HAL
open science

On the standing waves solutions to a nonlocal, nonlinear Schrödinger equation occuring in plasma Physics

Thierry Colin

To cite this version:

Thierry Colin. On the standing waves solutions to a nonlocal, nonlinear Schrödinger equation occuring in plasma Physics. Physica D: Nonlinear Phenomena, 1993, 64, pp.215-236. 10.1016/0167-2789(93)90256-Z . hal-00848085

HAL Id: hal-00848085

https://inria.hal.science/hal-00848085v1

Submitted on 25 Jul 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the standing waves solutions to a nonlocal, nonlinear Schrödinger equation occuring in plasma Physics

T. Colin*
Centre de Mathématiques et leurs Applications,
ENS Cachan et CNRS,
61 avenue du Président Wilson,
94235 Cachan Cedex, France.

Abstract

In this work, our goal is to study the standing waves solutions to some generalization of the following system.

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^2\nabla\psi), \\ \Delta\psi = \phi. \end{cases}$$

We shall prove the existence of standing waves, solution to this equations and provide some stability results.

*This work was partially supported by a grant from the Ministère de la Recherche et de la Technologie, France, Contrat no 90S0315.

Contents

1 Introduction and statement of the results. 2
1.1 Introduction. 2
1.2 Statement of the results. 6

2 Standing waves solutions 8
2.1 Concentration-compactness lemma 9
2.2 Application of the concentration-compactness lemma. 13
2.3 Radial standing waves. 18

3 Stability of standing waves 22
3.1 Proof of the propositions. 23
3.1.1 First proposition. 23
3.1.2 Second proposition. 23
3.2 Equivalent problems 24
3.3 Stability for $\sigma < 4/3$. 28
3.4 Instability for $\sigma > 4/3$. 28

A Proof of proposition 2. 33

1 Introduction and statement of the results.

1.1 Introduction.

In a previous work ([4]), we study the Cauchy problem for the following familly of system:

$$\begin{cases} i\phi_t + \sum_{k,l=1}^{3} a_{kl} \frac{\partial^2 \phi}{\partial x_k \partial x_l} = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \\ \phi(x,0) = \phi_0(x), \end{cases} \tag{1}$$

where $a_{kl} = a_{lk}$ are real constants, the matrix (a_{kl}) being nonsingular and $\sigma > 0$.
These systems are generalizations of the following ones which occures in Plasma Physics (see [6]).

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^2\nabla\psi), \\ \Delta\psi = \phi. \end{cases} \tag{2}$$

If ϕ is a solution of (1), one can show that the following quantities are invariants of the motion:

$$m(t) = \int_{R^3} |\nabla\psi(t)|^2 dx = m(0), \tag{3}$$

$$E(t) = \int_{R^3} (\frac{1}{2}\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2})dx = E(0), \tag{4}$$

where q is the following hermitian form:

$$q(\mathbf{u}) = \sum_{k,l=1}^{3} a_{kl}u_k\bar{u}_l.$$

Indeed, multiplying the first equation of (1) by $\bar{\psi}$ leads, after integration, to:

$$-i\int \nabla\psi_t.\nabla\bar{\psi} + \int a_{kl}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}\frac{\partial^2\psi}{\partial x_i\partial x_i} = \int |\nabla\psi|^{\sigma+2}. \tag{5}$$

But $\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}$ is real, indeed an integration by parts gives:

$$\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l} = \int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_l}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_i}.$$

Hence, taking the imaginary part of (5) leads to:

$$Re\int \nabla\psi_t.\nabla\bar{\psi} = 0$$

and we obtain (3). On the other hand, multiplying the first equation of (1) by $\bar{\psi}_t$ and using the same method, we arrive at (with the summation convention):

$$-i\int \nabla\psi_t.\nabla\bar{\psi}_t + \int a_{kl}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l}\frac{\partial^2\psi}{\partial x_i\partial x_i} = \int |\nabla\psi|^{\sigma}\nabla\psi.\nabla\bar{\psi}_t. \tag{6}$$

But:

$$\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l}=\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_l}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_i},$$

therefore,

$$Re(\int a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}_t}{\partial x_k\partial x_l})=\frac{1}{2}\int\frac{d}{dt}a_{kl}\frac{\partial^2\psi}{\partial x_i\partial x_i}\frac{\partial^2\bar{\psi}}{\partial x_k\partial x_l}$$

$$=\frac{1}{2}\frac{d}{dt}\int\sum_i q(\nabla\frac{\partial\psi}{\partial x_i}).$$

Since

$$Re(\int|\nabla\psi|^\sigma\nabla\psi.\nabla\bar{\psi}_t)=(\sigma+2)\frac{d}{dt}(\int|\nabla\psi|^{\sigma+2}),$$

we now obtain (4) using (6).

Let us recall the results that we have obtained concerning the Cauchy problem in [4].

We introduce

$$H=\{\psi\in L^6\cap C_0(R^3),\nabla\psi\in H^1\},$$

endowed with the norm $||\nabla\psi||_{H^1}$.

Theorem 1. *Let $0<\sigma<4$.*

$*$*Let $\psi_0\in H$, then there exists an unique maximal solution on $[0,T(\psi_0)[$ $\psi\in C([0,T(\psi_0)[,H)$ to:*

$$\begin{cases} i(\nabla\psi)_t+L(\nabla\psi)=\nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi)\\ \psi(x,0)=\psi_0(x)\end{cases}$$

$*$*Moreover, $\phi=\Delta\psi\in L^r(0,t,L^{\sigma+2})$ with $\frac{2}{r}+\frac{3}{\sigma+2}=\frac{3}{2}$ for all $t<T$.*

$*$*The function ψ is a solution to:*

$$\begin{cases} i\phi_t+L\phi=-div(|\nabla\psi|^\sigma\nabla\psi)\\ \nabla\psi=\phi\\ \phi(x,0)=\phi_0(x)\end{cases}$$

and if $0<\sigma<3$, it is the only solution in $C([0,T[,H)$.

$*$*The solution ψ depends continuously on ψ_0 in $C([0,T[,H)$ in the following sense: if $\psi_0^n\to\psi_0$ in H then for all $T<T(\psi_0)$, if n is sufficiently large, the corresponding solutions exist on a common interval $[0,T]$ and $\psi_n\to\psi$ in $C([0,T],H)$.*

We have a result of regularity:

Theorem 2. *If $\psi_0 \in H$ with $\nabla\psi_0 \in H^2$, then the solution given by Theorem 1 satisfies*

$$\nabla\psi \in C([0, T(\psi_0)[, H^2).$$

In some cases, we are able to prove that the solution exists globally for all $t > 0$.

Theorem 3. *a)Let $\psi_0 \in H$, then the solution of (1) satisfies:*

$$m(t) = \int_{R^3} |\nabla\psi(t)|^2 dx = m(0),$$

$$E(t) = \int_{R^3} (\frac{1}{2}\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2})dx = E(0),$$

where q is the following hermitian form:

$$q(u) = \sum_{k,l=1}^{3} a_{kl} u_k \bar{u}_l.$$

b)If the matrix (a_{kl}) is negative, then $||\psi||_H$ remains bounded and the solution is global in time.

c)If the matrix (a_{kl}) is positive, then if $\sigma < \frac{4}{3}$, $||\psi||_H$ remains bounded and the solution is global in time. If $\sigma \geq \frac{4}{3}$, then if $||\psi_0||_H$ is sufficiently small, $||\psi||_H$ remains bounded and the solution is global in time.

d)If ψ is a solution of (2), if $\nabla\psi_0 \in H^{m+1} \cap W^{m+1,\frac{6}{5}}$ for $m \geq 4$, then there exists $\delta > 0$ such that, if

$$||\nabla\psi_0||_{H^{m+1}} + ||\nabla\psi_0||_{W^{m+1,\frac{6}{5}}} < \delta,$$

then the solution is global in time and there exists $\bar{\phi}$ satisfying

$$\begin{cases} i\bar{\phi}_t + \Delta\bar{\phi} = 0 \\ \Delta\bar{\psi} = \bar{\phi}, \end{cases}$$

such that:

$$\begin{cases} ||\nabla\psi||_{W^{m-2,6}} \leq \frac{C}{1+t} \\ ||\nabla\psi||_{H^{m+1}} \leq C \\ ||\nabla\psi - \nabla\bar{\psi}||_{H^{m+1}} \leq \frac{C}{1+t}. \end{cases}$$

We have the following finite-time blow-up result:

Theorem 4. *Let $\psi_0 \in H$ be such that $\int |x|^2 |\nabla\psi|^2 < \infty$, then the solution of (1) with $L = \Delta$ satisfies*

$$|x|\ |\nabla\psi| \in L^\infty(0,t,L^2) \cap L^r(0,t,L^{\sigma+2}),$$

for all $t < T(\psi_0)$.

There exists some radial initial values such that the corresponding solutions blow up in finite time.

1.2 Statement of the results.

The main results of this work are:

Theorem 5. *Let us suppose that the matrix a_{ij} is positive, and let $0 < \sigma < 4$. For every $\omega > 0$, there exists a function $\psi_\omega \in H$ satisfying:*

$$\begin{cases} -\omega\phi_\omega + L\phi_\omega = -div(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega) \\ \Delta\psi_\omega = \phi_\omega \end{cases}$$

i.e. $e^{i\omega t}\psi_\omega(x)$ satisfies (1).

Moreover, ψ_ω is a solution to the following minimization problem:

$$(P1)\ inf\{-\int |\nabla\psi|^{\sigma+2}, \psi \in H, \int |\nabla\psi|^2 = \lambda, \int \sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) = \mu\}$$

for convenient λ, μ and $\nabla\psi \in \mathcal{C}^2(R^3)$.

Moreover

$$\int |\nabla\psi_\omega|^{\sigma+2}, \int \sum_{i=1}^{3} q(\nabla\frac{\partial\psi_\omega}{\partial x_i})\ and\ \int |\nabla\psi_\omega|^2$$

are independent of the solution ψ_ω of $(P1)$ that we consider.

We will prove this theorem in next section, adapting to our functionnal setting the concentration-compactness arguments of P.L. Lions [5].
About standing waves, let us mention the following result.

Theorem 6. *For all $\omega > 0$ and $0 < \sigma < 4$, there exists radial functions ψ_ω such that $\frac{\partial}{\partial r}\psi_\omega \in \mathcal{C}^2(R^3)$ satisfying:*

$$\begin{cases} -\omega\phi_\omega + \Delta\phi_\omega = -div(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega) \\ \Delta\psi_\omega = \phi_\omega. \end{cases}$$

We prove this theorem using compactness properties of spaces of radial functions (see Strauss [8]).
Moreover, these radial standing waves are related to the finite time blow up by the following proposition.

Proposition 1 *Let $\sigma = 4/3$, then for all $t_0 > 0$, there exists a radial function ψ_0 so that the corresponding solution to:*

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^{4/3}\nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

blows up exactly at $t = t_0$.

We can now state some stability results for these standing waves:

Theorem 7. $*$*If $0 < \sigma < \frac{4}{3}$, we consider*

$$(P2)\ inf\{\frac{1}{2}\int\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int|\nabla\psi|^2, \int|\nabla\psi|^2 = \int|\nabla\psi_\omega|^2\}$$

where ψ_ω is a solution of (P1). Then (P1) and (P2) are equivalent.

$*$*The set S_ω of standing waves (solutions of (P1) or (P2)) is stable,ie for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\forall\psi_0 \in H,\ inf_{\psi_\omega\in S_\omega}||\psi_0 - \psi_\omega||_H < \delta \to inf_{\psi_\omega\in S_\omega}||\psi(.,t) - \psi_\omega||_H \leq \epsilon.$$

The first part of this theorem will be proved using concentration-compactness methods, the second part adapting the method of Cazenave-Lions [2].

Theorem 8. $*$*For $0 < \sigma < 4$ we consider:*

$$(P3)\ inf\{\frac{1}{2}\int\sum_{i=1}^{3} q(\nabla\frac{\partial\psi}{\partial x_i}) - \frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int|\nabla\psi|^2,$$

$$\int \sum_{i=1}^{3} q(\nabla \frac{\partial \psi}{\partial x_i}) = \int \sum_{i=1}^{3} q(\nabla \frac{\partial \psi_\omega}{\partial x_i})\}$$

where ψ_ω is a solution of (P1). Then (P1) and (P3) are equivalent.

$*$If $\sigma > \frac{4}{3}$, the orbite $e^{i\theta}\psi_\omega$ is unstable, i.e. there exists $\epsilon_0 > 0$ and $\psi_n^0 \to \psi_\omega$ in H such that:

$$sup_{t>0} inf_{\theta \in R} ||\psi_n(.,t) - e^{i\theta}\psi_\omega||_H \geq \epsilon_0.$$

The first part of this theorem will be proved using concentration-compactness methods. For the second part, we will prove that our equations enter in the framework of Shatah-Strauss [7].
The results of this paper were announced in [3].

2 Standing waves solutions

We want to find solutions to

$$\begin{cases} i\phi_t + L\phi = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

of the form $e^{i\omega t}u(x)$, where L is elliptic. Therefore if we make a scaling, we can restrict ourself to the case where $L = \Delta$; i.e. we want to solve:

$$\begin{cases} -\omega u + \Delta u = -div(|\nabla v|^\sigma \nabla v) \\ \Delta v = u \end{cases} \tag{7}$$

In order to solve (7), we introduce the following minimization problem:

$$inf\{-\int |\nabla\psi|^{\sigma+2},\ \phi \in H,\ \int |\nabla\psi|^2 = 1,\ \int |\Delta\psi|^2 = 1\},$$

where $H = \{\psi \in L^6,\ \nabla\psi \in H^1\}$.

If ψ is a solution of the minimization problem, then there exists two numbers α and β (Lagrange mulipliers) such that:

$$\beta\Delta^2\psi - \alpha\Delta\psi = -div(|\nabla\psi|^\sigma \nabla\psi).$$

A suitable scaling will give a solution of (7).

2.1 Concentration-compactness lemma

A few notations are in order.
We set

$$I(\lambda,\mu) = inf(-\int |\nabla\psi|^{\sigma+2},\ \psi \in S_{\lambda,\mu}),$$

for $\lambda, \mu > 0$, and

$$S_{\lambda,\mu} = \{\psi \in H,\ \int |\nabla\psi|^2 = \lambda,\ \int |\Delta\psi|^2 = \mu\}.$$

We now have:

Lemma 1.

$$I(\lambda,\mu) = I(1,1)\lambda^{1-\sigma/4}\mu^{3\sigma/4}$$

and $-\infty < I(1,1) < 0$.

Proof: The Lemma follows obviously from the Gagliardo-Nirenberg's inequality:

$$|\nabla\psi|^{\sigma+2}_{L^{\sigma+2}} \leq C|\Delta\psi|^{3\sigma/2}_{L^2}|\nabla\psi|^{2-\sigma/2}_{L^2}.$$

■

We will use the concentration-compactness lemma of P.L. Lions (see [5]). The aim of this section is to prove the following version of this lemma:

Lemma 2. *1)Let* ρ_n *a sequence in* $L^1(R^3)$ *satisfying* $\rho_n \geq 0$ *in* R^3 *and* $\int \rho_n dx = \lambda$ *for a* $\lambda > 0$ *fixed. Then there exists a subsequence* ρ_{n_k} *satisfying one of the three following alternatives:*

i)(Compactness) : there exists $y_{n_k} \in R^3$ *such that* $\forall\epsilon > 0$, $\exists R < \infty$ *such that:*

$$\int_{y_{n_k}+B_R} \rho_{n_k}(x)dx \geq \lambda - \epsilon.$$

ii)(Vanishing) : $\forall R < \infty$

$$sup_{y\in R^3} \int_{y+B_R} \rho_{n_k} \to 0$$

when $k \to \infty$.

iii)(Dichotomy) : there exists $\alpha \in]0,\lambda[$ *such that* $\forall \epsilon > 0$, *there exists* $\rho^1_{n_k}, \rho^2_{n_k} \in L^1_+(R^3)$ *such that if* $k \geq k_0$ *:*

$$|\int \rho^1_{n_k} dx - \alpha| < \epsilon, \;\; |\rho_{n_k} - (\rho^1_{n_k} + \rho^2_{n_k})|_{L^1} \leq \epsilon,$$

$\rho^1_{n_k}$ *has compact support and* $dist(supp\rho^1_{n_k}, supp\rho^2_{n_k}) \to \infty$ *as* $k \to \infty$.

2)Moreover, if $\rho_n = |\nabla u_n|^2$, *with* $u_n \in H$, *if* u_n *is bounded in* H *and if iii) occures, then one can take* $\rho^i_{n_k} = (|\nabla u^i_{n_k}|^2)^2$ *for* $i = 1, 2$ *and* $u^i_{n_k}$ *are bounded in* H *and satisfy*

$$\int |\Delta u_{n_k}|^2 \geq \int (|\Delta u^1_{n_k}|^2 + |\Delta u^2_{n_k}|^2) - \epsilon$$

and

$$|\nabla u_{n_k} - (\nabla u^1_{n_k} + \nabla u^2_{n_k})|_{L^p} \leq \delta_p(\epsilon),$$

with $\delta_p(\epsilon) \to 0$ *when* $\epsilon \to 0$ *for* $2 \leq p < 6$.

Proof: The 1) is lemma I1 of P.L. Lions [5] partI. The 2) is an adaptation of lemma III1 in [5] partI.

1) Let us recall briefly how to prove the first part, (see [5] for details). We introduce:

$$Q_n(t) = sup_{y \in R^3} \int_{y+B_t} \rho_n(x) dx$$

The Q_n are nondecreasing, nonnegative functions, uniformly bounded on R^+. We can extract a subsequence $Q_{n_k}(t) \to Q(t)$ for every $t \geq 0$, where Q is a nondecreasing, nonnegative function. Let $\alpha = lim_{t \to \infty} Q(t) \in [0, \lambda]$.

If $\alpha = 0$ then $Q \equiv 0$ and vanishing occures.

If $\alpha = \lambda$, no mass disappears : compactness occures (see [5] for details).

If $0 < \alpha < \lambda$: some mass disappears, and dichotomy occures. Let us explain why: let $\epsilon > 0$, choose R such that $Q(R) > \alpha - \epsilon$. If k is sufficiently large we have: $\alpha - \epsilon < Q_{n_k}(R) < \alpha + \epsilon$. Moreover, one can find $R_k \to \infty$ such that $Q_{n_k}(R_k) \leq \alpha + \epsilon$. Finally, there exists $y_k \in R^3$ such that $\int_{y_k + B_R} \rho_{n_k}(x) dx \in]\alpha - \epsilon, \alpha + \epsilon[$. Let $\rho^1_{n_k} = \rho_{n_k} 1_{y_k + B_R}$ and $\rho^2_{n_k} = \rho_{n_k} 1_{R^3 - (y_k + B_{R_k})}$ then

$$\int \{\rho_{n_k} - \rho^1_{n_k} - \rho^2_{n_k}\} dx = \int_{R \leq |x - y_k| \leq R_k} \rho_{n_k} dx \leq (Q_{n_k}(R_k) + \epsilon) - (Q_{n_k}(R) - \epsilon) \leq 4\epsilon$$

The condition on the supports is satisfied and this implies that $\rho^1_{n_k} \rho^2_{n_k} = 0$ a.e.

2)To prove 2), we make a truncation of $\rho_{n_k} = |\nabla u_{n_k}|^2$ with regular functions: let $\epsilon > 0$, $\xi \in \mathcal{D}(R^3)$, $0 \leq \xi \leq 1$, with $\xi \equiv 1$ if $|x| \leq 1$ and $\xi \equiv 0$ if $|x| \geq 2$. Let $\phi = 1 - \xi$, $\xi_\mu = \xi(\frac{\cdot}{\mu})$ and $\phi_\mu = \phi(\frac{\cdot}{\mu})$. We take R_0 such that $Q(R) \leq \alpha - \epsilon$ for $R \geq R_0$. Let $v \in H$ such that $|v|_H \leq M$ where $M \geq sup_n |u_n|_H$.
We then have:

Lemma 3. *There exists a constant $C(M)$ such that:*

$$|\int |\Delta(\xi_R v)|^2 dx - \int \xi_R^2 |\Delta v|^2 dx| \leq \frac{C(M)}{R} \tag{8}$$

and

$$|\int |\nabla(\xi_R v)|^2 dx - \int \xi_R^2 |\nabla v|^2 dx| \leq \frac{C(M)}{R} \tag{9}$$

Proof: Let us prove the first one for example. We have:

$$|\int |\Delta(\xi_R v)|^2 dx - \int \xi_R^2 |\Delta v|^2 dx| = \int \Delta \xi_R^2 v^2 + 4 \int (\nabla \xi_R . \nabla v)^2$$

$$+4 \int \Delta \xi_R v \nabla \xi_R . \nabla v + 4 \int \nabla \xi_R . \nabla v \xi_R \Delta v + 2 \int \Delta \xi_R v \xi_R \Delta v,$$

using Hölder's and Cauchy-Schwartz inequalities we obtain that this last quantity is lower or equal to:

$$(\int (\Delta \xi_R)^3)^{1/3} (\int v^6)^{1/6} + 4 \int \nabla \xi_R^2 \nabla v^2 + 4 (\int \Delta \xi_R^3)^{1/3} (\int v^6)^{1/6} (\int \nabla \xi_R^2 \nabla v)^{1/2}$$

$$+4 (\int \nabla \xi_R^2 \nabla v^2)^{1/2} (\int \xi_R^2 \Delta v^2)^{1/2} + 2 (\int \Delta \xi_R^3)^{1/3} (\int v^6)^{1/6} (\int \xi_R^2 \Delta v^2)^{1/2}.$$

But $\nabla \xi_R(x) = \frac{1}{R} \nabla \xi(\frac{x}{R})$ and $\Delta \xi_R(x) = \frac{1}{R^2} \Delta \xi(\frac{x}{R})$. Moreover $\xi, \nabla \xi, \Delta \xi$ are bounded and $\int |\nabla v|^2, \int |\Delta v|^2 \leq M^2$. Since $(\int v^6)^{1/6} \leq C(\int |\nabla v|^2)^{1/2}$ for all $v \in H$, the above expression is bounded by $\frac{C(M)}{R}$ as claimed. ■

Choose $R_1 \geq R_0$ such that $\frac{C(M)}{R_1} \leq \epsilon$, thus

$$Q(R_1) \geq \alpha - \epsilon. \tag{10}$$

If k is sufficiently large, then

$$Q_{n_k}(R_1) \leq \int_{y_k + B_{R_1}} |\nabla u_{n_k}|^2 + \epsilon. \tag{11}$$

We define $u^1_{n_k} = \xi_{R_1}(. - y_k)u_{n_k} \in H$ and:

$$|\int |\nabla u^1_{n_k}|^2 dx - \alpha| \leq |\int |\nabla u^1_{n_k}|^2 dx - \int \xi^2_{R_1}(. - y_k)|\nabla u_{n_k}|^2 dx|$$

$$+|\int \xi^2_{R_1}(. - y_k)|\nabla u_{n_k}|^2 dx - \alpha|.$$

(9) implies that this last quantity is smaller than:

$$\epsilon + |\int_{y_k + B_{R_1}} |\nabla u_{n_k}|^2 - \alpha| \leq 3\epsilon,$$

by (10) and (11).
We may find $R_k \to \infty$ such that

$$Q_{n_k}(2R_k) \leq \alpha + 2\epsilon, \tag{12}$$

take $\phi_k = \phi_{R_k}(. - y_k)$ and $u^2_{n_k} = \phi_k u_{n_k} \in H$. We have the condition on the supports and

$$\int |\nabla u_{n_k} - (\nabla u^1_{n_k} + \nabla u^2_{n_k})|^2 dx = \int |\nabla(\{1 - \xi_{R_1}(x - y_k) - \phi_k\}u_{n_k})|^2 dx$$

$$\leq \frac{2C(M)}{R_1} + \int \{1 - \xi_{R_1}(. - y_k) - \phi_k\}^2 |\nabla u_{n_k}|^2 dx$$

$$\leq 2\epsilon + \int_{R_1 \leq |x - y_k| \leq 2R_k} |\nabla u_{n_k}|^2 dx,$$

using (11):

$$\leq 2\epsilon + Q_{n_k}(2R_k) - Q_{n_k}(R_1) + \epsilon,$$

by (12):

$$\leq 3\epsilon + \alpha + 2\epsilon - Q_{n_k}(R_1).$$

But if $k \geq k_0$, $|Q_{n_k}(R_1) - \alpha| \leq \epsilon$, this implies that

$$\int |\nabla u_{n_k} - (\nabla u^1_{n_k} + \nabla u^2_{n_k})|^2 dx \leq 6\epsilon$$

Now interpolating L^p between L^2 and L^6 for $2 \leq p < 6$ leads to

$$|\nabla u_{n_k} - (\nabla u^1_{n_k} + \nabla u^2_{n_k})|_{L^p} \leq \delta_p(\epsilon).$$

Moreover

$$|\int |\Delta u^1_{n_k}|^2 dx - \int \xi^2_{R_1}(x - y_k)|\Delta u_{n_k}|^2 dx| \leq \epsilon,$$

and lemma 2 is proved. ■

2.2 Application of the concentration-compactness lemma.

The aim of this section is to prove:

Proposition 2. *For all $\lambda, \mu > 0$, the following minimization problem:*

$$inf\{-\int |\nabla\psi|^{\sigma+2},\ \psi \in H,\ \int |\nabla\psi|^2 = \lambda,\ \int |\Delta\psi|^2 = \mu\}$$

has a solution.
Moreover, every minimizing sequence is relatively compact in H up to translations.

Proof: We follow [5]. Take $\psi_n \in H$ satisfying $-|\nabla\psi_n|^{\sigma+2}_{L^{\sigma+2}} \to I(\lambda,\mu)$ with $\int |\nabla\psi_n|^2 = \lambda$ and $\int |\Delta\psi_n|^2 = \mu$. Let us apply lemma 2 with $\rho_n = |\nabla\psi_n|^2$.

Suppose that iii) occures: We apply 2) of lemma 2: there exists $\psi^1_{n_k}, \psi^2_{n_k}$ bounded in H such that for $k \geq k_0$

$$|\nabla\psi_{n_k} - (\nabla\psi^1_{n_k} + \nabla\psi^2_{n_k})|_{L^{\sigma+2}} \leq \delta_{\sigma+2} \to 0.$$

On the other hand, since $\nabla\psi^i_{n_k}$ in bounded in H^1 and hence in $L^{\sigma+2}$, we see that

$$\int |\nabla\psi_{n_k}|^{\sigma+2} \leq K\delta_{\sigma+2}(\epsilon) + \int |\nabla\psi^1_{n_k} + \nabla\psi^2_{n_k}|^{\sigma+2}.$$

Moreover $dist(supp\nabla\psi^1_{n_k}, supp\nabla\psi^2_{n_k}) \to \infty$ so that $(\nabla\psi^1_{n_k})^2(\nabla\psi^2_{n_k})^2 = 0$ a.e. This leads to

$$\int (\nabla\psi^1_{n_k} + \nabla\psi^2_{n_k})^{\sigma+2} = \int |\nabla\psi^1_{n_k}|^{\sigma+2} + \int |\nabla\psi^2_{n_k}|^{\sigma+2}.$$

Finally we obtain

$$-\int |\nabla\psi_{n_k}|^{\sigma+2} \geq -K\delta_{\sigma+2}(\epsilon) - \int |\nabla\psi^1_{n_k}|^{\sigma+2} - \int |\nabla\psi^2_{n_k}|^{\sigma+2}.$$

Then:

$$\begin{aligned} -\textstyle\int |\nabla\psi_{n_k}|^{\sigma+2} \geq -K\delta_{\sigma+2}(\epsilon) + I(\textstyle\int |\nabla\psi^1_{n_k}|^2, \textstyle\int |\Delta\psi^1_{n_k}|^2) \\ +I(\textstyle\int |\nabla\psi^2_{n_k}|^2, \textstyle\int |\Delta\psi^2_{n_k}|^2), \end{aligned} \tag{13}$$

with

$$|\int |\nabla\psi^1_{n_k}|^2 - \alpha| \leq \epsilon,$$

$$|\int |\nabla \psi^2_{n_k}|^2 - (\lambda - \alpha)| \leq \epsilon$$

and

$$\mu = \int |\Delta \psi_{n_k}|^2 \geq \int |\Delta \psi^1_{n_k}|^2 + \int |\Delta \psi^2_{n_k}|^2 - \epsilon.$$

Hence possibly extracting subsequences:

$$\begin{cases} \int |\nabla \psi^1_{n_k}|^2 \to_{k\to\infty} \bar{\alpha}_\epsilon \\ \int |\nabla \psi^2_{n_k}|^2 \to_{k\to\infty} \bar{\beta}_\epsilon \\ \int |\Delta \psi^1_{n_k}|^2 \to_{k\to\infty} \mu^\epsilon_1 \\ \int |\Delta \psi^2_{n_k}|^2 \to_{k\to\infty} \mu^\epsilon_2, \end{cases}$$

with

$$|\bar{\alpha}_\epsilon - \alpha| \leq \epsilon$$

$$|\bar{\beta}_\epsilon - (\beta - \alpha)| \leq \epsilon$$

$$\mu \geq \mu^\epsilon_1 + \mu^\epsilon_2 - \epsilon.$$

We let $k \to \infty$ in (13) and we obtain:

$$I(\lambda, \mu) \geq -K\delta_{\sigma+2}(\epsilon) + I(\bar{\alpha}_\epsilon, \mu^\epsilon_1) + I(\bar{\beta}_\epsilon, \mu^\epsilon_2).$$

Since $\mu^\epsilon_1 + \mu^\epsilon_2 - \epsilon \leq \mu$, possibly extracting subsequences: $\mu^\epsilon_1 \to \mu_1$, $\mu^\epsilon_2 \to \mu_2$, with $0 \leq \mu_1 + \mu_2 \leq \mu$. This leads to

$$I(\lambda, \mu) \geq I(\alpha, \mu_1) + I(\lambda - \alpha, \mu_2).$$

Using the explicit value of $I(\lambda, \mu)$ given by lemma 1, we get that

$$\mu^{3\sigma/4} \leq (\frac{\alpha}{\lambda})^{1-\sigma/4}\mu_1^{3\sigma/4} + (1 - \frac{\alpha}{\lambda})^{1-\sigma/4}\mu_2^{3\sigma/4} < \mu_1^{3\sigma/4} + \mu_2^{3\sigma/4}$$

which is a contradiction since $\mu \geq \mu_1 + \mu_2$. So dichotomy does not occur.

Suppose that vanishing occurs: We use lemma I1 part II of P.L. Lions [5]:

Lemma 4. *Let $1 < p \leq \infty$, $1 \leq q < \infty$ with $q \neq \frac{Np}{N-p}$ if $p < N$. Suppose that u_n is bounded in $L^q(R^N)$, and ∇u_n is bounded in $L^p(R^n)$ and*

$$sup_{y\in R^N} \int_{y+B_R} |u_n|^q dx \to_{n\to\infty} 0$$

for one $R > 0$.
Then $u_n \to 0$ as $n \to \infty$ in $L^\alpha(R^N)$ for all $\alpha \in]q, \frac{Np}{N-p}[$

For the proof see [5].
In our case $p = 2$, $N = 3$, $q = 2$. Then $\nabla u_n \to 0$ in $L^{\sigma+2}$, this is a contradiction since $I(\lambda, \mu) < 0$, and vanishing does not occur.

Hence i) occurs:
$*$We define $\tilde{\psi}_n = \psi_n(. + y_n)$, possibly extracting a subsequence we have $\nabla\tilde{\psi}_n \rightharpoonup \nabla\tilde{\psi}$ in H^1 and $L^{\sigma+2}$ weak. Moreover $\lambda \geq \int_{B_R} |\nabla\tilde{\psi}_n|^2 dx \geq \lambda - \epsilon$; then since $\nabla\tilde{\psi}_n$ is bounded in H^1_{loc}, $\nabla\tilde{\psi}_n \to \nabla\tilde{\psi}$ in L^2_{loc} strong and this leads to:

$$\lambda \geq \int_{B_R} |\nabla\tilde{\psi}|^2 \geq \lambda - \epsilon.$$

Letting $R \to \infty$ and $\epsilon \to 0$ gives:

$$\int |\nabla\tilde{\psi}|^2 = \lambda = lim_n \int |\nabla\tilde{\psi}_n|^2.$$

This implies that $\nabla\tilde{\psi}_n \to \nabla\tilde{\psi}$ in L^2 strong.
$*$But $\nabla\tilde{\psi}_n$ is bounded in H^1 and hence in L^6, interpolation implies that: $\nabla\tilde{\psi}_n \to \nabla\tilde{\psi}$ in L^p strong for $2 \leq p < 6$ in particular for $p = \sigma + 2$, it follows that

$$I(\lambda, \mu) = -\int |\nabla\tilde{\psi}|^{\sigma+2}.$$

$*$In order to conclude, we still have to show that $\int |\Delta\tilde{\psi}|^2 = \mu$. We note: $\int |\Delta\tilde{\psi}|^2 = \nu$.
If $\nu = 0$, then $\tilde{\psi} = 0$, and this is a contradiction.
If $0 < \nu < \mu$, Gagliardo-Nirenberg's inequality implies:

$$-I(\lambda, \mu) \leq C\lambda^{1-\sigma/4}\nu^{3\sigma/4},$$

and

$$I(\lambda, \mu) \geq -C\lambda^{1-\sigma/4}\nu^{3\sigma/4} > -C\lambda^{1-\sigma/4}\nu^{3\sigma/4} = I(\lambda, \mu),$$

which is a contradiction, so that $\nu = \mu$, and $\Delta\tilde{\psi} \to \Delta\tilde{\psi}$ in L^2 strong. ■

We still have to prove the second part of Theorem 5. Indeed, we found ϕ satisfying

$$-div(|\nabla\psi|^\sigma \nabla\psi) = \alpha\Delta\phi + \beta\phi$$
$$\Delta\psi = \phi,$$

α and β being Lagrange multipliers. We introduce $\phi(x) = a\tilde{\phi}(bx)$ for $a, b \neq 0$. One obtains:

$$a(\alpha b^2 \Delta\tilde{\psi} + \beta\tilde{\phi}) = -a(\frac{|a|}{|b|})^\sigma div(|\nabla\tilde{\psi}|^\sigma \nabla\tilde{\psi}).$$

Let us take $a = b$ and $b^2 = \frac{1}{|\alpha|}$, we find:

$$\beta\tilde{\phi} + \nu\Delta\tilde{\phi} = -div(|\nabla\tilde{\psi}|^\sigma \nabla\tilde{\psi}),$$

where $\nu = \pm 1$.
We first prove a result of regularity:

Lemma 5. *If $\nabla\psi$ is a solution of:*

$$\begin{cases} \beta\tilde{\phi} + \nu\Delta\tilde{\phi} = -div(|\nabla\tilde{\psi}|^\sigma \nabla\tilde{\psi}) \\ \Delta\psi = \phi, \end{cases}$$

with $\psi \in H$, then $\nabla\psi \in \mathcal{C}^2(R^3)$.

Proof:Let us take $\nabla\psi \in H^s$ with $s \geq 1$, then $\nabla\psi \in L^z$ with $\frac{1}{z} = \frac{1}{2} - \frac{s}{3}$ so that

$$|\nabla\psi|^{\sigma+1} \in L^{\frac{6}{(3-2s)(\sigma+1)}} \subset H^{-s_0}$$

with $s_0 = \frac{(3-2s)(\sigma+1)-3}{2}$. Now $\nabla\psi$ satisfies:

$$\xi.\mathcal{F}(|\nabla\psi|^\sigma \nabla\psi) = \beta\mathcal{F}\phi + \nu|\xi|^2\mathcal{F}\phi$$

and

$$\nu|\xi|^{1-s_0}|\mathcal{F}\phi| \leq |\beta|\frac{|\mathcal{F}\phi|}{|\xi|^{1+s_0}} + |\xi|^{-s_0}|\mathcal{F}(|\nabla\psi|^\sigma \nabla\psi)$$

and all these terms are in $L^2(|\xi| \geq 1)$, hence $\phi \in H^{1-s_0}$ and $\nabla\psi \in H^{2-s_0} = H^{s_1}$. If now we define $s_j = \frac{4-3\sigma}{2} + s_{j-1}(\sigma+1)$ and $s_0 = 1$, then $s_j \to \infty$ therefore if j is sufficiently large, then $\nabla\psi \in H^{3/2+\epsilon}$ which is an algebra, we conclude that $div(|\nabla\psi|^\sigma \nabla\psi) \in H^{1/2+\epsilon}$ and $\phi \in H^{5/2+\epsilon}$; this leads to the result. ■

Lemma 6. *We have:*

$$-\beta \int |\nabla\psi|^2 + \nu \int |\phi|^2 = \int |\nabla\psi|^{\sigma+2}.$$

Proof: Let us multiply

$$\beta\phi + \nu\Delta\phi = -div(|\nabla\psi|^\sigma \nabla\psi)$$

by $v \in \mathcal{D}(R^3)$, after integration, one obtains:

$$-\beta \int \nabla\psi.\nabla v + \nu \int \phi\Delta v = \int |\nabla\psi|^\sigma \nabla\psi.\nabla v.$$

Let $v_n \in \mathcal{D}(R^3)$ such that $v_n \to \psi$ in H, passing to the limit leads to the lemma. ■

Proposition 3. *ψ satisfies*

$$\beta/2 \int |\nabla\psi|^2 + \nu/2 \int |\phi|^2 = (1 - \frac{3}{\sigma+2}) \int |\nabla\psi|^{\sigma+2}$$

Proof: This identity is called Pohozaev Identity, and we proceed as follows: we multiply the equation by $(x.\nabla)\psi$ and then integrate. However, this is not directly possible since we do not know the exact behavior of ψ at infinity. We will use $(x.\nabla\psi)e^{-\epsilon x^2}$ and then we make $\epsilon \to 0$. We leave the detail of the proof in the Appendix. ■

The relationships given by lemma 6 and lemma 3 lead to:

$$\beta \int |\nabla\psi|^2 = -\frac{4-\sigma}{2(\sigma+2)} \int |\nabla\psi|^{\sigma+2}$$

and $\beta < 0$. On the other hand:

$$\nu \int |\phi|^2 = 3(\frac{1}{2} - \frac{1}{\sigma+2}) \int |\nabla\psi|^{\sigma+2} > 0$$

and $\nu = +1$. Let $\omega = -\beta$, we have found ϕ satisfying

$$\begin{cases} -\omega\phi + \Delta\phi = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

and the equalities

$$\begin{cases} \omega \int |\nabla\psi|^2 + \int |\phi|^2 = \int |\nabla\psi|^{\sigma+2} \\ -\omega \int |\nabla\psi|^2 + \int |\phi|^2 = (2 - \frac{6}{\sigma+2}) \int |\nabla\psi|^{\sigma+2}. \end{cases} \tag{14}$$

So we have proved Theorem 5 for **one** ω, by scalling, using the homogeneous properties of the equation, we have a solution for every $\omega > 0$. Moreover, thanks to the relationships (14), the values of $\int |\nabla\psi|^{\sigma+2}$, $\int |\nabla\psi|^2$, $\int |\Delta\psi|^2$ are the same for all the solutions of

$$inf\{-\int |\nabla\psi|^{\sigma+2},\ \int |\nabla\psi|^2 = \lambda(\omega),\ \int |\Delta\psi|^2 = \mu(\omega)\}.$$

■

2.3 Radial standing waves.

The aim of this section is to prove that there exists some radial standing waves for the equation:

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^\sigma \nabla\psi) \\ \nabla\psi = \phi, \end{cases} \tag{15}$$

i.e. solution of the form $e^{i\omega t}\phi(x)$ with ϕ depending only on r. Then (15) becomes

$$\begin{cases} -\omega\phi + \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\phi}{dr}) = -\frac{1}{r^2}\frac{d}{dr}(r^2|\frac{d\psi}{dr}|^\sigma\frac{d\psi}{dr}) \\ \phi = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\psi}{dr}) \end{cases} \tag{16}$$

Multiplying by r^2 and then integrating leads to

$$-\omega\psi_r + \frac{d}{dr}(\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\psi}{dr})) = -|\psi_r|^\sigma\psi_r$$

Now let $f = \psi_r$, f satisfies:

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - (\omega + \frac{2}{r^2})f = -|f|^\sigma f. \tag{17}$$

In order to solve (17), we consider the following minimization problem:

$$I(\lambda,\mu) = inf\{-\int_0^\infty f^{\sigma+2}r^2dr,\ \int_0^\infty f'^2r^2dr + 2\int_0^\infty f^2dr = \mu,\ \int_0^\infty f^2r^2dr = \lambda\} \tag{18}$$

About (18), we first prove:

Lemma 7. $*$*For* $f \in H^1_r(R^3)$, $\int_0^\infty |f|^2 dr$ *makes sense and*

$$\int_0^\infty |f|^2 dr \leq 4 \int_0^\infty |\nabla f|^2 r^2 dr.$$

$*$ $I(\lambda, \mu) > -\infty$.

Proof: Since $f \in H^1_r(R^3)$, f as a function of one variable is in $H^1([\alpha, +\infty[)$ for all $\alpha > 0$, and

$$f^2(y) - f^2(x) = \int_x^y 2f(t)f'(t)dt \; \forall x, y.$$

Integrating for $x \in [\epsilon, y]$ leads to

$$f^2(y)(y-\epsilon) - \int_\epsilon^y f^2(x)dx = \int_\epsilon^y (\int_x^y 2f(t)f'(t)dt)dx.$$

We use Fubini's theorem in order to obtain

$$\int_\epsilon^y f^2(x)dx = f^2(y)(y-\epsilon) - \int_\epsilon^y 2f(t)f'(t)(t-\epsilon)dt.$$

Moreover,

$$\int_\epsilon^y |f^2(x)|dx \leq |f^2(y)(y-\epsilon)| + 2(\int_\epsilon^y |f^2(t)|dt)^{1/2}(\int_\epsilon^y (t-\epsilon)^2 |f'(t)|^2 dt)^{1/2}.$$

This implies that

$$\begin{array}{l} (\int_\epsilon^y |f^2(x)|dx)^{1/2} \leq (\int_\epsilon^y (t-\epsilon)^2 |f'(t)|^2 dt)^{1/2} \\ +(\int_\epsilon^y (t-\epsilon)^2 |f'(t)|^2 dt + |f^2(y)(y-\epsilon)|)^{1/2}. \end{array} \tag{19}$$

Now since $f \in H^1_r(R^3)$, we have $|f(y)| \leq C|y|^{-1}|f|_{H^1}$ (see Berestycki-Lions [1]). Letting $\epsilon \to 0$ and $y \to \infty$ in (19), we get

$$\int_0^\infty f(x)^2 dx \leq 4 \int_0^\infty |f'(t)|^2 t^2 dt$$

as claimed.
A Gagliardo-Nirenberg's inequality gives:

$$\int_0^\infty f^{\sigma+2} r^2 dr \leq C(\int_0^\infty f^2 r^2 dr)^{1-\sigma/4} (\int_0^\infty f'^2 r^2 dr)^{3\sigma/2},$$

therefore

$$I(\lambda, \mu) \geq -C\lambda^{1-\sigma/4}\mu^{3\sigma/4}$$

and $I(\lambda, \mu) > -\infty$. ■

We now solve problem (18).

Proposition 4. *The minimum:*

$$I(\lambda,\mu) = inf\{-\int f^{\sigma+2}r^2dr,\ \int f'^2r^2dr + 2\int f^2dr = \mu,\ \int f^2dr = \lambda\}$$

is attained.

Proof: Let us take a minimizing sequence f_n such that

$$-\int_0^\infty f_n^{\sigma+2}r^2dr \to I(\lambda,\mu)$$

and

$$\int_0^\infty f_n'^2r^2dr + 2\int_0^\infty f_n^2dr = \mu,\ \int_0^\infty f_n^2r^2dr = \lambda.$$

Then we can suppose that $f_n \rightharpoonup f$ in $H_r^1(R^3)$ weakly and since the injection of $H_r^1(R^3)$ in $L^{\sigma+2}(R^3)$ is compact (see Strauss [8]), then $f_n \to f$ in $L^{\sigma+2}$ strong, so that

$$-\int_0^\infty f_n^{\sigma+2}r^2dr \to -\int_0^\infty f^{\sigma+2}r^2dr.$$

Moreover $f \mapsto \int_0^\infty f^2dr$ is convex and continuous on $H_r^1(R^3)$, thus it is weakly lower semi-continuous and

$$b \equiv \int_0^\infty f'^2r^2dr + 2\int_0^\infty f^2dr \leq \mu,$$

$$a \equiv \int_0^\infty f^2r^2dr \leq \lambda.$$

We define $\tilde{f}(x) = \alpha f(\beta x)$, with α, β satisfying

$$\lambda = \int \tilde{f}^2r^2dr = \frac{\alpha^2}{\beta^3}a,$$

$$\mu = \int \tilde{f}'^2r^2dr + 2\int \tilde{f}^2dr = \frac{\alpha^2}{\beta}b.$$

We obtain

$$-\int \tilde{f}^{\sigma+2}r^2dr = -\frac{\alpha^{\sigma+2}}{\beta^3}I(\lambda,\mu).$$

By definition of $I(\lambda,\mu)$ we get

$$-\int \tilde{f}^{\sigma+2} r^2 dr \geq I(\lambda,\mu).$$

This implies that

$$\frac{\alpha^{\sigma+2}}{\beta^3} \leq 1, \tag{20}$$

since $I(\lambda,\mu) < 0$. Now $\frac{\alpha^4}{\beta^4} = \frac{\lambda\mu}{ab}$ and $\beta = (\frac{a\mu}{b\lambda})^{1/2}$ and $\frac{\alpha^{\sigma+2}}{\beta^3} = (\frac{\mu}{b})^{3\sigma/4}(\frac{\lambda}{a})^{1-\sigma/4} \geq 1$. Together with (20), this inequality implies that $\lambda = a$ and $\mu = b$ and the minimum is attained. ■

For all $v \in \mathcal{D}(R^+)$

$$-\int |f|^\sigma f v r^2 dr = 2\alpha \int f v r^2 dr + 2\beta \int \frac{df}{dr} r^2 v' dr + 4\beta \int f v dr,$$

so that we found a solution of (16).

We still have to prove Proposition 1.

Proposition 1. *Let $\sigma = 4/3$, then for all $t_0 > 0$, there exists a radial function ψ_0 so that the corresponding solution to:*

$$\begin{cases} i\phi_t + \Delta\phi = -div(|\nabla\psi|^{4/3}\nabla\psi) \\ \Delta\psi = \phi \end{cases}$$

blows up exactly at $t = t_0$.

Proof: If we restrict (1) to the radial functions, we obtain

$$iu_t + \Delta u - \frac{2}{|x|^2} u = -|u|^\sigma u, \tag{21}$$

where $u = \psi_r$. This last equation satisfies the pseudo-conformal transformation laws, i.e if $u(x,t)$ is a solution of (21), then the function

$$v(x,t) = \frac{1}{t^{3/2}} e^{\frac{i|x|^2}{4t}} \bar{u}(\frac{x}{t}, \frac{1}{t})$$

is a solution to (21) too. If we apply this transformation to any radial standing waves that we found, we obtain the Proposition. ■

3 Stability of standing waves

In the previous section, we found some functions ϕ_ω satisfying

$$\left\{ \begin{array}{c} \Delta\phi_\omega - \omega\phi_\omega = -div(|\nabla\psi_\omega|^\sigma \nabla\psi_\omega) \\ \Delta\psi_\omega = \phi_\omega. \end{array} \right. \tag{22}$$

$\nabla\psi_\omega$ are solutions of the following minimization problem

$$(P1)\ inf\{-\int |\nabla\psi|^{\sigma+2},\ \psi \in H\ \int |\nabla\psi|^2 = \lambda,\ \int |\Delta\psi|^2 = \mu\}.$$

Each solution of $(P1)$ satisfies:

$$\left\{ \begin{array}{c} \omega \int |\nabla\psi_\omega|^2 = (\frac{3}{\sigma+2} - \frac{1}{2}) \int |\nabla\psi_\omega|^{\sigma+2}, \\ \int |\phi_\omega|^2 = 3(\frac{1}{2} - \frac{1}{\sigma+2}) \int |\nabla\psi_\omega|^{\sigma+2} \end{array} \right. \tag{23}$$

and since $\nabla\psi_\omega$ is a solution of $(P1)$, the quantities $\int |\nabla\psi_\omega|^2$, $\int |\nabla\psi_\omega|^{\sigma+2}$ and $\int |\phi_\omega|^2$ do not depend on the solution of $(P1)$ that we consider, but only on ω. The aim of this section is to investigate the stability of these standing waves. A few notations are in order:

$$\left\{ \begin{array}{c} E(\psi) = \int (\frac{1}{2}|\Delta\psi|^2 - \frac{1}{\sigma+2}|\nabla\psi|^{\sigma+2})dx \\ Q(\psi) = \frac{1}{2}\int |\nabla\psi|^2. \end{array} \right. \tag{24}$$

So that (22) becomes $E'(\psi) + \omega Q'(\psi) = 0$. In order to prove the stability and instability properties, we introduce new minimization problems and we prove:

Proposition 5. *If $\sigma < 4/3$, then the following minimization problem has a solution*

$$(\tilde{P}2)\ min\{E(\psi),\ \psi \in H,\ \int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2\} = -a$$

with $a > 0$.
Moreover, let $\psi_n \in H$ such that $E(\psi_n) \to -a$ and $\int |\nabla\psi_n|^2 \to \int |\nabla\psi_\omega|^2$, then there exists $y_n \in R^3$ such that $\psi_n(. - y_n)$ is compact in H.

Proposition 6. *If $0 < \sigma < 4$, then the following minimization problem has a solution:*

$$(\tilde{P}3) Min\{-\frac{1}{\sigma+2}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\psi|^2,\ \psi \in H,\ \int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2\} = -b$$

with $b > 0$.

3.1 Proof of the propositions.

3.1.1 First proposition.

We fix $0 < \sigma < 4/3$. We first prove that the minimum is finite. Indeed, Gagliardo-Nirenberg's inequality gives:

$$E(\psi) \geq -C(\int |\Delta\psi|^2)^{3\sigma/4} + \frac{1}{2}\int |\Delta\psi|^2,$$

for $\psi \in H$ such that $\int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2$. Since $\sigma < 4/3$,

$$-C(\int |\Delta\psi|^2)^{3\sigma/4} + \frac{1}{2}\int |\Delta\psi|^2$$

is bounded from below and $-a > -\infty$.
Let us now show that $-a < 0$. Indeed, consider $\psi \in H$ such that $\int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2$ and $\tilde{\psi}(x) = \alpha\psi(\beta x)$ for $\alpha, \beta > 0$, then if $\alpha^2 = \beta$, $\int |\nabla\tilde{\psi}|^2 = \int |\nabla\psi|^2$ and

$$E(\tilde{\psi}) = -\beta^{3\sigma/2}\int \frac{|\nabla\psi|^{\sigma+2}}{\sigma+2} + \frac{\beta^2}{2}\int |\Delta\psi|^2.$$

Since $3\sigma/2 < 2$, for β small, $E(\tilde{\psi}) < 0$ and $-a < 0$.
Using the same technique as for the proof of Theorem 5, in particular the concentration-compactness lemma, one can prove that $(\tilde{P}2)$ has a solution. ■

3.1.2 Second proposition.

Gagliardo-Nirenberg's inequality yields:

$$-\frac{1}{\sigma+2}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\psi|^2 \geq -C(\int |\nabla\psi|^2)^{1-\sigma/4} + \frac{\omega}{2}\int |\nabla\psi|^2,$$

for all $\psi \in H$ such that

$$\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2.$$

Since $1 - \sigma/4 < 1$, we obtain that $-b > -\infty$.
On the other hand, for $\psi \in H$ with $\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2$, we define: $\tilde{\psi}(x) = \alpha\psi(\frac{x}{\alpha^2})$. We have $\int |\Delta\tilde{\psi}|^2 = \int |\Delta\psi_\omega|^2$, and

$$-\frac{1}{\sigma+2}\int |\Delta\tilde{\psi}|^{\sigma+2} + \frac{\omega}{2}\int |\nabla\tilde{\psi}|^2 = -\frac{1}{\sigma+2}\alpha^{4-\sigma}\int |\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\alpha^4\int |\nabla\psi|^2.$$

Thus if α is small enough, this last quantity is negative.
To prove that problem $(\tilde{P}2)$ has a solution, we proceed as for the proof of theorem 5. ■

3.2 Equivalent problems

Problem $(\tilde{P}2)$ is equivalent to:

$$(P2) inf\{\frac{1}{2}\int|\Delta\psi|^2 - \frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int|\nabla\psi|^2,\ \int|\nabla\psi|^2 = \int|\nabla\psi_\omega|^2\}$$

for $\sigma < \frac{4}{3}$.
Problem $(\tilde{P}3)$ is equivalent to:

$$(P3) inf\{\frac{1}{2}\int|\Delta\psi|^2 - \frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2} + \frac{\omega}{2}\int|\nabla\psi|^2,\ \int|\Delta\psi|^2 = \int|\Delta\psi_\omega|^2\}$$

for $\sigma < 4$.
We now investigate the relationships between $(P1), (P2)$ and $(P3)$.

Proposition 7. *Let $0 < \sigma < \frac{4}{3}$, then $(P1)$ and $(P2)$ are equivalent.*

Proof: Let us write the Euler-Lagrange equation corresponding to problem $(P2)$: there exists a Lagrange multiplier γ so that:

$$\Delta^2\psi + div(|\nabla\psi|^\sigma\nabla\psi) - \omega\Delta\psi = \gamma\Delta\psi. \tag{25}$$

It is roughly the same equation as in the proof of Theorem 5, so that the corresponding result of regularity apply and if we multiply (25) by $\bar{\psi}$ and then integrate, we obtain

$$\int|\Delta\psi|^2 - \int|\nabla\psi|^{\sigma+2} + (\omega+\gamma)\int|\nabla\psi|^2 = 0. \tag{26}$$

Multiplying (25) by $(x.\nabla)\psi$ and then integrating leads to:

$$\frac{1}{2}\int|\Delta\psi|^2 + (\frac{3}{\sigma+2} - 1)\int|\nabla\psi|^{\sigma+2} - \frac{\omega+\gamma}{2}\int|\nabla\psi|^2 = 0. \tag{27}$$

We make a combination of (26) and (27):

$$2\int|\Delta\psi|^2 = 3(1 - \frac{2}{\sigma+2})\int|\nabla\psi|^{\sigma+2}, \tag{28}$$

$$(1-\frac{6}{\sigma+2})\int|\nabla\psi|^{\sigma+2}=-2(\omega+\gamma)\int|\nabla\psi|^2. \tag{29}$$

We note $\lambda=\int|\nabla\psi_\omega|^2=\int|\nabla\psi|^2$, we have, using (28) and (29):

$$\int|\nabla\psi|^{\sigma+2}=\frac{2(\sigma+2)(\omega+\gamma)}{4-\sigma}\lambda \tag{30}$$

and

$$\int|\Delta\psi|^2=\frac{3\sigma(\omega+\gamma)}{4-\sigma}\lambda. \tag{31}$$

An analogue calculation, using the corresponding formulas for $\nabla\psi_\omega$ leads to

$$\int|\nabla\psi_\omega|^{\sigma+2}=\frac{2\omega(\sigma+2)}{4-\sigma}\lambda \tag{32}$$

and

$$\int|\Delta\psi_\omega|^2=\frac{3\sigma}{4-\sigma}\lambda. \tag{33}$$

Hence

$$-\frac{1}{\sigma+2}\int|\nabla\psi_\omega|^{\sigma+2}+\frac{1}{2}\int|\Delta\psi_\omega|^2=\frac{\omega\lambda(3\sigma-4)}{2(4-\sigma)} \tag{34}$$

while

$$-\frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2}+\frac{1}{2}\int|\Delta\psi|^2=\frac{(\omega+\gamma)\lambda(3\sigma-4)}{2(4-\sigma)}. \tag{35}$$

Since $3\sigma-4<0$ and $E(\psi_\omega)\geq E(\psi)$, (34) and (35) imply

$$\gamma\geq 0. \tag{36}$$

Set $\tilde{\psi}(x)=\alpha\psi(\alpha^2x)$, then $\int|\nabla\tilde{\psi}|^2=\int|\nabla\psi|^2$, and we choose α such that $\int|\Delta\tilde{\psi}|^2=\int|\Delta\psi|^2$, this gives:

$$\alpha^4=\frac{\omega}{\omega+\gamma} \tag{37}$$

Using (37), we obtain that

$$-\int|\nabla\tilde{\psi}|^{\sigma+2}=-\alpha^{3\sigma-4}\int|\nabla\psi_\omega|^{\sigma+2}.$$

If we use the definition of $(P1)$, we find

$$-\int |\nabla\psi_\omega|^{\sigma+2} \leq -\alpha^{3\sigma-4}\int |\nabla\psi_\omega|^{\sigma+2}$$

or equivalently

$$1 \geq \alpha,$$

and since $3\sigma - 4 < 0$, $\alpha \geq 1$ and $\gamma \leq 0$. Together with (36), this inequality implies

$$\gamma = 0.$$

In fact, we have proved that if ψ is a solution of $(P2)$, and ψ_ω is a solution of $(P1)$, then

$$\int |\nabla\psi|^2 = \int |\nabla\psi_\omega|^2,$$

$$\int |\Delta\psi|^2 = \int |\Delta\psi_\omega|^2,$$

$$\int |\nabla\psi|^{\sigma+2} = \int |\nabla\psi_\omega|^{\sigma+2},$$

so that $(P1)$ and $(P2)$ are equivalent, and the proposition is proved. ■

Proposition 8. *Let $0 < \sigma < 4$, then $(P1)$ and $(P2)$ are equivalent.*

Proof: Let us write the Euler-Lagrange equation corresponding to problem $(P3)$

$$div(|\nabla\psi|^\sigma\nabla\psi) - \omega\Delta\psi + \Delta^2\psi = \delta\Delta^2\psi, \tag{38}$$

where δ is a Lagrange multiplier. Multiplying (38) by $\bar{\psi}$ and then integrating leads to:

$$(1-\delta)\int |\Delta\psi|^2 + \omega\int |\nabla\psi|^2 = \int |\nabla\psi|^{\sigma+2}. \tag{39}$$

Multiplying (38) by $(x.\nabla)\psi$ and integrating gives

$$\frac{1-\delta}{2}\int |\Delta\psi|^2 - \frac{\omega}{2}\int |\nabla\psi|^2 = (1-\frac{3}{\sigma+2})\int |\nabla\psi|^{\sigma+2}. \tag{40}$$

Making linear combination of (39) and (40), we obtain

$$\omega\int |\nabla\psi|^2 = (\frac{3}{\sigma+2} - \frac{1}{2})\int |\nabla\psi|^{\sigma+2} \tag{41}$$

and

$$(1-\delta)\int|\phi|^2 = 3(\frac{1}{2}-\frac{1}{\sigma+2})\int|\nabla\psi|^{\sigma+2}. \tag{42}$$

We recall that for ψ_ω the solution of $(P1)$ we have

$$\omega\int|\nabla\psi_\omega|^2 = (\frac{3}{\sigma+2}-\frac{1}{2})\int|\nabla\psi|^{\sigma+2} \tag{43}$$

and

$$\int|\phi_\omega|^2 = 3(\frac{1}{2}-\frac{1}{\sigma+2})\int|\nabla\psi_\omega|^{\sigma+2}, \tag{44}$$

so that, using (43) and (44), we get that

$$-\frac{1}{\sigma+2}\int|\nabla\psi_\omega|^{\sigma+2}+\frac{\omega}{2}\int|\nabla\psi_\omega|^2 = -\frac{1}{6}\int|\Delta\psi_\omega|^2.$$

On the other hand, using (41) and (42)

$$-\frac{1}{\sigma+2}\int|\nabla\psi|^{\sigma+2}+\frac{\omega}{2}\int|\nabla\psi|^2 = -\frac{1-\delta}{6}\int|\Delta\psi|^2 = -\frac{1-\delta}{6}\int|\Delta\psi_\omega|^2$$

and necessarily

$$\delta \leq 0. \tag{45}$$

If we let $\mu = \int|\Delta\psi|^2 = \int|\Delta\psi_\omega|^2$, we get:

$$\int|\nabla\psi|^{\sigma+2} = \frac{2(1-\delta)(\sigma+2)}{3\sigma}\mu,\ \int|\nabla\psi_\omega|^{\sigma+2} = \frac{2(\sigma+2)}{3\sigma}\mu \tag{46}$$

and

$$\int|\nabla\psi|^2 = \frac{(1-\delta)(4-\sigma)}{3\omega\sigma},\ \int|\nabla\psi_\omega|^2 = \frac{4-\sigma}{3\omega\sigma}\mu. \tag{47}$$

Set $\tilde{\psi}(x) = \alpha\psi(\frac{x}{\alpha^2})$, then $\int|\Delta\psi|^2 = \int|\Delta\tilde{\psi}|^2$, and we choose α such that $\int|\nabla\tilde{\psi}|^2 = \int|\nabla\psi_\omega|^2$. Using (46) and (47), we obtain $\alpha^4 = \frac{1}{1-\delta}$. Then

$$-\int|\nabla\tilde{\psi}|^{\sigma+2} = -\alpha^{-\sigma}\int|\nabla\psi_\omega|^{\sigma+2},$$

so that

$$-\int|\nabla\psi_\omega|^{\sigma+2} \leq -\alpha^{-\sigma}\int|\nabla\psi_\omega|^{\sigma+2}$$

and $\alpha \geq 1$ therefore

$$\delta \geq 0.$$

This inequality with (45) implies that $\delta = 0$, and we conclued as for the previous proposition. ■

3.3 Stability for $\sigma < 4/3$.

Adapting the method of Cazenave-Lions [2], we may prove the second part of Theorem 7. Let S_ω the set of standing waves, solutions of $(P1)$ or $(P2)$. Let us suppose that this set is unstable under the flow of

$$i(\nabla\psi)_t + \Delta(\nabla\psi) = \nabla(-\Delta)^{-1}div(|\nabla\psi|^\sigma\nabla\psi). \tag{48}$$

There exists $\psi_n^0 \in H$, $t_n \in R^+$ and $\epsilon > 0$ such that

$$inf_{\psi_\omega \in S_\omega}|\psi_n^0 - \psi_\omega|_H \to 0$$

and the solution $\psi_n(t)$ of (48) with $\psi_n(0) = \psi_0$ satisfies

$$inf_{\psi_\omega \in S_\omega}|\psi_n(t_n, .) - \psi_\omega|_H \geq \epsilon.$$

We have that

$$\begin{cases} \int |\nabla\psi_n^0|^2 \to \int |\nabla\psi_\omega|^2, \\ \int |\nabla\psi_n^0|^{\sigma+2} \to \int |\nabla\psi_\omega|^{\sigma+2}, \\ \int |\Delta\psi_n^0|^2 \to \int |\Delta\psi_\omega|^2. \end{cases}$$

Since E and Q are conserved by the flow of (48), we know that

$$Q(\psi_n(t_n)) \to Q(\psi_\omega)$$

and

$$E(\psi_n(t_n)) \to E(\psi_\omega) = -a.$$

By proposition 5, $\psi_n(t_n)$ is compact up to translation and

$$inf_{\psi_\omega \in S_\omega}|\psi_n(t_n, .) - \psi_\omega|_H \to 0$$

which is a contradiction. ■

3.4 Instability for $\sigma > 4/3$.

The aim of this section is to show how it is possible to adapt the proof of J.Shatah and W.Strauss [7] (which works for the Klein-Gordon equation), to our problem. We introduce

$$d(\omega) = E(\psi_\omega) + \omega Q(\psi_\omega).$$

Then, since $(P1)$ and $(P3)$ are equivalent

$$d(\omega) = inf\{E(u) + \omega Q(u),\ u \in H,\ \int |\Delta u|^2 = \int |\Delta \psi_\omega|^2\}.$$

Using the relationships (32) and (33), we obtain that

$$d(\omega) = \frac{1}{3}\int |\Delta \psi_\omega|^2.$$

Let us now find the dependence of $d(\omega)$ in ω. Let $\psi_\omega(x) = \alpha\tilde{\psi}(\beta x)$, $\tilde{\psi}$ satisfies:

$$-\frac{\omega}{\alpha^\sigma \beta^\sigma}\Delta\tilde{\psi} + \frac{1}{\alpha^\sigma \beta^{\sigma-2}}\Delta^2\tilde{\psi} = -div(|\nabla\tilde{\psi}|^\sigma \nabla\tilde{\psi})$$

we take α and β such that $\alpha^\sigma\beta^{\sigma-2} = 1$ and $\alpha^\sigma\beta^\sigma = \omega$, this leads to:

$$\alpha = \omega^{\frac{1}{\sigma}-\frac{1}{2}},\ \beta = \omega^{1/2}.$$

Then $\tilde{\psi} = \psi_1$ and

$$d(\omega) = \frac{\omega^{\frac{2}{\sigma}-\frac{1}{2}}}{3}\int |\Delta \psi_1|^2.$$

This function is strictly concave if and only if $\frac{4}{3} < \sigma < 4$, and from now on, we fix such a σ.

Lemma 8.

$$d'(\omega) = Q(\psi_\omega).$$

Proof: Indeed, $d(\omega) = E(\psi_\omega) + \omega Q(\psi_\omega)$ and

$$d'(\omega) =< E'(\psi_\omega) + \omega Q'(\psi_\omega), \frac{\partial \psi_\omega}{\partial \omega} > +Q(\psi_\omega).$$

But

$$E'(\psi_\omega) + \omega Q'(\psi_\omega) = 0,$$

thereby proving the lemma. ■

Lemma 9. *Fix $\omega = \omega_0$ and $\psi_0 = \psi_{\omega_0}$. The for all C^2 curve $u(\lambda)$ such that $u(0) = \psi_0$ and $Q(u(\lambda)) = Q(\psi_0)$ then*

$$\frac{d^2}{dt^2}E(u(\lambda))|_{\lambda=0} =< (E''(\psi_0) + \omega_0 Q''(\psi_0))y_0, y_0 >,$$

where $y_0 = u'(0)$.

Proof:

$$\frac{d^2}{d\lambda^2}E(u(\lambda)) =< E''(u(\lambda))u', u' > + < E'(u), u'' > . \tag{49}$$

On the other hand, since Q is constant along the curve:

$$< Q''(u)u', u' > + < Q'(u), u'' >= 0 \tag{50}$$

and (49)+ω_0 (50)$|_{\lambda=0}$ gives the result, since $E'(\psi_0) + \omega_0 Q'(\psi_0) = 0$. ■

We introduce:

$$\nabla\chi(\omega, x) = (\nabla\psi_\omega)(\frac{x}{\lambda(\omega)}),$$

with

$$\lambda^3(\omega) = \frac{Q(\psi_0)}{Q(\psi_\omega)}. \tag{51}$$

Then we have $Q(\chi(\omega, x)) = Q(\psi_0)$.

Lemma 10.

$$\frac{d^2}{d\omega^2}E(\chi(\omega))|_{\omega=\omega_0} \leq d''(\omega_0).$$

Proof: we define $\alpha(\omega) = E(\chi(\omega)) - d(\omega) + \omega d'(\omega_0)$, then $\alpha(\omega_0) = 0$ by definition of E, Q and d and lemma 8. Therefore by the definition of χ and since $Q(\chi(\omega)) = Q(\psi_0)$, we have:

$$E(\chi(\omega)) = -\omega Q(\psi_0) + \frac{\lambda}{2}\int |\Delta\psi_\omega|^2 - \frac{\lambda^3}{\sigma+2}\int |\nabla\psi_\omega|^{\sigma+2} + \frac{\lambda^3\omega}{2}\int |\nabla\psi_\omega|^2.$$

Using the relationships (32) and (33), one obtain:

$$\omega\int |\nabla\psi_\omega|^2 = \frac{4-\sigma}{3\sigma}\int |\phi_\omega|^2, \tag{52}$$

$$\int |\nabla \psi_\omega|^{\sigma+2} = \frac{2(\sigma+2)}{3\sigma} \int |\phi_\omega|^2, \tag{53}$$

whence

$$E(\chi(\omega)) = -\omega Q(\psi_0) + (\frac{\lambda}{2} - \frac{\lambda^3}{6}) \int |\phi_0|^2.$$

Since $\frac{\lambda}{2} - \frac{\lambda^3}{6} \leq \frac{1}{3}$ for $\lambda \geq 0$ we have:

$$E(\chi(\omega)) \leq -\omega d'(\omega_0) + d(\omega_0)$$

and $\alpha(\omega) \leq 0$, therefore $\alpha(\omega)$ is maximum for $\omega = \omega_0$ and $\frac{\partial^2 \alpha}{\partial \omega^2}(\omega_0) \leq 0$. The lemma follows. ■

Lemma 11. *We have:*

$$E(\chi(\omega)) < E(\psi_0),$$

for $\omega \neq \omega_0$, ω near ω_0.

Proof: First we note that

$$\frac{d}{d\omega} E(\chi(\omega))|_{\omega=\omega_0} =< E'(\chi(\omega_0)), \frac{d\chi}{d\omega}(\omega_0) > . \tag{54}$$

Since $Q(\chi(\omega)) = Cte$,

$$< Q'(\chi(\omega_0)), \frac{d\chi}{d\omega}(\omega_0) >= 0. \tag{55}$$

(54)+ω_0(55) leads to

$$\frac{d}{d\omega} E(\chi(\omega)) =< E'(\psi_0) + \omega_0 Q'(\psi_0), \frac{d\psi}{d\omega}(\omega_0) >= 0.$$

Moreover $\frac{d^2}{d\omega^2} E(\chi(\omega))|_{\omega=\omega_0} \leq d''(\omega_0) < 0$ so that E is localy maximal in ψ_0 and the lemma is proved. ■

We define:

$$y_0 = \frac{\partial \nabla \chi(\omega)}{\partial \omega}|_{\omega=\omega_0}. \tag{56}$$

About y_0, we prove:

Lemma 12. *(a)*$< (E''(\psi_0) + \omega_0 Q''(\psi_0))y_0, y_0 > \leq d''(\omega_0)$.
(b) $< Q'(\psi_0), y_0 >= -\int \nabla\psi_0.\bar{y}_0 = 0$.
(c) $\int \Delta\psi_0 \; div(y_0) > 0$.

Proof: (a) follows from lemmas 9 and 10.
(b) is trivial.
(c) By definition of χ, $\int |\Delta\chi(\omega)|^2 dx = \lambda \int |\Delta\psi_\omega|^2$, hence by differentiation in $\omega = \omega_0$:

$$2\int \Delta\psi_0 \; div(y_0) = \lambda' \int |\Delta\psi_0|^2 + 3d'(\omega_0), \tag{57}$$

Since $d(\omega) = \frac{1}{3}\int |\Delta\psi_\omega|^2$. But $\lambda^3 = \frac{d'(\omega_0)}{d'(\omega)}$, therefore differentiation in $\omega = \omega_0$ leads to

$$3\lambda'(\omega_0) = -\frac{d''(\omega_0)}{d'(\omega_0)}.$$

But $d''(\omega_0) < 0$, so that $\lambda'(\omega_0) > 0$ and the lemma follows. ■

Set

$$G = \{f \in (H^1(R^3))^3, \; \exists\psi \in \mathcal{D}' \; such \; that \; f = \nabla\psi\}.$$

We know that (see [4])

$$G = \{\nabla\psi, \; \psi \in L^6 \cap C_0, \; \nabla\psi \in H^1\}.$$

Let I the following injection of G in G':

$$\begin{array}{c} G \to G' \\ \nabla u \to I(\nabla u) \end{array}$$

defined by:

$$< I(\nabla u), \nabla v >= Re \int \nabla u.\nabla\bar{v}.$$

With this preliminaries, we are in the mathematical setting of Shatah-Strauss [7] and we have the result of Theorem 8.

Appendix.

A Proof of proposition 2.

Proposition 3. *ψ satisfies*

$$\beta/2\int|\nabla\psi|^2+\nu/2\int|\phi|^2=(1-\frac{3}{\sigma+2})\int|\nabla\psi|^{\sigma+2}.$$

Proof: We multiply the equation:

$$\beta\phi+\tilde{\epsilon}\Delta\phi=-div(|\nabla\psi|^{\sigma}\nabla\psi)$$

by $(x.\nabla\psi)e^{-\epsilon x^2}$, and we compute every terms:

i) $\int\phi(x.\nabla)\psi e^{-\epsilon x^2}$.
By numerous integrations by parts we get that

$$\int\phi(x.\nabla\psi)e^{-\epsilon x^2}=\frac{1}{2}\int|\nabla\psi|^2e^{-\epsilon x^2}+2\epsilon\int e^{-\epsilon x^2}(x.\nabla\psi)^2-\epsilon\int|\nabla\psi|^2x^2e^{-\epsilon x^2}. \tag{58}$$

When we make $\epsilon\to 0$, $\int|\nabla\psi|^2e^{-\epsilon x^2}\to\int|\nabla\psi|^2$ by Lebesgue's theorem.
The other terms will be treated by:

Lemma 13. *Let $f\in L^1_+(R^3)$ then*

$$\epsilon\int fx^2e^{-\epsilon x^2}dx\to 0,$$

as $\epsilon\to 0$.

Proof: Consider $g(x)=x^2e^{-\epsilon x^2}$, then $|g(x)|\leq\frac{1}{\epsilon}e^{-1}$. Define $r_\epsilon=\frac{1}{\epsilon^{1/4}}$, then $g(r_\epsilon)=\frac{1}{\epsilon^{1/2}}e^{-\epsilon^{1/2}}\leq\frac{1}{\epsilon^{1/2}}$. We have

$$\epsilon\int fx^2e^{-\epsilon x^2}dx=\epsilon\int_{|x|<r_\epsilon}fx^2e^{-\epsilon x^2}dx+\epsilon\int_{|x|>r_\epsilon}fx^2e^{-\epsilon x^2}dx.$$

Hence

$$\epsilon \int f x^2 e^{-\epsilon x^2} dx \leq \epsilon^{1/2} \int_{|x|<r_\epsilon} f dx + \int_{|x|>r_\epsilon} f(x) e^{-1} dx$$
$$\leq \epsilon^{1/2} \int_{R^3} f dx + \int_{|x|>r_\epsilon} f(x) e^{-1} dx \to 0,$$

as $\epsilon \to 0$ since $r_\epsilon \to \infty$. And the lemma is proved.

■

Using lemma 13, and passing to the limit in (58) leads to:

$$\lim_{\epsilon \to 0} \int \phi (x.\nabla\psi) e^{-\epsilon x^2} = \frac{1}{2} \int |\nabla\psi|^2. \tag{59}$$

ii)$\int div(|\nabla\psi|^\sigma \nabla\psi)(x.\nabla\psi)e^{-\epsilon x^2}$.
By numerous integrations by parts, one obtain:

$$\begin{array}{c} \int div(|\nabla\psi|^\sigma \nabla\psi)(x.\nabla\psi)e^{-\epsilon x^2} = (\frac{3}{\sigma+2} - 1) \int |\nabla\psi|^{\sigma+2} \\ +2\epsilon \int (x.\nabla\psi)^2 |\nabla\psi|^\sigma e^{-\epsilon x^2} - \frac{1}{2}\epsilon \int x^2 |\nabla\psi|^{\sigma+2} e^{-\epsilon x^2}. \end{array} \tag{60}$$

Using lemma 13 and letting $\epsilon \to 0$ in (60) leads to:

$$\lim_{\epsilon \to 0} \int div(|\nabla\psi|^\sigma \nabla\psi)(x.\nabla\psi)e^{-\epsilon x^2} = (\frac{3}{\sigma+2} - 1) \int |\nabla\psi|^{\sigma+2}. \tag{61}$$

iii)$\int \Delta\phi (x.\nabla\psi)e^{-\epsilon x^2}$.
Numerous integrations by parts and the use of lemma 13 leads to:

$$\lim_{\epsilon \to 0} \int \Delta\phi (x.\nabla\psi) e^{-\epsilon x^2} = \frac{1}{2} \int |\phi|^2. \tag{62}$$

From (59), (61) and (62) we deduce

$$\frac{\beta}{2} \int |\nabla\psi|^2 + \frac{\tilde{\epsilon}}{2} \int |\phi|^2 = (1 - \frac{3}{\sigma+2}) \int |\nabla\psi|^{\sigma+2},$$

as claimed in the proposition. ■

Aknowledgements: I wish to thank Professor J.M.Ghidaglia for his help and Professor J.C. Saut for having drawn my attention on the work [6].

References

[1] **H.Berestycki, P.L.Lions** *Nonlinear scalar field equations.* **Arch. Rat. Mech. Anal. 82, 1983, p313-375.**

[2] **T.Cazenave, P.L.Lions** *Orbital stability of standing waves for some nonlinear Schrödinger equations .* **Comm. Math. Phys. 85, 1982, p549-561.**

[3] **T.Colin** *Sur une équation de Schrödinger non linéaire et non locale intervenant en Physique des plasmas.* **C.R. Acad. Sci. Paris, t 314, série I, 1992, p449-452.**

[4] **T.Colin** *On the Cauchy problem for a nonlocal, nonlinear Schrödinger equation occuring in Plasma Phisics.* **To appear.**

[5] **P.L.Lions** *The concentration compactness-principle in the calculus of variation.* **part1 : Ann. Inst. Henri Poincaré, analyse non linéaire, vol1, no2, 1984 p109-145. part2 : Ann. Inst. Henri Poincaré, analyse non linéaire, vol1, no4, 1984 p223-283.**

[6] **S.L.Musher, A.M.Rubenchick, V.E.Zakharov** *Hamiltonian approach to the description of nonlinear plasma phenomena.* **Physics reports 129, no5, 1985, p285-366.**

[7] **J.Shatah, W.Strauss** *Instability of nonlinear bound states.* **Comm. Math. Phys. 100, 1985, p173-190.**

[8] **W.Strauss** *Existence of solitary waves in higher dimensions.* **Comm. Math. Phys. 55, 1977, p149-162.**