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AN OPTIMAL QUANTITATIVE TWO-SCALE EXPANSION IN STOCHASTIC HOMOGENIZATION OF DISCRETE ELLIPTIC EQUATIONS

ANTOINE GLORIA, STEFAN NEUKAMM & FELIX OTTO

Abstract. We establish an optimal, linear rate of convergence for the stochastic homogenization of discrete linear elliptic equations. We consider the model problem of independent and identically distributed coefficients on a discretized unit torus. We show that the difference between the solution to the random problem on the discretized torus and the first two terms of the two-scale asymptotic expansion has the same scaling as in the periodic case. In particular the L^2-norm in probability of the H^1-norm in space of this error scales like ε, where ε is the discretization parameter of the unit torus. The proof makes extensive use of previous results by the authors, and of recent annealed estimates on the Green's function by Marahrens and the third author.

Keywords: stochastic homogenization, homogenization error, quantitative estimate.

2010 Mathematics Subject Classification: 35B27, 39A70, 60H25, 60F99.

1. Introduction

We establish a linear rate of convergence for the stochastic homogenization of discrete linear elliptic equations, which is optimal. Before we turn to the stochastic case, let us recall some standard analogous results in the periodic case. Let A be a uniformly elliptic and bounded symmetric matrix field on the unit torus $\mathbb{T} := (\mathbb{R}/\mathbb{Z})^d$, $D \subset \mathbb{R}^d$ be a smooth domain, and f be a smooth function. Let $\varepsilon > 0$. It is well-known that the unique weak solution $u_\varepsilon \in H^1_0(D)$ of the linear elliptic equation

$$\left\{ \begin{array}{rcl} -\nabla \cdot A\Big(\frac{\cdot}{\varepsilon}\Big)\nabla u_\varepsilon & = & f \text{ in } D, \\ u_\varepsilon & = & 0 \text{ on } \partial D \end{array} \right. \tag{1.1}$$

converges weakly in $H^1(D)$ as $\varepsilon \to 0$ to the unique weak solution $u_{\rm hom} \in H^1_0(D)$ of the homogenized equation

$$\left\{ \begin{array}{rcl} -\nabla \cdot A_{\rm hom}\nabla u_{\rm hom} & = & f \text{ in } D, \\ u_{\rm hom} & = & 0 \text{ on } \partial D. \end{array} \right. \tag{1.2}$$

The homogenized matrix $A_{\rm hom}$ is symmetric and characterized for all $\xi \in \mathbb{R}^d$ by

$$\xi \cdot A_{\rm hom}\xi \,=\, \int_{\mathbb{T}} (\nabla\phi_\xi + \xi) \cdot A(\nabla\phi_\xi + \xi),$$

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where ϕ_ξ is the unique weak solution in $H^1(\mathbb{T})$ with vanishing mean of the periodic corrector equation in direction ξ:

$$-\nabla \cdot A(\nabla\phi_\xi + \xi) \,=\, 0 \text{ in } \mathbb{T}.$$

From a two-scale expansion, one formally expects for all x in the interior of D:

$$u_\varepsilon(x) \,=\, u_{\rm hom}(x) + \varepsilon \sum_{j=1}^d \phi_j(\tfrac{x}{\varepsilon})\nabla_j u_{\rm hom}(x) + o(\varepsilon), \tag{1.3}$$

where the ϕ_j are the correctors in the canonical directions $\mathbf{e}_j$ (extended by periodicity to $\mathbb{R}^d$). This identity cannot hold at the boundary since the correctors ϕ_j do not satisfy the homogeneous Dirichlet boundary conditions: There is indeed a boundary layer. Yet, for all $\tilde{D}$ compactly supported in D, Allaire and Amar [1, Theorem 2.3] proved the following rigorous version of (1.3) if the coefficients A are Hölder continuous:

$$\|u_\varepsilon - u_{\rm hom} - \varepsilon \sum_{j=1}^d \phi_j(\tfrac{\cdot}{\varepsilon})\nabla_j u_{\rm hom}\|_{H^1(\tilde{D})} \,\leq\, C\varepsilon, \tag{1.4}$$

where the multiplicative constant C only depends on $\tilde{D}$, D, the Hölder exponent and norm of A, and on the C^3-norm of $u_{\rm hom}$ (which is smooth since f and D are smooth). This result relies on a previous work by Avellaneda and Lin [2, Theorem 5], who proved under similar assumptions that

$$\|u_\varepsilon - u_{\rm hom}\|_{L^\infty(D)} \,\leq\, C\varepsilon. \tag{1.5}$$

These are bounds on the homogenization error.

In the stochastic case, we consider a matrix field A that is stationary and ergodic, in place of periodic. We refer the reader to [18] for details. In order to obtain quantitative estimates in the spirit of (1.4) and (1.5), one has to make assumptions on the statistics of A in addition to ergodicity. There are few results in the literature on quantitative estimates of the homogenization error for elliptic equations in divergence form in the stochastic case. In [19, Theorem 3.1], Yurinskiĭ proved for algebraically decaying correlations that for all $d > 2$, there exists some Hölder exponent $\gamma > 0$ and a function T of ε such that

$$\left\langle \|u_\varepsilon - u_{\rm hom} - \varepsilon \sum_{j=1}^d \phi_{T(\varepsilon),j}(\tfrac{\cdot}{\varepsilon})\nabla_j u_{\rm hom}\|^2_{H^1(D)} \right\rangle^{1/2} \,\leq\, C\varepsilon^\gamma, \tag{1.6}$$

where $\phi_{T,j}$ is the modified corrector, which is the stationary, almost sure solution to

$$T^{-1}\phi_{T,j} - \nabla \cdot A(\nabla\phi_{T,j} + \mathbf{e}_j) \,=\, 0 \text{ in } \mathbb{R}^d. \tag{1.7}$$

This equation is an approximation of the corrector equation when $T \to \infty$. This is the first quantitative result in stochastic homogenization. Note that a formal linearization in the case of small ellipticity contrast $\lambda \uparrow 1$ yields $\gamma = 1$ for $d > 2$. Besides not covering dimension $d = 2$, the work by Yurinskiĭ does not allow to reach the scaling $\gamma = 1$, even in the case of small ellipticity constrast (and for domains $\tilde{D}$ compactly included in D).

In this article, we simplify the above setting with three respects:

(a) We treat **finite-difference equations** instead of continuum partial differential equations. More precisely, we replace the differential operator in (1.1) by the finite difference operator $\nabla_\varepsilon^* \boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla_\varepsilon$ which acts on discrete functions defined on the scaled lattice $\varepsilon\mathbb{Z}^d$, $\varepsilon > 0$. Here ∇_ε and ∇_ε^* are finite difference approximations of the continuum

gradient ∇ and continuum divergence operator $-\nabla\cdot$, and are defined for scalar fields $v : \varepsilon\mathbb{Z}^d \to \mathbb{R}$ and vector fields, $g = (g_1, \dots, g_d) : \varepsilon\mathbb{Z}^d \to \mathbb{R}^d$ as follows:

$$\begin{aligned} &\nabla_{\varepsilon,i} v(y) = \varepsilon^{-1}(v(y+\varepsilon\mathbf{e}_i) - v(y)), \quad \nabla^*_{\varepsilon,i} v(y) = \varepsilon^{-1}(v(y-\varepsilon\mathbf{e}_i) - v(y)), \\ &\nabla_\varepsilon v = (\nabla_{\varepsilon,1} v, \dots, \nabla_{\varepsilon,d} v), \qquad \nabla^*_\varepsilon g = \sum_{i=1}^d \nabla^*_{\varepsilon,i} g_i. \end{aligned} \tag{1.8}$$

The coefficients of the operator $\nabla^*_\varepsilon \boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla_\varepsilon$ are given by a randomly chosen matrix field $\boldsymbol{a} : \mathbb{Z}^d \to \mathbb{R}^{d\times d}$. We assume that $\boldsymbol{a}$ is diagonal and uniformly elliptic in the sense that

$$\boldsymbol{a}(x) \in \Omega_0 := \left\{ \begin{pmatrix} \boldsymbol{a}_1 & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{a}_d \end{pmatrix} \,\middle|\, \lambda \le \boldsymbol{a}_j \le 1,\ j = 1, \dots, d \right\}$$

for all $x \in \mathbb{Z}^d$. Above, $\lambda > 0$ is a (deterministic) ellipticity constant and fixed throughout the paper.

(b) In order to avoid boundary layers, we consider a problem on the (discretized) unit **torus** $\mathbb{T}_\varepsilon := (\varepsilon\mathbb{Z}/\mathbb{Z})^d$ and assume w. l. o. g. that the grid size ε is the inverse of a large integer $L := \varepsilon^{-1} \in \mathbb{N}$. The equation under consideration is the following finite difference equation with random coefficients

$$\nabla^*_\varepsilon \boldsymbol{a}(\tfrac{\cdot}{\varepsilon})\nabla_\varepsilon u_\varepsilon \;=\; f_\varepsilon \text{ on } \mathbb{T}_\varepsilon, \qquad \sum_{\mathbb{T}_\varepsilon} u_\varepsilon = 0, \tag{1.9}$$

where $f_\varepsilon : \mathbb{T}_\varepsilon \to \mathbb{R}$ is a deterministic r. h. s. with zero mean, and should be viewed as a discretization of a continuum r. h. s. $f \in L^2(\mathbb{T})$. Note that on $\mathbb{T}_\varepsilon$ we have by periodicity the *discrete integration by parts formula*

$$\sum_{\mathbb{T}_\varepsilon} \nabla_\varepsilon v \cdot g = \sum_{\mathbb{T}_\varepsilon} v \nabla^*_\varepsilon g \qquad \text{for all } v : \mathbb{T}_\varepsilon \to \mathbb{R} \text{ and } g : \mathbb{T}_\varepsilon \to \mathbb{R}^d.$$

(c) We consider the simplest possible statistics, namely **independent and identically distributed** (i. i. d.) coefficients. Since (1.9) is posed on the discretized unit torus $\mathbb{T}_\varepsilon$, $\varepsilon^{-1} = L \in \mathbb{N}$, we in fact consider the *periodic i. i. d. ensemble* which is constructed as follows: For a fixed measure β on our matrix space Ω_0 ("single-site measure") and for fixed size L of the discrete, rescaled torus $\mathbb{T}_L := (\mathbb{Z}/L\mathbb{Z})^d = \frac{1}{\varepsilon}\mathbb{T}_\varepsilon$, we shall throughout the paper denote by $\langle\cdot\rangle$ the *periodic i. i. d. ensemble* on

$$\Omega_L = \{\boldsymbol{a} \colon \mathbb{Z}^d \to \Omega_0 \,|\, \forall z \in \mathbb{Z}^d\ \boldsymbol{a}(\cdot + Lz) = \boldsymbol{a}\} \mathrel{\widehat{=}} \Omega_0^{\mathbb{T}_L}.$$

For a random variable, i. e. a measurable function $\zeta \colon \Omega_L \to \mathbb{R}$, it is given by the product measure

$$\langle\zeta\rangle = \prod_{x\in\mathbb{T}_L} \int_{\Omega_0} \zeta(\boldsymbol{a})\, \beta(d\boldsymbol{a}(x)). \tag{1.10}$$

Evidently, our i. i. d. ensemble is *stationary* in the sense that for all shifts $y \in \mathbb{Z}^d$ the random (periodic) tensor fields $\boldsymbol{a}$ and $\boldsymbol{a}(\cdot + y)$ have the same distribution in Ω_L.

In the above setting the qualitative theory of stochastic homogenization applies (see e. g. Kozlov [12], Papanicolaou and Varadhan [18] and Künnemann [13, Theorem 4]): There exists a deterministic, symmetric, positive definite $d \times d$ matrix $\boldsymbol{a}_{\text{hom}}$ (only depending on the single-site probability measure β) such that the following statement is true: Suppose that f_ε converges (in a discrete H^{-1}-norm) to some function $f \in L^2(\mathbb{T})$ with $\int_{\mathbb{T}} f = 0$. Let

u_ε be the unique (random) solution with vanishing mean to (1.9), and let $u_{\rm hom} \in H^1(\mathbb{T})$ be the unique solution to the continuum homogenized equation

$$-\nabla \cdot \boldsymbol{a}_{\rm hom} \nabla u_{\rm hom} \;=\; f \text{ in } \mathbb{T}, \qquad \int_{\mathbb{T}} u_{\rm hom} = 0. \tag{1.11}$$

Then u_ε converges almost surely (weakly in the discrete H^1-norm) to $u_{\rm hom}$.

We are interested in estimates on the speed of convergence of u_ε to $u_{\rm hom}$. The error $u_\varepsilon - u_{\rm hom}$ consists of two contributions that are of different nature, namely the *homogenization error* and the *discretization error*. The latter is purely deterministic and introduced by approximating the deterministic, *continuum*, homogenized equation (1.11) by the deterministic, homogenized, *finite difference* equation

$$\nabla_\varepsilon^* \boldsymbol{a}_{\rm hom} \nabla_\varepsilon u_{0,\varepsilon} \;=\; f_\varepsilon \text{ in } \mathbb{T}_\varepsilon, \qquad \sum_{\mathbb{T}_\varepsilon} u_{0,\varepsilon} = 0, \tag{1.12}$$

where $f_\varepsilon \in L^2(\mathbb{T}_\varepsilon)$ denotes a suitable approximation of the continuum r. h. s. f. The discretization error is well-studied and well-understood (see for instance [11]).

In this paper we focus on the homogenization error that monitors the difference between the random, variable-coefficient equation (1.9) and the deterministic, constant-coefficient equation (1.12). The main result of this paper is the upcoming quantitative two-scale expansion estimate. We quantify the error in the discrete $L^2(\mathbb{T}_\varepsilon)$- and $H^1(\mathbb{T}_\varepsilon)$-norms, which are defined for $v : \mathbb{T}_\varepsilon \to \mathbb{R}$ as

$$\|v\|_{L^2(\mathbb{T}_\varepsilon)} = \sqrt{\varepsilon^d \sum_{\mathbb{T}_\varepsilon} v^2}, \qquad \|v\|_{H^1(\mathbb{T}_\varepsilon)} = \sqrt{\varepsilon^d \sum_{\mathbb{T}_\varepsilon} (v^2 + |\nabla_\varepsilon v|^2)}.$$

Theorem 1. *Let $d \geq 2$ and a be i. i. d. coefficients. Then there exists a deterministic, symmetric, positive definite matrix $\boldsymbol{a}_{\rm hom} \in \mathbb{R}^{d\times d}$ (only depending on β and d) with the following property.*
Given $\varepsilon > 0$ with $\varepsilon^{-1} = L \in \mathbb{N}$, and a r. h. s.

$$f_\varepsilon : \mathbb{T}_\varepsilon \to \mathbb{R} \qquad \text{with } \sum_{\mathbb{T}_\varepsilon} f_\varepsilon = 0,$$

let $u_\varepsilon : \Omega_L \times \mathbb{T}_\varepsilon \to \mathbb{R}$ and $u_{0,\varepsilon} : \mathbb{T}_\varepsilon \to \mathbb{R}$ be the unique solutions to (1.9) and (1.12), respectively. Then

$$\left\langle \|u_\varepsilon - u_{0,\varepsilon} - \varepsilon \sum_{j=1}^d \phi_j(\tfrac{\cdot}{\varepsilon}) \nabla_{\varepsilon,j} u_{0,\varepsilon}\|^2_{H^1(\mathbb{T}_\varepsilon)} \right\rangle^{1/2} \lesssim \varepsilon \, ||f_\varepsilon||_{L^2(\mathbb{T}_\varepsilon)} \begin{cases} (\ln \frac{1}{\varepsilon})^{1/2} & \text{for } d = 2 \\ 1 & \text{for } d > 2 \end{cases}, \tag{1.13}$$

where $\phi_1, \ldots, \phi_d$ are the periodic correctors associated with the periodic i. i. d. ensemble $\langle\cdot\rangle$ via the periodic corrector equation (see (2.3) below). The multiplicative constant in (1.13) only depends on the constant of ellipticity λ, and the dimension d.

As a corollary we get:

Corollary 1. *In the situation of Theorem 1 we have*

$$\left\langle \|u_\varepsilon - u_{0,\varepsilon}\|^2_{L^2(\mathbb{T}_\varepsilon)} \right\rangle^{1/2} \lesssim \varepsilon \|f_\varepsilon\|_{L^2(\mathbb{T}_\varepsilon)} \begin{cases} (\ln \frac{1}{\varepsilon})^{1/2} & \text{for } d=2 \\ 1 & \text{for } d>2 \end{cases}, \tag{1.14}$$

The multiplicative constant in this estimate only depends on λ and d.

Note that a bound similar to (1.14) (with however a suboptimal exponent) was recentlty obtained by Conlon and Spencer in [4], in the case of the whole space and a massive term. This result complements the estimates of the quantity $u_\varepsilon - \langle u_\varepsilon \rangle$ obtained in [5, Theorem 2], and in their optimal form in [14, Corollary 2 and Corollary 3]. In particular, the annealed estimates of [14, Theorem 1] (see Lemma 3.2 below) on the gradient of the (periodic) Green's function, combined with the argument leading to [5, Theorem 2], allow one to prove that

$$\left\langle \|u_\varepsilon - \langle u_\varepsilon \rangle\|^2_{L^2(\mathbb{T}_\varepsilon)} \right\rangle^{1/2} \lesssim \varepsilon \|f_\varepsilon\|_{L^2(\mathbb{T}_\varepsilon)} \begin{cases} (\ln \frac{1}{\varepsilon})^{1/2} & \text{for } d=2 \\ 1 & \text{for } d>2 \end{cases}.$$

The combination of this estimate with Corollary 1 then yields an estimate of the difference between the solution $u_{0,\varepsilon}$ of the problem with constant coefficients and the expectation $\langle u_\varepsilon \rangle$ of the solution of the original problem — i. e. the systematic error:

$$\|u_{0,\varepsilon} - \langle u_\varepsilon \rangle\|_{L^2(\mathbb{T}_\varepsilon)} \lesssim \varepsilon \|f_\varepsilon\|_{L^2(\mathbb{T}_\varepsilon)} \begin{cases} (\ln \frac{1}{\varepsilon})^{1/2} & \text{for } d=2 \\ 1 & \text{for } d>2 \end{cases}.$$

It is rather surprising that we have to go through the H^1-norm in order to control this systematic error, or conversely that we do not have to estimate this term in order to prove the (seemingly stronger) statement of Theorem 1. Compared to the work [19] by Yurinskiĭ, Theorem 1 covers dimension $d = 2$ and gives optimal estimates (in terms of scaling in ε) in any dimension, as can be seen by considering the regime of small ellipticity contrast. Note that for $d = 1$, the scaling is different: (1.14) is expected to be replaced by

$$\left\langle \|u_\varepsilon - u_{0,\varepsilon}\|^2_{L^2(\mathbb{T}_\varepsilon)} \right\rangle^{1/2} \lesssim \sqrt{\varepsilon} \|f_\varepsilon\|_{L^2(\mathbb{T}_\varepsilon)},$$

as explicity checked in the continuum setting by Bourgeat and Piatnitski in [3]. As opposed to these works, the present analysis heavily relies on the use of a spectral gap estimate in the probability space. We refer the reader to [6] for relevant references on the subject.

Remark 1.1. The reason why we consider the discrete setting is the following: In the proof of Theorem 1 we make extensive use of recent, quantitative results that we obtained in a series of paper [9, 10, 6, 14] in the discrete setting. The extension of some results to the continuum setting is currently under investigation, see [8].

Remark 1.2. (Rescaling). For the proof of Theorem 1 it is convenient to rescale the discretized torus $\mathbb{T}_\varepsilon$ so that the grid size becomes 1. Recall that $\varepsilon = \frac{1}{L}$ for some integer $L \in \mathbb{N}$. Hence, the L-rescaled version of $\mathbb{T}_\varepsilon$ yields the discrete rescaled torus $\mathbb{T}_L = (\mathbb{Z}/L\mathbb{Z})^d = \frac{1}{\varepsilon}\mathbb{T}_\varepsilon$. In analogy with (1.8) we introduce discrete derivatives ∇ and ∇^* acting on scalar fields $v : \mathbb{T}_L \to \mathbb{R}$ and vector fields $g = (g_1, \dots, g_d) : \mathbb{T}_L \to \mathbb{R}^d$ as follows:

$$\begin{aligned} &\nabla_i v(x) = v(x + \mathbf{e}_i) - v(x), \quad \nabla_i^* v(x) = v(x - \mathbf{e}_i) - v(x), \\ &\nabla v = (\nabla_1 v, \dots, \nabla_d v), \qquad \nabla^* g = \sum_{i=1}^d \nabla_i^* g_i. \end{aligned} \tag{1.15}$$

In order to state Theorem 1 in its rescaled version, we set

$$u(x) := u_\varepsilon(\varepsilon x), \qquad u_0(x) := u_{0,\varepsilon}(\varepsilon x), \qquad \tilde{f}(x) := \varepsilon^2 f_\varepsilon(\varepsilon x) \qquad \text{for } x \in \mathbb{Z}^d.$$

So defined, u, u_0 and $\tilde{f}$ are functions on the rescaled torus $\mathbb{T}_L$ with vanishing mean, and the finite difference equations (1.9) and (1.12) turn into

$$\nabla^* \boldsymbol{a} \nabla u = \tilde{f} \text{ on } \mathbb{T}_L, \qquad \sum_{\mathbb{T}_L} u = 0, \tag{1.16}$$

$$\nabla^* \boldsymbol{a}_{\mathrm{hom}} \nabla u_0 = \tilde{f} \text{ on } \mathbb{T}_L, \qquad \sum_{\mathbb{T}_L} u_0 = 0. \tag{1.17}$$

Furthermore, the two-scale expansion in (1.13) takes the form

$$z := u - u_0 - \sum_{j=1}^d \phi_j \, \nabla_j u_0, \tag{1.18}$$

and estimate (1.13) of Theorem 1 can be reformulated as

$$\left\langle \sum_{\mathbb{T}_L} (z^2 + L^2 |\nabla z|^2) \right\rangle^{\frac{1}{2}} \lesssim L \mu_d^{\frac{1}{2}}(L) \left(\sum_{\mathbb{T}_L} \tilde{f}^2 \right)^{\frac{1}{2}}, \tag{1.19}$$

where we set for abbreviation

$$\mu_d(L) = \begin{cases} \ln L & \text{for } d = 2, \\ 1 & \text{for } d > 2. \end{cases}$$

In fact we shall establish (1.19) (and thus (1.13)) in form of the estimate

$$\left\langle \sum_{\mathbb{T}_L} (z^2 + L^2 |\nabla z|^2) \right\rangle^{\frac{1}{2}} \lesssim L \mu_d^{\frac{1}{2}}(L) \left(\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2 \right)^{\frac{1}{2}}, \tag{1.20}$$

where $\nabla^2 u_0(x)$ denotes the discrete Hessian of u_0 at x and is given by the $d \times d$ matrix with entries $-\nabla_i^* \nabla_j u_0(x)$. Note that (1.20) indeed implies (1.19), since u_0 (as a solution to the constant-coefficient difference equations (1.17)) satisfies the a priori estimate $\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2 \lesssim \sum_{\mathbb{T}_L} f^2$ up to a multiplicative constant that only depends on λ and d.

Notation. Throughout this article, we use the following notation:

- $d \geq 2$ is the dimension;
- $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ denotes the canonical basis of $\mathbb{Z}^d$;
- $\mathbb{T}_L = (\mathbb{Z}/L\mathbb{Z})^d$ denotes the discretized L-rescaled torus;
- $x \mod L$ denotes the unique point in $([0, L) \cap \mathbb{Z})^d$ with $x = (x \mod L) + Lx'$ for some $x' \in \mathbb{Z}^d$;
- $\lesssim$ and $\gtrsim$ stand for $\leq$ and $\geq$ up to a multiplicative constant which only depends on the quantities specified in the context;
- when both $\lesssim$ and $\gtrsim$ hold, we simply write $\sim$;
- β denotes a single-site probability measure on Ω_0, see Section 2.1;
- $\langle \cdot \rangle$ denotes the L-periodic i. i. d. ensemble on Ω_L associated with β, see Section 2.1;
- $\mathrm{cov}\,[\cdot;\cdot]$ denotes the covariance associated with $\langle \cdot \rangle$;

- we denote the (i,j)-th entry of a $d\times d$-matrix $\boldsymbol{b}$ by $\boldsymbol{b}^{ij}$ and write ":" for the inner product in $\mathbb{R}^{d\times d}$, i. e. $\boldsymbol{a}:\boldsymbol{b}=\sum_{i,j=1}^d \boldsymbol{a}^{ij}\boldsymbol{b}^{ij}$;
- for all $L>0$, $\mu_d(L)=\ln L$ for $d=2$ and $\mu_d(L)=1$ for $d>2$.

2. Assumptions on the ensemble and the notion of the corrector

In this section we introduce and motivate the requi assumptions on the ensemble, the definition of the corrector and the homogenized coefficients. We recall some recent quantitative estimates from stochastic homogenization that are at the basis of the proof of Theorem 1. Finally, we comment on the role played by the i. i. d. assumption.

2.1. **Assumptions on the ensemble.** Recall that $\langle\cdot\rangle$ denotes the i. i. d. ensemble associated with the single-site measure β via (1.10). Our theory involves only two probabilistic ingredients: A covariance estimate and a Logarithmic Sobolev Inequality (LSI) for $\langle\cdot\rangle$. We start with the covariance estimate, which is explicitly used in the proof of this paper. We shall comment on the LSI in the following section.

Lemma 2.1. *Let $\langle\cdot\rangle$ denote the periodic i. i. d. ensemble (see* (1.10)*). Then we have for any two random variables ζ, $\tilde\zeta$:*

$$\operatorname{cov}\Big[\zeta;\tilde\zeta\Big]\le\sum_{y\in\mathbb{T}_L}\langle(\frac{\partial\zeta}{\partial y})^2\rangle^{\frac12}\langle(\frac{\partial\tilde\zeta}{\partial y})^2\rangle^{\frac12}. \tag{2.1}$$

Here

$$\operatorname{cov}\Big[\zeta;\tilde\zeta\Big]:=\langle(\zeta-\langle\zeta\rangle)(\tilde\zeta-\langle\tilde\zeta\rangle)\rangle$$

denotes the covariance of ζ and $\tilde\zeta$. Furthermore, for a site $y\in\mathbb{T}_L$, the random variable $\frac{\partial\zeta}{\partial y}$ is defined by

$$(\frac{\partial\zeta}{\partial y})(\boldsymbol{a}):=\zeta(\boldsymbol{a})-\int_{\Omega_0}\zeta(\boldsymbol{a})\,\beta(d\boldsymbol{a}(y)).$$

Like the classical partial derivatives $\{\frac{\partial\zeta}{\partial\boldsymbol{a}_{kk}(y)}\}_{k=1,\cdots,d}$, the function $\frac{\partial\zeta}{\partial y}$ measures how sensitively ζ depends on the variable $\boldsymbol{a}(y)=\{\boldsymbol{a}_{kk}(y)\}_{k=1,\cdots,d}\in\Omega_0$. For reasons explained in [6] we call these derivatives *vertical*. For obvious reasons, $\frac{\partial\zeta}{\partial y}$ is called the *discrete* vertical derivative of ζ at y.

We note that for $\zeta=\tilde\zeta$, (2.1) turns into the Spectral Gap Estimate (SG) with constant 1, i. e.

$$\langle\zeta^2\rangle\le\langle\sum_{y\in\mathbb{T}_L}(\frac{\partial\zeta}{\partial y})^2\rangle \tag{2.2}$$

for any random variable ζ with vanishing expectation $\langle\zeta\rangle=0$. We refer to [6, Lemma 7] for a proof in the present context and note that (2.2) is extensively used in the proof of Lemma 2.2 as only probabilistic ingredient. This estimate is reminiscent of the Brascamp-Lieb inequality used by Naddaf and Spencer in [16]. In a slightly different context, a covariance estimate like (2.1) was established in [10, Lemma 3]. For the convenience of the reader, we present the elementary proof of Lemma 2.1.

2.2. **Corrector.** We now introduce the important concept of the *corrector*. Since we only have to deal with the *periodic* (as opposed to the infinite) ensemble $\langle\cdot\rangle$, we can avoid discussing all technicalities. Indeed, for any realization of $\boldsymbol{a}$ according to $\langle\cdot\rangle$, that is, for any periodic coefficient field $\boldsymbol{a} \in \Omega_L$ and for any coordinate direction $j = 1, \cdots, d$, there exists a unique scalar field $\phi_j(\boldsymbol{a}, \cdot)\colon \mathbb{T}_L \to \mathbb{R}$ characterized by

$$\nabla^* \boldsymbol{a}(\nabla\phi_j + \mathbf{e}_j) = 0 \text{ on } \mathbb{T}_L \quad \text{and} \quad \sum_{\mathbb{T}_L} \phi_j = 0. \tag{2.3}$$

Here $\mathbf{e}_j$ denotes the unit vector in direction of the j-th coordinate axis, and the discrete derivatives are defined in (1.15). Clearly, for every $j = 1, \cdots, d$, this defines a *random* scalar field ϕ_j. Evidently, this random field is *stationary* in the sense that for any shift $y \in \mathbb{R}^d$ one has $\phi_j(\boldsymbol{a}(\cdot + y), \cdot) = \phi_j(\boldsymbol{a}, \cdot + y)$. The periodic function $\phi_j : \mathbb{Z}^d \to \mathbb{R}$ "corrects" the affine function $x \mapsto x_j$ such that the resulting function $\mathbb{Z}^d \ni x \mapsto \phi_j(x) + x_j$ is a-harmonic. In this sense, $(\phi_1, \cdots, \phi_d)$ provide $\boldsymbol{a}$-harmonic coordinates for $\mathbb{T}_L$. We thus call the ϕ_j's the (periodic) *corrector*.

A crucial ingredient in the proof of our main result is the following boundedness estimate on the moments of the corrector:

Lemma 2.2. *(Gloria, Neukamm & Otto,* [6]*). For $j = 1, \ldots, d$ we have*

$$\langle |\phi_j|^2 \rangle \lesssim \begin{cases} \ln L & \text{for } d = 2 \\ 1 & \text{for } d > 2 \end{cases}, \tag{2.4}$$

$$\langle |\nabla\phi_j|^4 \rangle \lesssim 1. \tag{2.5}$$

The multiplicative constants in these estimates only depend on λ and d.

We remark that the previous estimate indeed holds for the more general class of stationary ensembles which satisfy SG. For the question in which sense $\lim_{L\uparrow\infty} \phi_j$ exists, which is however is not relevant for this paper, we refer to [6, Chapter 6].

2.3. **Homogenized coefficient.** Let us recall that a coefficient field

$$\boldsymbol{a} \in \Omega := \{\boldsymbol{a} : \mathbb{Z}^d \to \Omega_0\} = \Omega_0^{\mathbb{Z}^d}$$

can be seen as a description of a network of resistors: Suppose $u\colon \mathbb{Z}^d \to \mathbb{R}$ is an $\boldsymbol{a}$-harmonic function; if $u(x)$ is interpreted as the potential at vertex x and $\boldsymbol{a}_{kk}(x)$ as conductivity along the edge joining x to $\mathbf{e}_k + x$, then $\boldsymbol{a}_{kk}(x)\nabla_k u(x)$ can be interpreted as the (stationary) current along this edge. In this sense, the merit of the homogenized coefficient $\boldsymbol{a}_{\rm hom}$ is that, almost surely, it relates the spatially averaged potential gradient $\xi = \sum \nabla u$ to the spatially averaged current $\boldsymbol{a}_{\rm hom}\xi = \sum \boldsymbol{a}\nabla u$. For our special $\boldsymbol{a}$-harmonic function $u = \phi_j + x_j$, the spatially averaged (over one period cell) potential gradient is given by

$$L^{-d} \sum_{\mathbb{T}_L} (\nabla\phi_j + e_j) = e_j,$$

whereas the spatially averaged current is given by

$$L^{-d} \sum_{\mathbb{T}_L} \boldsymbol{a}(\nabla\phi_j + e_j).$$

Hence the expected value of the latter, that is,

$$\langle L^{-d} \sum_{\mathbb{T}_L} \boldsymbol{a}(\nabla\phi_j + e_j)\rangle \overset{\text{stationarity}}{=} \langle \boldsymbol{a}(0)(\nabla\phi_j(0) + e_j)\rangle =: \boldsymbol{a}_{\mathrm{hom},L} e_j \tag{2.6}$$

is a good (and computable) proxy for $\boldsymbol{a}_{\mathrm{hom}}$ as $L \uparrow \infty$. Qualitative homogenization theory ensures that it is indeed true that

$$\lim_{L\uparrow\infty} \boldsymbol{a}_{\mathrm{hom},L} = \boldsymbol{a}_{\mathrm{hom}}, \tag{2.7}$$

see for instance [17, Theorem 4.6]. For us, this has the convenient side-effect that we don't have to give the technically more demanding, independent definition of $\boldsymbol{a}_{\mathrm{hom}}$. For the latter, we refer to [6, Introduction] for instance. Moreover — and this is a second important ingredient for our result — the quantitative theory in [6] provides an optimal estimate of this "systematic error" in the case of the (infinite) i. i. d. ensemble.

Lemma 2.3. *(Gloria, Neukamm & Otto,* [6, Proposition 5]*). Consider the i. i. d. ensemble. Then we have*

$$|\boldsymbol{a}_{\mathrm{hom},L} - \boldsymbol{a}_{\mathrm{hom}}| \lesssim L^{-d} \ln^d L,$$

where the multiplicative constant depends only on d and λ.

As opposed to the other important ingredients for our result, the annealed estimates on the corrector and on the Green's function in Lemma 2.2 and in Lemma 3.2 below, the above ingredient relies on the i. i. d. property in a subtle way. In fact, if we were not dealing with the i. i. d. ensemble $\langle\cdot\rangle$, but with a more general infinite ensemble (that we call $\langle\cdot\rangle$ for the purpose of the discussion in this paragraph only), the choice of the periodic ensemble $\langle\cdot\rangle_L$ (that we endow with the index L for the purpose of this discussion only) is more subtle — and of practical importance: On the one hand, in view of Lemma 2.2 and of Lemma 3.2 below in particular, $\langle\cdot\rangle_L$ should satisfy a Logarithmic Sobolev Inequality (LSI) uniformly in $L \uparrow \infty$. On the other hand, in view of Lemma 2.3, $\langle\cdot\rangle_L$ should be well-coupled to $\langle\cdot\rangle$ in order to avoid a large systematic error. If the ensemble is not i. i. d. , the definition of $\langle\cdot\rangle_L$ through a "brutal" periodization, which is obtained by *restricting* $\boldsymbol{a}(\cdot + y)$ to $\mathbb{T}_L$, with a random shift $y \in \mathbb{T}_L$ to retain stationarity, seems both unnatural and difficult to control. It seems more natural and promising to us to define $\langle\cdot\rangle_L$ as the distribution of $\boldsymbol{a}$ under $\langle\cdot\rangle$ *conditioned* on the $\mathbb{T}_L$-periodicity of $\boldsymbol{a}$. However, this conditioning is singular; and we only expect control if the ensemble $\langle\cdot\rangle$ can be characterized by a sufficiently short-range (translation invariant) Hamiltonian. In this case, we expect that Dobrushin-Shlosman criteria (which as uniform mixing conditions ensure near-independence of $\boldsymbol{a}(x)$ and $\boldsymbol{a}(y)$ for $|x - y| \gg 1$ for all conditional measures) conveniently provide LSI for $\langle\cdot\rangle_L$ uniform in L. The extension of Lemma 2.3 to such a situation is investigated in a forthcoming work, see [7]. Let us also point out that the present proof relies on the covariance estimate of Lemma 2.1. It is not yet clear to us whether such a covariance estimate can survive beyond the i. i. d. case.

3. Proofs of Theorem 1 and Corollary 1

In this section we present the proofs of Theorem 1 and Corollary 1. Beforehand, we recall some auxiliary estimates on the elliptic Green's function that we need in the proof.

Structure of the proof and auxiliary estimates on the Green's function. The starting point to prove Theorem 1 is the same as in the proof of [18, Theorem 3] by Papanicolaou and Varadhan. Recall that z is given by (1.18). By uniform ellipticity and L-periodicity of $\boldsymbol{a}$, an integration by parts on $\mathbb{T}_L$ yields

$$\lambda \left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle \leq \left\langle \sum_{\mathbb{T}_L} \nabla z \cdot \boldsymbol{a} \nabla z \right\rangle = \left\langle \sum_{\mathbb{T}_L} z \, \nabla^* \boldsymbol{a} \nabla z \right\rangle. \tag{3.1}$$

As we shall see below, in *Step 1* of the proof of Theorem 1, an application of $\nabla^* \boldsymbol{a} \nabla$ to z yields the decomposition

$$\nabla^* \boldsymbol{a} \nabla z = \nabla^* g + r_1 + r_2, \tag{3.2}$$

where the random vector field $g : \mathbb{T}_L \to \mathbb{R}^d$, the deterministic scalar field $r_1 : \mathbb{T}_L \to \mathbb{R}$, and the random scalar field $r_2 : \mathbb{T}_L \to \mathbb{R}$ are given by

$$\begin{aligned} g_i &= -\sum_{j=1}^{d} \boldsymbol{a}^{ii} \, \phi_j(\cdot + \mathbf{e}_i) \, \nabla_i \nabla_j u_0, \\ r_1 &= (\boldsymbol{a}_{\mathrm{hom},L} - \boldsymbol{a}_{\mathrm{hom}}) : \nabla^* \nabla u_0, \\ r_2 &= (\boldsymbol{b} - \boldsymbol{a}_{\mathrm{hom},L}) : \nabla^* \nabla u_0. \end{aligned} \tag{3.3}$$

Above, $\boldsymbol{a}_{\mathrm{hom},L}$ is defined via (2.6), $\boldsymbol{a}_{\mathrm{hom}}$ is defined via (2.7), and $\boldsymbol{b} : \mathbb{T}_L \to \mathbb{R}^{d \times d}$ denotes the matrix field with entries

$$\boldsymbol{b}^{ij} := \boldsymbol{a}^{ii}(\cdot - \mathbf{e}_i)\big(\nabla_i \phi_j(\cdot - \mathbf{e}_i) + \delta(i-j)\big) \qquad \text{for } i, j = 1, \ldots, d. \tag{3.4}$$

As we shall see below (in *Step 3* of the proof of Theorem 1), the scalar fields r_1 and r_2 satisfy

$$\sum_{\mathbb{T}_L} r_1 = 0 \qquad \text{and} \qquad \langle r_2(x) \rangle = 0 \text{ for all } x \in \mathbb{T}_L. \tag{3.5}$$

By combining (3.1) and (3.2), the bound (1.20) follows from estimates of the terms $\left\langle \sum_{\mathbb{T}_L} \nabla z \cdot g \right\rangle$, $\left\langle \sum_{\mathbb{T}_L} (z - \bar{z}) \, r_1 \right\rangle$ and $\left\langle \sum_{\mathbb{T}_L} z \, r_2 \right\rangle$, an estimate of the spatial mean

$$\bar{z} := L^{-d} \sum_{\mathbb{T}_L} z,$$

and the discrete Poincaré inequality on the torus.

The most intricate estimate is the one of $\left\langle \sum_{\mathbb{T}_L} z \, r_2 \right\rangle$, see *Step 5* in the proof of Theorem 1 below. Since the term $r_2(x)$ has vanishing expectation for all $x \in \mathbb{T}_L$ one may write it as a covariance

$$\left\langle \sum_{x \in \mathbb{T}_L} z(x) r_2(x) \right\rangle = \sum_{x \in \mathbb{T}_L} \operatorname{cov}\left[z(x); r_2(x) \right].$$

In order to benefit from this, we appeal to the covariance estimate of Lemma 2.1 and to a vertical derivative calculus on coefficient fields that we introduced in [6]. The process of estimating the vertical derivatives of r_2 and z involves the (periodic) Green's function:

Definition 3.1. The L-periodic Green's function $G_L : \mathbb{T}_L \times \mathbb{T}_L \times \Omega_L \to \mathbb{R}$ is defined as follows. For all $y \in \mathbb{T}_L$ and $\boldsymbol{a} \in \Omega_L$ the function $G_L(\cdot, y; \boldsymbol{a})$ is the unique L-periodic mean free solution to

$$\nabla^* \boldsymbol{a} \nabla G_L(\cdot, y; \boldsymbol{a}) = \delta(\cdot - y) - L^{-d} \qquad \text{on } \mathbb{T}_L, \tag{3.6}$$

where δ is the Dirac mass at zero.

When no confusion occurs we use the shorthand notation $G_L(x,y)$ for $G_L(x,y;\boldsymbol{a})$. We shall use both quenched (i. e. pointwise deterministic) and annealed (i. e. statistically averaged) estimates on $|\nabla G_L|$. The pointwise estimates rely on the De Giorgi-Nash-Moser Hölder regularity theory (and are standard in the continuum case):

Lemma 3.1. *There exists $\gamma > 0$ depending only on λ and d such that for all $\boldsymbol{a} \in \Omega_L$ and $L \in \mathbb{N}$, the Green's function $G_L(\cdot,\cdot;\boldsymbol{a})$ satisfies the following quenched estimate:*

$$|\nabla_x G_L(x,y;\boldsymbol{a})|, |\nabla_y G_L(x,y;\boldsymbol{a})| \lesssim (|x-y \mod L|+1)^{2-d-\gamma}. \tag{3.7}$$

In the estimate the multiplicative constant only depends on λ and d. (Note that G_L is symmetric, so that the estimate (3.7) does not depend on the variable with respect to which we differentiate.)

See Appendix A for the proof. The crucial other ingredient is the recent annealed estimate of [14] by Marahrens and the third author, which we recall below in a version for the L-periodic Green's functions:

Lemma 3.2. *(Marahrens & Otto,*[14, Theorem 1]*). The periodic Green's function G_L satisfies the following annealed estimates:*

$$\left\langle |\nabla G_L(x,y)|^4 \right\rangle^{\frac{1}{4}} \lesssim (|x-y \mod L|+1)^{1-d}, \tag{3.8}$$

and

$$\left\langle |\nabla_x \nabla_y G_L(x,y)|^4 \right\rangle^{\frac{1}{4}} \lesssim (|x-y \mod L|+1)^{-d}, \tag{3.9}$$

where the multiplicative constants only depend on λ and d.

Let us mention that the proof of the annealed estimates on the derivatives of the Green's functions relies on a strengthened version of the spectral gap estimate (2.2), namely a Logarithmic-Sobolev Inequality. We refer the reader to [14] for details. Note that as for the variance estimate by Naddaf and Spencer in [16], an optimal control of the fourth moment is enough for our quantitative expansion.

Proof of Theorem 1. We prove the estimate of Theorem 1 in its rescaled formulation (1.20). Note that the identity

$$\sum_{\mathbb{T}_L} \sum_{i,j=1}^{d} |\nabla_i^* \nabla_j u_0|^2 = \sum_{\mathbb{T}_L} \sum_{i,j=1}^{d} |\nabla_i \nabla_j u_0|^2 = \sum_{\mathbb{T}_L} |\nabla^2 u_0|^2 \tag{3.10}$$

follows from periodicity and the elementary identity $|\nabla_i^* \nabla_j u_0(\cdot)| = |\nabla_i \nabla_j u_0(\cdot - \mathbf{e}_i)|$. The argument is divided into five steps. In the first step we derive the decomposition (3.2) with property (3.5). In Steps 2 and 3 we argue that it suffices to prove

$$\left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle \lesssim \mu_d(L) \sum_{\mathbb{T}_L} |\nabla^2 u_0|^2. \tag{3.11}$$

In the remaining steps we prove (3.11) starting with the inequality $\lambda \left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle \leq \left\langle \sum_{\mathbb{T}_L} z\, \nabla^* \boldsymbol{a} \nabla z \right\rangle$. To that end we appeal to the representation formula (3.2) which is a sum of a term in divergence form $\nabla^* g$, a deterministic term r_1 and a remainder with vanishing expectation r_2. The first two terms are estimated in Step 4. The third term r_2,

which has vanishing expectation, is controlled using the covariance estimate of Lemma 2.1 and will be treated in Steps 5a–5d.

Step 1. Derivation of the decomposition (3.2) with property (3.5).

Let us show that (3.2) holds with g, r_1 and r_2 given by (3.3). By the defining equation (1.9) for u,

$$\begin{aligned}\nabla^*\boldsymbol{a}\nabla z &\overset{(1.18)}{=} \nabla^*\boldsymbol{a}\nabla u - \nabla^*\boldsymbol{a}\nabla u_0 - \sum_{j=1}^d \nabla^*\boldsymbol{a}\nabla(\phi_j\nabla_j u_0)\\ &\overset{(1.9)}{=} \tilde{f} - \underbrace{\nabla^*\boldsymbol{a}\nabla u_0}_{=:\,I} - \underbrace{\sum_{j=1}^d \nabla^*\boldsymbol{a}\nabla(\phi_j\nabla_j u_0)}_{=:\,II}\,. \end{aligned} \tag{3.12}$$

We shall use the following discrete Leibniz rule: For all $\zeta_1, \zeta_2 : \mathbb{T}_L \to \mathbb{R}$,

$$\nabla_i(\zeta_1\zeta_2) \;=\; (\nabla_i\zeta_1)\,\zeta_2 + \zeta_1(\cdot+\mathbf{e}_i)\nabla_i\zeta_2, \quad \nabla_i^*(\zeta_1\zeta_2) \;=\; (\nabla_i^*\zeta_1)\,\zeta_2 + \zeta_1(\cdot-\mathbf{e}_i)\nabla_i^*\zeta_2. \tag{3.13}$$

For the first term this yields

$$I \;=\; \sum_{i=1}^d (\nabla_i^*\boldsymbol{a}^{ii})\,\nabla_i u_0 + \sum_{i=1}^d \boldsymbol{a}^{ii}(\cdot-\mathbf{e}_i)\,\nabla_i^*\nabla_i u_0,$$

while for the second term we obtain

$$\begin{aligned} II &= \sum_{i,j=1}^d \nabla_i^*\Big(\boldsymbol{a}^{ii}\nabla_i(\phi_j\nabla_j u_0)\Big)\\ &\overset{(3.13)}{=} \sum_{i,j=1}^d \nabla_i^*\Big(\boldsymbol{a}^{ii}(\nabla_i\phi_j\nabla_j u_0 + \phi_j(\cdot+\mathbf{e}_i)\nabla_i\nabla_j u_0)\Big)\\ &\overset{(3.13)}{=} \sum_{i,j=1}^d \nabla_i^*(\,\boldsymbol{a}^{ii}\nabla_i\phi_j\,)\nabla_j u_0 \;+\; \sum_{i,j=1}^d \boldsymbol{a}^{ii}(\cdot-\mathbf{e}_i)\nabla_i\phi_j(\cdot-\mathbf{e}_i)\nabla_i^*\nabla_j u_0\\ &\quad + \sum_{i,j=1}^d \nabla_i^*\Big(\boldsymbol{a}^{ii}\,\phi_j(\cdot+\mathbf{e}_i)\nabla_i\nabla_j u_0\Big). \end{aligned}$$

Because of the periodic corrector equation (2.3), the first term of the r. h. s. turns into

$$\sum_{i,j=1}^d \nabla_i^*(\,\boldsymbol{a}^{ii}\nabla_i\phi_j\,)\,\nabla_j u_0 = -\sum_{j=1}^d \big(\,\nabla_j^*\boldsymbol{a}^{jj}\,\big)\nabla_j u_0.$$

Hence, the terms in $I + II$ that involve the *first* derivative of u_0 cancel, so that

$$\begin{aligned} I + II \;&=\; \sum_{i,j=1}^d \boldsymbol{a}^{ii}(\cdot-\mathbf{e}_i)\big(\,\delta(i-j) + \nabla_i\phi_j(\cdot-\mathbf{e}_i)\,\big)\nabla_i^*\nabla_j u_0\\ &\quad + \sum_{i,j=1}^d \nabla_i^*\Big(\boldsymbol{a}^{ii}\,\phi_j(\cdot+\mathbf{e}_i)\nabla_i\nabla_j u_0\Big). \end{aligned} \tag{3.14}$$

The last term on the r. h. s. is precisely $-\nabla^* g$, the first term is $\boldsymbol{b} : \nabla^2 u_0$, cf. (3.3) and (3.4). The claim (3.2) then follows from (3.12), (3.14) and identity (1.17) which can be written in the form $\tilde{f} = \boldsymbol{a}_{\mathrm{hom}} : \nabla^* \nabla u_0$ since $\boldsymbol{a}_{\mathrm{hom}}$ is constant.
To conclude this step, we prove (3.5). The first identity simply follows from the L-periodicity of u_0. The second identity can be seen as follows: By the definition of $\boldsymbol{b}$, the stationarity of $\boldsymbol{a}$ and ϕ_j, and the definition of $\boldsymbol{a}_{\mathrm{hom},L}$ via (2.6) we have

$$\big\langle \boldsymbol{b}^{ij}(\cdot) \big\rangle = \Big\langle \boldsymbol{a}^{ii}(0) \Big(\delta(i-j) + \nabla_i \phi_j(0) \Big) \Big\rangle = \boldsymbol{a}^{ij}_{\mathrm{hom},L},$$

so that

$$\big\langle \boldsymbol{b} : \nabla^2 u_0 \big\rangle = \langle \boldsymbol{b} \rangle : \nabla^2 u_0 = \boldsymbol{a}_{\mathrm{hom},L} : \nabla^2 u_0$$

as desired.

Step 2. Reduction to an estimate for ∇z.

We claim that (1.20) (and thus the statement of Theorem 1, see Remark 1.2) follows from (3.11). Indeed, by the discrete Poincaré inequality $\sum_{\mathbb{T}_L} z^2 \lesssim L^2 \sum_{\mathbb{T}_L} |\nabla z|^2 + L^d \bar{z}^2$ we only need to prove that

$$\big\langle \bar{z}^2 \big\rangle \lesssim L^{2-d} \mu_d(L) \sum_{\mathbb{T}_L} |\nabla^2 u_0|^2.$$

Since the spatial means of u and u_0 vanish by definition, we have $\bar{z} = -L^{-d} \sum_{\mathbb{T}_L} \sum_{j=1}^d \phi_j \nabla_j u_0$, so that

$$\big\langle \bar{z}^2 \big\rangle \;=\; L^{-2d} \sum_{x \in \mathbb{T}_L} \sum_{x' \in \mathbb{T}_L} \Bigg\langle \Big(\sum_{i=1}^d \phi_i(x) \nabla_i u_0(x) \Big) \Big(\sum_{j=1}^d \phi_j(x') \nabla_j u_0(x') \Big) \Bigg\rangle .$$

We expand the square on the r. h. s.. Since u_0 is deterministic we get

$$\big\langle \bar{z}^2 \big\rangle \;=\; L^{-2d} \sum_{x \in \mathbb{T}_L} \sum_{x' \in \mathbb{T}_L} \sum_{i,j=1}^d \nabla_i u_0(x) \nabla_j u_0(x') \big\langle \phi_i(x) \phi_j(x') \big\rangle .$$

By Cauchy-Schwarz' inequality, stationarity of the correctors, and the bounds of Lemma 2.2,

$$\big\langle \phi_i(x) \phi_j(x') \big\rangle \;\leq\; \max_{k=1,\dots,d} \big\langle \phi_k^2 \big\rangle \;\lesssim\; \mu_d(L).$$

Hence,

$$\big\langle \bar{z}^2 \big\rangle \;\lesssim\; \mu_d(L) \left(L^{-d} \sum_{x \in \mathbb{T}_L} |\nabla u_0| \right)^2 .$$

The desired estimate then follows from Jensen's and Poincaré's inequalities.

Step 3. Reduction based on the decomposition (3.2).

In this step we argue that the desired estimate reduces to the following statement: Suppose that the functions g, r_1 and r_2 of the decompositon (3.2) satisfy the following estimates

in addition to (3.5):

$$\left\langle \sum_{\mathbb{T}_L} |g|^2 \right\rangle \lesssim \mu_d(L) \sum_{\mathbb{T}_L} |\nabla^2 u_0|^2, \tag{3.15}$$

$$\sum_{\mathbb{T}_L} r_1^2 \lesssim \tfrac{\ln^d L}{L^d} \sum_{\mathbb{T}_L} |\nabla^2 u_0|^2, \tag{3.16}$$

$$\left\langle \sum_{\mathbb{T}_L} z\, r_2 \right\rangle \lesssim \left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle^{\frac{1}{2}} \left(\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2 \right)^{\frac{1}{2}} + \mu_d(L) \sum_{\mathbb{T}_L} |\nabla^2 u_0|^2, \tag{3.17}$$

then estimate (1.20) (and thus Theorem 1) holds.

By Step 2 we just have to check (3.11). Indeed, by combining (3.1), the decomposition (3.2), the triangle inequality, an integration by parts, and (3.5) we get

$$\lambda \left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle \leq \left| \left\langle \sum_{\mathbb{T}_L} \nabla z \cdot g \right\rangle \right| + \left| \left\langle \sum_{\mathbb{T}_L} (z - \bar{z}) r_1 \right\rangle \right| + \left| \left\langle \sum_{\mathbb{T}_L} z\, r_2 \right\rangle \right|.$$

The first term is estimated by (3.15) and the Cauchy-Schwarz inequality:

$$\left| \left\langle \sum_{\mathbb{T}_L} \nabla z \cdot g \right\rangle \right| \lesssim \mu_d^{\frac{1}{2}}(L) \left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle^{\frac{1}{2}} \left(\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2 \right)^{\frac{1}{2}}.$$

The second term is estimated by (3.16), the Cauchy-Schwarz inequality, the Poincaré inequality for functions on $\mathbb{T}_L$ with zero mean, and the elementary estimate $\frac{\ln^d L}{L^{d-1}} \lesssim 1$ for $d > 1$:

$$\left| \left\langle \sum_{\mathbb{T}_L} (z - \bar{z}) r_1 \right\rangle \right| \lesssim \left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle^{\frac{1}{2}} \left(\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2 \right)^{\frac{1}{2}}.$$

The combination of (3.17) with the previous three inequalities yields

$$\left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle \lesssim \mu_d^{\frac{1}{2}}(L) \left\langle \sum_{\mathbb{T}_L} |\nabla z|^2 \right\rangle^{\frac{1}{2}} \left(\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2 \right)^{\frac{1}{2}} + \mu_d(L) \sum_{\mathbb{T}_L} |\nabla^2 u_0|^2,$$

which implies (3.11) by Young's inequality.

Step 4. Proof of the estimates (3.15) and (3.16).

Estimate (3.15) follows from the definition of g, cf. (3.3), the bound (2.4) of Lemma 2.2 on the second moment of the stationary ϕ_j and identity (3.10). Similarly, (3.16) follows from the definition of r_1 and the optimal bound on the error $|\boldsymbol{a}_{\mathrm{hom},L} - \boldsymbol{a}_{\mathrm{hom}}|$ of Lemma 2.3.

In the last step we prove (3.17). Since this step is rather long, we subdivide it further.

Step 5a. Application of the covariance estimate.

Since by (3.5), $\langle r_2(x)\rangle = 0$ for all $x \in \mathbb{T}_L$, we have

$$\begin{aligned}\left|\left\langle \sum_{x\in\mathbb{T}_L} z(x) r_2(x)\right\rangle\right| &= \left|\sum_{x\in\mathbb{T}_L} \operatorname{cov}\left[z(x); r_2(x)\right]\right| \\ &\overset{(3.3)}{=} \left|\sum_{x\in\mathbb{T}_L}\sum_{i,j=1}^{d} \nabla_i^*\nabla_j u_0(x) \operatorname{cov}\left[z(x); \boldsymbol{a}^{ij}_{\mathrm{hom},L} - \boldsymbol{b}^{ij}(x)\right]\right| \\ &\overset{(3.10)}{\leq} \left(\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2\right)^{\frac12} \left(\sum_{x\in\mathbb{T}_L}\sum_{i,j=1}^{d} \left(\operatorname{cov}\left[z(x); \boldsymbol{a}^{ij}_{\mathrm{hom},L} - \boldsymbol{b}^{ij}(x)\right]\right)^2\right)^{\frac12}.\end{aligned}$$

With the covariance estimate in Lemma 2.1 the r. h. s. is bounded by

$$\left(\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2\right)^{\frac12} \left(\sum_{x\in\mathbb{T}_L}\sum_{i,j=1}^{d} \left(\sum_{y\in\mathbb{T}_L} \left\langle \left(\frac{\partial z(x)}{\partial y}\right)^2\right\rangle^{\frac12} \left\langle \left(\frac{\partial \boldsymbol{b}^{ij}(x)}{\partial y}\right)^2\right\rangle^{\frac12}\right)^2\right)^{\frac12}.$$

Hence, for (3.17) it suffices to prove that

$$\begin{aligned}&\left(\sum_{x\in\mathbb{T}_L} \left(\sum_{y\in\mathbb{T}_L} \left\langle \left(\frac{\partial z(x)}{\partial y}\right)^2\right\rangle^{\frac12} \left\langle \left(\frac{\partial \boldsymbol{b}^{ij}(x)}{\partial y}\right)^2\right\rangle^{\frac12}\right)^2\right)^{\frac12} \\ &\quad \leq \left\langle \sum_{\mathbb{T}_L} |\nabla z|^2\right\rangle^{\frac12} + \mu_d(L) \left(\sum_{\mathbb{T}_L} |\nabla^2 u_0|^2\right)^{\frac12} \qquad \text{for } i,j = 1,\dots,d.\end{aligned} \tag{3.18}$$

To estimate the vertical derivatives we need to identify $\frac{\partial z}{\partial y}$ and $\frac{\partial \boldsymbol{b}^{ij}}{\partial y}$. This is done by appealing to the elliptic equations (1.16) and (2.3) and vertical differential calculus. Since the basic argument is simple, but polluted due to the discrete nature of the vertical and spatial derivatives, we will first present a *formal* calculation where the vertical derivative $\frac{\partial}{\partial y}$ is replaced by the classical partial derivative $\frac{\partial}{\partial \boldsymbol{a}_{kk}(y)}$ (defined for differentiable functions on Ω_L). The *rigorous* argument is then carried out in Step 5c and Step 5d below.

Step 5b. *Formal* derivation of formulas for the vertical derivatives.

We first (formally) identify $\frac{\partial u}{\partial \boldsymbol{a}_{kk}(y)}$ and $\frac{\partial \phi_i(x)}{\partial \boldsymbol{a}(x)}$. Applying $\frac{\partial}{\partial \boldsymbol{a}_{kk}(y)}$ to the elliptic equations (1.16) and (2.3) yields

$$\begin{aligned}\nabla^*\boldsymbol{a}(x)\nabla\frac{\partial u(x)}{\partial \boldsymbol{a}_{kk}(y)} &= -\nabla^*\frac{\partial \boldsymbol{a}(x)}{\partial \boldsymbol{a}_{kk}(y)}\nabla u(x), \\ \nabla^*\boldsymbol{a}(x)\nabla\frac{\partial \phi_j(x)}{\partial \boldsymbol{a}_{kk}(y)} &= -\nabla^*\frac{\partial \boldsymbol{a}(x)}{\partial \boldsymbol{a}_{kk}(y)}(\nabla\phi_j(x) + \mathbf{e}_j),\end{aligned}$$

using that $\frac{\partial}{\partial \boldsymbol{a}_{kk}(y)}$ and ∇ commute. Since $\frac{\partial \boldsymbol{a}(x)}{\partial \boldsymbol{a}_{kk}(y)} = (\mathbf{e}_k \otimes \mathbf{e}_k)\delta(x-y \mod L)$ for all $x, y \in \mathbb{T}_L$, the Green representation formula yields

$$\begin{aligned} \frac{\partial u(x)}{\partial \boldsymbol{a}_{kk}(y)} &= -\nabla_{y_k} G_L(x,y) \nabla_k u(y), \\ \frac{\partial \phi_j(x)}{\partial \boldsymbol{a}_{kk}(y)} &= -\nabla_{y_k} G_L(x,y)(\nabla_k \phi_j(y) + \delta(k-j)). \end{aligned} \tag{3.19}$$

Next we identify $\frac{\partial \boldsymbol{b}^{ij}}{\partial \boldsymbol{a}_{kk}(y)}$. We apply $\frac{\partial}{\partial \boldsymbol{a}_{kk}(y)}$ to the definition (3.4) of $\boldsymbol{b}^{ij}$ and use the identity above in the form of

$$\frac{\partial \nabla_i \phi_j(x)}{\partial \boldsymbol{a}_{kk}(y)} = -\nabla_{x_i} \nabla_{y_k} G_L(x,y)(\nabla_k \phi_j(y) + \delta(k-j)). \tag{3.20}$$

Rearranging the terms yields the identity

$$\begin{aligned} \frac{\partial \boldsymbol{b}^{ij}(x+\mathbf{e}_i)}{\partial \boldsymbol{a}_{kk}(y)} =& \delta(k-i)\delta(x-y \mod L)(\nabla_i \phi_j(y) + \delta(i-j)) \\ &- \boldsymbol{a}^{ii}(x) \nabla_{x_i} \nabla_{y_k} G_L(x,y)(\nabla_k \phi_j(y) + \delta(k-j)). \end{aligned} \tag{3.21}$$

Likewise, for the identification of $\frac{\partial z(x)}{\partial \boldsymbol{a}_{kk}(y)}$ we apply $\frac{\partial}{\partial \boldsymbol{a}_{kk}(y)}$ to (1.18):

$$\begin{aligned} \frac{\partial z(x)}{\partial \boldsymbol{a}_{kk}(y)} &= \frac{\partial u(x)}{\partial \boldsymbol{a}_{kk}(y)} - \sum_{j=1}^d \frac{\partial \phi_j(x)}{\partial \boldsymbol{a}_{kk}(y)} \nabla_j u_0(x) \\ &\overset{(3.19)}{=} -\nabla_{y_k} G_L(x,y) \left(\nabla_k u(y) - \sum_{j=1}^d (\nabla_k \phi_j(y) + \delta(k-j)) \nabla_j u_0(x) \right). \end{aligned}$$

Since we want to make ∇z appear, we substitute ∇u by the following expression

$$\nabla u(y) = \nabla z(y) + \sum_{j=1}^d (\nabla \phi_j(y) + \mathbf{e}_j) \nabla_j u_0(y) + \sum_{j=1}^d \phi_j(y) \nabla \nabla_j u_0(y), \tag{3.22}$$

which can formally be obtained by applying ∇ to (1.18) and using the continuum Leibniz rule $\nabla(\phi_j \nabla_j u_0) = \nabla \phi_j \nabla_j u_0 + \phi_j \nabla \nabla_j u$. We then get

$$\begin{aligned} \frac{\partial z(x)}{\partial \boldsymbol{a}_{kk}(y)} = -\nabla_{y_k} G_L(x,y) \Bigg(& \nabla_k z(y) + \sum_{j=1}^d (\nabla_k \phi_j(y) + \delta(k-j))(\nabla_j u_0(y) - \nabla_j u_0(x)) \\ & + \sum_{j=1}^d \phi_j(y) \nabla_k \nabla_j u_0(y) \Bigg). \end{aligned} \tag{3.23}$$

Let us stress the fact that the expression in the brackets on the r. h. s. is $\nabla z(y)$ plus terms that vanish if u_0 is affine. This will be crucial in order to obtain an optimal estimate.

Step 5c. *Rigorous* derivation of formulas for the vertical derivatives.

We now derive rigorous versions of (3.21) and (3.23), which will lead to the desired estimate (3.18). For the rigorous argument $\frac{\partial}{\partial \boldsymbol{a}_{kk}(y)}$ has to be replaced by the discrete vertical

derivative $\frac{\partial}{\partial y}$ for which the Leibniz rule is not valid. The main ingredient is the following rigorous version of (3.19): For $j = 1, \dots, d$ and $x, y \in \mathbb{T}_L$ we have

$$\frac{\partial \phi_j(x)}{\partial y} = -\nabla_y G_L(x,y) \cdot [\nabla \phi_j(y) + \mathbf{e}_j]_y, \tag{3.24}$$

$$\frac{\partial \nabla_i \phi_j(x)}{\partial y} = -\nabla_{x_i} \nabla_y G_L(x,y) \cdot [\nabla \phi_j(y) + \mathbf{e}_j]_y \tag{3.25}$$

$$\frac{\partial u(x)}{\partial y} = -\nabla_y G_L(x,y) \cdot [\nabla u(y)]_y, \tag{3.26}$$

where $[\cdot]_y$ denotes the commutator of the multiplication with $\boldsymbol{a}(y)$ and $\frac{\partial}{\partial y}$, i. e.

$$[F]_y := \frac{\partial(\boldsymbol{a}(y)F)}{\partial y} - \boldsymbol{a}(y)\frac{\partial F}{\partial y} = \boldsymbol{a}(y)\left\langle F\right\rangle_y - \left\langle \boldsymbol{a}(y)F\right\rangle_y \tag{3.27}$$

for all random vectors F. By Jensen's inequality, the commutator satisfies the following estimate: For all $1 \le q < \infty$,

$$\left\langle |[F]_y|^q \right\rangle \le 2^q \left\langle |F|^q \right\rangle, \tag{3.28}$$

which we will use in the sequel for $q = 2$ and $q = 4$. Note that for all $x, y \in \mathbb{T}_L$, since the coefficients are i. i. d.,

$$\frac{\partial(\boldsymbol{a}(x)F)}{\partial y} - \boldsymbol{a}(x)\frac{\partial F}{\partial y} = [F]_y \delta(x - y \mod L). \tag{3.29}$$

Here comes the argument for (3.24). By (2.3) we have

$$\begin{aligned} 0 &= \nabla^* \boldsymbol{a}(x)(\nabla\phi_j(x) + \mathbf{e}_j) - \left\langle \nabla^* \boldsymbol{a}(x)(\nabla\phi_j(x) + \mathbf{e}_j)\right\rangle_y \\ &= \nabla^* \Big(\boldsymbol{a}(x)(\nabla\phi_j(x) + \mathbf{e}_j) - \left\langle \boldsymbol{a}(x)(\nabla\phi_j(x) + \mathbf{e}_j)\right\rangle_y \Big) \\ &= \nabla^* \Big(\boldsymbol{a}(x)\frac{\partial \nabla\phi_j(x)}{\partial y} + \boldsymbol{a}(x)\left\langle \nabla\phi_j(x) + \mathbf{e}_j\right\rangle_y - \left\langle \boldsymbol{a}(x)(\nabla\phi_j(x) + \mathbf{e}_j)\right\rangle_y \Big) \end{aligned}$$

Using then (3.27) and (3.29) for $F = \nabla\phi_j(x) + \mathbf{e}_j$, this turns into

$$\nabla^* \Big(\boldsymbol{a}(x)\nabla\frac{\partial \phi_j(x)}{\partial y}\Big) = -\nabla^* \Big([\nabla\phi_j(x) + \mathbf{e}_j]_y \delta(x - y \mod L)\Big),$$

from which (3.24) follows by the Green representation formula and an integration by parts. Identity (3.25) follows from applying ∇_x to (3.24). The argument for (3.26) is similar to the one for (3.24) and left to the reader.

We are now in position to derive the rigorous versions of (3.21) and (3.23), and start with b^{ij}. We claim that

$$\frac{\partial \boldsymbol{b}^{ij}(x + \mathbf{e}_i)}{\partial y} = \Big(\delta(x - y \mod L)\mathbf{e}_i - \boldsymbol{a}^{ii}(x)\nabla_{x_i}\nabla_y G_L(x,y)\Big) \cdot [\nabla\phi_j(y) + \mathbf{e}_j]_y, \tag{3.30}$$

Indeed, using again (3.27) for $F = \nabla\phi_j(x) + \mathbf{e}_j$, we have

$$\begin{aligned} \frac{\partial(\boldsymbol{a}(x)(\nabla\phi_j(x) + \mathbf{e}_j))}{\partial y} &= \boldsymbol{a}(x)\nabla\frac{\partial\phi_j(x)}{\partial y} + \boldsymbol{a}(x)\left\langle \nabla\phi_j(x) + \mathbf{e}_j\right\rangle_y - \left\langle \boldsymbol{a}(x)(\nabla\phi_j(x) + \mathbf{e}_j)\right\rangle_y \\ &= \boldsymbol{a}(x)\frac{\partial\nabla\phi_j(x)}{\partial y} + \delta(x - y \mod L)[\nabla\phi_j(y) + \mathbf{e}_j]_y, \end{aligned}$$

which, combined with (3.25), yields (3.30).

We then turn to z and claim that

$$\frac{\partial z(x)}{\partial y} = -\nabla_y G_L(x,y)\cdot\Big(F_1(y)+F_2(y,x)+F_3(y)\Big), \tag{3.31}$$

where

$$\begin{aligned} F_1(y) &:= [\nabla z(y)]_y, \qquad F_2(y,x) := \sum_{j=1}^d \big(\nabla_j u_0(y)-\nabla_j u_0(x)\big)[\nabla\phi_j(y)+\mathbf{e}_j]_y,\\ F_3(y) &:= \sum_{i,j=1}^d \nabla_i\nabla_j u_0(y)[\phi_j(y+\mathbf{e}_i)\mathbf{e}_i]_y. \end{aligned}$$

We first apply $\frac{\partial}{\partial y}$ to (1.18) and use (3.24) and (3.26):

$$\frac{\partial z(x)}{\partial y} = -\nabla_y G_L(x,y)\cdot\Big([\nabla u(y)]_y - \sum_{j=1}^d \nabla_j u_0(x)\,[\nabla\phi_j(y)+\mathbf{e}_j]_y\Big). \tag{3.32}$$

We then wish to substitute the term $\nabla u(y)$ by an expression that involves $\nabla z(y)$. To that end we apply ∇ to (1.18) and get with the help of the discrete Leibniz rule (3.13)

$$\begin{aligned} \nabla u(y) &= \nabla z(y)+\nabla u_0(y)+\sum_{j=1}^d \nabla_j u_0(y)\nabla\phi_j(y)+\sum_{i,j=1}^d \nabla_i\nabla_j u_0(y)\,\phi_j(y+\mathbf{e}_i)\mathbf{e}_i\\ &= \nabla z(y)+\sum_{j=1}^d \nabla_j u_0(y)(\nabla\phi_j(y)+\mathbf{e}_j)+\sum_{i,j=1}^d \nabla_i\nabla_j u_0(y)\,\phi_j(y+\mathbf{e}_i)\mathbf{e}_i. \end{aligned}$$

Combined with (3.32) the desired identity (3.31) follows.

Step 5d. Estimates of the vertical derivatives of $\boldsymbol{b}^{ij}$ and z.

We claim that

$$\left\langle\left(\frac{\partial \boldsymbol{b}^{ij}(x)}{\partial y}\right)^2\right\rangle^{1/2} \lesssim (|y-x \mod L|+1)^{-d} \tag{3.33}$$

and

$$\left\langle\left(\frac{\partial z(x)}{\partial y}\right)^2\right\rangle^{1/2} \lesssim I_1+I_2, \tag{3.34}$$

where

$$\begin{aligned} I_1 &:= (|y-x \mod L|+1)^{2-d-\gamma}\left(\left\langle|\nabla z(y)|^2\right\rangle^{1/2}+\mu_d(L)|\nabla^2 u_0(y)|\right),\\ I_2 &:= (|y-x \mod L|+1)^{1-d}|\nabla u_0(x)-\nabla u_0(y)|, \end{aligned}$$

for some $\gamma>0$ depending only on λ and d.

We start with (3.33). By the Cauchy-Schwarz inequality in probability, (3.30) turns into

$$\begin{aligned} |[\text{L. H. S. of (3.33)}]| \lesssim\ & \left\langle\delta(x-\mathbf{e}_i-y \mod L)+|\nabla_{x_i}\nabla_y G_L(x-\mathbf{e}_i,y)|^4\right\rangle^{1/4}\\ &\times\left\langle|[\nabla\phi_j(y)+\mathbf{e}_j]_y|^4\right\rangle^{1/4}. \end{aligned}$$

The first term of the r. h. s. is estimated by the annealed estimate (3.9) of Lemma 3.2. For the second term we appeal to (3.28) with $q = 4$, and to the bound (2.5) in Lemma 2.2 on the quartic moment of $\nabla\phi_j$:

$$\left\langle |[\nabla\phi_j(y) + \mathbf{e}_j]_y|^4 \right\rangle \leq 2^4 \left\langle \left|\nabla\phi_j(y) + \mathbf{e}_j\right|^4 \right\rangle \lesssim \left\langle |\nabla\phi_j|^4 \right\rangle + 1 \lesssim 1. \tag{3.35}$$

The desired estimate (3.33) follows.

We then turn to (3.34). Based on (3.31), we first bound the l. h. s. in (3.34) by the sum of three terms:

$$\begin{aligned} |[\text{L. H. S. of (3.34)}]| \lesssim & \left\langle |\nabla_y G_L(x,y)|^2 |F_1(y)|^2 \right\rangle^{1/2} \\ & + \left\langle |\nabla_y G_L(x,y)|^2 |F_2(y,x)|^2 \right\rangle^{1/2} + \left\langle |\nabla_y G_L(x,y)|^2 |F_3(y)|^2 \right\rangle^{1/2}. \end{aligned} \tag{3.36}$$

We estimate the first and third terms on the r. h. s. by appealing to the quenched estimate of Lemma 3.1 and Jensen's inequality in probability:

$$\begin{aligned} & \left\langle |\nabla_y G_L(x,y)|^2 |F_1(y)|^2 \right\rangle^{1/2} + \left\langle |\nabla_y G_L(x,y)|^2 |F_3(y)|^2 \right\rangle^{1/2} \\ & \lesssim \ (|x-y \mod L| + 1)^{2-d-\gamma} \left(\left\langle |\nabla z(y)|^2 \right\rangle^{1/2} + \max_{j=1,\dots,d} \left\langle \phi_j^2 \right\rangle^{1/2} |\nabla^2 u_0(y)| \right). \end{aligned}$$

Due to (2.4) in Lemma 2.2 the second moment of ϕ_j is bounded by $\mu_d(L)$, and thus we can control the r. h. s. by I_1. It remains to estimate the second term in (3.36). By appealing to the definition of F_2, the Cauchy-Schwarz inequality in probability, and (3.35) we have

$$\left\langle |\nabla_y G_L(x,y)|^2 |F_2(y,x)|^2 \right\rangle^{1/2} \lesssim \left\langle |\nabla_y G_L(x,y)|^4 \right\rangle^{1/4} |\nabla u_0(x) - \nabla u_0(y)|.$$

In view of the annealed estimate (3.8) of Lemma 3.2 the r. h. s. is controlled by I_2 as desired.

Step 5e. Proof of (3.18) and thus (3.17).

By combining (3.34) and (3.33) we get

$$\begin{aligned} & [\text{L. H. S. of (3.18)}] \\ & \lesssim \left(\sum_{x\in\mathbb{T}_L} \left(\sum_{y\in\mathbb{T}_L} (|y-x \mod L| + 1)^{2(1-d)-\gamma} \left(\left\langle |\nabla z(y)|^2 \right\rangle^{1/2} + \mu_d(L) |\nabla^2 u_0(y)| \right) \right)^2 \right)^{1/2} \\ & \quad + \left(\sum_{x\in\mathbb{T}_L} \left(\sum_{y\in\mathbb{T}_L} (|y-x \mod L| + 1)^{1-2d} |\nabla u_0(x) - \nabla u_0(y)| \right)^2 \right)^{1/2}. \end{aligned}$$

Since $\gamma > 0$, the discrete convolution kernel $x \mapsto (|x \mod L| + 1)^{2(1-d)-\gamma}$ has $\ell^1(\mathbb{T}_L)$-norm bounded independently of L (even for $d = 2$), i. e. $\sum_{\mathbb{T}_L} (|x \mod L| + 1)^{2(1-d)-\gamma} \lesssim 1$. Hence, by the convolution estimate w. r. t. the $\ell^2(\mathbb{T}_L)$-norm, the first term on the r. h. s. is controlled by the r. h. s. of (3.18). It remains to treat the second sum and suffices to

show that

$$\left(\sum_{x\in\mathbb{T}_L}\left(\sum_{y\in\mathbb{T}_L}(|y-x \mod L|+1)^{1-2d}|\nabla u_0(x)-\nabla u_0(y)|\right)^2\right)^{1/2} \lesssim \mu_d(L)\left(\sum_{\mathbb{T}_L}|\nabla^2 u_0|^2\right)^{1/2}. \tag{3.37}$$

By the definition of the discrete gradient and periodicity we have for $i=1,\dots,d$

$$\left(\sum_{x\in\mathbb{T}_L}|\nabla u_0(x+\mathbf{e}_i)-\nabla u_0(x)|^2\right)^{1/2} = \left(\sum_{\mathbb{T}_L}|\nabla_i\nabla u_0|^2\right)^{1/2},$$

which combined with the triangle inequality and periodicity yields

$$\left(\sum_{x\in\mathbb{T}_L}|\nabla u_0(x+z)-\nabla u_0(x)|^2\right)^{1/2} \lesssim |z \mod L|\left(\sum_{\mathbb{T}_L}|\nabla^2 u_0|^2\right)^{1/2} \tag{3.38}$$

for all $\mathbb{Z}^d$. We then use the triangle inequality in the form of

$$\left(\sum_x\left(\sum_z X_{xz}\right)^2\right)^{1/2} \le \sum_z\left(\sum_x X_{xz}^2\right)^{1/2},$$

so that

$$\begin{aligned}
&[\text{L. H. S. of (3.37)}]\\
&= \left(\sum_{x\in\mathbb{T}_L}\left(\sum_{z\in\mathbb{T}_L}(|z \mod L|+1)^{2(1-d)}\frac{|\nabla u_0(x)-\nabla u_0(x+z)|}{(|z \mod L|+1)}\right)^2\right)^{1/2}\\
&\overset{\Delta\text{-inequality}}{\le} \sum_{z\in\mathbb{T}_L}\left((|z \mod L|+1)^{4(1-d)}\sum_{x\in\mathbb{T}_L}\frac{|\nabla u_0(x)-\nabla u_0(x+z)|^2}{(|z \mod L|+1)^2}\right)^{1/2}\\
&\overset{(3.38)}{\le} \sum_{z\in\mathbb{T}_L}(|z \mod L|+1)^{2(1-d)}\left(\sum_{\mathbb{T}_L}|\nabla^2 u_0|^2\right)^{1/2}.
\end{aligned}$$

Evaluating the sum in z on the r. h. s. yields the claimed estimate (3.37), recalling that $\mu_d(L)=\ln L$ for $d=2$ and $\mu_d(L)=1$ for $d>2$. This proves (3.18), and therefore the desired estimate (3.17) by Step 5a.

□

3.1. **Proof of Corollary 1.** Estimate (1.14) is a direct consequence of (1.13) and the estimate

$$\max_{j=1,\dots,d}\langle|\phi_j|^2\rangle \lesssim \begin{cases}\ln L & \text{for } d=2\\ 1 & \text{for } d>2\end{cases}$$

of Lemma 2.2. □

Appendix A. Proofs of the auxiliary lemmas

A.1. **Proof of Lemma 2.1.** We adapt the arguments of [6, Proof of Lemma 7], as for the proof of [10, Lemma 3] starting from [9, Lemma 2.3]. We first introduce a couple of notations: Let $\{y_n\}_{n=1,\dots,N}$, $N := L^d$, be an enumeration of $\mathbb{T}_L$. For $n = 1, \cdots, N$ define the average

$$\langle\cdot\rangle_{\leq n} := \prod_{1\leq k\leq n} \int_{\Omega_0} \beta(d\boldsymbol{a}(y_k)).$$

Set $\zeta_n := \langle\zeta\rangle_{\leq n}$, $\tilde{\zeta}_n := \left\langle\tilde{\zeta}\right\rangle_{\leq n}$ for $n \geq 1$ and $\zeta_0 := \langle\zeta\rangle_{\leq 0} := \zeta$, $\tilde{\zeta}_0 := \left\langle\tilde{\zeta}\right\rangle_{\leq 0} := \tilde{\zeta}$. W. l. o. g. we assume that both ζ and $\tilde{\zeta}$ have zero expectation. We split the proof into two steps.

Step 1. Martingale decomposition.

We claim that

$$\left\langle\zeta\tilde{\zeta}\right\rangle = \sum_{n=1}^{N} \left\langle(\zeta_n - \zeta_{n-1})(\tilde{\zeta}_n - \tilde{\zeta}_{n-1})\right\rangle. \tag{A.1}$$

Here comes the argument: Since $\{y_1, \dots, y_N\} = \mathbb{T}_L$ we have $\langle\cdot\rangle = \langle\cdot\rangle_{\leq N}$ by (1.10), and thus $\zeta_N = \langle\zeta\rangle = 0$, $\tilde{\zeta}_N = \left\langle\tilde{\zeta}\right\rangle = 0$ and $\zeta_N\tilde{\zeta}_N = 0$. Moreover, by construction we have $\zeta_0 = \zeta$ and $\tilde{\zeta}_0 = \tilde{\zeta}$, and thus

$$\zeta\tilde{\zeta} = \sum_{n=1}^{N} \zeta_{n-1}\tilde{\zeta}_{n-1} - \zeta_n\tilde{\zeta}_n. \tag{A.2}$$

The identity (A.1) then follows from taking the expectation of (A.2) provided we prove that

$$\left\langle(\zeta_n - \zeta_{n-1})(\tilde{\zeta}_n - \tilde{\zeta}_{n-1})\right\rangle = \left\langle\zeta_{n-1}\tilde{\zeta}_{n-1}\right\rangle - \left\langle\zeta_n\tilde{\zeta}_n\right\rangle. \tag{A.3}$$

This last identity can be seen as follows: By definition we have

$$\tilde{\zeta}_n = \left\langle\tilde{\zeta}\right\rangle_{\leq n} = \prod_{1\leq k\leq n} \int_{\Omega_0} \tilde{\zeta}\,\beta(d\boldsymbol{a}(y_k)) = \int_{\Omega_0} \tilde{\zeta}_{n-1}\,\beta(d\boldsymbol{a}(y_n)).$$

Since ζ_n does not depend on $y_1, \dots, y_n$, we have

$$\zeta_n\tilde{\zeta}_n = \int_{\Omega_0} \zeta_n\tilde{\zeta}_{n-1}\,\beta(d\boldsymbol{a}(y_n)).$$

Integrating both sides w. r. t. $\boldsymbol{a}(y_k)$, $k \neq n$, yields $\left\langle\zeta_n\tilde{\zeta}_n\right\rangle = \left\langle\zeta_n\tilde{\zeta}_{n-1}\right\rangle$ and thus by symmetry $\left\langle\zeta_n\tilde{\zeta}_n\right\rangle = \left\langle\zeta_{n-1}\tilde{\zeta}_n\right\rangle$, so that we obtain (A.3).

Step 2. Conclusion.

From [6, Step 2, Proof of Lemma 7], since $\langle\cdot\rangle$ is an L-periodic i. i. d. measure, for all $n \in \{1, \dots, N\}$ we have

$$\left\langle(\zeta_{n-1} - \zeta_n)^2\right\rangle \leq \left\langle\left(\frac{\partial\zeta}{\partial y_n}\right)^2\right\rangle, \quad \left\langle(\tilde{\zeta}_{n-1} - \tilde{\zeta}_n)^2\right\rangle \leq \left\langle\left(\frac{\partial\tilde{\zeta}}{\partial y_n}\right)^2\right\rangle.$$

Hence the claim follows from (A.1) and Cauchy-Schwarz' inequality.

□

A.2. **Proof of Lemma 3.1.** It suffices to consider the case $y = 0$, since $G_L(x, y; \boldsymbol{a}) = G_L(x - y, 0; \boldsymbol{a}(\cdot - y))$ and due to the fact that the asserted estimate depends on $\boldsymbol{a}$ only through its ellipticity constant λ. We shall write $G_L(x) := G_L(x, 0; \boldsymbol{a})$ for brevity. We divide the proof into three steps. First we derive quenched estimates on the Green's function for $d > 2$ from the quenched estimates of [6, Lemma 24 and (219)] on the corresponding parabolic Green's functions. In the second step we prove the desired estimate for $d > 2$ using the De Giorgi-Nash-Moser Hölder estimate, see [15, Theorem 5.2] in the discrete setting. We then deduce the estimate for $d = 2$ from the estimate for $d = 3$ using an argument by Avellaneda and Lin, see [2, Proof of Theorem 13].

Step 1. Quenched estimate on G_L for $d > 2$.

Let $G'_L(t, y)$ denote the periodic, parabolic Green's function considered in [6, Definition 2]; it is characterized as follows: For all $x \in \mathbb{Z}^d$ the function G'_L is the unique $C^\infty(\mathbb{R}, \ell^\infty(\mathbb{Z}^d))$ solution of

$$\left\{ \begin{array}{rcl} \partial_t G'_L(t, x) - \nabla \cdot \boldsymbol{a} \nabla G'_L(t, x) & = & 0, \\ G'_L(0, x) & = & \delta(x \mod L). \end{array} \right. \tag{A.4}$$

Since both the initial data and the coefficient field are L-periodic, G'_L is in fact of class $C^\infty(\mathbb{R}, \ell^1(\mathbb{T}_L))$. The elliptic Green's function G_L can be formally obtained by integrating in time the mean free version of the parabolic Green's function:

$$G_L(x) = \int_0^\infty (G'_L(t, x) - L^{-d}) dt.$$

To turn this into a rigorous argument one needs to prove that $t \mapsto G'_L(t, x) - L^{-d} \in L^1(0, +\infty)$. This is a direct consequence of [6, eq. (219) in the proof of the discrete version of Lemma 24]: For all $\alpha > 0$ we have

$$\begin{aligned} 0 \le G'_L(t, x) &\lesssim (t + 1)^{-\frac{d}{2}} \Big(\frac{(|x \mod L| + 1)^2}{t + 1} + 1 \Big)^{-\frac{\alpha}{2}} \quad \text{for } t \lesssim L^2, \\ |G'_L(t, x) - L^{-d}| &\lesssim L^{-d} \exp(-c_0 \frac{t}{L^2}) \quad \text{for } t \gtrsim L^2. \end{aligned}$$

For $d > 2$, this directly yields

$$|G'_L(x)| \le \int_0^\infty |G_L(t, x) - L^{-d}| dt \lesssim (1 + |x \mod L|)^{2-d}. \tag{A.5}$$

Step 2. Quenched estimate on ∇G_L for $d > 2$.

We now deduce (3.7) from (A.5) for $d > 2$. Recall that it suffices to prove the estimate for $y = 0$ and that we write $G_L(x) = G_L(x, 0)$ for brevity. Fix a radius $2 \le R \sim 1 \ll L$. For $|x \mod L| \le 2R$ the r. h. s. of (3.7) is of order 1 so that the estimate directly follows from the combination of (A.5) with the discrete estimate $|\nabla_i G_L(x)| \le |G_L(x + e_i)| + |G_L(x)|$. Hence, it suffices to consider the case $|x \mod L| \ge 2R$. For all $y' \in \mathbb{T}_L$ and $r > 0$, set $B_r(y') = \{y'' \in \mathbb{T}_L, |y' - y'' \mod L| \le r\}$. Using the elementary inequality

$$|\nabla G_L(x)| \le \sqrt{d} \operatorname*{osc}_{B_1(x)} G_L \,,$$

this turns into an estimate of the oscillation of G_L on $B_1(x)$. Noting that G_L satisfies

$$\nabla^* \boldsymbol{a} \nabla G_L = L^{-d} \tag{A.6}$$

in the set $B_{|x \mod L|/2}(x)$ (which does not contain $0 \mod L$), one may appeal to the De Giorgi-Nash-Moser Hölder estimate. In particular, by [15, Theorem 5.2], there exists $0 < \gamma < 1$ depending only on λ and d such that

$$\operatorname*{osc}_{B_1(x)} G_L \lesssim \left(|x \mod L|^{-\gamma} \max_{B_{|x \mod L|/2}(x)} |G_L| \right) + L^{-d}, \tag{A.7}$$

Combined with (A.5), this yields (3.7) for $d > 2$.

Step 3. Quenched estimate on ∇G_L for $d = 2$.

Following [2, Proof of Theorem 13] we derive the estimates for $d = 2$ from the estimate for $d = 3$. To distinguish quantities in different dimensions we use the superscripts $^{(2)}$ and $^{(3)}$; e.g. $\mathbb{T}_L^{(2)}$ denotes the 2-dimensional torus. To a given two-dimensional, L-periodic coefficient field $\boldsymbol{a}^{(2)} \in \Omega_L^{(2)}$ we associate a three-dimensional coefficient-field $\boldsymbol{a}^{(3)} \in \Omega_L^{(3)}$ via

$$\boldsymbol{a}^{(3)}(x, x_3) := \operatorname{diag}\left[[\boldsymbol{a}^{(2)}(x)]_1, [\boldsymbol{a}^{(2)}(x)]_2, 1 \right] \qquad (x, x_3) \in \mathbb{Z}^2 \times \mathbb{Z} = \mathbb{Z}^3.$$

In the following we use the shorthand notation $G_L^{(2)}(x) := G_L^{(2)}(x, 0; \boldsymbol{a}^{(2)})$ and $G_L^{(3)}(x, x_3) := G_L^{(3)}((x, x_3), (0, 0); \boldsymbol{a}^{(3)})$. It is elementary to check that

$$G_L^{(2)}(x) = \sum_{x_3 \in ([0,L) \cap \mathbb{Z})} G_L^{(3)}(x, x_3),$$

and thus

$$\nabla_i G_L^{(2)}(x) = \sum_{x_3 \in ([0,L) \cap \mathbb{Z})} \nabla_i G_L^{(3)}(x, x_3) \qquad \text{for } i = 1, 2.$$

The quenched estimate (3.7) for $d = 3$ then turns into

$$\begin{aligned} |\nabla_i G_L^{(2)}(x)| &\leq \sum_{x_3 \in ([0,L) \cap \mathbb{Z})} |\nabla_i G_L^{(3)}(x, x_3)| \\ &\lesssim \sum_{x_3 \in ([0,L) \cap \mathbb{Z})} (1 + |(x, x_3) \mod L|)^{-1-\gamma} \\ &\lesssim \sum_{x_3 \in ([0,\infty) \cap \mathbb{Z})} (1 + |x \mod L| + x_3)^{-1-\gamma} \\ &\lesssim (1 + |x \mod L|)^{-\gamma} \lesssim (1 + |x \mod L|)^{-\gamma} \end{aligned}$$

which is nothing but (3.7) for $d = 2$. □

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