{1}{2}}^\varepsilon(t_I) \neq 0 \quad \text{and} \quad |\Psi[\rho_*^\varepsilon](t_I+)| < v \left(\rho_{-\frac{1}{2}}^\varepsilon(t_I) \right) + v \left(\rho_{\frac{1}{2}}^\varepsilon(t_I) \right). \quad (25)$$

In this case $\rho_{\pm\frac{1}{2}}^\varepsilon(t_I+) = 0$ and $\rho_{\pm\frac{3}{2}}^\varepsilon(t_I+) = \rho_{\pm\frac{1}{2}}^\varepsilon(t_I)$.

4. ρ_ξ^ε consists of a shock front on the left of the turning curve if and only if either

$$0 = \rho_{\frac{1}{2}}^\varepsilon(t_I) < \rho_{-\frac{1}{2}}^\varepsilon(t_I) \quad \text{and} \quad \Psi[\rho_*^\varepsilon](t_I+) > -v \left(\rho_{-\frac{1}{2}}^\varepsilon(t_I) \right) - v \left(\rho_{\frac{1}{2}}^\varepsilon(t_I) \right), \quad (26)$$

or

$$0 < \rho_{\frac{1}{2}}^\varepsilon(t_I) < \rho_{-\frac{1}{2}}^\varepsilon(t_I) \quad \text{and} \quad \Psi[\rho_*^\varepsilon](t_I+) \geq v \left(\rho_{-\frac{1}{2}}^\varepsilon(t_I) \right) + v \left(\rho_{\frac{1}{2}}^\varepsilon(t_I) \right), \quad (27)$$

or

$$0 < \rho_{-\frac{1}{2}}^\varepsilon(t_I) < \rho_{\frac{1}{2}}^\varepsilon(t_I) \quad \text{and}$$

$$v \left(\rho_{-\frac{1}{2}}^\varepsilon(t_I) \right) + v \left(\rho_{\frac{1}{2}}^\varepsilon(t_I) \right) \leq \Psi[\rho_*^\varepsilon](t_I+) < \Phi_\xi \left(\rho_{-\frac{1}{2}}^\varepsilon(t_I), \rho_{\frac{1}{2}}^\varepsilon(t_I) \right). \quad (28)$$

In the case (26) we have $\rho_{\pm\frac{1}{2}}^\varepsilon(t_I+) = 0 < \rho_{-\frac{3}{2}}^\varepsilon(t_I+) = \rho_{-\frac{1}{2}}^\varepsilon(t_I)$. In the cases (27), (28) we have $\rho_{-\frac{1}{2}}^\varepsilon(t_I+) < \min \left\{ \rho_{-\frac{3}{2}}^\varepsilon(t_I+) = \rho_{-\frac{1}{2}}^\varepsilon(t_I), \rho_{\frac{1}{2}}^\varepsilon(t_I+) = \rho_{\frac{1}{2}}^\varepsilon(t_I) \right\}$ and $\rho_{-\frac{1}{2}}^\varepsilon(t_I+) = 0$ if and only if $\Psi[\rho_*^\varepsilon](t_I+) = v \left(\rho_{-\frac{1}{2}}^\varepsilon(t_I) \right) + v \left(\rho_{\frac{1}{2}}^\varepsilon(t_I) \right)$.

5. ρ_ξ^ε consists of rarefaction fronts on the left of the turning curve if and only if

$$0 \leq \rho_{-\frac{1}{2}}^\varepsilon(t_I) < \rho_{\frac{1}{2}}^\varepsilon(t_I) \quad \text{and} \quad \Psi[\rho_*^\varepsilon](t_I+) > \Phi_\xi \left(\rho_{-\frac{1}{2}}^\varepsilon(t_I), \rho_{\frac{1}{2}}^\varepsilon(t_I) \right). \quad (29)$$

In this case $\rho_{-\frac{1}{2}}^\varepsilon(t_I) < \rho_{-\frac{1}{2}}^\varepsilon(t_I+) < \rho_{\frac{1}{2}}^\varepsilon(t_I+) = \rho_{\frac{1}{2}}^\varepsilon(t_I)$.

6. ρ_ξ^ε consists of the turning curve alone if and only if

$$\rho_{\pm\frac{1}{2}}^\varepsilon(t_I) = 0, \quad (30)$$

or

$$\rho_{-\frac{1}{2}}^\varepsilon(t_I) \neq \rho_{\frac{1}{2}}^\varepsilon(t_I) \quad \text{and} \quad \Psi[\rho_*^\varepsilon](t_I+) = \Phi_\xi\left(\rho_{-\frac{1}{2}}^\varepsilon(t_I), \rho_{\frac{1}{2}}^\varepsilon(t_I)\right). \quad (31)$$

In both cases $\rho_{\pm\frac{1}{2}}^\varepsilon(t_I+) = \rho_{\pm\frac{1}{2}}^\varepsilon(t_I)$.

PROOF. For notational convenience, we omit the dependence on ε , write $\rho_{\pm\frac{1}{2}} = \rho_{\pm\frac{1}{2}}^\varepsilon(t_I)$ and denote $g(\rho_i^\varepsilon) = g_i$, $g(\rho_i^\xi) = g_i^\xi$ for any function $g : [0, 1] \rightarrow \mathbb{R}$.

(1, “ \Leftarrow ”) If ρ_ξ consists of rarefaction fronts x_i^ξ , $i = 1, \dots, m$, between the states $\rho_{\frac{1}{2}}^\xi, \dots, \rho_{m-\frac{1}{2}}^\xi, \rho_{\frac{1}{2}}$, on the right of the turning curve, then

$$\rho_{\frac{1}{2}} < \rho_{m-\frac{1}{2}}^\xi < \dots < \rho_{\frac{1}{2}}^\xi \quad \text{and} \quad \dot{\xi}(t_I+) < \dot{x}_1^\xi(t_I+) < \dots < \dot{x}_m^\xi(t_I+),$$

where $\dot{x}_i^\xi(t_I+) = \lambda_r\left(\rho_{i-\frac{1}{2}}^\xi, \rho_{i+\frac{1}{2}}^\xi\right)$, $i = 1, \dots, m-1$, $\dot{x}_m^\xi(t_I+) = \lambda_r\left(\rho_{m-\frac{1}{2}}^\xi, \rho_{\frac{1}{2}}^\xi\right)$. In particular $\rho_{\frac{1}{2}}^\xi \neq 0$ and therefore, by (12a), we have $\rho_{\frac{1}{2}}^\xi \neq \rho_{-\frac{1}{2}}$ and $\dot{\xi}(t_I+) = \lambda_\xi\left(\rho_{-\frac{1}{2}}, \rho_{\frac{1}{2}}^\xi\right)$. Moreover, we have $\rho_{\frac{1}{2}}^\xi(t_I) < \rho_{-\frac{1}{2}}$ because, by Lemma 8 and (17), for all $\rho < \rho_{\frac{1}{2}}^\xi$

$$\dot{x}_1^\xi(t_I+) = \lambda_r\left(\rho_{\frac{1}{2}}^\xi, \rho_{\frac{3}{2}}^\xi\right) < v_{\frac{1}{2}}^\xi < \lambda_\xi\left(\rho, \rho_{\frac{1}{2}}^\xi\right).$$

As a consequence the first condition in (21) is satisfied. We then observe that $\Phi_0[\rho](t_I+) = \Phi_\xi\left(\rho_{-\frac{1}{2}}, \rho_{\frac{1}{2}}^\xi\right)$ and

$$\begin{aligned} \Psi[\rho](t_I+) &= \Psi[\rho_*](t_I+) + \sum_{i=1}^{m-1} \Phi_r\left(\rho_{i-\frac{1}{2}}^\xi, \rho_{i+\frac{1}{2}}^\xi\right) + \Phi_r\left(\rho_{m-\frac{1}{2}}^\xi, \rho_{\frac{1}{2}}^\xi\right) \\ &= \Psi[\rho_*](t_I+) + q_{\frac{1}{2}}^\xi - q_{\frac{1}{2}}. \end{aligned}$$

By (13) we have that

$$\Phi_\xi\left(\rho_{-\frac{1}{2}}, \rho_{\frac{1}{2}}^\xi\right) = \Psi[\rho_*](t_I+) + q_{\frac{1}{2}}^\xi - q_{\frac{1}{2}}$$

and therefore by Lemma 9

$$\Psi[\rho_*](t_I+) = \Phi_\xi\left(\rho_{-\frac{1}{2}}, \rho_{\frac{1}{2}}^\xi\right) - q_{\frac{1}{2}}^\xi + q_{\frac{1}{2}} < \Phi_\xi\left(\rho_{-\frac{1}{2}}, \rho_{\frac{1}{2}}\right).$$

Thus also the second condition in (21) is satisfied.

(2, “ \Leftarrow ”) If ρ_ξ consists of a shock front x_1^ξ on the right of the turning curve between the states $\rho_{\frac{1}{2}}^\xi$ and $\rho_{\frac{1}{2}}$, then

$$\rho_{\frac{1}{2}} > \rho_{\frac{1}{2}}^\xi, \quad \dot{\xi}(t_I+) < \dot{x}_1^\xi(t_I+) = \lambda_s\left(\rho_{\frac{1}{2}}^\xi, \rho_{\frac{1}{2}}\right)$$

and

$$\Phi_0[\rho](t_I+) = \left[c_{-\frac{1}{2}} + c_{\frac{1}{2}}^\xi\right] \dot{\xi}(t_I+), \quad \Psi[\rho](t_I+) = \Psi[\rho_*](t_I+) + \Phi_s\left(\rho_{\frac{1}{2}}^\xi, \rho_{\frac{1}{2}}\right).$$

By (13) we have

$$\Psi[\rho_*](t_I+) = \left[c_{-\frac{1}{2}} + c_{\frac{1}{2}}^\xi\right] \dot{\xi}(t_I+) - \Phi_s\left(\rho_{\frac{1}{2}}^\xi, \rho_{\frac{1}{2}}\right).$$

By (18) it must be $\rho_{-\frac{1}{2}} \geq \rho_{\frac{1}{2}}^\xi$. In order to proceed with the proof, we have to distinguish the following cases:

- If $\rho_{-\frac{1}{2}} = \rho_{\frac{1}{2}}^\xi$, then by (12a) we have $\rho_{-\frac{1}{2}} = \rho_{\frac{1}{2}}^\xi = 0$. As a consequence

$$\dot{\xi}(t_I+) < \lambda_s\left(\rho_{\frac{1}{2}}^\xi, \rho_{\frac{1}{2}}\right) = v_{\frac{1}{2}}, \quad \Phi_s\left(\rho_{\frac{1}{2}}^\xi, \rho_{\frac{1}{2}}\right) = v_{\frac{1}{2}} - 1$$

and

$$\Psi[\rho_*](t_I+) < 2v_{\frac{1}{2}} - \left[v_{\frac{1}{2}} - 1\right] = v_{-\frac{1}{2}} + v_{\frac{1}{2}}.$$

- If $\rho_{-\frac{1}{2}} \neq \rho_{\frac{1}{2}}^\xi$, then $\dot{\xi}(t_I+) = \lambda_\xi\left(\rho_{-\frac{1}{2}}, \rho_{\frac{1}{2}}^\xi\right)$ and therefore $\Psi[\rho_*](t_I+) = \psi\left(\rho_{\frac{1}{2}}^\xi\right)$, where

$$\psi\left(\rho_{\frac{1}{2}}^\xi\right) = -\frac{2\rho_{\frac{1}{2}}^\xi}{\rho_{-\frac{1}{2}} - \rho_{\frac{1}{2}}^\xi} \left[v_{-\frac{1}{2}}c_{\frac{1}{2}}^\xi + 1\right] - \left[v_{-\frac{1}{2}} + v_{\frac{1}{2}}\right]c_{\frac{1}{2}}^\xi - \left[c_{-\frac{1}{2}} + c_{\frac{1}{2}}\right]\rho_{\frac{1}{2}}^\xi.$$

Clearly the map $\rho \mapsto \psi(\rho)$ is decreasing and

$$\psi(0) = -v_{-\frac{1}{2}} - v_{\frac{1}{2}}, \quad \lim_{\rho \uparrow \rho_{-\frac{1}{2}}} \psi(\rho) = -\infty, \quad \lim_{\rho \uparrow \rho_{\frac{1}{2}}} \psi(\rho) = \Phi_\xi\left(\rho_{-\frac{1}{2}}(t_I), \rho_{\frac{1}{2}}(t_I)\right).$$

Recalling that $\rho_{\frac{1}{2}}^\xi < \min \left\{ \rho_{-\frac{1}{2}}, \rho_{\frac{1}{2}} \right\}$, we deduce (23) and (24).

(3, “ \Leftarrow ”) If ρ_ξ consists of a shock front x_{-1}^ξ on the left of the turning curve between the states $\rho_{-\frac{1}{2}}$ and $\rho_{-\frac{1}{2}}^\xi$, and a shock front x_1^ξ on the right of the turning curve between the states $\rho_{\frac{1}{2}}^\xi$ and $\rho_{\frac{1}{2}}$, then

$$\begin{aligned} \rho_{-\frac{1}{2}}^\xi &< \rho_{-\frac{1}{2}}, & \rho_{\frac{1}{2}}^\xi &< \rho_{\frac{1}{2}}, \\ \dot{x}_{-1}^\xi(t_I+) &= -\lambda_s \left(\rho_{-\frac{1}{2}}, \rho_{-\frac{1}{2}}^\xi \right) < \dot{\xi}(t_I+) < \dot{x}_1^\xi(t_I+) &= \lambda_s \left(\rho_{\frac{1}{2}}^\xi, \rho_{\frac{1}{2}} \right). \end{aligned}$$

By (18) we have $\lambda_\xi \left(\rho, \rho_{\frac{1}{2}}^\xi \right) > \lambda_s \left(\rho_{\frac{1}{2}}^\xi, \rho_{\frac{1}{2}} \right)$ for all $\rho < \rho_{\frac{1}{2}}^\xi$, and therefore $\rho_{\frac{1}{2}}^\xi \leq \rho_{-\frac{1}{2}}^\xi$. Analogously, we have $-\lambda_s \left(\rho_{-\frac{1}{2}}, \rho_{-\frac{1}{2}}^\xi \right) > \lambda_\xi \left(\rho_{-\frac{1}{2}}^\xi, \rho \right)$ for all $\rho < \rho_{-\frac{1}{2}}^\xi$, and therefore $\rho_{\frac{1}{2}}^\xi \geq \rho_{-\frac{1}{2}}^\xi$. In conclusion we proved that $\rho_{\frac{1}{2}}^\xi = \rho_{-\frac{1}{2}}^\xi$ and this, by (12a), implies that $\rho_{\pm\frac{1}{2}}^\xi = 0$. Hence

$$\begin{aligned} \Phi_0[\rho](t_I+) &= 2\dot{\xi}(t_I+), \\ \Psi[\rho](t_I+) &= \Psi[\rho_*](t_I+) + \Phi_s \left(\rho_{-\frac{1}{2}}, 0 \right) + \Phi_s \left(0, \rho_{\frac{1}{2}} \right) = \Psi[\rho_*](t_I+) - v_{-\frac{1}{2}} + v_{\frac{1}{2}} \end{aligned}$$

and by (13) we have

$$\Psi[\rho_*](t_I+) = 2\dot{\xi}(t_I+) + v_{-\frac{1}{2}} - v_{\frac{1}{2}}.$$

Thus (25) holds true because $-v_{-\frac{1}{2}} < \dot{\xi}(t_I+) < v_{\frac{1}{2}}$.

(4, “ \Leftarrow ”) & (5, “ \Leftarrow ”) In accordance to the symmetry of the problem, the proofs are analogous to that for the cases (2) and (1), respectively.

(6, “ \Leftarrow ”) If ρ_ξ is given by the turning curve only, then (30) and (31) follow from (12a) and (13).

(“ \Rightarrow ”) The converse is obvious by the above construction of the solutions. Indeed, for any given $\rho_{-\frac{1}{2}}, \rho_{\frac{1}{2}}, \Psi[\rho_*](t_I+)$, only one of the conditions (21)-(31) is satisfied because $\Phi_\xi(\alpha, \beta) > v(\alpha) + v(\beta)$ for any $0 < \alpha < \beta$ and $\Phi_\xi(0, \beta) = 1 + v(\beta)$. Furthermore, ρ^ξ never consists of rarefaction fronts on both sides of the turning curve. Indeed, as already proved, the presence of rarefaction fronts on the left of the turning curve implies that $\dot{\xi}(t_I+) > 0$, as well as the presence of rarefaction fronts on the right of the turning curve implies that $\dot{\xi}(t_I+) < 0$.

The proof of Theorem 6 is then complete.

3. Proof of Theorem 3

In this section we give a proof of Theorem 3. As proved in Theorem 6, in general new fronts can start from the turning curve, with the result that ρ^ε may well take values outside the grid \mathcal{G}^ε . However, as we will see in the next theorem, condition (7) ensures that the ε -approximate solution takes values always in \mathcal{G}^ε .

Theorem 7. *If the initial datum satisfies the condition (7), then there exists an ε -admissible approximate solution of (3) in the sense of Definition 5 with values in \mathcal{G}^ε and defined globally in time.*

PROOF. Consider $\bar{\xi}^\varepsilon$ given by (9), $\rho_L^\varepsilon(t, \cdot)$ and $\rho_R^\varepsilon(t, \cdot)$ as in Section 2. Recalling (15) and that $|\bar{x}_i^\varepsilon| \leq 1$ for $i \neq 0$, we notice that

$$|\Phi_i[\rho^\varepsilon](t)| \leq \left| c\left(\rho_{i-\frac{1}{2}}^\varepsilon\right) - c\left(\rho_{i+\frac{1}{2}}^\varepsilon\right) \right|, \quad i \neq 0.$$

Now let $\bar{\rho}_\infty = \|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})}$. Since $v(\rho) = 1 - \rho$ and using the assumption (7), we have

$$\begin{aligned} |\Psi[\rho_*^\varepsilon](0+)| &\leq \text{TV}(c(\rho_L^\varepsilon(0+))) + \text{TV}(c(\rho_R^\varepsilon(0+))) \\ &\leq \text{TV}(c(\bar{\rho})) + [c(\bar{\rho}(-1+)) - c(1/2)]_+ + [c(\bar{\rho}(1-)) - c(1/2)]_+ \\ &< 2 - 3\bar{\rho}_\infty \leq v\left(\bar{\rho}_{-\frac{1}{2}}^\varepsilon\right) + v\left(\bar{\rho}_{\frac{1}{2}}^\varepsilon\right). \end{aligned}$$

Above, the terms $[c(\bar{\rho}(\pm 1 \mp)) - c(1/2)]_+$ account for the possible rarefactions arising at the boundaries at time $t = 0$. Thus, by Theorem 6, only cases (22), (25), (26) and (30) can hold true. In all of these cases, we have $\rho_{\pm\frac{1}{2}}^\varepsilon(0+) = 0$. Moreover, as a consequence of the maximum principle $\|\rho^\varepsilon(t)\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \|\bar{\rho}^\varepsilon\|_{\mathbf{L}^\infty(\Omega; \mathbb{R})} \leq \bar{\rho}_\infty < 1$ for any $t > 0$ sufficiently small. Thus, for $t > 0$ sufficiently small we have $\rho^\varepsilon \equiv 0$ in $|x - \bar{\xi}^\varepsilon| < v(\bar{\rho}_\infty)t$.

At each interaction we apply the algorithm and Theorem 6 to extend ρ^ε in time as described in Section 2. We want to prove that, at each interaction, the condition (30) holds true by showing that no front can reach the turning curve.

Assume that, for some $\tilde{t} > 0$, one has $\rho^\varepsilon(t, \xi^\varepsilon(t) \pm) = 0$ for all $t \in]0, \tilde{t}[$. Since no interactions with the ξ^ε have occurred up to time \tilde{t} , then

$$\begin{aligned} &\text{TV}(c(\rho^\varepsilon(t, \cdot))) \leq \text{TV}(c(\rho^\varepsilon(0+, \cdot))) \\ &\leq \text{TV}(c(\bar{\rho})) + [c(\bar{\rho}(-1+)) - c(1/2)]_+ + [c(\bar{\rho}(1-)) - c(1/2)]_+ + 2\bar{\rho}_\infty. \end{aligned} \quad (32)$$

Above, the last term accounts for the possible jumps arising at $x = \bar{\xi}^\varepsilon$.

More precisely we find that

$$\begin{aligned} \sum_{|i| \geq 2} |\Phi_i[\rho^\varepsilon](\tilde{t}-)| &\leq \sum_{|i| \geq 2} \left| c\left(\rho_{i+\frac{1}{2}}^\varepsilon\right) - c\left(\rho_{i-\frac{1}{2}}^\varepsilon\right) \right| \\ &\leq \text{TV}(c(\bar{\rho})) + [c(\bar{\rho}(-1+)) - c(1/2)]_+ + [c(\bar{\rho}(1-)) - c(1/2)]_+. \end{aligned} \quad (33)$$

Indeed, recall that the indexes are possibly rearranged at each interaction; the above sum decreases in all possible interactions, that is: when a front leaves the domain Ω ; when two or more fronts interact, with all indexes different from ± 1 ; when the interaction involves a ± 1 front, in which case the resulting front will inherit the ± 1 index.

Clearly, $x_{\pm 1}^\varepsilon$ are shock fronts with speeds $\dot{x}_{\pm 1}^\varepsilon(\tilde{t}-) = \pm v\left(\rho_{\pm \frac{3}{2}}^\varepsilon(\tilde{t}-)\right)$. Recalling that $c(\rho) \cdot v(\rho) = 1$, we can write

$$\begin{aligned} \Phi_{-1}[\rho^\varepsilon](\tilde{t}-) + \Phi_1[\rho^\varepsilon](\tilde{t}-) &= \\ &= \left[c\left(\rho_{-\frac{3}{2}}^\varepsilon(\tilde{t}-)\right) - 1 \right] v\left(\rho_{-\frac{3}{2}}^\varepsilon(\tilde{t}-)\right) + \left[1 - c\left(\rho_{\frac{3}{2}}^\varepsilon(\tilde{t}-)\right) \right] v\left(\rho_{\frac{3}{2}}^\varepsilon(\tilde{t}-)\right) \\ &= \rho_{-\frac{3}{2}}^\varepsilon - \rho_{\frac{3}{2}}^\varepsilon. \end{aligned}$$

Thus $|\Phi_{-1}[\rho^\varepsilon](\tilde{t}-) + \Phi_1[\rho^\varepsilon](\tilde{t}-)| \leq \bar{\rho}_\infty$. By virtue of (13), (15), (33) and (7) we have that

$$\begin{aligned} 2|\dot{\xi}^\varepsilon(\tilde{t}-)| &= \left| \sum_{i \neq 0} \Phi_i[\rho^\varepsilon](\tilde{t}-) \right| \leq \bar{\rho}_\infty + \sum_{|i| \geq 2} |\Phi_i[\rho^\varepsilon](\tilde{t}-)| \\ &\leq \bar{\rho}_\infty + \text{TV}(c(\bar{\rho})) + [c(\bar{\rho}(-1+)) - c(1/2)]_+ + [c(\bar{\rho}(1-)) - c(1/2)]_+ \\ &< 2(1 - \bar{\rho}_\infty) \leq 2|\dot{x}_{\pm 1}^\varepsilon(\tilde{t}-)| \end{aligned}$$

and therefore the turning curve does not reach $x_{\pm 1}^\varepsilon$ at time $t = \tilde{t}$. As a consequence, no new front starts from the turning curve and we can apply the standard theory of wave-front tracking to prolong ρ^ε after any interaction and to ensure its global existence.

Finally we prove that, up to a subsequence, as ε goes to zero ρ^ε converges to a solution of (2), (3). By construction, see the proof of Theorem 7, we

immediately have that for a.e. $t > 0$

$$\begin{aligned} \|\rho^\varepsilon(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} &\leq \|\bar{\rho}^\varepsilon\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \leq \|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}, \\ \text{TV}(\rho^\varepsilon(t)) &\leq \text{TV}(\bar{\rho}^\varepsilon) + [\bar{\rho}^\varepsilon(-1+) - 1/2]_+ + [\bar{\rho}^\varepsilon(1-) - 1/2]_+ + 2\|\bar{\rho}^\varepsilon\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \\ &\leq L, \end{aligned}$$

where $L = \text{TV}(\bar{\rho}) + [\bar{\rho}(-1+) - 1/2]_+ + [\bar{\rho}(1-) - 1/2]_+ + 2\|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}$. Finally observe that

$$\int_{-1}^1 |\rho^\varepsilon(t, x) - \rho^\varepsilon(s, x)| dx \leq L |t - s|.$$

Indeed, if no interaction occurs for times between t and s , then

$$\begin{aligned} \int_{-1}^1 |c(\rho^\varepsilon(t, x)) - c(\rho^\varepsilon(s, x))| dx &\leq \sum_{i \neq 0} \left| (t - s) \dot{x}_i^\varepsilon(t) \left(\rho_{i-\frac{1}{2}}^\varepsilon - \rho_{i+\frac{1}{2}}^\varepsilon \right) \right| \\ &\leq |t - s| \text{TV}(\rho^\varepsilon(t)). \end{aligned}$$

The case when one or more interactions take place for times between t and s is similar, because by (12) the map $t \mapsto \rho^\varepsilon(t)$ is \mathbf{L}^1 -continuous across interaction times.

Thus, by applying Helly's Theorem in the form [7, Theorem 2.4], there exists a function $\rho \in \mathbf{L}_{\text{loc}}^1([0, +\infty[\times \Omega; [0, 1])$ and a subsequence, still denoted ρ^ε , such that

$$\begin{aligned} \rho^\varepsilon &\rightarrow \rho \text{ in } \mathbf{L}_{\text{loc}}^1([0, +\infty[\times \Omega; \mathbb{R}) \text{ as } \varepsilon \downarrow 0, \\ \text{TV}(\rho(t)) &\leq L, \\ \|\rho(t)\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} &\leq \|\bar{\rho}\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}, \\ \|\rho(t) - \rho(s)\|_{\mathbf{L}^1(\Omega;\mathbb{R})} &\leq L |t - s| \text{ for all } t, s \geq 0. \end{aligned}$$

Regarding ξ^ε , we observe that the associated sequence is bounded and uniformly equicontinuous, because of the uniform Lipschitz constant. Therefore, by Ascoli-Arzelà theorem, we can extract a subsequence uniformly converging to some $\xi \in W^{1,1}([0, T])$, for any $T > 0$, with the same Lipschitz constant $(1 - \bar{\rho}_\infty)$. In particular, ξ evolves in a cone where ρ is zero, and it is straightforward to show that (3b) holds.

We want to prove that ρ is an entropy weak solution of the initial-boundary value problem (2), (3) in the sense of Definition 1. For any fixed

$k \in [0, 1]$ and $\psi \in \mathbf{C}_c^\infty(\mathbb{R} \times \Omega; [0, +\infty[)$, we have to prove that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left\{ \int_0^{+\infty} \int_{-1}^1 [|\rho^\varepsilon - k| \partial_t \psi + \mathcal{F}(t, x, \rho^\varepsilon, k) \partial_x \psi] dx dt \right. \\ \left. + \int_{-1}^1 |\bar{\rho}^\varepsilon(x) - k| \psi(0, x) dx + \int_0^{+\infty} [f(\rho^\varepsilon(t, -1+)) - f(k)] \psi(t, -1) dt \right. \\ \left. + \int_0^{+\infty} [f(\rho^\varepsilon(t, 1-)) - f(k)] \psi(t, 1) dt \right\} + 2 \int_0^{+\infty} f(k) \psi(t, \xi(t)) dt \geq 0. \end{aligned}$$

Since the shock fronts and the turning curve satisfy the Rankine-Hugoniot condition, it is sufficient to prove the above estimate for a test function ψ whose support is a subset of $]t_1, t_2[\times]x_1, x_2[$ and for a rarefaction front x_i^ε that is the only front in $]t_1, t_2[\times]x_1, x_2[$. It is not limitative to assume that $i > 1$. In this case, the above quantity writes

$$\begin{aligned} & \int_{t_1}^{t_2} \int_{x_1}^{x_2} [|\rho^\varepsilon - k| \partial_t \psi + \operatorname{sgn}(\rho^\varepsilon - k) [f(\rho^\varepsilon) - f(k)] \partial_x \psi] dx dt \\ = & \int_{t_1}^{t_2} \int_{x_1}^{x_i^\varepsilon(t)} [|\rho_{i-\frac{1}{2}}^\varepsilon - k| \partial_t \psi + \operatorname{sgn}(\rho_{i-\frac{1}{2}}^\varepsilon - k) [f(\rho_{i-\frac{1}{2}}^\varepsilon) - f(k)] \partial_x \psi] dx dt \\ & + \int_{t_1}^{t_2} \int_{x_i^\varepsilon(t)}^{x_2} [|\rho_{i+\frac{1}{2}}^\varepsilon - k| \partial_t \psi + \operatorname{sgn}(\rho_{i+\frac{1}{2}}^\varepsilon - k) [f(\rho_{i+\frac{1}{2}}^\varepsilon) - f(k)] \partial_x \psi] dx dt \\ = & \int_{t_1}^{t_2} [-|\rho_{i-\frac{1}{2}}^\varepsilon - k| \dot{x}_i^\varepsilon(t) + \operatorname{sgn}(\rho_{i-\frac{1}{2}}^\varepsilon - k) [f(\rho_{i-\frac{1}{2}}^\varepsilon) - f(k)]] \psi(t, x_i^\varepsilon(t)) dt \\ & + \int_{t_1}^{t_2} [|\rho_{i+\frac{1}{2}}^\varepsilon - k| \dot{x}_i^\varepsilon(t) - \operatorname{sgn}(\rho_{i+\frac{1}{2}}^\varepsilon - k) [f(\rho_{i+\frac{1}{2}}^\varepsilon) - f(k)]] \psi(t, x_i^\varepsilon(t)) dt \\ = & \int_{t_1}^{t_2} \Theta(\rho_{i-\frac{1}{2}}^\varepsilon, \rho_{i+\frac{1}{2}}^\varepsilon, k) \psi(t, x_i^\varepsilon(t)) dt, \end{aligned}$$

where

$$\begin{aligned} \Theta(\rho_{i-\frac{1}{2}}^\varepsilon, \rho_{i+\frac{1}{2}}^\varepsilon, k) = & \left[|\rho_{i+\frac{1}{2}}^\varepsilon - k| - |\rho_{i-\frac{1}{2}}^\varepsilon - k| \right] \dot{x}_i^\varepsilon(t) \\ & + \operatorname{sgn}(\rho_{i-\frac{1}{2}}^\varepsilon - k) [f(\rho_{i-\frac{1}{2}}^\varepsilon) - f(k)] \\ & - \operatorname{sgn}(\rho_{i+\frac{1}{2}}^\varepsilon - k) [f(\rho_{i+\frac{1}{2}}^\varepsilon) - f(k)]. \end{aligned}$$

By studying the various cases corresponding to the possible values of $k \in [0, 1]$ with respect to $\rho_{i+\frac{1}{2}}^\varepsilon < \rho_{i-\frac{1}{2}}^\varepsilon$ and recalling that $|\dot{x}_i^\varepsilon(t)| \leq 1$ for $i \neq 0$, we find

that

$$\left| \Theta \left(\rho_{i-\frac{1}{2}}^\varepsilon, \rho_{i+\frac{1}{2}}^\varepsilon, k \right) \right| \leq \varepsilon (2 + \varepsilon) .$$

Indeed, if $k < \rho_{i+\frac{1}{2}}^\varepsilon$ or $k > \rho_{i-\frac{1}{2}}^\varepsilon$, then

$$\begin{aligned} \left| \Theta \left(\rho_{i-\frac{1}{2}}^\varepsilon, \rho_{i+\frac{1}{2}}^\varepsilon, k \right) \right| &= \left| \left(\rho_{i-\frac{1}{2}}^\varepsilon - \rho_{i+\frac{1}{2}}^\varepsilon \right) \dot{x}_i^\varepsilon(t) + f \left(\rho_{i+\frac{1}{2}}^\varepsilon \right) - f \left(\rho_{i-\frac{1}{2}}^\varepsilon \right) \right| \\ &\leq 2 \left(\rho_{i-\frac{1}{2}}^\varepsilon - \rho_{i+\frac{1}{2}}^\varepsilon \right) , \end{aligned}$$

while, if $k = \alpha \rho_{i+\frac{1}{2}}^\varepsilon + (1 - \alpha) \rho_{i-\frac{1}{2}}^\varepsilon$ for an $\alpha \in [0, 1]$, then

$$\begin{aligned} &\left| \Theta \left(\rho_{i-\frac{1}{2}}^\varepsilon, \rho_{i+\frac{1}{2}}^\varepsilon, k \right) \right| = \\ &= \left| (2\alpha - 1) \left(\rho_{i+\frac{1}{2}}^\varepsilon - \rho_{i-\frac{1}{2}}^\varepsilon \right) \left(1 + \dot{x}_i^\varepsilon(t) - 2\rho_{i-\frac{1}{2}}^\varepsilon \right) + \left(\rho_{i-\frac{1}{2}}^\varepsilon - \rho_{i+\frac{1}{2}}^\varepsilon \right)^2 (2\alpha^2 - 1) \right| \\ &\leq \left(\rho_{i-\frac{1}{2}}^\varepsilon - \rho_{i+\frac{1}{2}}^\varepsilon \right) \left(2 + \rho_{i-\frac{1}{2}}^\varepsilon - \rho_{i+\frac{1}{2}}^\varepsilon \right) . \end{aligned}$$

Thus, letting ε go to zero we have that ρ is indeed an entropy weak solution of the initial-boundary value problem (2), (3) in the sense of Definition 1.

4. Technical section

Lemma 8 (Estimates for σ_r). *For any $\alpha, \beta \in [0, 1]$ with $\beta < \alpha$ we have*

$$\lambda_r(\alpha, \beta) < v(\alpha) .$$

PROOF. Observe that

$$\log(x) \leq \frac{x^2 - 1}{2x} .$$

Indeed the above estimate holds for $x = 1$ and $\frac{d}{dx} \left[\frac{x^2 - 1}{2x} - \log(x) \right] = \frac{(x-1)^2}{2x^2} \geq 0$. Therefore

$$\lambda_r \left(1 - \frac{1}{x}, 1 - \frac{1}{y} \right) < \lambda_r \left(1 - \frac{1}{x}, 0 \right) = \frac{2 \log(x)}{x - 1} - 1 \leq \frac{1}{x} = v \left(1 - \frac{1}{x} \right) .$$

Then, it is sufficient to take $x = c(\alpha)$ and $y = c(\beta)$ to complete the proof.

Lemma 9 (Estimates for Φ_ξ). *For any $\alpha \in [0, 1]$ we have*

1. $\gamma \mapsto \Phi_\xi(\alpha, \gamma) - q(\gamma)$ *is decreasing in $[0, \alpha]$;*
2. $\gamma \mapsto \Phi_\xi(\gamma, \alpha) + q(\gamma)$ *is increasing in $[0, \alpha]$.*

PROOF. We first observe that

$$\Phi_\xi\left(1 - \frac{1}{x}, 1 - \frac{1}{y}\right) - q\left(1 - \frac{1}{y}\right) = [x + y] \left[\frac{1}{y} - \frac{1}{x} + \frac{x + y - 2}{y - x} \right] + y - 2 \log y.$$

and that the derivative with respect to y of the above function is

$$(x, y) \mapsto - \frac{(x^2 - y^2 + 2xy)(x^2 - y^2 + 2xy(y - 1))}{x(x - y)^2 y^2}$$

and is negative for all $x > y \geq 1$. Then, it is sufficient to take $x = c(\alpha)$ and $y = c(\beta)$ to complete the proof of the first point (1). The second point follows from the previous one by observing that Φ_ξ is anti-symmetric.

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References

- [1] Adimurthi, R. Dutta, S.S. Ghoshal, G.D. Veerappa Gowda, Existence and nonexistence of TV bounds for scalar conservation laws with discontinuous flux, *Comm. Pure Appl. Math.* 64 (2011) 84–115.
- [2] D. Amadori, M. Di Francesco, The one-dimensional Hughes model for pedestrian flow: Riemann-type solutions, *Acta Math. Sci. Ser. B Engl. Ed.* 32 (2012) 259–280.
- [3] B. Andreianov, K.H. Karlsen, N.H. Risebro, A theory of L^1 -dissipative solvers for scalar conservation laws with discontinuous flux, *Arch. Ration. Mech. Anal.* 201 (2011) 27–86.

- [4] B. Andreianov, K. Sbihi, Well-posedness of general boundary-value problems for scalar conservation laws, *Trans. AMS* (2014) accepted.
- [5] E. Audusse, B. Perthame, Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies, *Proc. Roy. Soc. Edinburgh Sect. A* 135 (2005) 253–265.
- [6] C. Bardos, A.Y. le Roux, J.C. Nédélec, First order quasilinear equations with boundary conditions, *Comm. Partial Differential Equations* 4 (1979) 1017–1034.
- [7] A. Bressan, *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2000.
- [8] J. Carrillo, Conservation laws with discontinuous flux functions and boundary condition, *Journal of Evolution Equations* 3 (2003) 283–301.
- [9] R.M. Colombo, M.D. Rosini, Well posedness of balance laws with boundary, *J. Math. Anal. Appl.* 311 (2005) 683–702.
- [10] C.M. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, *J. Math. Anal. Appl.* 38 (1972) 33–41.
- [11] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften*, third ed., Springer-Verlag, Berlin, 2010.
- [12] M. Di Francesco, P.A. Markowich, J.F. Pietschmann, M.T. Wolfram, On the Hughes’ model for pedestrian flow: the one-dimensional case, *J. Differential Equations* 250 (2011) 1334–1362.
- [13] N. El-Khatib, P. Goatin, M. Rosini, On entropy weak solutions of Hughes’ model for pedestrian motion, *Z. Angew. Math. Phys.* 64 (2013) 223–251.
- [14] M. Garavello, R. Natalini, B. Piccoli, A. Terracina, Conservation laws with discontinuous flux, *Netw. Heterog. Media* 2 (2007) 159–179.

- [15] P. Goatin, M. Mimault, The wave-front tracking algorithm for Hughes' model of pedestrian motion, *SIAM J. Sci. Comput.* 35 (2013) B606–B622.
- [16] R.L. Hughes, A continuum theory for the flow of pedestrians, *Transportation Research Part B* 36 (2002) 507–535.
- [17] R.L. Hughes, The flow of human crowds, in: *Annual review of fluid mechanics*, Vol. 35, Annual Reviews, Palo Alto, CA, 2003, pp. 169–182.
- [18] S.N. Kružhkov, First order quasilinear equations with several independent variables., *Mat. Sb. (N.S.)* 81 (123) (1970) 228–255.
- [19] F. Otto, Initial-boundary value problem for a scalar conservation law, *C. R. Acad. Sci. Paris Sér. I Math.* 322 (1996) 729–734.
- [20] E.Y. Panov, Existence of strong traces for quasi-solutions of multidimensional conservation laws, *J. Hyperbolic Differ. Equ.* 4 (2007) 729–770.
- [21] E.Y. Panov, On existence and uniqueness of entropy solutions to the Cauchy problem for a conservation law with discontinuous flux, *J. Hyperbolic Differ. Equ.* 06 (2009) 525–548.
- [22] A. Vasseur, Strong traces for solutions of multidimensional scalar conservation laws, *Arch. Ration. Mech. Anal.* 160 (2001) 181–193.