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# An Alternative Axiomatization of DEL and Its Applications*

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## Abstract

In this paper, we provide a new axiomatization of the event-model-based Dynamic Epistemic Logic, based on the completeness proof method proposed in [Wang and Cao, 2013]. This axiomatization does not use any of the standard reduction axioms, but naturally captures the essence of the update product. We demonstrate the use of our new axiomatization and the corresponding proof techniques by three sets of results: characterization theorems of the update operations, representation theorems of the DEL-generatable epistemic temporal structures given a fixed event model, and a complete axiomatization of DEL on models with protocols.

## 1 Introduction

Dynamic Epistemic Logic (DEL) and Epistemic Temporal Logic (ETL) are two major semantics-driven logical approaches for modelling knowledge and interactions in multi-agent settings (cf. e.g., [Baltag *et al.*, 1998; van Ditmarsch *et al.*, 2007] for DEL and [Fagin *et al.*, 1995; Parikh and Ramanujam, 1985] for ETL). In a nutshell, the DEL approach updates the epistemic structures by executing events, while the ETL approach computes the epistemic structure on the given temporal structure of the system.

It is not hard to see that iterated updates in DEL construct the temporal structure based on an initial epistemic model. Thus it is natural to cast the DEL-generated temporal epistemic structures as a particular class of ETL models, as studied in [van Benthem *et al.*, 2009]. This point of view turned out to be very useful in solving technical questions in the field (cf. e.g., [Aucher and Herzig, 2011; Holliday *et al.*, 2012]). Wang and Cao [2013] tried to make this view more explicit in logic, by giving a non-standard ETL-style axiomatization of Public Announcement Logic proposed in [Plaza, 1989; Gerbrandy and Groeneveld, 1997]. The new axiomatization differs from the standard reduction-axiom-based systems not only in the shape of the axioms but more importantly in the corresponding proof method for completeness. It was argued that the new axioms naturally specify the essence of the update, and can be applied to other dynamic epistemic logics where the standard reduction technique is not applicable.

In this work, we follow the proposal made in [Wang and Cao, 2013] and take the analysis to the event-model-based DEL [Baltag *et al.*, 1998]. Importantly, we demonstrate the usefulness of our new axioms and the related proof techniques by transparently unifying many existing/new results about DEL in the same picture.

Our main contributions are summarized as follows:

• We provide a **new axiomatization** $\mathcal{DEN}$ of DEL based on a revision of the proof strategy proposed in [Wang and Cao, 2013] using no reduction axioms.

• The new **axioms** characterize the update product (Theorem 11), which is the DEL version of the characterization result of PAL relativization in [van Benthem, 2011, Ch 3.8].

• The corresponding **properties** to the new axioms characterize the DEL-generatable epistemic temporal structures given an event model $\mathcal{U}$ (Theorem 13 and 22). We deviate from the representation theorems in [van Benthem *et al.*, 2009] and [van Benthem and Liu, 2004] by considering a fixed event model and allowing epistemic temporal models that are not in the tree-like shape.

• By weakening the **proof system** $\mathcal{DEN}$, we provide a complete axiomatization of DEL on epistemic models with state-dependent protocols (Theorem 21). Our logic turns out to be equivalent to the TDEL on tree-like generatable epistemic temporal structures discussed in [Hoshi and Yap, 2009], which is a well-known example of a dynamic epistemic-style logic which cannot be reduced to epistemic logic.

The readers are suggested to go through Section 2 even when they are familiar with DEL, since some crucial assumptions and observations are explained there about our exposition of DEL. In Section 3 and 4 we give the new axiomatization and related main results respectively.

## 2 Preliminaries

Given a non-empty set $\mathbf{P}$ of basic proposition letters, and a non-empty set $\mathbf{\Sigma}$ of basic events, the *dynamic epistemic language* **LDEL** is defined as follows:

$$\phi \quad ::= \quad \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid [e]\phi$$

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where $p \in \mathbf{P}$ and $e \in \mathbf{\Sigma}$. We call the $[e]$-free part of **LDEL** the *epistemic language* (**LEL**). As usual, we define $\bot$, $\phi \vee \psi$, $\phi \to \psi$, $\Diamond\phi$, and $\langle e\rangle\phi$ as the abbreviations of $\neg\top$, $\neg(\neg\phi \wedge \neg\psi)$, $\neg\phi \vee \psi$, $\neg\Box\neg\phi$, and $\neg[e]\neg\phi$ respectively.

For simplicity, we only consider the single agent case in this paper, although all of our results and techniques apply to the multi-agent case as well.

The language **LDEL** is usually interpreted on *epistemic models*[1] which are triples in the form of $(S, \to, V)$ where

- $S$ is a non-empty set of possible worlds,
- $\to \subseteq S \times S$ is a binary relation over $S$,
- $V : \mathbf{P} \to 2^S$ is a valuation function assigning each basic proposition letter a set of worlds where it is true.

In the sequel, given a model $\mathcal{M}$, we use $S_{\mathcal{M}}, \to_{\mathcal{M}}$ and $V_{\mathcal{M}}$ to denote the corresponding components of $\mathcal{M}$. We use $\mathbb{M}$ to denote the class of all the epistemic models.

To interpret the $[e]$ operators, the event models are introduced as below:

**Definition 1 (Event Model).** Given $\mathbf{\Sigma}$, an event model $\mathcal{U}$ is a tuple $(\mathbf{\Sigma}, \rightarrowtail, \mathit{Pre})$ where:

- $\mathbf{\Sigma}$ is a non-empty (countable) set of events.
- $\rightarrowtail \subseteq \mathbf{\Sigma} \times \mathbf{\Sigma}$ is a binary relation on $\mathbf{\Sigma}$.
- $\mathit{Pre} : \mathbf{\Sigma} \to \mathbf{LEL}$ is a function assigning each event a precondition (an **LEL** formula).

A pointed event model $\mathcal{U}, e$ is an event model with a designated point $e$ in $\mathbf{\Sigma}$. We call $\mathcal{U}$ an *image-finite* event model if $\{f \mid e \rightarrowtail f\}$ is finite for each $e$. $\lhd$

**Important:** In this paper, we assume that there is a single fixed event model $\mathcal{U}$ which is *image-finite* but can be infinite.[2] The readers who are familiar with DEL may find this assumption and our exposition of DEL slightly different from the usual ones. We will come back to this in Remark 1.

Given an epistemic model $\mathcal{M} = (S, \to, V)$, the truth value of DEL formulas at a state $s$ in $\mathcal{M}$ is defined as follows:

$$\boxed{\begin{array}{rcl} \mathcal{M}, s \vDash \top & \Leftrightarrow & \text{always} \\ \mathcal{M}, s \vDash p & \Leftrightarrow & s \in V(p) \\ \mathcal{M}, s \vDash \neg\phi & \Leftrightarrow & \mathcal{M}, s \nvDash \phi \\ \mathcal{M}, s \vDash \phi \wedge \psi & \Leftrightarrow & \mathcal{M}, s \vDash \phi \text{ and } \mathcal{M}, s \vDash \psi \\ \mathcal{M}, s \vDash \Box\psi & \Leftrightarrow & \forall t : s \to t \text{ implies } \mathcal{M}, t \vDash \psi \\ \mathcal{M}, s \vDash [e]\phi & \Leftrightarrow & \mathcal{M}, s \vDash \mathit{Pre}(e) \\ & & \quad \text{implies } \mathcal{M} \otimes \mathcal{U}, (s, e) \vDash \phi \end{array}}$$

where the updated model is defined as follows:

**Definition 2 (Update Product $\otimes$).** Given an epistemic model $\mathcal{M} = (S, \to, V)$ and an event model $\mathcal{U} = (\mathbf{\Sigma}, \rightarrowtail, \mathit{Pre})$, the product model is an epistemic model $(\mathcal{M} \otimes \mathcal{U}) = (S', \to', V')$ where:

$$\begin{array}{rcl} S' & = & \{(s, e) \mid \mathcal{M}, s \vDash \mathit{Pre}(e)\} \\ \to' & = & \{((s, e), (s', e')) \mid s \to s' \text{ and } e \rightarrowtail e'\} \quad \lhd \\ V'((s, e)) & = & V(s) \end{array}$$

[1]In this paper, we *do not* restrict ourselves to S5 models unless specified. We call the models *epistemic* due to the origin of DEL and its usual applications.

[2]Aucher and Herzig [2011] also fix an event model in their discussion, but explores the use of converse modality in the language.

**Important:** A semantics-driven logic can be represented as a triple $\langle$language, class of models, satisfaction relation$\rangle$, thus the standard DEL can be viewed as $\langle \mathbf{LDEL}, \mathbb{M}, \vDash\rangle$ while EL is $\langle \mathbf{LEL}, \mathbb{M}, \vDash\rangle$. This view is important since in the later part of the paper we will introduce many other semantics of the language **LDEL** on other classes of structures. We write $\mathbb{M} \vDash \phi$ if $\phi$ is valid in all the models in $\mathbb{M}$.

*Remark* 1. It is the time to explain our deviations from the standard exposition of DEL:

$[e]$ **vs.** $[\mathcal{U}, e]$. Although $[\mathcal{U}, e]$ is taken as an innocent operator in most of the DEL literature, a technically satisfactory treatment has to be quite complicated as demonstrated by [Baltag and Moss, 2004], due to the mixture of syntax and semantics introduced by $[\mathcal{U}, e]$ operators.[3] Here we make a simple separation of the syntax from the semantics by using atomic events w.r.t. a fixed event model. Note that we allow the event model to be infinite (but image-finite for the later axiomatization) thus this single event model can also be seen as a disjoint union of many finite event models in the standard setting. This simpler language is easier to handle when many alternative semantics are introduced later on.

**LEL vs. LDEL in preconditions**: Instead of arbitrary **LDEL** formulas, we only allow **LEL** formulas as preconditions. It is not an essential restriction, for DEL is equally expressive as epistemic logic (due to Theorem 1 below). On the other hand, this change can simplify the proofs considerably, though our results also hold in the standard setting.

The following axiomatization is based on the expositions in [Baltag and Moss, 2004; van Ditmarsch *et al.*, 2007]:[4]

System $\mathcal{DE}$

| Axiom schemata | | Rules | |
|---|---|---|---|
| TAUT | all the instances of tautologies | MP | $\dfrac{\phi, \phi \to \psi}{\psi}$ |
| DISTK | $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ | NECK | $\dfrac{\phi}{\Box\phi}$ |
| UATOM | $[e]p \leftrightarrow (\mathit{Pre}(e) \to p)$ | RE | $\dfrac{\phi \leftrightarrow \chi}{\psi \leftrightarrow \psi[\chi/\phi]}$ |
| UNEG | $[e]\neg\phi \leftrightarrow (\mathit{Pre}(e) \to \neg[e]\phi)$ | | |
| UCON | $[e](\phi \wedge \chi) \leftrightarrow ([e]\phi \wedge [e]\chi)$ | | |
| UK | $[e]\Box\phi \leftrightarrow (\mathit{Pre}(e) \to \bigwedge_{f:e\rightarrowtail f} \Box[f]\phi)$ | | |

where $\phi, \psi, \chi$ denote arbitrary formulas, $p \in \mathbf{P}$, and $e, f \in \mathbf{\Sigma}$. RE is the rule of replacement of equivalents. We call UATOM, UNEG UCON, UK the *reduction axioms*. We use $\mathcal{EL}$ to denote the system $\mathcal{DE}$ without the reduction axioms.

The following result is well-known:

[3]The readers may not agree with this claim at the first glance, but they may face troubles when spelling out all the details, e.g., arguing the proof system is recursively enumerable.

[4]In $\mathcal{DE}$, we use RE instead of the *composition axiom schema* in [van Ditmarsch *et al.*, 2007]. Note that the latter schema requires that a certain composition of two event models is again a well-formed event model, while it may not be possible in **LDEL** where the event model is fixed. In this light, we think $\mathcal{DE}$ is more robust than the axiomatization using composition axiom schema since $\mathcal{DE}$ works even when the space of event models is not composition closed.

**Theorem 1 ([Plaza, 1989]).** DEL *is equally expressive as* EL *and there is a translation* $t$ *which can turn a* ***LDEL*** *formula into an equivalent* ***LEL*** *formula w.r.t.* $\vDash$.

The standard reduction technique to show the completeness of $\mathcal{DE}$ can be summarized as follows:

$$\mathbb{M} \vDash \phi \iff \mathbb{M} \vDash t(\phi) \implies \vdash_{\mathcal{EL}} t(\phi) \implies \vdash_{\mathcal{DE}} t(\phi) \implies \vdash_{\mathcal{DE}} \phi.$$

The second step is due to the completeness of $\mathcal{EL}$ and the last step requires $\vdash_{\mathcal{DE}} \phi \leftrightarrow t(\phi)$. There are different ways to make the proof system meet the requirement. In our setting the last step relies on RE and works in an inside-out fashion (cf. [Wang and Cao, 2013] for a detailed discussion). Based on this proof strategy we can prove the completeness of $\mathcal{DE}$ (cf. e.g. [van Ditmarsch *et al.*, 2007]):

**Theorem 2.** $\mathcal{DE}$ *is sound and complete for* $\langle$***LDEL***$, \mathbb{M}, \vDash\rangle$.

## 3 Axiomatization $\mathcal{DEN}$

We propose the following alternative axiomatization $\mathcal{DEN}$:

System $\mathcal{DEN}$

| Axiom schemata | | Rules | |
|---|---|---|---|
| TAUT | all the instances of tautologies | MP | $\dfrac{\phi, \phi \to \psi}{\psi}$ |
| DISTK | $\Box(\phi \to \chi) \to (\Box\phi \to \Box\chi)$ | NECK | $\dfrac{\phi}{\Box\phi}$ |
| DISTU | $[e](\phi \to \chi) \to ([e]\phi \to [e]\chi)$ | NECU | $\dfrac{\phi}{[e]\phi}$ |
| INV | $(p \to [e]p) \wedge (\neg p \to [e]\neg p)$ | | |
| PRE | $\langle e\rangle\top \leftrightarrow \mathit{Pre}(e)$ | | |
| NM | $\Diamond\langle f\rangle\phi \to [e]\Diamond\phi$ (if $e \rightarrowtail f$) | | |
| PR | $\langle e\rangle\Diamond\phi \to \bigvee_{f: e\rightarrowtail f} \Diamond\langle f\rangle\phi$ | | |

where $p \in \mathbf{P}$, and $e, f \in \mathbf{\Sigma}$.[5]

Intuitively, DISTU and NECU are analogues of DISTK and NECK. INV expresses the stability of the valuation. PRE specifies the precondition of an event. NM and PR are variations of the axiom schemata of *no learning* and *perfect recall* in ETL (cf. e.g., [Halpern *et al.*, 2004] for discussion).

In their system $\mathcal{PAN}$ for PAL, Wang and Cao [2013] used an axiom schema PFUNC (partial functionality), which can be translated into the following axiom schema in our setting: $\langle e\rangle\phi \leftrightarrow (\mathit{Pre}(e) \wedge [e]\phi)$. It is easy to see that this axiom schema is essentially a reformulation of the reduction axiom for negation (UNEG). To make a neat system, instead of PFUNC or UNEG, we have a much weaker axiom schema PRE which is also present in [Aucher and Herzig, 2011]. To derive UNEG based on PRE, we need the functionality axiom schema $\langle e\rangle\phi \to [e]\phi$ whose instances are indeed derivable in $\mathcal{DEN}$ but in a highly non-trivial way.[6]

The soundness of $\mathcal{DEN}$ is straightforward. Note that although all the other reduction axioms are derivable in $\mathcal{DEN}$ without much effort,[7] it is quite involved to derive UNEG in $\mathcal{DEN}$ as we mentioned in footnote 6. Thus we cannot conclude the completeness of $\mathcal{DEN}$ immediately based on the completeness of $\mathcal{DE}$. In this paper, we follow the general ideas in [Wang and Cao, 2013] to prove the completeness of $\mathcal{DEN}$ directly, without referring to $\mathcal{DE}$ nor $\mathcal{EL}$. We revise the proof strategy of [Wang and Cao, 2013] as follows to make it closer to the standard reduction approach:

1. Define an auxiliary semantics $\Vdash$ of **LDEL** on epistemic temporal models with $e$-transitions.[8]
2. Find a class $\mathbb{C}$ of epistemic temporal models such that for any **LDEL** formula $\phi : \mathbb{M} \vDash \phi \implies \mathbb{C} \Vdash \phi$.
3. Show that $\mathcal{DEN}$ completely axiomatizes the valid **LDEL** formulas on $\mathbb{C}$ w.r.t. $\Vdash$.

In sum, we proceed as follows (from left to right):

$$\mathbb{M} \vDash \phi \implies \mathbb{C} \Vdash \phi \implies \vdash_{\mathcal{DEN}} \phi.$$

Compared to the standard reduction technique, here we keep the formulas intact but change the semantics, and reduce the original completeness to the completeness w.r.t. the new semantics on a class of temporal epistemic models. The new completeness question is much easier to solve using standard techniques in modal logic.

Let us start with the new semantics.

**Definition 3 (Extended model).** Given an event model $\mathcal{U}$, an $\mathcal{U}$*-extended model* $\mathcal{N}$ is a tuple:

$$(S, \to, \{\xrightarrow{e} \mid e \in \mathbf{\Sigma}\}, V)$$

where:

- $(S, \to, V)$ is an epistemic model,
- For each $e \in \mathbf{\Sigma}$, $\xrightarrow{e}$ is a (possibly empty) binary relation over $S$.

**Important notation:** We call $(S, \to, \{\xrightarrow{e} \mid e \in \mathbf{\Sigma}\})$ the *extended frame* of $\mathcal{N}$, and $(S, \to, V)$ the *epistemic core* of $\mathcal{N}$ (notation $\mathcal{N}^-$). As usual, we say that $\mathcal{N}$ is *image-finite* if for each $s$ in $\mathcal{N}$, $s$ has finitely many $\to$ successors *and* finitely many $e$-successors for each $e \in \mathbf{\Sigma}$. For a sequence of events $h = e_0 \dots e_k$ we write $t \xrightarrow{h} t'$ for $t \xrightarrow{e_0} \dots \xrightarrow{e_k} t'$. $\lhd$

Since we fix a $\mathcal{U}$, we will just call the above models *extended models*. Note that extended models can be viewed as temporal epistemic models with the extra information of $\mathcal{U}$. The components of $\mathcal{U}$ will be used later when defining the special class $\mathbb{C}$.

We now define an auxiliary semantics $\Vdash$ for the language **LDEL** on extended models (we omit the clauses which are the same as for $\vDash$):

[5] We can rewrite NM into $\Diamond\langle f\rangle\phi \to \bigwedge_{e: e\rightarrowtail f} [e]\Diamond\phi$ if for each $f$, $\{e \mid e \rightarrowtail f\}$ is finite.

[6] By a simultaneous induction on the structure of $\phi$ we can prove the following two claims at the same time: (1) $\vdash_{\mathcal{DEN}} \langle e\rangle\phi \to [e]\phi$ and (2) $[e]\phi$ is $\mathcal{DEN}$-provably equivalent to an **LEL** formula. The case of $\phi = \neg\psi$ for (2) needs the induction hypothesis of (1), and the case of $\phi = [e]\psi$ for (1) requires the IH of (2) and RE which is an admissible rule in $\mathcal{DEN}$ (cf. [Wang and Cao, 2013] for discussion of RE in PAL). We omit the details due to space limitation.

[7] UCON is derivable from NECU and DISTU, UATOM is derivable from PRE, INV, and UK is drivable from PRE, PR, and NM.

[8] Note that, as in [Aucher and Herzig, 2011], the epistemic temporal models here are *not* necessarily tree-like structures as in [van Benthem *et al.*, 2009].

$$\boxed{\mathcal{M}, s \Vdash [e]\phi \quad \Leftrightarrow \quad \forall t : s \xrightarrow{e} t \text{ implies } \mathcal{M}, t \Vdash \phi}$$

By definition, the two semantics coincide on **LEL** formulas:

**Proposition 3.** *For any **LEL** formula $\phi$ and any pointed extended model $\mathcal{M}, s$: $\mathcal{M}, s \Vdash \phi \iff \mathcal{M}^-, s \vDash \phi$.*

However, $\vDash$ and $\Vdash$ differ on extended models for formulas involving $[e]$, unless restricted to special classes of models.[9]

**Definition 4 (Normal extended model).** An extended model

$$\mathcal{N} = (S, \to, \{\xrightarrow{e} \mid e \in \mathbf{\Sigma}\}, V)$$

is *normal* if the following properties hold for any $s, t$ in $\mathcal{N}$:[10].

**Pre** $s$ has $e$-successors iff $\mathcal{N}, s \Vdash \mathit{Pre}(e)$.

**Inv** if $s \xrightarrow{e} t$ then for all $p \in \mathbf{P} : t \in V(p) \iff s \in V(p)$.

**Nm (no miracles)** if $s \to s'$ and $s' \xrightarrow{f} t'$ then for all $e$ and $t$ such that $s \xrightarrow{e} t$ and $e \rightarrowtail f$, we have $t \to t'$.

**Pr (perfect recall)** if $s \xrightarrow{e} t$ and $t \to t'$ then there exists an $s'$ such that $s \to s'$ and $s' \xrightarrow{f} t'$ for some $f$ such that $e \rightarrowtail f$ in $\mathcal{U}$.

Note that **Pre**, **Nm** and **Pr** all refer to components of $\mathcal{U}$. The last two properties are best illustrated by the following diagrams of commutativity:

$$\begin{tikzcd}
s \arrow[r] \arrow[d, "e"'] & s' \arrow[d, "f"] \\
t & t'
\end{tikzcd}
\;\overset{\texttt{NM}}{\Longrightarrow}\;
\begin{tikzcd}
s \arrow[r] \arrow[d, "e"'] & s' \arrow[d, "f"] \\
t \arrow[r] & t'
\end{tikzcd}
\;\overset{\texttt{PR}}{\Longleftarrow}\;
\begin{tikzcd}
s \arrow[d, "e"'] & \\
t \arrow[r] & t'
\end{tikzcd}
\qquad (s \rightarrowtail f)$$

Before going further we need notions of bisimulation over epistemic models and extended models.

**Definition 5 (Bisimulation).** A binary relation $Z$ is called a *bisimulation* between two pointed epistemic models $\mathcal{M}, s$ and $\mathcal{N}, t$, if $sZt$ and whenever $wZv$ the following hold:

**Invariance** for all $p \in \mathbf{P}$, $w \in V_{\mathcal{M}}(p)$ iff $v \in V_{\mathcal{N}}(p)$,

**Zig** if $w \to w'$ for some $w'$ in $\mathcal{M}$ then there is a $v' \in S_{\mathcal{N}}$ with $v \to v'$ and $w'Zv'$,

**Zag** if $v \to v'$ for some $v'$ in $\mathcal{N}$ then there is a $w' \in S_{\mathcal{M}}$ with $w \to w'$ and $w'Zv'$.

A binary relation $Z$ is called a $\mathbf{\Sigma}$-*bisimulation* between two pointed extended models $\mathcal{M}, s$ and $\mathcal{N}, t$, if $sZt$ and whenever $wZv$ the following hold **besides** Invariance, Zig and Zag:

**$\mathbf{\Sigma}$-Zig** if $w \xrightarrow{e} w'$ for some $w'$ in $\mathcal{M}$ then there is a $v' \in S_{\mathcal{N}}$ with $v \xrightarrow{e} v'$ and $w'Zv'$,

**$\mathbf{\Sigma}$-Zag** if $v \xrightarrow{e} v'$ for some $v'$ in $\mathcal{N}$ then there is a $w' \in S_{\mathcal{M}}$ with $w \xrightarrow{e} w'$ and $w'Zv'$.

[9]More precisely, we say $\Vdash$ and $\vDash$ *coincide* on an extended model $\mathcal{N}$ if for any $\phi \in$ **LDEL**, any $s$ in $\mathcal{N}$, $\mathcal{N}, s \Vdash \phi \iff \mathcal{N}^-, s \vDash \phi$.

[10]In [Aucher and Herzig, 2011], properties involving 'backward looking' were discussed on EDL models which are similar to our extended models.

We say $\mathcal{M}, s$ and $\mathcal{N}, t$ are *bisimilar* (resp. $\mathbf{\Sigma}$-*bisimilar*) if there is a bisimulation (resp. $\mathbf{\Sigma}$-bisimulation) between them. We use $\leftrightarrows$ (resp. $\leftrightarrows_{\mathbf{\Sigma}}$) for the bisimilarity relation (resp. $\mathbf{\Sigma}$-bisimilarity). $\lhd$

It is well-known that the **LDEL** formulas are preserved under bisimulation over epistemic models w.r.t. $\vDash$ (cf. e.g., [van Ditmarsch *et al.*, 2007]), and **LDEL** formulas are preserved under $\mathbf{\Sigma}$-bisimulation over extended models w.r.t. $\Vdash$ (cf. e.g., [Blackburn *et al.*, 2002]).

Here comes the crucial lemma saying that in a normal extended model, *product updating* with $(\mathcal{U}, e)$ has the same effect as *moving* along the $e$-transitions.

**Lemma 4.** *If $w \xrightarrow{e} v$ in a normal extended model $\mathcal{M}$, then*

$$\mathcal{M}^- \otimes \mathcal{U}, (w, e) \leftrightarrows \mathcal{M}^-, v.$$

*Proof.* Let $Z$ be the binary relation between $\mathcal{M}^- \otimes \mathcal{U}$ and $\mathcal{M}^-$ such that $(s, f)Zt$ iff $s \xrightarrow{f} t$ in $\mathcal{M}$ for some $f$ in $\mathcal{U}$.

Clearly, $Z$ is non-empty since $w \xrightarrow{e} v$ in $\mathcal{M}$. Now suppose $sZt$ (thus $s \xrightarrow{f} t$ in $\mathcal{M}$ for some $f$), we need to check the three conditions of bisimulation.

The invariance condition is immediate due to the definition of $\otimes$ and **Inv**. For **Zig**, suppose $(s, f) \to (s', f')$ in $\mathcal{M}^- \otimes \mathcal{U}$, then it is clear that $s \to s'$ in $\mathcal{M}^-$, $f \rightarrowtail f'$ in $\mathcal{U}$, and $\mathcal{M}^-, s' \vDash \mathit{Pre}(f')$. According to Proposition 3 and the fact that the preconditions are **LEL** formulas, $\mathcal{M}, s' \Vdash \mathit{Pre}(f')$. Thus from **Pre**, there is a $t'$ such that $s' \xrightarrow{f'} t'$ namely $(s', f')Zt'$. Now $s \xrightarrow{f} t$, $s \to s'$, $s' \xrightarrow{f'} t'$ and $f \rightarrowtail f'$, thus from **Nm** we have $t \to t'$ in $\mathcal{M}$ and thus in $\mathcal{M}^-$. Finally for **Zag**, suppose $t \to t'$ for some $t'$ in $\mathcal{M}^-$. From **Pr**, there is an $s'$ in $\mathcal{M}$ such that $s \to s'$ and $s' \xrightarrow{f'} t'$ for some $f'$ such that $f \rightarrowtail f'$. By **Pre**, $\mathcal{M}, s' \Vdash \mathit{Pre}(f')$. According to Proposition 3 again, $\mathcal{M}^-, s' \vDash \mathit{Pre}(f')$ thus $(s, f) \to (s', f')$ exists in $\mathcal{M}^- \otimes \mathcal{U}$ and $(s', f')Zt'$. $\square$

Now we are ready to prove that the two semantics do coincide on normal extended models. Almost all of our main results rely on this crucial theorem and its variations.

**Theorem 5.** *For any **LDEL** formula $\phi$ and any pointed normal extended model $\mathcal{M}, s$:*

$$\mathcal{M}, s \Vdash \phi \iff \mathcal{M}^-, s \vDash \phi$$

*Proof.* We prove it by induction on the structure of the formulas. The cases for Boolean combinations and $\Box\phi$ are trivial due to Proposition 3 and the induction hypothesis. For the case of $[e]\phi$, we distinguish two cases depending on the truth value of $\mathit{Pre}(e)$. Suppose $\mathcal{M}, s \nVdash \mathit{Pre}(e)$ then by the induction hypothesis (IH) $\mathcal{M}^-, s \nvDash \mathit{Pre}(e)$ thus $\mathcal{M}^-, s \vDash [e]\phi$. Since $\mathcal{M}$ is normal and $\mathcal{M}, s \nVdash \mathit{Pre}(e)$, by **Pre** there is no outgoing $e$-transition from $s$ in $\mathcal{M}$, therefore $\mathcal{M}, s \Vdash [e]\phi$. Now we consider the case when $\mathcal{M}, s \Vdash \mathit{Pre}(e)$. By **Pre**, there must be at least one $e$-successor of $s$ in $\mathcal{M}$. Now take an arbitrary $e$-successor of $s$, call it $t$. From IH and Lemma 4, $\mathcal{M}^- \otimes \mathcal{U}, (s, e) \leftrightarrows \mathcal{M}^-, t$. Since **LDEL** formulas are invariant under bisimulation w.r.t. $\vDash$, for any **LDEL** formula $\phi$: $\mathcal{M}^- \otimes \mathcal{U}, (s, e) \vDash \phi \iff \mathcal{M}^-, t \vDash \phi$. From IH,

$\mathcal{M}^- \otimes \mathcal{U}, (s,e) \models \phi \iff \mathcal{M}, t \Vdash \phi$. According to the fact that the selection of $t$ is arbitrary, it is clear that: $\mathcal{M}, s \Vdash [e]\phi \iff \mathcal{M}^-, s \models [e]\phi$. □

Let $\mathbb{C}_n$ be the class of normal extended models. The following corollary is immediate based on Theorem 5.

**Corollary 1.** *For any $\phi \in$ **LDEL**: $\mathbb{M} \models \phi \implies \mathbb{C}_n \Vdash \phi$.*

Now we only need to show that $\mathcal{DEN}$ completely axiomatize the valid **LDEL** formulas on $\mathbb{C}_n$ w.r.t. $\Vdash$.

**Theorem 6.** *For any $\phi \in$ **LDEL**: $\mathbb{C}_n \Vdash \phi \iff \vdash_{\mathcal{DEN}} \phi$. Namely, $\mathcal{DEN}$ is sound and complete for $\langle$**LDEL**$, \mathbb{C}_n, \Vdash\rangle$.*

*Proof.* We only sketch the proof since it is rather routine. The soundness is straightforward. For the completeness, first note that the proof system induces a normal modal logic[11], due to DISTK, DISTU, NECK, and NECU. The proof starts by building the canonical extended model with both $\to$ and $\xrightarrow{e}$ transitions in the usual way for normal modal logics with $\Box$ and $[e]$ modalities. The truth lemma is immediate, and we just need to show the canonical extended model is indeed in $\mathbb{C}_n$. Note that PR and NM are Sahlqvist formulas if we replace the proposition variable $\phi$ by $p$, and $\mathbf{Pr}$ and $\mathbf{Nm}$ are exactly the frame properties that PR and NM defined. Then by the Sahlqvist completeness theorem, the canonical extended model has $\mathbf{Pr}$ and $\mathbf{Nm}$ (cf. e.g., [Blackburn *et al.*, 2002]). On the other hand it is also easy to show that INV and PRE make sure the canonical model has $\mathbf{Inv}$ and $\mathbf{Pre}$ respectively. □

From Corollary 1 and Theorem 6 it follows:

**Theorem 7.** *$\mathcal{DEN}$ is complete for $\langle$**LDEL**$, \mathbb{M}, \models\rangle$.*

Similarly we can show:

**Theorem 8.** *$\mathcal{DEN}$+T, $\mathcal{DEN}$+T+4 and $\mathcal{DEN}$+T+4+5 are sound and complete w.r.t. the standard semantics of **LDEL** on the class of all **T**, **S4**, **S5** frames.*

## 4 Applications

Recall that $\mathbb{M}$ is the class of all the epistemic models.

**Definition 6.** Given an event model $\mathcal{U}$, a *model transformer* $\circledast$ w.r.t. $\mathcal{U}$ is a function: $\mathbb{M} \to \mathbb{M}$ such that $S_{\circledast(\mathcal{M})} \subseteq S_{\mathcal{M}} \times \mathbf{\Sigma}$. Following the convention, we write $\mathcal{M} \circledast \mathcal{U}$ for $\circledast(\mathcal{M})$ and write $\mathcal{M} \circledast \mathcal{U}^i$ for $\mathcal{M} \circledast \underbrace{\mathcal{U} \ldots \circledast \mathcal{U}}_{i}$, with $\mathcal{M} \circledast \mathcal{U}^0 = \mathcal{M}$.

A model transformer $\circledast$ is a (partial) *tensor product* if for all $(s,e),(t,f) \in S_{\mathcal{M}\circledast\mathcal{U}}$:

- $(s,e) \to_{\mathcal{M}\circledast\mathcal{U}} (t,f)$ iff $s \to t$ in $\mathcal{M}$ and $e \rightarrowtail f$ in $\mathcal{U}$.

As before, we fix $\mathcal{U}$ thus when we mention a model transformer we mean a model transformer w.r.t. $\mathcal{U}$.

Following [van Benthem *et al.*, 2009], we define **LDEL**-protocols:

**Definition 7.** Let $\mathbf{\Sigma}^*$ be the set of finite (possibly empty) sequences of the events in $\mathbf{\Sigma}$. A (state-dependent) protocol w.r.t. $\mathcal{M}$ is a function $\rho : S_{\mathcal{M}} \to 2^{\mathbf{\Sigma}^*}$ such that for any $s \in S_{\mathcal{M}} : \rho(s) \subseteq \mathbf{\Sigma}^*$ is closed under prefixes. Thus the empty string $\epsilon$ is in $\rho(s)$ for any protocol $\rho$. We use $\rho_u$ to denote the *universal protocol*, i.e., for all $s \in S_{\mathcal{M}} : \rho_u(s) = \mathbf{\Sigma}^*$.

Now we generalize the DEL-forest defined in [van Benthem *et al.*, 2009] to *arbitrary* model transformer $\circledast$:

**Definition 8 ($\circledast$-forest).** Given an epistemic model $\mathcal{M}$ and a protocol $\rho$, let

$$\mathcal{F}^{\circledast}(\mathcal{M}, \rho) = (S, \to, \{\xrightarrow{e} \mid e \in \mathbf{\Sigma}\}, V)$$

be the extended model generated by executing $\rho$ where:

- $S = \{(s, e_1, \ldots, e_k) \mid 0 \le k$ and $(s, e_1, \ldots, e_k)$ exists in $\mathcal{M} \circledast \mathcal{U}^k$ and $e_1 \ldots e_k \in \rho(s)\}$.
- $(s, e_1, \ldots, e_k) \to (t, e'_1, \ldots, e'_j)$ iff $k = j$, and $(s, e_1, \ldots, e_k) \to (t, e'_1, \ldots, e'_k)$ in $\mathcal{M} \circledast \mathcal{U}^k$.
- $(s, e_1, \ldots, e_k) \xrightarrow{e} (t, e'_1, \ldots, e'_j)$ iff $s = t$ and $e'_1 \ldots e'_j = e_1 \ldots e_k e$
- $V((s, e_1, \ldots, e_k)) = V_{\mathcal{M}\circledast\mathcal{U}^k}(s, e_1, \ldots, e_k)$

**Important notation:** We often omit the universal protocol and write $\mathcal{F}^{\circledast}(\mathcal{M})$ for $\mathcal{F}^{\circledast}(\mathcal{M}, \rho_u)$. ◁

Intuitively, $\mathcal{F}^{\circledast}(\mathcal{M}, \rho)$ is a (synchronous) *update universe* consisting of the updated models in the form of $\mathcal{M} \circledast \mathcal{U}^k$ linked by update transitions $\xrightarrow{e}$ under the constraint of $\rho$.

In the sequel, we write $\mathcal{M}, s \equiv_{\mathbf{LDEL}} \mathcal{N}, t$ if $\mathcal{M}, s$ and $\mathcal{N}, t$ satisfy the same set of **LDEL** formulas.[12]

Clearly, the update product $\otimes$ is a model transformer. It is not hard to show the following by the definition of $\otimes$:

**Proposition 9.** *For any pointed epistemic model $\mathcal{M}, s$, $\mathcal{F}^{\otimes}(\mathcal{M})$ is normal and $\mathcal{M}, s \equiv_{\mathbf{LDEL}} \mathcal{F}^{\otimes}(\mathcal{M}), s$.*

*Proof.* The normality is straightforward based on the definition of $\mathcal{F}^{\otimes}(\mathcal{M})$. Since $\mathcal{M}, s$ is actually the "ground floor" of $(\mathcal{F}^{\otimes}(\mathcal{M}))^-$, which is isolated from the other part of $(\mathcal{F}^{\otimes}(\mathcal{M}))^-$, we have $\mathcal{M}, s \equiv_{\mathbf{LDEL}} (\mathcal{F}^{\otimes}(\mathcal{M}))^-, s$. Since $\mathcal{F}^{\otimes}(\mathcal{M})$ is normal, due to Theorem 5 we have $(\mathcal{F}^{\otimes}(\mathcal{M}))^- \equiv_{\mathbf{LDEL}} \mathcal{F}^{\otimes}(\mathcal{M}), s$. Thus $\mathcal{M}, s \equiv_{\mathbf{LDEL}} \mathcal{F}^{\otimes}(\mathcal{M}), s$. □

From the above proposition, for each pointed epistemic model $\mathcal{M}, s$ we have an **LDEL**-equivalent normal extended model $\mathcal{F}^{\otimes}(\mathcal{M})$. Corollary 1 can then be strengthened:

**Proposition 10.** *For any $\phi \in$ **LDEL**: $\mathbb{M} \models \phi \iff \mathbb{C}_n \Vdash \phi$.*

### 4.1 Characterization theorems under universal protocol

Let $\mathbb{M}^{\circledast}$ be the class of all the $\mathcal{F}^{\circledast}(\mathcal{M})$ based on some $\mathcal{M} \in \mathbb{M}$ under the universal protocol. We say $\mathbb{M}^{\circledast}$ *frame-validates* an axiom schema $\phi$ if all the frames underlying the models in $\mathbb{M}^{\circledast}$ validates all the instances of $\phi$ w.r.t. $\Vdash$. We say $\mathbb{M}^{\circledast}$ *model-validates* an axiom $\phi$ if all the models in $\mathbb{M}^{\circledast}$ validate all the instances of $\phi$ w.r.t. $\Vdash$.

**Theorem 11.** *(1) $\mathbb{M}^{\circledast}$ frame-validates* NM *and* PR *iff $\circledast$ is a tensor product. (2) $\mathbb{M}^{\circledast}$ frame-validates* NM *and* PR*, and model-validates* INV *and* PRE *iff $\circledast$ is the update product.*

[11] Except that $\mathcal{DEN}$ is not closed under uniform substitution (US): the application of US to INV does not preserve validity.

[12] Note that $\mathcal{M}, s$ and $\mathcal{N}, t$ may be of different types, e.g. one extended model and one epistemic model. In such cases we evaluate **LDEL** formulas according to $\Vdash$ and $\models$ respectively.

*Proof.* (Sketch) We only prove (1): It is a standard exercise in modal logic to verify that any extended frame validating NM (resp. PR) w.r.t. $\Vdash$ iff it has the $\mathbf{Nm}$ (resp. $\mathbf{Pr}$) property. Now, to complete the proof, we break the requirement of a tensor product into two parts:
I: $(s,e) \to_{\mathcal{M}\circledast\mathcal{U}} (t,f)$ implies $s \to t$ in $\mathcal{M}$ and $e \rightarrowtail f$ in $\mathcal{U}$.
II: If $(s,e)$ and $(t,f)$ exist in $\mathcal{M}\circledast\mathcal{U}$, $s \to t$ in $\mathcal{M}$, and $e \rightarrowtail f$ in $\mathcal{U}$, then $(s,e) \to_{\mathcal{M}\circledast\mathcal{U}} (t,f)$.

We just need to show $\mathbf{Pr}$ corresponds to (I) and $\mathbf{Nm}$ corresponds to (II) which is quite straightforward. □

In the following, we prove the characterization results of extended models which are generatable by DEL updates.

**Definition 9 (Generatable model *modulo* $\mathbf{\Sigma}$*-bisimulation*).** Given a model transformer $\circledast$, a pointed extended model $\mathcal{N}, t$ is $\circledast$*-generatable* if there is a pointed epistemic model $\mathcal{M}, s$ such that $\mathcal{N}, t \leftrightarrows_{\mathbf{\Sigma}} \mathcal{F}^{\circledast}(\mathcal{M}), s$. A pointed extended model $\mathcal{N}, t$ is $\circledast$*-prot-generatable* if there is an epistemic model $\mathcal{M}, s$ and a protocol $\rho$ such that $\mathcal{N}, t \leftrightarrows_{\mathbf{\Sigma}} \mathcal{F}^{\circledast}(\mathcal{M}, \rho), s$.

Given a *pointed* extended model $\mathcal{N}, t$, its $\mathbf{\Sigma}$-bisimulation contraction (notation $\mathcal{N}^t_C$) is the $|t|$-generated submodel of the $\mathbf{\Sigma}$-bisimulation contraction of $\mathcal{N}$, where $|t|$ is the equivalence class of $t$ w.r.t. $\leftrightarrows_{\mathbf{\Sigma}}$. We can show:

**Proposition 12.** *For any pointed extended model $\mathcal{N}, t$: $\mathcal{N}, t$ is $\mathbf{\Sigma}$-bisimilar to a pointed normal model iff $\mathcal{N}^t_C$ is normal.*

*Proof.* $\Leftarrow$ is trivial, we only show $\Longrightarrow$ : Suppose $\mathcal{N}, t \leftrightarrows_{\mathbf{\Sigma}} \mathcal{N}', t'$ and $\mathcal{N}'$ is normal. Clearly $\mathcal{N}^t_C, |t| \leftrightarrows_{\mathbf{\Sigma}} \mathcal{N}', t'$. $\mathbf{Inv}$ and $\mathbf{Pre}$ can be verified easily. For $\mathbf{Pr}$: suppose $|w| \xrightarrow{e} |u| \to |v|$, then since $\mathcal{N}^t_C, |t| \leftrightarrows_{\mathbf{\Sigma}} \mathcal{N}', t'$ and the fact that all the non-$|t|$ nodes are connected with $|t|$, there are $w', u'$ and $v'$ in $\mathcal{N}'$ such that $w' \xrightarrow{e} u' \to v'$ and $\mathcal{N}^t_C, |w| \leftrightarrows_{\mathbf{\Sigma}}, w'$, $\mathcal{N}^t_C, |u| \leftrightarrows_{\mathbf{\Sigma}}, u'$ and $\mathcal{N}^t_C, |v| \leftrightarrows_{\mathbf{\Sigma}}, v'$. Since $\mathcal{N}'$ is normal there is a $r'$ in $\mathcal{N}'$ such that $w' \to r' \xrightarrow{f} v'$ for some $f$ with $e \rightarrowtail f$ in $\mathcal{U}$. Since $\mathcal{N}^t_C, |w| \leftrightarrows_{\mathbf{\Sigma}}, w'$ then there is a $|r|$ in $\mathcal{N}^t_C$ such that $|w| \to |r|$ and $\mathcal{N}^t_C, |r| \leftrightarrows_{\mathbf{\Sigma}} \mathcal{N}', r'$. Since $r' \xrightarrow{f} v'$, there is an $|o|$ in $\mathcal{N}^t_C$ such that $|r'| \xrightarrow{f} |o|$ and $\mathcal{N}^t_C, |o| \leftrightarrows_{\mathbf{\Sigma}} \mathcal{N}', v'$. Since $\mathcal{N}^t_C, |v| \leftrightarrows_{\mathbf{\Sigma}}, v'$, we have $\mathcal{N}^t_C, |v| \leftrightarrows_{\mathbf{\Sigma}} \mathcal{N}^t_C, |o|$. Now since $\mathcal{N}^t_C$ is $\mathbf{\Sigma}$-bisimulation contracted $|o| = |v|$. Therefore there is a $|r|$ in $\mathcal{N}^t_C$ such that $|w| \to |r| \xrightarrow{f} |v|$. $\mathbf{Nm}$ can be proved similarly using the properties of the contraction. □

**Theorem 13.** *For any image-finite pointed extended model $\mathcal{N}, t$: $\mathcal{N}, t$ is $\otimes$-generatable iff $\mathcal{N}^t_C$ is normal.*

*Proof.* Suppose $\mathcal{N}, t$ is $\otimes$-generatable, then $\mathcal{N}, t \leftrightarrows \mathcal{F}^{\otimes}(\mathcal{M}), s$ for some model $\mathcal{M}, s$. From Proposition 9, $\mathcal{F}^{\otimes}(\mathcal{M})$ is normal. Now from Proposition 12, $\mathcal{N}^t_C$ is normal.

For the other direction, suppose the $\mathcal{N}^t_C$ is normal. We first show the following:

$$\mathcal{N}, t \equiv_{\mathbf{LDEL}} \mathcal{N}^t_C, |t| \equiv_{\mathbf{LDEL}} (\mathcal{N}^t_C)^-, t \equiv_{\mathbf{LDEL}} \mathcal{F}^{\otimes}((\mathcal{N}^t_C)^-), |t|$$

The first equivalence is due to the bisimilarity between $\mathcal{N}, t$ and $\mathcal{N}^t_C$ and the fact that **LDEL** is preserved under $\mathbf{\Sigma}$-bisimulation w.r.t. $\Vdash$ (cf. e.g., [Blackburn *et al.*, 2002]). The second equivalence is due to the normality of $\mathcal{N}^t_C$ and Theorem 5. The third equivalence is due to Proposition 9. Now let $\mathcal{N}' = (\mathcal{N}^t_C)^-$. Since $\mathcal{N}, t$ and $\mathcal{U}$ are image-finite, it is not hard to show by induction on the length of the sequence $h$ that for each $(h,e)$ in $\mathcal{F}^{\otimes}(\mathcal{N}')$ there are only finitely many $(h',e')$ such that $(h,e) \to (h',f)$. Moreover, each $s$ in $\mathcal{F}^{\otimes}(\mathcal{N}')$ has at most one $f$-successor for each $f \in \mathbf{\Sigma}$. Therefore $\mathcal{F}^{\otimes}(\mathcal{N}')$ is also image-finite. Now by the well-known Hennessy-Milner theorem (cf. e.g., [Blackburn *et al.*, 2002]), $\mathcal{N}, t \leftrightarrows_{\mathbf{\Sigma}} \mathcal{F}^{\otimes}(\mathcal{N}'), |t|$. □

Compared to the representation results in [van Benthem *et al.*, 2009] and [van Benthem and Liu, 2004], there are two major differences in our setting: 1. Our notion of generatable extended models is modulo $\mathbf{\Sigma}$-bisimulation, thus allowing models which are not in the tree-like shape; 2. More importantly, we fix an event model $\mathcal{U}$ and consider the generatable models w.r.t. $\mathcal{U}$.[13] In particular, $\mathbf{Nm}$ and $\mathbf{Pr}$ also refer to the structure of $\mathcal{U}$. Due to such differences, our result is not a special case of Theorem 2 in [van Benthem *et al.*, 2009].

## 4.2 DEL with protocols

In [Hoshi and Yap, 2009], the authors discussed the axiomatization of **LDEL** on generatable history-based epistemic temporal models, i.e., $\langle \mathbf{LDEL}, \mathbb{M}^{\otimes}_{prot}, \Vdash \rangle$ where $\mathbb{M}^{\otimes}_{prot}$ is the class of all the $\mathcal{F}^{\otimes}(\mathcal{M}, \rho)$. This looks more like an epistemic temporal logic rather than a dynamic epistemic logic. In this paper, inspired by [van Ditmarsch *et al.*, 2011], we give a new 'dynamic' semantics of **LDEL** directly on epistemic models with protocols $(\mathcal{M}, \rho)$, by using the transformer $\odot$ defined below (the clauses that coincide with the standard semantics $\models$ are omitted):

$$\boxed{\begin{array}{rcl} \mathcal{M}, \rho, s \Vdash [e]\phi & \Leftrightarrow & \mathcal{M}, \rho, s \Vdash \mathit{Pre}(e) \textbf{ and } e \in \rho(s) \\ & & \text{implies } (\mathcal{M}, \rho) \odot \mathcal{U}, (s,e) \models \phi \end{array}}$$

where $\mathcal{M}, \rho \odot \mathcal{U} = (S', \to', V', \rho')$ with:

$$\begin{array}{rcl} S' & = & \{(s,e) \mid \mathcal{M}, s \Vdash \mathit{Pre}(e) \textbf{ and } e \in \rho(s)\} \\ \to' & = & \{((s,e),(s',e')) \mid s \to s' \text{ and } e \rightarrowtail e'\} \\ V'((s,e)) & = & V(s) \\ \rho'((s,e)) & = & \{h \mid eh \in \rho(s)\} \end{array}$$

Note that the protocol $\rho$ constrains the domain of the updated model. Moreover, after executing an event $e$, the current protocol has to be updated (the reader can verify that $\rho'$ is indeed a well-defined protocol). Let $\mathbb{PM}$ be the class of all the epistemic models with protocols then the above logic can be denoted as $\langle \mathbf{LDEL}, \mathbb{PM}, \Vdash \rangle$.

A moment of reflection should confirm the following observation where $(\mathcal{M}, \rho)$ at the right-hand-side below is taken as a single model:[14]

**Proposition 14.** *$\mathcal{F}^{\otimes}(\mathcal{M}, \rho)$ is exactly $\mathcal{F}^{\odot}((\mathcal{M}, \rho))$.*

We also need a revised notion of bisimulation between epistemic models with protocols:

**Definition 10 (Prot-bisimulation).** A binary relation $Z$ is called a *prot-bisimulation* between two pointed epistemic

[13] [van Benthem and Liu, 2004] shows a result in the shape of: an epistemic temporal model is generatable by **some** event model iff it satisfies certain properties. Here we replace *some* by a fixed $\mathcal{U}$.

[14] Here we also need to relax the definition of model transformer to allow epistemic models with protocols.

models with protocols $\mathcal{M}, \rho, s$ and $\mathcal{N}, \gamma, t$, if $sZt$ and whenever $wZv$ the following holds **besides** Invariance, Zig, Zag:

**Prot-invariance** $\rho(w) = \gamma(v)$.

We use $\leftrightarrows_{prot}$ to denote the prot-bisimilarity. $\lhd$

It is rather routine to show that prot-bisimulation preserves truth of **LDEL** formulas under $\Vdash$:

**Proposition 15.** *If $\mathcal{M}, \rho, s \leftrightarrows_{prot} \mathcal{N}, \gamma, t$ then for all **LDEL** formula $\phi$ : $\mathcal{M}, \rho, s \equiv_{\textbf{LDEL}} \mathcal{N}, \gamma, t$.*

**Important notation:** Given an extended model $\mathcal{N}$ and a state $t$ in it, let $path(t)$ be $\{h \mid t \xrightarrow{h} t' \text{ for some } t'\}$. The *characteristic protocol* of $\mathcal{N}$ (notation $\rho_{\mathcal{N}}$) is a protocol for $\mathcal{N}^-$ such that $\rho_{\mathcal{N}}(s) = path(s)$ for each $s \in S_{\mathcal{N}} = S_{\mathcal{N}^-}$.

**Definition 11.** An extended model is *semi-normal* if it satisfies $\mathbf{Inv}$, $\mathbf{Nm}$, $\mathbf{Pr}$ and the following two for all the $e, s$:

$\mathbf{PPre}$ $s$ has $e$-successors implies $\mathcal{M}, s \Vdash \mathit{Pre}(e)$;

$\mathbf{Det}$ if $s$ has two $e$-successors $t, t'$ then $path(t) = path(t')$.

$\mathbf{PPre}$ is clearly weaker than $\mathbf{Pre}$ due to the protocol constraints. $\mathbf{Det}$ says the protocol is not really branching. To capture these two properties, we propose the following two axiom schemata (where $e \in \mathbf{\Sigma}$ and $h \in \mathbf{\Sigma}^*$):

$$\texttt{PPRE} : \langle e \rangle \top \to \mathit{Pre}(e) \qquad \texttt{DET} : \langle e \rangle \langle h \rangle \top \to [e] \langle h \rangle \top$$

Let $\mathcal{PDEN}$ be $\mathcal{DEN}$-PRE+PPRE+DET and $\mathbb{C}_{sn}$ be the class of all the semi-normal extended models. Note that DET is a special case of $\langle e \rangle \phi \to [e]\phi$, and PPRE is clearly weaker than PRE, thus it can be shown that $\mathcal{PDEN}$ is strictly weaker than $\mathcal{DEN}$(similarly, semi-normality is weaker than normality). We will show that $\mathcal{PDEN}$ completely axiomatizes **LDEL** on models in $\mathbb{PM}$. Again we follow the completeness proof strategy as before:

$$\mathbb{PM} \Vdash\!\!\!\Vdash \phi \implies \mathbb{C}_{sn} \Vdash \phi \implies \vdash_{\mathcal{PDEN}} \phi.$$

It is not hard to show that:

**Theorem 16.** *For any $\phi \in$ **LDEL**:$\mathbb{C}_{sn} \Vdash \phi \iff \vdash_{\mathcal{PDEN}} \phi$.*

To complete the proof we only need to show $\mathbb{PM} \Vdash\!\!\!\Vdash \phi \implies \mathbb{C}_{sn} \Vdash \phi$. Following the same strategy, we need the analogue of Lemma 4:

**Lemma 17.** *For any semi-normal extended model $\mathcal{M}$, if $w \xrightarrow{e} v$ in $\mathcal{M}$, we have :*

$$(\mathcal{M}^-, \rho_{\mathcal{M}}) \odot \mathcal{U}, (w, e) \leftrightarrows_{prot} \mathcal{M}^-, \rho_{\mathcal{M}}, v.$$

*Proof.* (Sketch) Most of the proof is identical to the proof to Lemma 4. There are two changes: 1. We need to verify the extra Prot-invariance; 2. We cannot use $\mathbf{Pre}$ but only $\mathbf{PPre}$. Prot-invariance can be guaranteed by $\mathbf{Det}$ and we do not need the full $\mathbf{Pre}$ under the presence of the protocol information. We omit the detailed proof due to the space limit. $\square$

Based on the above lemma, we can prove analogues of Theorem 5 and Proposition 9 by almost identical proofs:

**Theorem 18.** *For any **LDEL** formula $\phi$ and any semi-normal extended model $\mathcal{M}$:*

$$\mathcal{M}, s \Vdash \phi \iff \mathcal{M}^-, \rho_{\mathcal{M}}, s \Vdash\!\!\!\Vdash \phi$$

**Proposition 19.** *For any epistemic model $\mathcal{M}$ and any protocol $\rho$: $\mathcal{F}^{\otimes}(\mathcal{M}, \rho)$ is semi-normal and thus $\mathcal{F}^{\otimes}(\mathcal{M}, \rho), s \equiv_{\textbf{LDEL}} \mathcal{M}, \rho, s$.*

We can then prove the following proposition, which is the (strengthened) last piece for the completeness proof.

**Proposition 20.** *For any $\phi \in$ **LDEL**: $\mathbb{PM} \Vdash\!\!\!\Vdash \phi \iff \mathbb{C}_{sn} \Vdash \phi \iff \mathbb{M}^{\otimes}_{prot} \Vdash \phi$.*

**Theorem 21.** *$\mathcal{PDEN}$ is sound and complete for $\langle$**LDEL**$, \mathbb{PM} \Vdash\!\!\!\Vdash \rangle$ and $\langle$**LDEL**$, \mathbb{M}^{\otimes}_{prot} \Vdash \rangle$.*

This result also shows that our logic $\langle \mathbf{LDEL}, \mathbb{PM} \Vdash\!\!\!\Vdash \rangle$ is essentially the same as the logic proposed in [Hoshi and Yap, 2009] which can be viewed as $\langle \mathbf{LDEL}, \mathbb{M}^{\otimes}_{prot} \Vdash \rangle$.

Using Propositions 19 and 18 we can show for any pointed extended model $\mathcal{N}, t$:

$\mathcal{N}, t \equiv_{\textbf{LDEL}} \mathcal{N}^t_C, |t| \equiv_{\textbf{LDEL}} \mathcal{N}', \gamma, |t| \equiv_{\textbf{LDEL}} \mathcal{F}^{\otimes}(\mathcal{N}', \gamma), |t|$ where $\gamma = \rho_{\mathcal{N}^t_C}$ and $\mathcal{N}' = (\mathcal{N}^t_C)^-$. Based on this, an analogue of Theorem 13 follows due to an almost identical proof:

**Theorem 22.** *An image-finite pointed extended model $\mathcal{N}, t$ is $\otimes$-prot-generatable iff $\mathcal{N}^t_C$ is semi-normal.*

## 5 Conclusion

Let us sum up the results once again for clarity since we have been talking many different semantics:

- $\mathcal{DEN}$ completely axiomatizes $\langle \mathbf{LDEL}, \mathbb{C}_n, \Vdash \rangle$ and $\langle \mathbf{LDEL}, \mathbb{M}, \vDash \rangle$ (Theorems 6 and 7 respectively).
- $\mathcal{PDEN}$ completely axiomatizes $\langle \mathbf{LDEL}, \mathbb{C}_{sn}, \Vdash \rangle$, $\langle \mathbf{LDEL}, \mathbb{PM}, \Vdash\!\!\!\Vdash \rangle$, and $\langle \mathbf{LDEL}, \mathbb{M}^{\otimes}_{prot}, \Vdash \rangle$ (Theorems 16 and 21).
- PRE, INV, NM, and PR characterize update product while the first two axioms characterize (partial) tensor product (Theorem 11).
- $\mathbf{Pre}$, $\mathbf{Inv}$, $\mathbf{Nm}$ and $\mathbf{Pr}$ characterize the $\otimes$-generatable extended models under universal protocols (Theorem 13), while $\mathbf{PPre}$, $\mathbf{Det}$, $\mathbf{Inv}$, $\mathbf{Nm}$ and $\mathbf{Pr}$ characterizes the $\otimes$-generatable extended models under arbitrary protocols (Theorem 22).

In our view, the essential proof strategy for the completeness of a DEL-like logic (w.r.t. some proof system), is to reduce the completeness of the original logic to the completeness of epistemic temporal logic on certain class of epistemic temporal models. The latter problem is usually much easier to prove, compared to the original one. Therefore the most crucial step is to make the transition from the validity w.r.t. dynamic epistemic semantics to the validity w.r.t. epistemic temporal semantics on certain class of models (cf. Propositions 10 and 20). To do so, we can 'flatten' the dynamics in certain epistemic temporal structures (cf. Lemmata 4 and 17).

We believe that our method can be applied to other dynamic epistemic logics. In particular, the discussion on DEL with protocols shows the possibility to handle logics which cannot be reduced to epistemic logic. The same discussion also demonstrates that we can dynamify the logic satisfying certain epistemic temporal properties (cf. Theorem 21). For future work, we would like to look at the dynamic epistemic logics with iteration operators (e.g., common knowledge and Kleene star of the updates).

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