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Value at Risk with tempered multistable motions

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Abstract

This is an empirical work showing numerically the importance of taking into account non-stationarities in CGMY and similar models for computing VaRs. More precisely we model S&P 500 logs with a CGMY model, but where the parameters are allowed to vary. We show that C , G , M and Y indeed display significant variations across time. Most relevant is the variation of Y , which implies that the correct modelling frame is the one of tempered multistable motions, a recently introduced class of stochastic processes. We provide experimental evidence that these variations have a notable impact on VaR estimation, and thus should be taken into account for proper risk evaluation.

1 Background and Motivation

One of the reasons why the Black and Scholes model is unable to reproduce empirical Values at Risk (VaR) or implied volatility patterns is its stationary character.

Although stationary refinements of this model, and most prominently ones based on Lévy processes, permit a more faithful description of the reality of financial markets, they are still not totally satisfactory [5]. The most popular ways to introduce non-stationarities are through the use of a time-varying volatility. Two major classes of such models are stochastic volatility [9] and local volatility [3] ones. In the Gaussian and jump-diffusion frames, this amounts to replacing the instantaneous variance either by a stochastic process or by a function of the asset. In pure jump processes such as the CGMY model [4], this translates into letting the scale parameter C evolve. Quoting [4]: "For example, if one were to construct a model with a stochastic aggregate activity rate, then one could model C as an independent positive process, possibly following a square root law of its own". However, since the main feature of CGMY and similar processes is a fine modelling of jumps, and since it is precisely by taking into account jumps that these models improve upon diffusion ones, one may wonder why the parameter responsible for the jump intensity should be considered constant, while C would be allowed to vary. A truly non-stationary CGMY model should let all parameters evolve in time, and in particular the Y parameter, that finely tunes the jumps. Analogously to stochastic volatility and local volatility models, one should then design "stochastic jump exponent" and "local jump exponent" models. This is a long-term program. In this work, we content ourselves with empirical studies showing 1) that the parameters C, G, M, Y calibrated on the S&P 500 do evolve significantly in time, and 2) that these evolutions, in particular the one of Y , have a notable impact on VaR. Impact on pricing, as well as more theoretical studies, will be addressed elsewhere [12, 16]. We choose to concentrate on VaR in this preliminary study, as this is a simple object which has become, for better or for worse, a standard tool for risk management [10], in particular in the frame of Solvency II [21]. In practice, VaR is computed either in a non-parametric way from historical logs or based on a model which is calibrated on market data [11]. In most studies, and in particular in the frame of Gaussian models, VaR is essentially governed by volatility, *i.e.* instantaneous variance. Our point is that instantaneous variance is but one dimension of risk. In the frame of CGMY and similar models, it condenses into one value influences of the four parameters C, G, M and Y . A fine study of VaR should rather isolate the contributions of each of these parameters. In this work, we study the impact on VaR of variations of the overall level of activity, as measured by C , as well as of the intensity of jumps, as controlled by Y (studying the effects of G and M would of course also be interesting, but falls outside the scope of this paper). Our main motivation is that the presence of jumps on financial records is a well recognized fact [1], that have a strong impact

on VaR: Section 4 provides a precise quantification of this influence. It thus seems important to check whether the intensity of jumps is constant in time, and, if this is not the case, to model its evolution. The recently introduced class of (*tempered*) *multistable processes* [7, 8, 14, 17], which generalize tempered stable processes studied in [19], is well fitted for this purpose.

The remaining of this work is organized as follows: background on multistable processes and their tempered versions is provided in Section 2. Section 3 presents numerical studies showing that the parameters of a CGMY model, when calibrated on market data, do vary in a significant manner with time. In Section 4, we provide computer experiments highlighting the impact of such non-stationarities on VaR. Finally, Section 5 briefly explains how one can model a varying Y .

2 Multistable processes and their tempered versions

2.1 Multistable processes

Multistable processes extend the well-known stable processes (see, *e.g.* [20]) by letting the stability index α evolve in “time”. The idea is that, in many applications, it is likely that non stationary behaviours occur, which will typically impact the local intensity of jumps. This is for instance clear on any sufficiently long financial log. However, non stationarity is a “non-property”, and is thus hard to deal with directly. One way to model it in a practical way is to define processes that locally “resemble” known stationary ones. More precisely, we shall deal with *localisable* processes [6]. Recall that a process $Y = \{Y(t) : t \in \mathbb{R}\}$ is said to be *h-localisable* at u if there exists an $h \in \mathbb{R}$ and a non-trivial limiting process Y'_u (called the tangent process) such that

$$\lim_{r \rightarrow 0} \frac{Y(u + rt) - Y(u)}{r^h} = Y'_u(t). \quad (1)$$

For a multistable process Y , Y'_u will be a stable process with exponent depending on u .

The simplest multistable processes are symmetric Lévy multistable motions. Two versions exist: the first one is obtained from a field of Lévy stable processes [14]. More precisely, let $\alpha : [0, 1] \rightarrow [c, d]$ be a C^2 function, where $[c, d] \subset (1, 2)$, $(\Gamma_i)_{i \geq 1}$ be a sequence of arrivals of a Poisson process with intensity one, $(V_i)_{i \geq 1}$ be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$, and $(\gamma_i)_{i \geq 1}$ be a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i =$

$-1) = 1/2$. The sequences $(\Gamma_i)_{i \geq 1}$, $(V_i)_{i \geq 1}$, and $(\gamma_i)_{i \geq 1}$ are taken independent. The symmetric Lévy stable field is defined on $[0, 1]^2$ as follows [14] :

$$X(u, v) = C_{\alpha(v)}^{1/\alpha(v)} \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(v)} \mathbf{1}_{[0, u]}(V_i),$$

where

$$C_u = \left(\int_0^{\infty} x^{-u} \sin(x) dx \right)^{-1}. \quad (2)$$

The Lévy multistable motion associated to this field is by definition:

$$Z_1(t) = X(t, t).$$

Note that, when $\alpha(t)$ equals the constant α for all t , Z_1 is simply a series representation of α -stable Lévy motion.

Both the above field and process admit a Poisson type representation [8]: let Π be a Poisson process on $[0, 1] \times \mathbb{R}$, with mean measure $\mathcal{L}[0, 1] \times \mathcal{L}(\mathbb{R})$, where \mathcal{L} is Lebesgue measure. We denote (X, Y) a random point of Π in $[0, 1] \times \mathbb{R}$. A representation of the field, which is equivalent in law to the one above, is:

$$X(u, v) = \sum_{(X, Y) \in \Pi} \mathbf{1}_{[0, u]}(X) Y^{<-1/\alpha(v)>}$$

where $a^{} = \text{sign}(a)|a|^b$, and where the sum above means:

$$X(u, v) = \lim_{n \rightarrow \infty} \sum_{(X, Y) \in \Pi \cap R_n} \mathbf{1}_{[0, u]}(X) Y^{<-1/\alpha(v)>},$$

when $\alpha(v) \geq 1$, where R_n is the rectangle $\{(x, y) : 0 \leq x \leq 1, |y| \leq n\} \subset \mathbb{R}^2$.

Z_1 is tangent, at each t , to usual Lévy stable motion with index $\alpha(t)$, denoted $L_{\alpha(t)}$, i.e.:

$$\lim_{r \rightarrow 0} \frac{Z_1(u + rt) - Z_1(u)}{r^h} = L_{\alpha(t)}(t),$$

where the limit is in the sense of finite dimensional distributions.

The joint characteristic function of Z_1 equals:

$$\mathbb{E} \left(e^{i \sum_{j=1}^m \theta_j Z_1(t_j)} \right) = \exp \left(-2 \int_{[0, T]} \int_0^{+\infty} \sin^2 \left(\sum_{j=1}^m \theta_j a(t_j) \frac{C_{\alpha(t_j)}^{1/\alpha(t_j)}}{2y^{1/\alpha(t_j)}} \mathbf{1}_{[0, t_j]}(x) \right) dy dx \right) \quad (3)$$

where $m \in \mathbb{N}$, $(\theta_1, \dots, \theta_m) \in \mathbb{R}^m$, $(t_1, \dots, t_m) \in \mathbb{R}^m$.

A different multistable Lévy motion, that we shall denote Z_2 , may be defined from its joint characteristic function [7]:

$$\mathbb{E} \left(\exp \left(i \sum_{j=1}^d \theta_j Z_2(t_j) \right) \right) = \exp \left(- \int \left| \sum_{j=1}^d a(t_j) \theta_j \mathbf{1}_{[0, t_j]}(s) \right|^{\alpha(s)} ds \right).$$

As Z_1 , Z_2 is tangent at each point t to a Lévy stable motion with exponent $\alpha(t)$.

Z_2 also admits Fergusson-Klass-Le Page and Poisson representations:

$$Z_2(t) = a(t) \sum_{i=1}^{\infty} C_{\alpha(V_i)}^{1/\alpha(V_i)} \gamma_i \Gamma_i^{-1/\alpha(V_i)} \mathbf{1}_{(V_i \leq t)}. \quad (4)$$

$$Z_2(t) = a(t) \sum_{(X, Y) \in \Pi} C_{\alpha(X)}^{1/\alpha(X)} \mathbf{1}_{[0, t]}(X) Y^{<-1/\alpha(X)>}. \quad (5)$$

There is a strong difference between Z_2 and Z_1 : the former is a process with independent increments while the latter is not even a Markov process. Z_2 is thus easier to work with in some respects. However, it also has certain drawbacks as compared to Z_1 . For instance, Z_1 coincides, at each fixed time, with a Lévy motion. This is not true of Z_2 .

In [15], it is shown that both processes are semi-martingales. This is rather obvious for Z_2 , but less so for Z_1 , which is not a pure jump process. More precisely, the following relation between the Fergusson-Klass-LePage series representations of the two processes holds almost surely for all t :

$$Z_1(t) = A(t) + Z_2(t), \quad (6)$$

where A is the finite variation process defined by:

$$A(t) = \int_0^t \sum_{i=1}^{+\infty} \gamma_i \frac{d \left(C_{\alpha(\cdot)}^{1/\alpha(\cdot)} \Gamma_i^{-1/\alpha(\cdot)} \right)}{dt}(s) \mathbf{1}_{[0, s]}(V_i) ds. \quad (7)$$

2.2 Tempered multistable processes

In financial applications, the fact that (multi-)stable processes have infinite variance raises some problems. This is why tempered versions of these processes are

often considered instead (see [19] for a general exposition on tempered stable processes). One of the most popular such model is the CGMY one [4]. This is a pure jump Lévy process Z with marginal characteristic function at time t equal to:

$$\varphi_{Z(t)}(u) = \mathbb{E}(\exp(iuZ(t))) = \exp\left(tC\Gamma(-Y)\left[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right]\right). \quad (8)$$

The parameter $C > 0$ is a scaling parameter. $G \geq 0$ (resp. $M \geq 0$) governs the tempering of the negative (resp. positive) jumps. Finally, $Y < 2$ controls the intensity of jumps: for $G = M = 0$ and $Y \in (0, 2)$, Z is simply a Y -stable symmetric Lévy motion. Two significant facts are that, contrarily to stable processes, Y is not restricted to range in $(0, 2)$ but may take arbitrary values in $(-\infty, 2)$. Also, Z has finite moments of all order (and even exponential moments of sufficiently low order).

Tempered versions of both Z_1 and Z_2 , which generalize the CGMY process, are obtained by modifying in an adequate way the definition of the previous subsection [17]. More precisely, an independent increments multistable version Z_{II} of Z may be defined through its marginal characteristic function (here and below, Y is always assumed to be a continuous function ranging in $(-\infty, 2)$). It reads:

$$\varphi_{Z_{II}(t)}(u) = \exp\left\{C \int_0^t \Gamma(-Y(v)) \left[(M-iu)^{Y(v)} - M^{Y(v)} + (G+iu)^{Y(v)} - G^{Y(v)}\right] dv\right\} \quad (9)$$

A tempered analogue Z_{FB} of Z_1 has marginal characteristic function:

$$\varphi_{Z_{FB}(t)}(u) = \exp\left\{tC\Gamma(-Y(t)) \left[(M-iu)^{Y(t)} - M^{Y(t)} + (G+iu)^{Y(t)} - G^{Y(t)}\right]\right\}. \quad (10)$$

Both processes admit series representations [17]. In addition, both Z_{FB} and Z_{II} are localisable under mild regularity conditions on Y , with tangent process a stable motion.

In the sequel, we will use Z_{FB} as a model for the logarithm of financial assets. Similar results would be obtained with Z_{II} .

3 Numerical evidences of non-stationarity

We analyse in this section logs of S&P 500, and estimate the functional parameters $C(t)$, $G(t)$, $M(t)$ and $Y(t)$ of a Z_{FB} model as well as the ones of a plain CGMY model for comparison. Calibration is made using maximum likelihood. For the

CGMY model, this is well-known. For the Z_{FB} model, this is done simply using the fact that, at each time t , Z_{FB} is tangent to a CGMY process with parameters $C(t), G(t), M(t), Y(t)$: estimation is then performed using a sliding window. In details, to obtain the parameters for sample i , we use samples $i - 2000, \dots, i$ (that is, roughly eight years of historical data), and perform maximum likelihood estimation on this set of data.

S&P 500 data between 01/03/2002 and 01/02/2012 are shown on Figure 1. Figure 2 shows the instantaneous C, G, M and Y parameters.

As one can see it from the graphs, all parameters do vary significantly through time. Several interesting information may be drawn from those figures:

1. Except for the 2008-2009 crisis, movements on the parameters cannot be easily deduced from ones of the index log: the time evolution of the parameters reveal information that is concealed on the price variation.
2. Relative variations of the parameters are very large, with C ranging in $[0.02, 0.7]$, G in $[28, 134]$, M in $[26, 134]$ and Y in $[0.1, 1.15]$.
3. Consistent patterns appear: most notably, an increase in C always corresponds to a decrease in Y , while C, G and M tend to be positively correlated. In other words, an increase of overall activity coincides with a larger intensity of jumps, which at the same time undergo more tempering.
4. The graphs may be roughly split into three zones (this is specially apparent from the evolution of C): from 01/03/2002 to 01/09/2006, characterized by small values of C , moderate values of G and M , and large values of Y ; from 02/09/2006 to 01/12/2008, where C reaches large values, as do G and M , while Y drops to small ones; and finally from 02/12/2008 to 01/02/2012, where C, G and M all assume their lowest values as Y increases notably. We do not have, at this time, an explanation of the empirical observation.

Since the three zones mentioned in item 4 are notably different and each one is more or less homogeneous, it seems relevant to analyse them separately. Global estimates for the three zones are as follows:

	C	G	M	Y
Zone 1	0.03	84	99	0.67
Zone 2	0.16	68	87	0.30
Zone 3	0.03	50	63	0.58

To assess the goodness of fit, comparisons between historical data and globally fitted CGMY models for each of the three zones are displayed through densities and QQ plots on Figures 3 to Figure 5.

4 Impact on VaR of varying C and Y

Characterizing the impact on VaR of the various parameters in the CGMY model is a complex issue: it depends on the confidence threshold and on a complex interplay between the values of the parameters. For instance, it would seem intuitive that decreasing Y should increase VaR. However this is only the case when the threshold is in a certain range, which in turn depends on the values of the other parameters. Since VaR measures extreme events, one might think of using the asymptotic formulas (7.6) and (7.7) of [2] (see also [13]). However, for typical parameters values observed on markets, and at the threshold of, say, 1%, it appears that we are not yet in the domain where these formulas provide a reasonable approximation. We must therefore resort to numerical estimations.

The aim of this section is to provide graphical illustrations of how changes of C and Y impact VaR, and to compare the influences of these two parameters. Indeed, we have seen in Section 3 that both C and Y vary in time. A natural question is then whether such variations have a significant impact on VaR, or if they may be safely ignored. In order to elucidate this point, we shall make use of the empirical results of the previous section to set the ranges of variation of the parameters. It occurs that using the whole ranges between 01/01/2002 and 31/12/2012 results in unrealistically large variations of VaR. For this reason, we will rather work with the three roughly homogeneous zones identified above.

We draw two kinds of graphs;

- VaR as a function of C when Y is kept constant, and as a function of Y when C is kept constant. The value of the parameter which is kept constant is set to the one estimated globally on the considered region. The varying parameter spans the range of the local estimates in the region. This is on Figures 6 to 8.
- VaR either as a function of C when the variance is kept constant or as a function of Y when the variance is kept constant: recall that the variance of a CGMY process is equal to

$$\sigma^2 = C\Gamma(2 - Y) (G^{Y-2} + M^{Y-2}). \quad (11)$$

Varying only C or Y thus induces a variation of instantaneous variance. As a consequence, the impact on VaR in the previous experiment is a "mixed effect". This is why we perform this second experiment: the variance is first computed using the values of the parameters that have been estimated globally on the region. Then, when, e.g., C is varied in the range of the locally estimated $C(t)$ in the region, Y is adjusted so that σ , as computed with (11), remains constant. See Figures 9 to 11.

We display two more graphs: Figure 12 plots the evolution of C as a function of Y that yields a constant VaR (the bounds on Y are the ones of region 2): this is intended to stress the fact that a given fixed VaR may be obtained with significantly different levels of global activity C when Y varies in a realistic range. Figure 13 shows how the variance evolves when C spans its range of local estimates in region 1 and Y is adjusted accordingly so as to keep the VaR constant (equal to the one estimated with the global parameters in region 1). This is to emphasize that the instantaneous variance alone is not sufficient to characterize VaR: here, the same VaR is obtained with a variance undergoing a variation of 40%.

5 Modelling the evolution of Y

The results of the previous section show that taking into account non-stationarities of C and Y is important for accurate VaR evaluation. In practice, the future evolutions of these parameters are of course unknown, and one needs to model them. For C , this is similar to well-known methods dealing with time-dependent volatility. The case of Y is less classical. The aim of this section is to briefly present two ways of doing so. No details are given, but the interested reader is referred to articles presenting these methods.

1. Autonomous evolution of Y

The first approach is similar in principle to stochastic volatility models, with the difference that the evolution of Y , rather than the one of volatility, is modelled by a stochastic differential equation (SDE). In other words, one writes a system of coupled SDEs, one for the log-price, and a second one for Y . Different evolution equations (e.g. Hull and White or Heston) may be used. See [12] for more.

2. Self-stabilizing processes

The second method is in the spirit of local volatility models: instead of having the local volatility depend on the price, it is the local intensity of jumps that does so. This approach is further motivated by an intriguing empirical feature that appears when one analyses certain records such as the S&P500 index: it seems that there exists a relation between the value of the index and the one of Y . This calls for the development of "self-stabilizing models", *i.e.* processes X verifying a functional equation of the form: $Y(t) = g(X(t))$, almost surely for all t , where g is a smooth deterministic function. All the information concerning the future evolution of Y is then contained in g , which may be estimated from historical data under the assumption that the relation between X and Y does not vary in time.

Self-stabilizing processes may be obtained in three ways: the first one is a fixed point approach starting from a Lévy stable field. The second one uses a wavelet construction. Finally, the third one builds the process as a solution of an SDE. See [16] for more.

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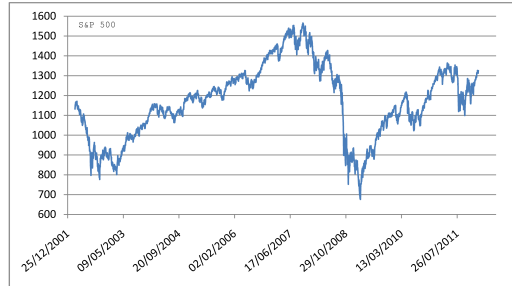


Figure 1: S&P 500 index between 01/03/2002 and 01/02/2012.

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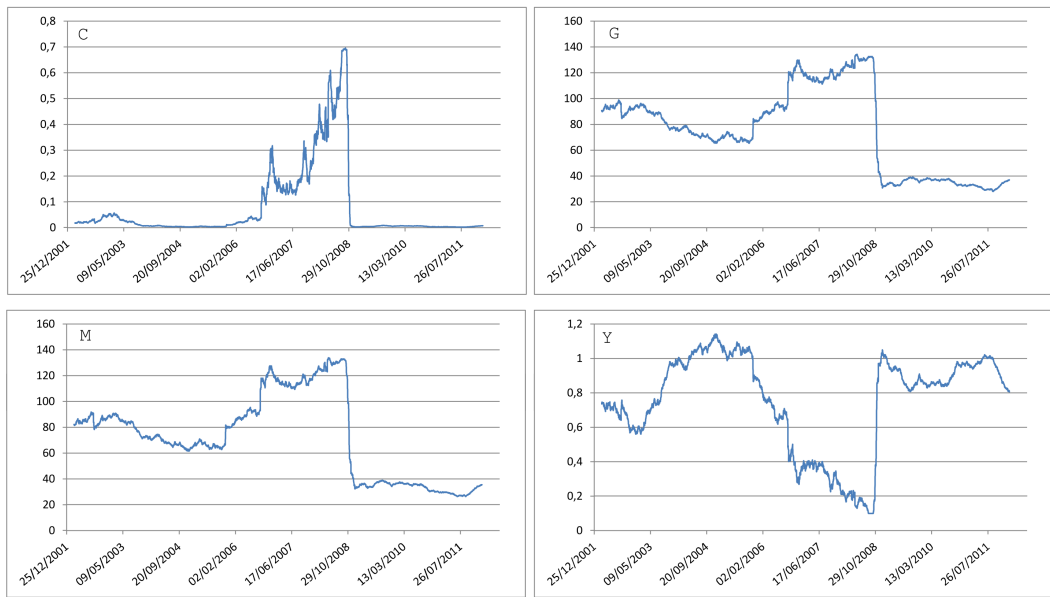


Figure 2: Evolution of C (top left), G (top right), M (bottom left) and Y (bottom right) for the S&P 500 index between 01/01/2002 and 31/12/2012.

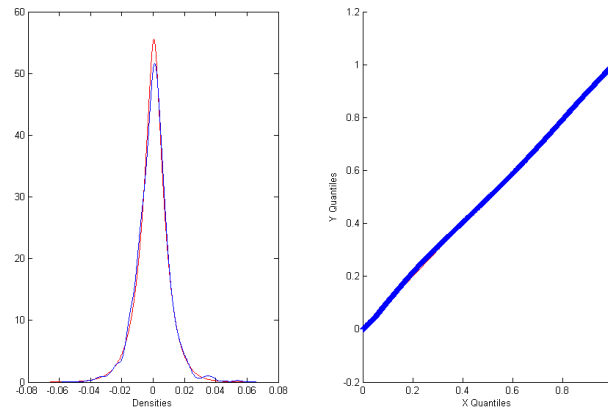


Figure 3: Densities (left) and QQ plots for zone 1; blue: historical data; red: CGMY model.

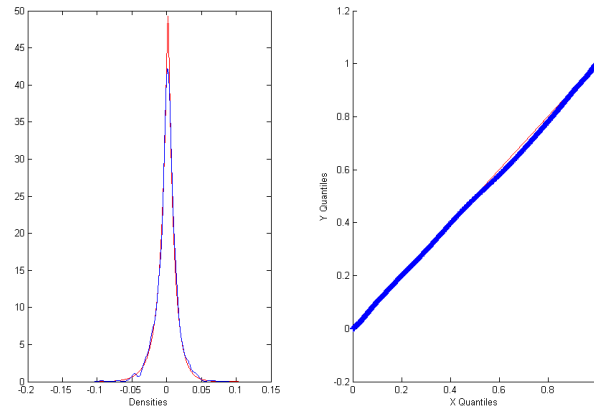


Figure 4: Densities (left) and QQ plots for zone 2; blue: historical data; red: CGMY model.

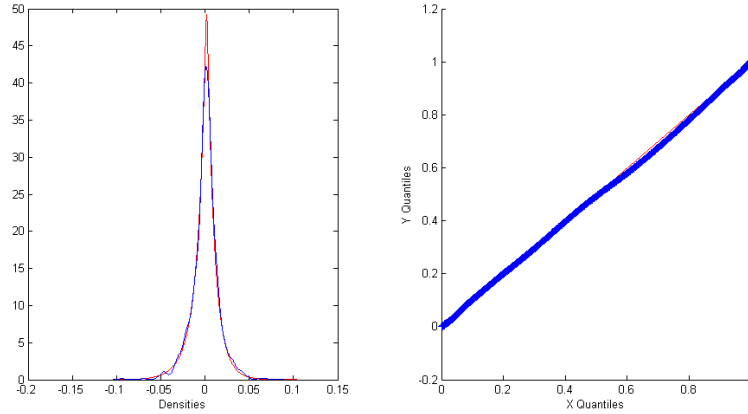


Figure 5: Densities (left) and QQ plots for zone 3; blue: historical data; red: CGMY model.

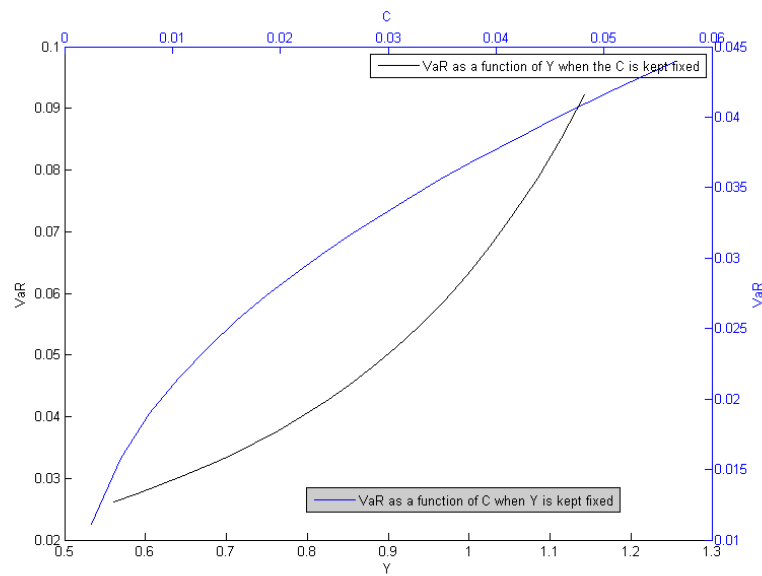


Figure 6: VaR as a function of Y (black) and C (blue), when all other parameters are kept constant, region 1.

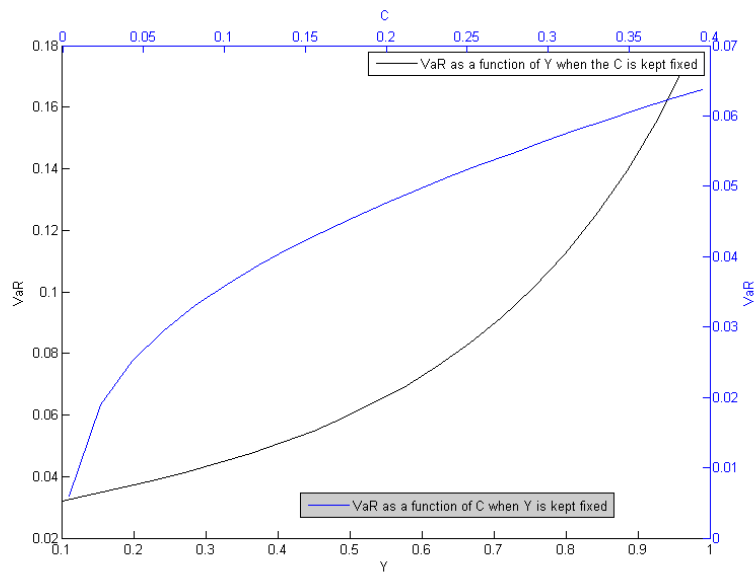


Figure 7: VaR as a function of Y (black) and C (blue), when all other parameters are kept constant, region 2.

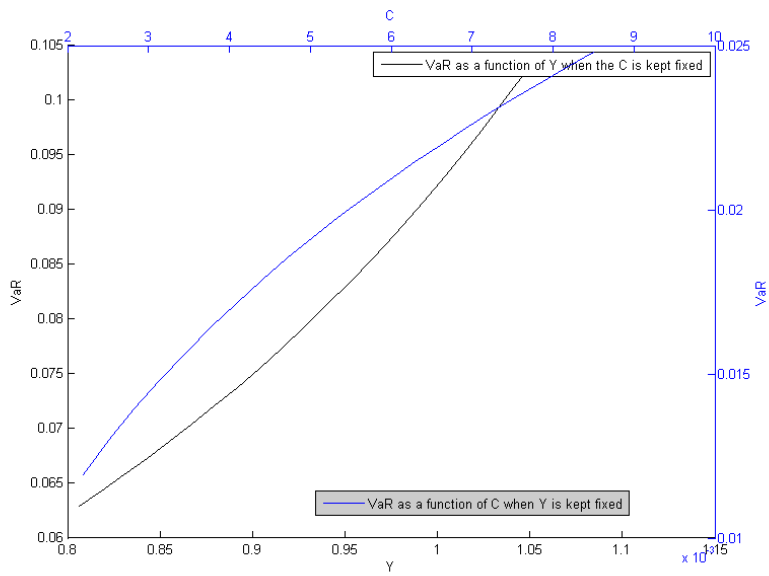


Figure 8: VaR as a function of Y (black) and C (blue), when all other parameters are kept constant, region 3.

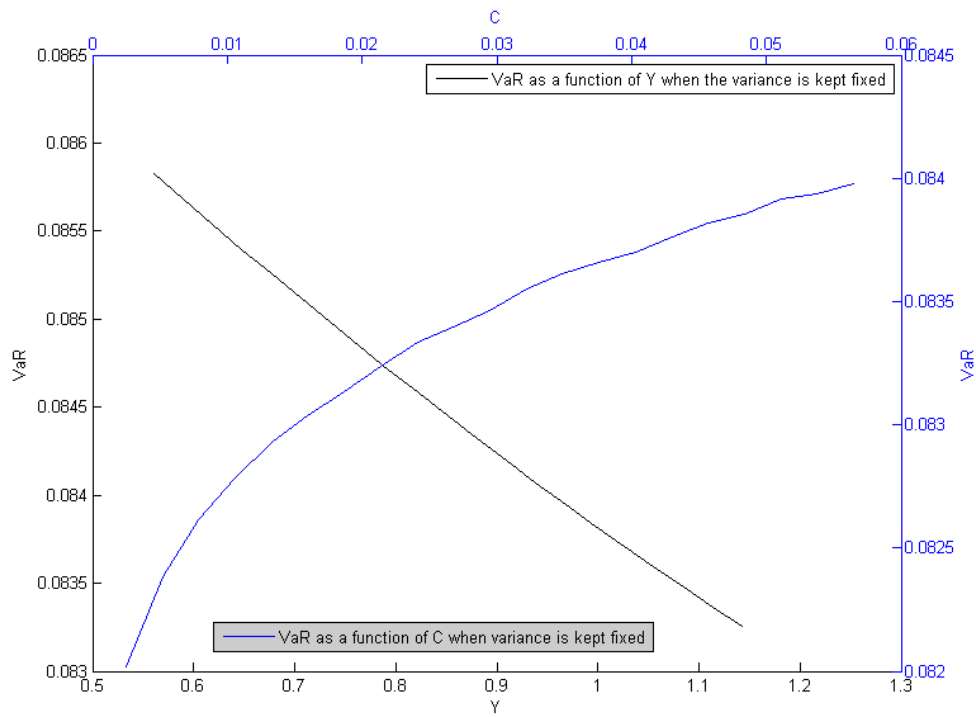


Figure 9: VaR as a function of Y (black) and C (blue), when the variance is kept constant, region 1.

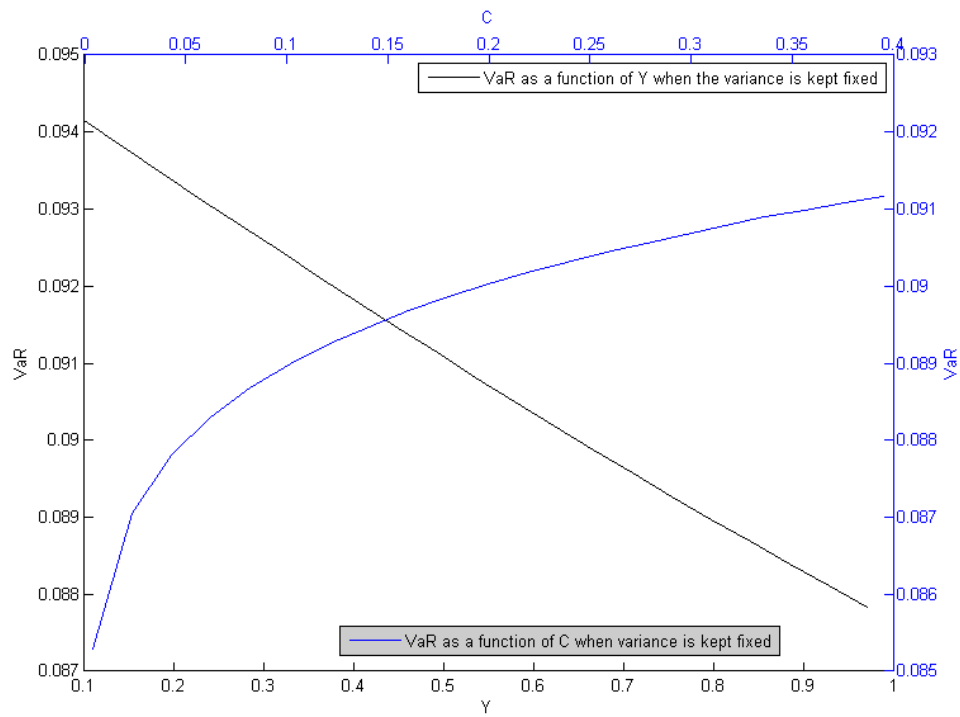


Figure 10: VaR as a function of Y (black) and C (blue), when the variance is kept constant, region 2.

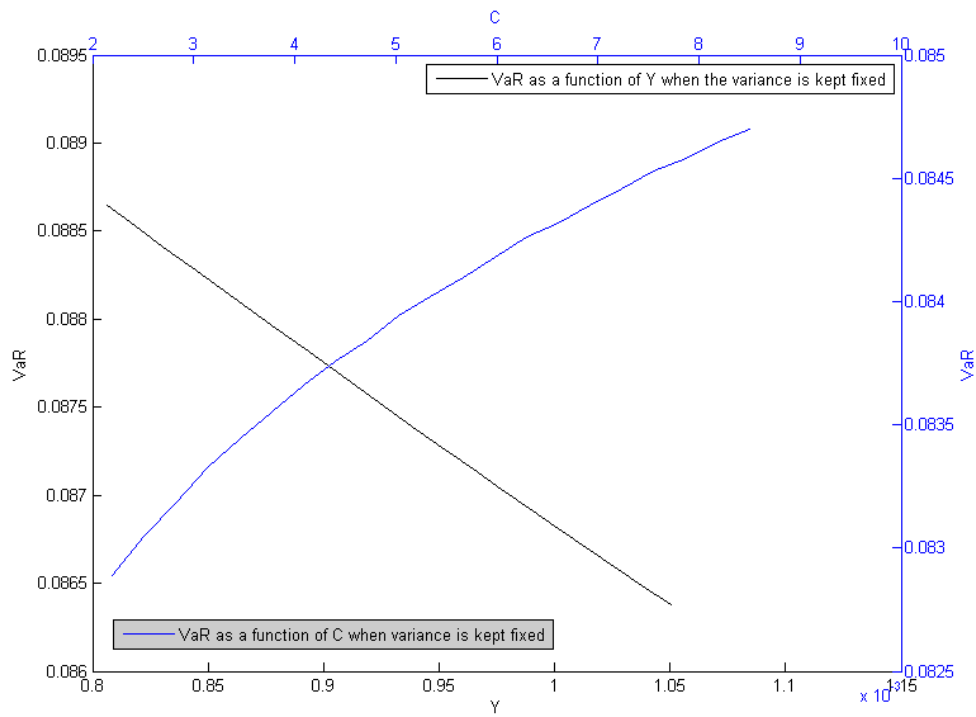


Figure 11: VaR as a function of Y (black) and C (blue), when the variance is kept constant, region 3.

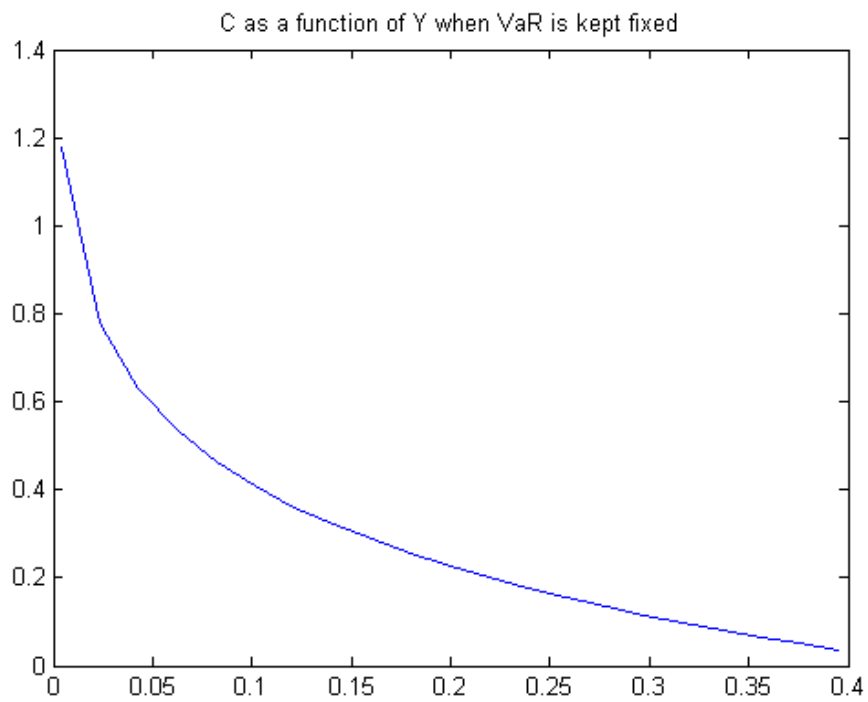


Figure 12: C as a function of Y that yields a constant VaR (bounds of region 2).

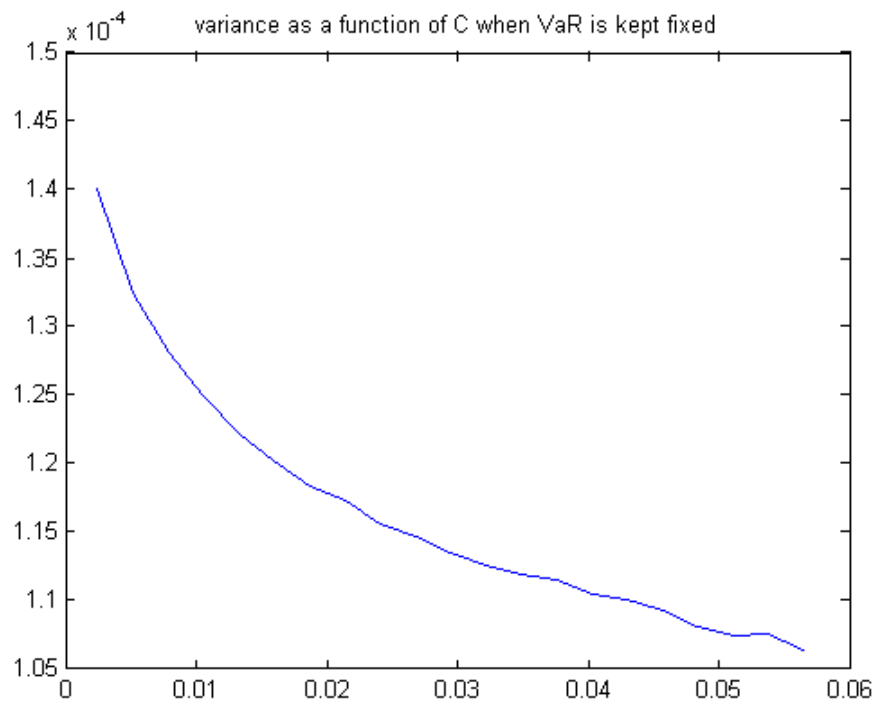


Figure 13: Variance as a function of C , where Y is adjusted so that VaR remains constant (bounds of region 1).

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