

Addendum to
*Weak rate of convergence of the Euler-Maruyama
scheme for stochastic differential equations with
non-regular drift:*
Detailed proof of Proposition 3

Arturo Kohatsu-Higa* Antoine Lejay† Kazuhiro Yasuda‡

August 31, 2016

Abstract

This note presents the detailed proof that the weak rate of convergence of the Euler-Maruyama scheme is of order 1 when the diffusion and drift coefficients are continuously differentiable in time and three times continuously differentiable in space. For the sake of our purpose, we also give explicit bounds on the constants which control the convergence of the scheme.

In the present note, we provide details of the proof of Proposition 3 in the accompanying article “Weak rate of convergence of the Euler-Maruyama scheme for stochastic differential equations with non-regular drift” by Arturo Kohatsu-Higa, Antoine Lejay and Kazuhiro Yasuda [5].

In the context and the notations, we refer to [5]. We recall however

$$\|f\|_{r,q,n} := \left(\int_0^T \left(\int_{\mathbb{R}^d} |f(t,x)|^r dt \right)^{q/r} \right)^{1/q} + \frac{1}{\sqrt{n}} |f(0,x)|,$$

$$S(f)(s,x) := \frac{f(s,x) - f(\phi(s),x)}{s - \phi(s)} \text{ and } \mathfrak{S}(f)(s,x) := \sup_{t \in (\phi(s),s)} |S(f)(t,x)|$$

with $\phi(s) = \sup\{kT/n \in [0, T]; k = 0, 1, \dots, n\}$. Besides, the process X^ϵ is the solution to the Stochastic Differential Equation

$$X_t^\epsilon = x + \int_0^t \sigma(s, X_s^\epsilon) dW_s + \int_0^t b_\epsilon(s, X_s^\epsilon) ds$$

for a Brownian motion W , while \bar{X}^ϵ is its continuous Euler-Maruyama scheme

$$\bar{X}^\epsilon = x + \int_0^t \sigma(\phi(s), \bar{X}_{\phi(s)}^\epsilon) dW_s + \int_0^t b_\epsilon(\phi(s), \bar{X}_{\phi(s)}^\epsilon) ds.$$

*Department of Mathematical Sciences — Ritsumeikan University and Japan Science and Technology Agency— 1-1-1 Nojihigashi, Kusatsu, Shiga, 525-8577, Japan.

†Université de Lorraine, IECL, UMR 7502, Vandœuvre-lés-Nancy, F-54500, France & CNRS, IECL, UMR 7502, Vandœuvre-lés-Nancy, F-54500, France & Inria, Villers-lés-Nancy, F-54600, France.

‡Faculty of Science and Engineering — Hosei University — 3-7-2, Kajino-cho, Koganei-shi, Tokyo, 184-8584, Japan.

Besides, we set

$$\mathbf{b}_\epsilon := (b_\epsilon, b_\epsilon^1) \text{ with } b_\epsilon^1 = b_\epsilon^* a^{-1} b_\epsilon.$$

Remark 1. The theorem involves a drift b and a family of drift b_ϵ which are smooth approximations of b . In the proof, we are only working in b_ϵ . We could have used directly a drift of class $\mathcal{C}_b^{1,3}$. Yet we keep the formulation of Proposition 3 in [5] to show that the constants which appears in the bounds do not depend on ϵ . This is the key point of this proposition.

Proposition 3 in [5]. Assume $\sigma \in \mathcal{C}_b^{1,3}([0, T] \times \mathbb{R}^d)$ be such that

$$\lambda|\xi|^2 \leq \xi^* a(t, x) \xi \leq \Lambda|\xi|^2 \text{ for all } (t, x, \xi) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \text{ with } a := \sigma \sigma^*,$$

b be measurable and bounded by Λ and for any $\epsilon > 0$, $b_\epsilon \in \mathcal{C}_b^{1,3}([0, T] \times \mathbb{R}^d)$ be bounded by Λ . We consider f in $\mathcal{C}_{S^1}^3(\mathbb{R}^d)$. Then there exists index sets $\mathcal{A}(\ell)$ such that the following bound is satisfied:

$$\begin{aligned} & |\mathbb{E}[f(X_T^\epsilon)] - \mathbb{E}[f(\bar{X}_T^\epsilon)]| \\ & \leq \frac{C(\|b_\epsilon\|_\infty)}{n} \left(\sum_{\substack{\ell \in \{0, \dots, 3\}^4 \\ \ell_1 + \dots + \ell_4 \leq 4}} \sum_{(r, q) \in \mathcal{A}(\ell)} \prod_{i=1}^4 \|\partial_x^{\ell_i} \mathbf{b}_\epsilon\|_{r_i, q_i, n} + \|\mathcal{S}(\mathbf{b}_\epsilon)\|_{r, q, n} \times (1 + \|\partial_x \mathbf{b}_\epsilon\|_{r, q, n}) \right). \end{aligned}$$

The index sets $\mathcal{A}(\ell)$ are non-empty and can be chosen freely so that they satisfy the following conditions.

In the first sum, the set $\mathcal{A}(\ell)$ is a finite combination of indices (r_i, q_i) , $i = 1, 2, 3, 4$ such that they satisfy that $q_i, r_i \in (p_i \vee 2, \infty]$ with $\frac{d}{2q_i} + \frac{1}{r_i} < \frac{1}{p_i \vee 2}$ for any $(p_1, \dots, p_4) \in (1, \infty)^4$ which are almost conjugate, that is $1/p_1 + \dots + 1/p_4 < 1$. In the particular case that $\ell_1 + \dots + \ell_4 = 4$ then at least one of these indices satisfy that $q, r \in (p, \infty]$, with $\frac{d}{2q} + \frac{1}{r} < \frac{1}{p}$ for any $p > 1$. This condition is also satisfied by the indices (q, r) which appear in the last term above.

1 Fundamental expansions

In order to make the proof as simple as possible we will only do it here in the one dimensional case. The multi-dimensional case follows similarly. Set $\Gamma^\epsilon(s) = b_\epsilon \sigma^{-1}(s, Y_s^\epsilon)$ and $\tilde{\Gamma}^\epsilon(s) = b_\epsilon \sigma^{-1}(\phi(s), \tilde{Y}_{\phi(s)})$ and in general $\nabla^i \Gamma^\epsilon(s) = \nabla^i (b_\epsilon \sigma^{-1})(s, Y_s^\epsilon)$ and $\nabla^i \tilde{\Gamma}^\epsilon(s) = \nabla^i (b_\epsilon \sigma^{-1})(\phi(s), \tilde{Y}_{\phi(s)})$. Let θ be a 1-dimensional $[0, 1]$ -uniform random variable which is independent of any other random variables. This notation will be used in many other variants in the calculation to follow without any further explanation. For example, θ_z, θ' etc. are also uniform random variables independent of all other random variables in the problem.

$$\begin{aligned} & \mathbb{E}[f(X_T^\epsilon)] - \mathbb{E}[f(\bar{X}_T^\epsilon)] \\ & = \mathbb{E}[\mathcal{Z}(\Gamma^\epsilon)_T f(Y_T)] - \mathbb{E}[\mathcal{Z}(\tilde{\Gamma}^\epsilon)_T f(\bar{Y}_T)] \\ & = \mathbb{E}[\mathcal{Z}(\Gamma^\epsilon)_T (f(Y_T) - f(\bar{Y}_T))] - \mathbb{E}[(\mathcal{Z}(\Gamma^\epsilon)_T - \mathcal{Z}(\tilde{\Gamma}^\epsilon)_T) f(\bar{Y}_T)] \\ & = \mathbb{E}[\mathcal{Z}(\Gamma^\epsilon)_T \nabla f(\theta Y_T + (1 - \theta) \bar{Y}_T)(Y_T - \bar{Y}_T)] \\ & \quad - \mathbb{E}[\exp(\theta_z \log \mathcal{Z}(\Gamma^\epsilon)_T + (1 - \theta_z) \log \mathcal{Z}(\tilde{\Gamma}^\epsilon)_T) (\log \mathcal{Z}(\Gamma^\epsilon)_T - \log \mathcal{Z}(\tilde{\Gamma}^\epsilon)_T) f(\bar{Y}_T)] \\ & = \mathbb{E}[F \mathcal{E}_T] - \mathbb{E}[F_1 (A_1 + A_2 + A_3 + A_4)], \end{aligned} \tag{1}$$

where we have used the notation

$$\begin{aligned}
F &= \mathcal{Z}(\Gamma^\epsilon)_T \nabla f(\theta Y_T + (1 - \theta) \bar{Y}_T), \\
F_1 &= \exp(\theta_z \log \mathcal{Z}(\Gamma^\epsilon)_T + (1 - \theta_z) \log \mathcal{Z}(\tilde{\Gamma}^\epsilon)_T) f(\bar{Y}_T), \\
\mathcal{E}_t &= Y_t - \bar{Y}_t, \\
A_1 &= \int_0^T (\nabla(b_\epsilon \sigma^{-1})(s, \theta' Y_s + (1 - \theta') \bar{Y}_s)) \mathcal{E}_s dW_s, \\
A_2 &= \int_0^T (b_\epsilon \sigma^{-1}(s, \bar{Y}_s) - b_\epsilon \sigma^{-1}(\phi(s), \bar{Y}_{\phi(s)})) dW_s, \\
A_3 &= -\frac{1}{2} \int_0^T (\nabla(b_\epsilon a^{-1} b_\epsilon)(s, \dot{\theta} Y_s + (1 - \dot{\theta}) \bar{Y}_s)) \mathcal{E}_s ds, \\
A_4 &= -\frac{1}{2} \int_0^T (b_\epsilon a^{-1} b_\epsilon(s, \bar{Y}_s) - b_\epsilon a^{-1} b_\epsilon(\phi(s), \bar{Y}_{\phi(s)})) ds.
\end{aligned}$$

The error process may be written as follows:

$$\mathcal{E}_t = \int_0^t \alpha(s) \mathcal{E}_s dW_s + \int_0^t G_s dW_s, \quad (2)$$

where

$$G_s = \sigma(s, \bar{Y}_s) - \sigma(\phi(s), \bar{Y}_{\phi(s)}), \quad (3)$$

and $\alpha(s)$ is given by

$$\alpha(s) = \int_0^1 \nabla \sigma(s, \theta Y_s + (1 - \theta) \bar{Y}_s) d\theta.$$

Then we can rewrite

$$\mathcal{E}_t = U_t \int_0^t U_s^{-1} G_s dW_s - U_t \int_0^t U_s^{-1} \alpha(s) G_s ds,$$

where U and U^{-1} are the processes which are solutions of the stochastic differential equation

$$U_t = 1 + \int_0^t \alpha(s) U_s dW_s \text{ and } U_t^{-1} = 1 - \int_0^t U_s^{-1} \alpha(s) dW_s.$$

And

$$\begin{aligned}
G_s &= \sigma(s, \bar{Y}_{\phi(s)}) - \sigma(\phi(s), \bar{Y}_{\phi(s)}) + \sigma(s, \bar{Y}_s) - \sigma(s, \bar{Y}_{\phi(s)}) \\
&= \int_{\phi(s)}^s \partial_r \sigma(r, \bar{Y}_{\phi(s)}) dr + \underbrace{\nabla \sigma(s, \theta \bar{Y}_s + (1 - \theta) \bar{Y}_{\phi(s)})}_{=: \sigma(\phi(s), \bar{Y}_s, \theta)} (\bar{Y}_s - \bar{Y}_{\phi(s)}) \\
&= \int_{\phi(s)}^s \partial_r \sigma(r, \bar{Y}_{\phi(s)}) dr + (\nabla \sigma(s, \bar{Y}_s, \theta)) \sigma(\phi(s), \bar{Y}_{\phi(s)}) \int_{\phi(s)}^s dW_r.
\end{aligned}$$

We remark at this point that is well known (see e.g. [6], [3], [4], [1], [2]) that once the Girsanov transformation is used in order to have a smooth diffusion Y and its approximation \bar{Y} they satisfy

that for any non-negative integers p, r there exists a constant $C_{p,r} > 0$ such that

$$\sup_{n,s \in [0,T]^r} \mathbb{E} \left[\sup_{t \in [0,T]} |D_s^r Y_t|^p + |D_s^r \bar{Y}_t|^p \right] \leq C_{p,r}, \quad (4)$$

$$\sup_{n,t \in [0,T]} \mathbb{E} \left[\sup_{s \in [0,t]^r} |D_s^r Y_t|^p + |D_s^r \bar{Y}_t|^p \right] \leq C_{p,r}. \quad (5)$$

Note that furthermore one has that $D_s Y_t = A_s B_t$ and $D_s \bar{Y}_t = \bar{A}_s \bar{B}_t$ for some appropriate smooth adapted processes A, B, \bar{A} and \bar{B} which satisfy that $\sup_n \mathbb{E}[\sup_s |X_s|^p] \leq C_p$ for a positive constant C_p and $X = A, B, \bar{A}$ and \bar{B} .

In the above sense many terms will be computed using a symbolic notation as follows.

Definition 1. Under the assumption that $\sigma \in \mathfrak{D}$ (that is, $\|\sigma\|_{1,3} < \infty$ where $\|\cdot\|_{i,j}$ denotes sup-norm for up to (i, j) derivative). We will say that a term is negligible of order (i, j) and denoted by $N^{(i,j)}$ if it is a random variable that may depend on various time parameters and it satisfies that for any non-negative integer p , there exists a constants $C_p > 0$ such that for $j + r \leq 3$

$$\sup_{n,s \in [0,T]^r, t} \mathbb{E} \left[|D_s^r N_t^{(i,j)}|^p \right] \leq C_p (\|\sigma\|_{i,j+r}).$$

Note that t represent the time parameter(s)¹ related to N and they may vary depending on the situation. The constants above may depend on the norm $\|\sigma\|_{i,j+r}$. We will use this definition for $i = 0, 1$ and $j = 0, 1, 2, 3$. To further simplify notation we let $N \equiv N^{(0,0)}$. Also in many situations the time variable is not important and therefore we also use the following reduction in notation $N^i \equiv N^{(0,i)}$.

Note that with this definition, we have the following properties

Lemma 1. • $N^{(i_1, j_1)} \times N^{(i_2, j_2)} + N^{(i_3, j_3)} = N^{(i, j)}$ where $i = \max\{i_1, i_2, i_3\}$ and similarly for j .

- $DN^{(i,j)} = N^{(i,j+1)}$.
- $\alpha(s) = N_s^{(0,1)}$, $U_t = N_t^{(0,1)}$, $U_t^{-1} = N_t^{(0,1)}$, $D_s Y_t = N_s^{(0,1)} N_t^{(0,1)}$.

To proceed with this symbolic notation, we define

Definition 2. Under the assumption that $\sigma \in \mathfrak{D}$ (that is, $\|\sigma\|_{1,3} < \infty$). We will say that a term is negligible of order (i, j) and denoted by $\bar{N}^{(i,j)}$ which satisfies

$$\bar{N}^{(i,j)} = \exp \left(A \int_0^T \Gamma_u^\epsilon dW_u - B \int_0^T (\Gamma_u^\epsilon)^2 du \right) N^{(i,j)}$$

for some positive random variables A and B which are independent of W . Note that as before $\bar{N}^{(i,j)}$ may depend on various time parameters and it satisfies that for any non-negative integer p , there exists a constants $C_p > 0$ such that for $j \leq 3$

$$\sup_{n,t, \epsilon \in (0,1)} \mathbb{E} \left[|\bar{N}_t^{(i,j)}|^p \right] \leq C_p (\|\sigma\|_{i,j}) \exp(C_p (\|\sigma\|_{0,0}) \|b_\epsilon\|_\infty).$$

A similar definition related to $\tilde{\Gamma}_\epsilon$ will be used, originating the notation $\tilde{N}^{(i,j)}$ and define $\hat{N}^{(i,j)} = \bar{N}^{(i_1, j_1)} \bar{N}^{(i_2, j_2)}$ where $i = i_1 \vee i_2$ and $j = j_1 \vee j_2$. Also the simplified notation \bar{N}^j , \tilde{N}^j and \hat{N}^j will be used.

¹That is, t may be a vector.

Then we have

$$\begin{aligned} F\mathcal{E}_T &= F \left[U_T \int_0^T U_s^{-1} G_s dW_s - U_T \int_0^T U_s^{-1} \alpha(s) G_s ds \right] \\ &= F U_T \int_0^T U_s^{-1} \int_{\phi(s)}^s \partial_t \sigma(r, \bar{Y}_{\phi(s)}) dr dW_s \end{aligned} \quad (6)$$

$$+ F U_T \int_0^T U_s^{-1} (\nabla \sigma_{i,j}(s, \bar{Y}_s, \underline{\theta})) \sigma(\phi(s), \bar{Y}_{\phi(s)}) \int_{\phi(s)}^s dW_r dW_s \quad (7)$$

$$- F U_T \int_0^T U_s^{-1} \alpha(s) \int_{\phi(s)}^s \partial_t \sigma(r, \bar{Y}_{\phi(s)}) dr ds \quad (8)$$

$$- F U_T \int_0^T U_s^{-1} \alpha(s) (\nabla \sigma(s, \bar{Y}_s, \underline{\theta})) \sigma(\phi(s), \bar{Y}_{\phi(s)}) \int_{\phi(s)}^s dW_r ds. \quad (9)$$

Next we have, as above,

$F_1 A_1$

$$= F_1 \int_0^T (\nabla(b_\epsilon \sigma^{-1})(s, Y_s, \bar{Y}_s, \theta')) U_s \int_0^s U_r^{-1} \int_{\phi(r)}^r \partial_t \sigma(u, \bar{Y}_{\phi(r)}) du dW_r dW_s \quad (10)$$

$$+ F_1 \int_0^T (\nabla(b_\epsilon \sigma^{-1})(s, Y_s, \bar{Y}_s, \theta')) U_s \int_0^s U_r^{-1} (\nabla \sigma(r, \bar{Y}_r, \underline{\theta})) \sigma(\phi(r), \bar{Y}_{\phi(r)}) \int_{\phi(r)}^r dW_u dW_r dW_s \quad (11)$$

$$- F_1 \int_0^T (\nabla(b_\epsilon \sigma^{-1})(s, Y_s, \bar{Y}_s, \theta')) U_s \int_0^s U_r^{-1} \alpha(r) \int_{\phi(r)}^r \partial_t \sigma(u, \bar{Y}_{\phi(r)}) du dr dW_s \quad (12)$$

$$- F_1 \int_0^T (\nabla(b_\epsilon \sigma^{-1})(s, Y_s, \bar{Y}_s, \theta')) U_s \int_0^s U_r^{-1} \alpha(r) (\nabla \sigma(r, \bar{Y}_r, \underline{\theta})) \sigma(\phi(r), \bar{Y}_{\phi(r)}) \int_{\phi(r)}^r dW_u dr dW_s. \quad (13)$$

Set $H_s = b_\epsilon \sigma^{-1}(s, \bar{Y}_s) - b_\epsilon \sigma^{-1}(\phi(s), \bar{Y}_{\phi(s)})$. As before,

$$\begin{aligned} H_s &= b_\epsilon \sigma^{-1}(s, \bar{Y}_{\phi(s)}) - b_\epsilon \sigma^{-1}(\phi(s), \bar{Y}_{\phi(s)}) + b_\epsilon \sigma^{-1}(s, \bar{Y}_s) - b_\epsilon \sigma^{-1}(s, \bar{Y}_{\phi(s)}) \\ &= \int_{\phi(s)}^s \partial_t b_\epsilon \sigma^{-1}(r, \bar{Y}_{\phi(s)}) dr + \underbrace{\nabla(b_\epsilon \sigma^{-1})(s, \underline{\theta}' \bar{Y}_s + (1 - \underline{\theta}') \bar{Y}_{\phi(s)})}_{=:(b_\epsilon \sigma^{-1})(\phi(s), \bar{Y}_s, \underline{\theta}')} (\bar{Y}_s - \bar{Y}_{\phi(s)}) \\ &= \int_{\phi(s)}^s \partial_t b_\epsilon \sigma^{-1}(r, \bar{Y}_{\phi(s)}) dr + (\nabla(b_\epsilon \sigma^{-1})(s, \bar{Y}_s, \underline{\theta}')) \int_{\phi(s)}^s \sigma(\phi(s), \bar{Y}_{\phi(r)}) dW_r \\ &= \int_{\phi(s)}^s \partial_t b_\epsilon \sigma^{-1}(r, \bar{Y}_{\phi(s)}) dr + (\nabla(b_\epsilon \sigma^{-1})(s, \bar{Y}_s, \underline{\theta}')) \sigma(\phi(s), \bar{Y}_{\phi(s)}) \int_{\phi(s)}^s dW_r. \end{aligned}$$

From now on, we will use the symbolic notation N in order to simplify expressions once the orders of the terms are clearly obtained. Then we have

$$\begin{aligned} F_1 A_2 &= F_1 \int_0^T (b_\epsilon \sigma^{-1}(s, \bar{Y}_s) - b_\epsilon \sigma^{-1}(\phi(s), \bar{Y}_{\phi(s)})) dW_s \\ &= F_1 \int_0^T \int_{\phi(s)}^s \partial_t (b_\epsilon \sigma^{-1})(r, \bar{Y}_{\phi(s)}) dr dW_s \end{aligned} \quad (14)$$

$$+ F_1 \int_0^T (\nabla(b_\epsilon \sigma^{-1})(s, \bar{Y}_s, \underline{\theta}')) N_s \int_{\phi(s)}^s dW_r dW_s. \quad (15)$$

We have from the explicit solution for \mathcal{E}

$$\begin{aligned}
& F_1 A_3 \\
&= -\frac{1}{2} F_1 \int_0^T \nabla \underbrace{(b_\epsilon a^{-1} b_\epsilon)(s, \dot{\theta} Y_s + (1 - \dot{\theta}) \bar{Y}_s)}_{=:(b_\epsilon a^{-1} b_\epsilon)(s, Y_s, \bar{Y}_s, \dot{\theta})} \mathcal{E}_s ds \\
&= -\frac{1}{2} F_1 \int_0^T (\nabla(b_\epsilon a^{-1} b_\epsilon)(s, Y_s, \bar{Y}_s, \dot{\theta})) U_s \int_0^s U_r^{-1} \int_{\phi(r)}^r N_{u,r}^{(1,0)} du dW_r ds \tag{16}
\end{aligned}$$

$$-\frac{1}{2} F_1 \int_0^T (\nabla(b_\epsilon a^{-1} b_\epsilon)(s, Y_s, \bar{Y}_s, \dot{\theta})) U_s \int_0^s U_r^{-1} N_r^{0,1} \int_{\phi(r)}^r dW_u dW_r ds \tag{17}$$

$$+\frac{1}{2} F_1 \int_0^T (\nabla(b_\epsilon a^{-1} b_\epsilon)(s, Y_s, \bar{Y}_s, \dot{\theta})) U_s \int_0^s U_r^{-1} N_r^{(0,1)} \int_{\phi(r)}^r N_{u,r}^{(1,0)} dudr ds \tag{18}$$

$$+\frac{1}{2} F_1 \int_0^T (\nabla(b_\epsilon a^{-1} b_\epsilon)(s, Y_s, \bar{Y}_s, \dot{\theta})) U_s \int_0^s U_r^{-1} N_r^{(0,1)} \int_{\phi(r)}^r dW_u dr ds. \tag{19}$$

Finally set $I_s = b_\epsilon a^{-1} b_\epsilon(s, \bar{Y}_s) - b_\epsilon a^{-1} b_\epsilon(\phi(s), \bar{Y}_{\phi(s)})$, and we have

$$\begin{aligned}
I_s &= b_\epsilon a^{-1} b_\epsilon(s, \bar{Y}_{\phi(s)}) - b_\epsilon a^{-1} b_\epsilon(\phi(s), \bar{Y}_{\phi(s)}) + b_\epsilon a^{-1} b_\epsilon(s, \bar{Y}_s) - b_\epsilon a^{-1} b_\epsilon(s, \bar{Y}_{\phi(s)}) \\
&= \int_{\phi(s)}^s \partial_t b_\epsilon a^{-1} b_\epsilon(r, \bar{Y}_{\phi(s)}) dr + \nabla \underbrace{(b_\epsilon a^{-1} b_\epsilon)(s, \underline{\theta}'' * \bar{Y}_s + (1 - \underline{\theta}'') \bar{Y}_{\phi(s)})}_{=:(b_\epsilon a^{-1} b_\epsilon)(\phi(s), \bar{Y}_s, \underline{\theta}'')} (\bar{Y}_s - \bar{Y}_{\phi(s)}) \\
&= \int_{\phi(s)}^s \partial_t b_\epsilon a^{-1} b_\epsilon(r, \bar{Y}_{\phi(s)}) dr + (\nabla(b_\epsilon a^{-1} b_\epsilon)(s, \bar{Y}_s, \underline{\theta}'')) N_s \int_{\phi(s)}^s dW_r.
\end{aligned}$$

Then we have

$$\begin{aligned}
F_1 A_4 &= -\frac{1}{2} F_1 \int_0^T (b_\epsilon a^{-1} b_\epsilon(s, \bar{Y}_s) - b_\epsilon a^{-1} b_\epsilon(\phi(s), \bar{Y}_{\phi(s)})) ds \\
&= -\frac{1}{2} F_1 \int_0^T \int_{\phi(s)}^s \partial_t (b_\epsilon a^{-1} b_\epsilon)(r, \bar{Y}_{\phi(s)}) dr ds \tag{20}
\end{aligned}$$

$$-\frac{1}{2} F_1 \int_0^T (\nabla(b_\epsilon a^{-1} b_\epsilon)(s, \bar{Y}_s, \underline{\theta}'')) N_s \int_{\phi(s)}^s dW_r ds. \tag{21}$$

By summing up the above results, that is, $F\mathcal{E}_T + F_1(A_1 + A_2 + A_3 + A_4)$, in (1), we have its integral expression given in our paper as equation (15):

$$\begin{aligned}
& \sum_{i,j=0}^1 c_{i,j}^0(T) \int_0^T c_{i,j}^1(t) \int_{\phi(t)}^t c_{i,j}^2(t, s) dW_s^i dW_t^j \\
& + \sum_{i,j,k=0}^1 c_{i,j,k}^0(T) \int_0^T c_{i,j,k}^1(t) \left[\int_0^t c_{i,j,k}^2(t, s) \left(\int_{\phi(s)}^s c_{i,j,k}^3(t, s, u) dW_u^i \right) dW_s^j \right] dW_t^k. \tag{22}
\end{aligned}$$

Note that from the above calculations for each term in (1), we have the following properties:

- When $i = 0$, $c_{0,j}^2$ is given as the derivative in t , and when $i \neq 0$, $c_{i,j}^2 = 1$. This property is also true for $c_{i,j,k}^3$, that is, when $i = 0$, $c_{0,j,k}^3$ is given as the derivative in t , and when $i \neq 0$, $c_{i,j,k}^3 = 1$. And when $i = 0$, we use the following: for a suitable function $c(t, x)$, using the notation in the paper,

$$\left| \int_{\phi(t)}^t \partial_t c(s, x) ds \right| = |c(t, x) - c(\phi(t), x)| = |S(c)(t, x)| |t - \phi(t)| \leq \frac{1}{n} S(c)(t, x). \tag{23}$$

- From above calculations, we find $c_{i,j}^0(T) = FU_T$ or F_1 , and $c_{i,j,k}^0(T) = F_1$. Note that U_T does not depend on b_ϵ . As we can see from the definition of FU_T and F_1 , they share the same properties as long as we only consider their L^p -norms, their stochastic derivatives and their corresponding norms. Without loss of generality, instead of considering the long expression for $c_{i,j}^0(T)$, we only consider the term FU_T . Note that the term F_1 is also included in $c_{i,j,k}^0(T) = F_1$.

2 Malliavin derivatives

We will now start to fully use the symbolic notation introduced by N . In fact, note that $\mathcal{Z}(\Gamma^\epsilon)_T = \bar{N}_T$. Therefore

$$D_s \mathcal{Z}(\Gamma^\epsilon)_T = \mathcal{Z}(\Gamma^\epsilon)_T \left(\Gamma_s^\epsilon + \int_s^T (D_s Y_u) \nabla \Gamma_u^\epsilon dW_u - \frac{1}{2} \int_s^T \nabla(\Gamma_u^\epsilon \Gamma_u^\epsilon) D_s Y_u du \right) = \gamma_s^{\epsilon,0} \bar{N}_T.$$

Using this, we have that $D_s \bar{N}_T = \gamma_s^{\epsilon,0} \bar{N}_T + \bar{N}_{s,T}^{(0,1)}$ where we will introduce the following symbolic notation

$$\gamma_{\mathbf{t},s}^{\epsilon,i} = \vec{\nabla}^i \Gamma_w^\epsilon \cdot \mathbf{N}_{\mathbf{t}}^i + \int_w^T \vec{\nabla}^{i+1} \Gamma_u^\epsilon \cdot \mathbf{N}_{\mathbf{t},s,u}^{i+1} dW_u + \int_w^T \vec{\nabla}^{i+1} (\Gamma_u^\epsilon)^2 \cdot \mathbf{N}_{\mathbf{t},s,u}^{i+1} du. \quad (24)$$

Here $w \equiv w(\mathbf{t}, s) = \max\{\mathbf{t}, s\}$ (meaning that this is the biggest component in the time vector \mathbf{t} and s) and $\vec{\nabla}^i \Gamma^\epsilon = (\partial_x^i \Gamma^\epsilon, \dots, \partial_x^i \Gamma^\epsilon)$ for $i \geq 1$ and $\vec{\nabla}^0 \Gamma^\epsilon = \Gamma^\epsilon$. The inner product is denoted by \cdot and the fact that here \mathbf{N}^i represents a vector of negligible components of order $(0, j)$ with $j \leq i$ such that each product inside the integrals of the form $\vec{\nabla}^j \Gamma^\epsilon N^k$ has to satisfy the condition that $j+k = i+2$, $j, k \geq 1$. For the first terms without integrals each product has to satisfy that $j+k = i+1$, $j, k \geq 1$. In the particular case that $i = 0$, then we allow $N^{-1} \equiv 1$ so that we have that $\vec{\nabla}^0 \Gamma^\epsilon N^{-1} = \Gamma^\epsilon$. Similarly, we also introduce the notation $\tilde{\gamma}_t^{\epsilon,i}$ for $\tilde{\Gamma}^\epsilon$. It is not difficult to check $D_t^i \gamma_s^{\epsilon,0} = \gamma_{\mathbf{t},s}^{\epsilon,i}$. It follows from (24) that

$$\gamma_s^{\epsilon,0} = \Gamma_s^\epsilon + N_s^1 \int_s^T N_u^1 \nabla \Gamma_u^\epsilon dW_u - \frac{N_s^1}{2} \int_s^T \nabla(\Gamma_u^\epsilon)^2 N_u^1 du. \quad (25)$$

In this particular case note that we also have that $\mathbb{E}[\sup_s |N_s^1|^p] \leq C_p$. In order to further simplify notation, and together with the above understanding we let: $\mathbf{N}^i = (N^i, \dots, N^1)$ and for a vector \mathbf{t} ,

$$\begin{aligned} I^0(\vec{\nabla}^i(\Gamma^\epsilon)^2 \cdot \mathbf{N}_{\mathbf{t}}^i)_w &:= \int_w^T \vec{\nabla}^i(\Gamma_u^\epsilon)^2 \cdot \mathbf{N}_{\mathbf{t},u}^i du, \\ I^1(\vec{\nabla}^i \Gamma^\epsilon \cdot \mathbf{N}_{\mathbf{t}}^i)_w &:= \int_w^T \vec{\nabla}^i \Gamma_u^\epsilon \cdot \mathbf{N}_{\mathbf{t},u}^i dW_u, \\ \mathbf{I}(\vec{\nabla}^i \Gamma^\epsilon \cdot \mathbf{N}_{\mathbf{t}}^i)_w &:= I^0(\vec{\nabla}^i(\Gamma^\epsilon)^2 \cdot \mathbf{N}_{\mathbf{t}}^i)_w + I^1(\vec{\nabla}^i \Gamma^\epsilon \cdot \mathbf{N}_{\mathbf{t}}^i)_w. \end{aligned}$$

As before, we also introduce similar notation for $\tilde{\gamma}_{\mathbf{t},s}^{\epsilon,i}$.

This leads to the following notation:

$$\mathbf{I}(\vec{\nabla}^i \tilde{\Gamma}^\epsilon \cdot \mathbf{N}_{\mathbf{t}}^i)_w := I^0(\vec{\nabla}^i(\tilde{\Gamma}^\epsilon)^2 \cdot \mathbf{N}_{\mathbf{t}}^i)_w + I^1(\vec{\nabla}^i \tilde{\Gamma}^\epsilon \cdot \mathbf{N}_{\mathbf{t}}^i)_w.$$

For further simplification we also let

$$\begin{aligned}\Gamma^\epsilon &:= (\Gamma^\epsilon, (\Gamma^\epsilon)^2), \\ \tilde{\Gamma}^\epsilon &:= (\tilde{\Gamma}^\epsilon, (\tilde{\Gamma}^\epsilon)^2), \\ \hat{\Gamma}^\epsilon &:= \Gamma^\epsilon + \tilde{\Gamma}^\epsilon, \\ (\hat{\Gamma}^\epsilon)^2 &:= (\Gamma^\epsilon)^2 + (\tilde{\Gamma}^\epsilon)^2.\end{aligned}$$

This leads to the notation

$$\mathbf{I}(\vec{\nabla}^i \hat{\Gamma}^\epsilon \cdot \mathbf{N}_t^i)_w := I^0(\vec{\nabla}^i (\hat{\Gamma}^\epsilon)^2 \cdot \mathbf{N}_t^i)_w + I^1(\vec{\nabla}^i \hat{\Gamma}^\epsilon \cdot \mathbf{N}_t^i)_w.$$

With this notation, we have now the following equalities:

$$\gamma_{t,s}^{\epsilon,i} = \vec{\nabla}^i \Gamma_w^\epsilon \cdot \mathbf{N}_t^i + \mathbf{I}(\vec{\nabla}^{i+1} \Gamma^\epsilon \cdot \mathbf{N}_{t,s}^{i+1})_w. \quad (26)$$

Note that we can also introduce the notation

$$\begin{aligned}\hat{\gamma}_{t,s}^{\epsilon,i} &:= \vec{\nabla}^i \hat{\Gamma}_w^\epsilon \cdot \mathbf{N}_t^i + \mathbf{I}(\vec{\nabla}^{i+1} \hat{\Gamma}^\epsilon \cdot \mathbf{N}_{t,s}^{i+1})_w \\ &:= \gamma_{t,s}^{\epsilon,i} + \tilde{\gamma}_{t,s}^{\epsilon,i}.\end{aligned}$$

With this notation, we have

$$D_t^2 \mathcal{Z}(\Gamma^\epsilon)_T = \sum_{\substack{j,k \geq -1 \\ (j+1)+(k+1)=2 \\ \mathbf{t}_1 \cup \mathbf{t}_2 = \mathbf{t}}}^1 \gamma_{\mathbf{t}_1}^{\epsilon,j} \gamma_{\mathbf{t}_2}^{\epsilon,k} \bar{N}_T,$$

and

$$D_t^3 \mathcal{Z}(\Gamma^\epsilon)_T = \sum_{\substack{j,k,l \geq -1 \\ (j+1)+(k+1)+(l+1)=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \mathbf{t}}}^2 \gamma_{\mathbf{t}_1}^{\epsilon,j} \gamma_{\mathbf{t}_2}^{\epsilon,k} \gamma_{\mathbf{t}_3}^{\epsilon,l} \bar{N}_T.$$

Here, we use the convention that $\gamma_{\mathbf{t}}^{\epsilon,-1} \equiv 1$ and the cardinality of the sets \mathbf{t}_1 , \mathbf{t}_2 and \mathbf{t}_3 should correspond to $j+1$, $k+1$ and $l+1$. Similar convention is used in the second sum and the ones to follow. Note that

$$\begin{aligned}c_{i_0 j_0}^0(T) &= \mathcal{Z}(\Gamma^\epsilon)_T U_T = \mathcal{Z}(\Gamma^\epsilon)_T N_T^1, \\ c_{i_0 j_0 k_0}^0(T) &= \mathcal{Z}(\Gamma^\epsilon)_T^{\theta_z} \mathcal{Z}(\tilde{\Gamma}^\epsilon)_T^{1-\theta_z} N_T.\end{aligned}$$

Therefore one has

$$D_t c_{i_0 j_0}^0(T) = \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \gamma_t^{\epsilon,j} \bar{N}_{t,T}^k, \quad (27)$$

$$D_t^2 c_{i_0 j_0}^0(T) = \sum_{\substack{j,k \geq -1 \\ l \geq 1, (j+1)+(k+1)+l=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \mathbf{t}}}^1 \gamma_{\mathbf{t}_1}^{\epsilon,j} \gamma_{\mathbf{t}_2}^{\epsilon,k} \bar{N}_{\mathbf{t}_3, T}^l, \quad (28)$$

and

$$D_{\mathbf{t}}^3 c_{i_0 j_0}^0(T) = \sum_{\substack{j,k,l \geq -1 \\ m \geq 1, (j+1)+(k+1)+(l+1)+m=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 \cup \mathbf{t}_4 = \mathbf{t}}} \gamma_{\mathbf{t}_1}^{\epsilon,j} \gamma_{\mathbf{t}_2}^{\epsilon,k} \gamma_{\mathbf{t}_3}^{\epsilon,l} \bar{N}_{\mathbf{t}_4, T}^m.$$

The same formulas as above are also satisfied by $c_{i_0 j_0 k_0}^0$ with the corresponding modification due to the last term N_T and replacing γ by $\hat{\gamma}$. That is,

$$D_{\mathbf{t}} c_{i_0 j_0 k_0}^0(T) = \sum_{\substack{j \geq -1 \\ k \geq 0, (j+1)+k=1}} \hat{\gamma}_{\mathbf{t}}^{\epsilon,j} \hat{N}_{\mathbf{t}, T}^k,$$

$$D_{\mathbf{t}}^2 c_{i_0 j_0 k_0}^0(T) = \sum_{\substack{j,k \geq -1 \\ l \geq 0, (j+1)+(k+1)+l=2 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \mathbf{t}}} \hat{\gamma}_{\mathbf{t}_1}^{\epsilon,j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon,k} \hat{N}_{\mathbf{t}_3, T}^l,$$

and

$$D_{\mathbf{t}}^3 c_{i_0 j_0 k_0}^0(T) = \sum_{\substack{j,k,l \geq -1 \\ m \geq 0, (j+1)+(k+1)+(l+1)+m=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 \cup \mathbf{t}_4 = \mathbf{t}}} \hat{\gamma}_{\mathbf{t}_1}^{\epsilon,j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon,k} \hat{\gamma}_{\mathbf{t}_3}^{\epsilon,l} \hat{N}_{\mathbf{t}_4, T}^m.$$

2.1 Malliavin derivatives

From now on, we always assume $0 \leq u \leq r \leq s \leq T$, which comes from (22). Now we start to calculate Malliavin derivatives which are used in the following sections.

The following is used in (17) below for $\mathbb{E}[F\mathcal{E}_T]$.

$$D_s \bar{N}_T^{(0,1)} = \gamma_s^{\epsilon,0} \bar{N}_T^{(0,1)} + \bar{N}_{s,T}^{(0,2)} = \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}} \gamma_s^{\epsilon,j} \bar{N}_{s,T}^{(0,k)}.$$

The following is used in (45) below for $\mathbb{E}[F\mathcal{E}_T]$.

$$\begin{aligned} D_r \left((D_s \bar{N}_T^{(0,1)}) N_s^{(0,1)} \right) &= D_r \left(\gamma_s^{\epsilon,0} \bar{N}_{s,T}^{(0,1)} + \bar{N}_{s,T}^{(0,2)} \right) \\ &= \gamma_{r,s}^{\epsilon,1} \bar{N}_{s,T}^{(0,1)} + \gamma_s^{\epsilon,0} \left(\gamma_r^{\epsilon,0} \bar{N}_{s,T}^{(0,1)} + \bar{N}_{r,s,T}^{(0,2)} \right) + \gamma_r^{\epsilon,0} \bar{N}_{s,T}^{(0,2)} + \bar{N}_{r,s,T}^{(0,3)} \\ &\approx \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3 \\ (j+1)+(k+1)+l=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r,s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}} \gamma_{\mathbf{t}_1}^{\epsilon,j} \gamma_{\mathbf{t}_2}^{\epsilon,k} \bar{N}_{\mathbf{t}_3, s, T}^{(0,l)}, \end{aligned}$$

where $\#\mathbf{t}$ denotes the number of elements in a set \mathbf{t} and $\#\mathbf{t} = 0$ is defined as $\mathbf{t} = \emptyset$. Also the notation \approx means that the right side may have more terms than needed but this is done to avoid writing all terms explicitly. Essentially this means that some terms may be zero.

The following is used in (47) below for $\mathbb{E}[F\mathcal{E}_T]$.

$$D_r \left(\bar{N}_T^{(0,1)} N_s^{(0,1)} \right) = D_r \bar{N}_{s,T}^{(0,1)} = \gamma_r^{\epsilon,0} \bar{N}_{s,T}^{(0,1)} + \bar{N}_{r,s,T}^{(0,2)} = \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}} \gamma_r^{\epsilon,j} \bar{N}_{r,s,T}^{(0,k)}.$$

The following is used in (48) below for $|\mathbb{E}[F_1 A_1]|$. Set $\nabla^i \bar{\Gamma}_s^\epsilon = \nabla^i (b_\epsilon \sigma^{-1})(s, Y_s, \bar{Y}_s, \theta')$.

$$\begin{aligned}
& D_r \left((D_s \hat{N}_T) \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \right) \\
&= D_r \left(\left(\hat{\gamma}_s^{\epsilon,0} \hat{N}_T^{(0,1)} + \hat{N}_{s,T}^{(0,2)} \right) \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \right) \\
&= D_r \left(\left(\hat{\gamma}_s^{\epsilon,0} \hat{N}_{s,T}^{(0,1)} + \hat{N}_{s,T}^{(0,2)} \right) \nabla \bar{\Gamma}_s^\epsilon \right) \\
&\approx \nabla \bar{\Gamma}_s^\epsilon \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3 \\ (j+1)+(k+1)+l=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r,s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \hat{\gamma}_{\mathbf{t}_1}^{\epsilon,j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon,k} \hat{N}_{\mathbf{t}_3,s,T}^{(0,l)} + \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \hat{\gamma}_s^{\epsilon,j} \hat{N}_{s,T}^{(0,k)} (\nabla^2 \bar{\Gamma}_s^\epsilon) N_r^{(0,1)} \\
&= \nabla \bar{\Gamma}_s^\epsilon \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3 \\ (j+1)+(k+1)+l=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r,s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \hat{\gamma}_{\mathbf{t}_1}^{\epsilon,j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon,k} \hat{N}_{\mathbf{t}_3,s,T}^{(0,l)} + \nabla^2 \bar{\Gamma}_s^\epsilon \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \hat{\gamma}_s^{\epsilon,j} \hat{N}_{r,s,T}^{(0,k)} \\
&\approx \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r,s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \hat{\gamma}_{\mathbf{t}_1}^{\epsilon,j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon,k} \hat{N}_{\mathbf{t}_3,s,T}^{(0,l)} \nabla^i \bar{\Gamma}_s^\epsilon.
\end{aligned}$$

The following is used in (49) below for $|\mathbb{E}[F_1 A_1]|$.

$$\begin{aligned}
& D_u \left((D_r \left((D_s \hat{N}_T) \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \right)) N_r^{(0,1)} \right) \\
&\approx D_u \left(\sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r,s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \hat{\gamma}_{\mathbf{t}_1}^{\epsilon,j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon,k} \hat{N}_{\mathbf{t}_3,s,T}^{(0,l)} \nabla^i \bar{\Gamma}_s^\epsilon \right) \\
&\approx \sum_{\substack{-1 \leq j \leq 2, -1 \leq k, \ell \leq 1, 1 \leq m \leq 4, 1 \leq i \leq 3 \\ i+(j+1)+(k+1)+(\ell+1)+m=5 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 \cup \mathbf{t}_4 = \{u,r,s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1, \#\mathbf{t}_3 = \ell+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} + \#\mathbf{t}_4 \cap \{u\} = 1}}^2 \hat{\gamma}_{\mathbf{t}_1}^{\epsilon,j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon,k} \hat{\gamma}_{\mathbf{t}_3}^{\epsilon,\ell} \hat{N}_{\mathbf{t}_4,s,T}^{(0,m)} \nabla^i \bar{\Gamma}_s^\epsilon.
\end{aligned}$$

The following is used in (50) below for $|\mathbb{E}[F_1 A_1]|$.

$$D_s \hat{N}_T = \hat{\gamma}_s^{\epsilon,0} \hat{N}_T^{(0,1)} + \hat{N}_{s,T}^{(0,2)} = \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \hat{\gamma}_s^{\epsilon,j} \hat{N}_{s,T}^{(0,k)}.$$

The following is used in (51) below for $|\mathbb{E}[F_1 A_1]|$.

$$\begin{aligned}
D_u \left((D_s \hat{N}_T) (\vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1) N_r^{(0,1)} \right) &= D_u \left(\sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \hat{\gamma}_s^{\epsilon, j} \hat{N}_{s,T}^{(0,k)} (\vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1) N_r^{(0,1)} \right) \\
&= D_u \left(\sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \hat{\gamma}_s^{\epsilon, j} \hat{N}_{r,s,T}^{(0,k)} \nabla \bar{\Gamma}_s^\epsilon \right) \\
&\approx \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^1 \hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \hat{N}_{\mathbf{t}_3, r, T}^{(0, l)} \nabla^i \bar{\Gamma}_s^\epsilon.
\end{aligned}$$

The following is used in (52) below for $|\mathbb{E}[F_1 A_2]|$.

$$D_s \hat{N}_T = \hat{\gamma}_s^{\epsilon, 0} \hat{N}_T^{(0,1)} + \hat{N}_{s,T}^{(0,2)} = \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \hat{\gamma}_s^{\epsilon, j} \hat{N}_{s,T}^{(0,k)}.$$

The following is used in (53) below for $|\mathbb{E}[F_1 A_2]|$. Set $\nabla^i \bar{\Gamma}_{\phi(s)}^\epsilon = \nabla^i (b_\epsilon \sigma^{-1})(s, \bar{Y}_s, \underline{\theta}')$.

$$\begin{aligned}
D_r \left((D_s \hat{N}_T) \vec{\nabla}^1 \bar{\Gamma}_{\phi(s)}^\epsilon \cdot \mathbf{N}_s^1 \right) &= D_r \left(\sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \hat{\gamma}_s^{\epsilon, j} \hat{N}_{s,T}^{(0,k)} (\vec{\nabla}^1 \bar{\Gamma}_{\phi(s)}^\epsilon \cdot \mathbf{N}_s^1) \right) \\
&= D_r \left(\sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \hat{\gamma}_s^{\epsilon, j} \hat{N}_{s,T}^{(0,k)} \nabla \bar{\Gamma}_{\phi(s)}^\epsilon \right) \\
&\approx \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \hat{N}_{\mathbf{t}_3, T}^{(0, l)} \nabla^i \bar{\Gamma}_{\phi(s)}^\epsilon.
\end{aligned}$$

The following is used in (54) below for $|\mathbb{E}[F_1 A_3]|$. Set $\nabla^i (\bar{\Gamma}_s^\epsilon)^2 = \nabla^i (b_\epsilon a^{-1} b_\epsilon)(s, Y_s, \bar{Y}_s, \theta)$.

$$\begin{aligned}
D_r \left(\hat{N}_T \vec{\nabla}^1 (\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1 \right) &= D_r \left(\hat{N}_{s,T}^{(0,1)} \nabla (\bar{\Gamma}_s^\epsilon)^2 \right) = \left(\hat{\gamma}_r^{\epsilon, 0} \hat{N}_{r,s,T}^{(0,1)} + \hat{N}_{r,s,T}^{(0,2)} \right) \nabla (\bar{\Gamma}_s^\epsilon)^2 + \hat{N}_{s,T}^{(0,1)} \nabla^2 (\bar{\Gamma}_s^\epsilon)^2 N_r^{(0,1)} \\
&= \left(\hat{\gamma}_r^{\epsilon, 0} \hat{N}_{r,s,T}^{(0,1)} + \hat{N}_{r,s,T}^{(0,2)} \right) \nabla (\bar{\Gamma}_s^\epsilon)^2 + \hat{N}_{r,s,T}^{(0,1)} \nabla^2 (\bar{\Gamma}_s^\epsilon)^2 \\
&= \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \hat{\gamma}_r^{\epsilon, j} \hat{N}_{r,s,T}^{(0,k)} \nabla^i (\bar{\Gamma}_s^\epsilon)^2.
\end{aligned}$$

The following is used in (55) below for $|\mathbb{E}[F_1 A_3]|$.

$$\begin{aligned}
D_u \left(\left(D_r \left(\hat{N}_T \vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1 \right) \right) N_r^{(0,1)} \right) &= D_u \left(\sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \hat{\gamma}_r^{\epsilon, j} \hat{N}_{r,s,T}^{(0,k)} \nabla^i (\bar{\Gamma}_s^\epsilon)^2 \right) \\
&\approx \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 3 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, r\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^1 \hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \hat{N}_{\mathbf{t}_3, s, T}^{(0, l)} \nabla^i (\bar{\Gamma}_s^\epsilon)^2.
\end{aligned}$$

The following is used in (57) below for $|\mathbb{E}[F_1 A_3]|$.

$$D_u \left(\hat{N}_T \left(\vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1 \right) N_r^{(0,1)} \right) = D_u \left(\hat{N}_{r,s,T}^{(0,1)} \nabla(\bar{\Gamma}_s^\epsilon)^2 \right) = \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \hat{\gamma}_u^{\epsilon, j} \hat{N}_{u,r,s,T}^{(0,k)} \nabla^i (\bar{\Gamma}_s^\epsilon)^2.$$

The following is used in (59) below for $|\mathbb{E}[F_1 A_4]|$. Set $\nabla^i (\bar{\Gamma}_{\phi(s)}^\epsilon)^2 = \nabla^i (b_\epsilon a^{-1} b_\epsilon)(\phi(s), \bar{Y}_s, \underline{\theta}')$.

$$D_r \left(\hat{N}_T \vec{\nabla}^1 \left(\bar{\Gamma}_{\phi(s)}^\epsilon \right)^2 \cdot \mathbf{N}_s \right) = D_r \left(\hat{N}_{s,T} \nabla \left(\bar{\Gamma}_{\phi(s)}^\epsilon \right)^2 \right) = \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \hat{\gamma}_r^{\epsilon, j} \hat{N}_{r,s,T}^{(0,k)} \nabla^i \left(\bar{\Gamma}_{\phi(s)}^\epsilon \right)^2.$$

3 Norm estimates

3.1 Some upper estimates

In this section, we consider L^p -norm estimates for $D_{\mathbf{t}}^3 c_{i_0, j_0, k_0}^0(T)$. We proceed in stages as before.

Here, recall the notation $\mathbf{b}_\epsilon = (b_\epsilon, b_\epsilon^1)$. Now, we extend/simplify the definitions using

$$\begin{aligned}\beta_t^\epsilon &:= \mathbf{b}_\epsilon(t, Y_t), & \vec{\nabla}^i \beta_t^\epsilon &:= \vec{\nabla}^i \mathbf{b}_\epsilon(t, Y_t), \\ \tilde{\beta}_t^\epsilon &:= \mathbf{b}_\epsilon(\phi(t), Y_{\phi(t)}), & \vec{\nabla}^i \tilde{\beta}_t^\epsilon &:= \vec{\nabla}^i \mathbf{b}_\epsilon(\phi(t), Y_{\phi(t)}), \\ \hat{\beta}_t^\epsilon &:= \beta_t^\epsilon + \tilde{\beta}_t^\epsilon, & \vec{\nabla}^i \hat{\beta}_t^\epsilon &:= \vec{\nabla}^i \beta_t^\epsilon + \vec{\nabla}^i \tilde{\beta}_t^\epsilon.\end{aligned}$$

To simplify the notation we further define $L^{p,q}[0, T] := L^p[0, T] \times L^q[0, T]$. That is, for i -dimensional stochastic process Y^1 and Y^2 ,

$$\begin{aligned}\|(Y^1, Y^2)\|_p &= \|(Y^1, Y^2)\|_{L^p(\Omega)} = \|Y^1\|_p + \|Y^2\|_p = \mathbb{E} \left[|Y_s^1|^p \right]^{1/p} + \mathbb{E} \left[|Y_s^2|^p \right]^{1/p}, \\ \|(Y^1, Y^2)\|_{L^{p,q}[0, T]} &= \|Y^1\|_{L^p[0, T]} + \|Y^2\|_{L^q[0, T]} = \left(\int_0^T |Y_s^1|^p ds \right)^{1/p} + \left(\int_0^T |Y_s^2|^q ds \right)^{1/q}.\end{aligned}$$

We start with a result which is clear from the definition of γ^ϵ , $\tilde{\gamma}^\epsilon$ and $\hat{\gamma}^\epsilon$

Lemma 2. For $\gamma = \gamma^\epsilon$, $\tilde{\gamma}^\epsilon$ and $\hat{\gamma}^\epsilon$ the following inequality is satisfied for any $p > 0$, for $1/p_1 + 1/p'_1 = 1$ with $p_1 \geq 1$,

$$\|\gamma_u\|_p \leq C_p(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{pp'_1}(\|\sigma\|_{0,1}) \|\vec{\nabla}^1 \hat{\beta}^\epsilon\|_{L^{2p_1 \cdot p_1}[0, T]; L^{pp_1}(\Omega)}.$$

In particular, we have that $C_\infty(\|\sigma\|_{0,j}) < \infty$ for $j = 0, 1$. As the upper bound above does not depend on u one clearly has that

$$\int_A du \|\gamma_u\|_p \leq |A| C_p(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + |A| C_{pp'_1}(\|\sigma\|_{0,1}) \|\vec{\nabla}^1 \hat{\beta}^\epsilon\|_{L^{2p_1 \cdot p_1}[0, T]; L^{pp_1}(\Omega)}. \quad (29)$$

Here $|A|$ is the Lebesgue measure of the set A .

From (24), we have that if we use norm properties (such as Hölder inequality), norm properties for N^i , BDG inequality to obtain that

$$\begin{aligned}\|\gamma_{\mathbf{t},s}^{\epsilon,i}\|_p &\leq C_{pp'_1}(\|\sigma\|_{0,i}) \|\vec{\nabla}^i \Gamma_w^\epsilon\|_{pp_1} + \|\mathbf{N}_{\mathbf{t},s}^{i+1}\|_{L^{2p'_1}[w, T]} \|\vec{\nabla}^{i+1} \Gamma_w^\epsilon\|_{L^{2p_1}[w, T]} \|p \\ &\quad + \|\mathbf{N}_{\mathbf{t},s}^{i+1}\|_{L^{p'_1}[w, T]} \|\vec{\nabla}^{i+1} (\Gamma_w^\epsilon)^2\|_{L^{p_1}[w, T]} \|p \\ &\leq C_{pp'_1}(\|\sigma\|_{0,i}) \|\vec{\nabla}^i \Gamma_w^\epsilon\|_{pp_1} + C_{pp'_1}(\|\sigma\|_{0,i+1}) \left(\|\vec{\nabla}^{i+1} \Gamma_w^\epsilon\|_{L^{2p_1}[w, T]} \|pp_1 + \|\vec{\nabla}^{i+1} (\Gamma_w^\epsilon)^2\|_{L^{p_1}[w, T]} \|pp_1 \right) \\ &=: C_{pp'_1}(\|\sigma\|_{0,i}) \|\vec{\nabla}^i \Gamma_w^\epsilon\|_{pp_1} + C_{pp'_1}(\|\sigma\|_{0,i+1}) \|\vec{\nabla}^{i+1} \Gamma_w^\epsilon\|_{L^{2p_1 \cdot p_1}[w, T]; L^{pp_1}(\Omega)}.\end{aligned}$$

Here note that $1/p_1 + 1/p'_1 = 1$, $p_1 \geq 1$.

Furthermore, note that for $\Gamma = \Gamma^\epsilon$ or $\tilde{\Gamma}^\epsilon$ and corresponding $\beta = \beta^\epsilon$ or $\tilde{\beta}^\epsilon$, we have

$$\begin{aligned}\|\vec{\nabla}^i \Gamma\|_{L^{2p_1 \cdot p_1}[w, T]; L^{pp_1}(\Omega)} &\leq C(\|\sigma\|_{0,i}) \|\vec{\nabla}^i \beta\|_{L^{2p_1 \cdot p_1}[w, T]; L^{pp_1}(\Omega)}, \\ \|\vec{\nabla}^i \Gamma_w^\epsilon\|_{pp_1} &\leq C(\|\sigma\|_{0,i}) \|\vec{\nabla}^i \beta_w^\epsilon\|_{pp_1}.\end{aligned} \quad (30)$$

Therefore we obtain the following estimates

$$\|\gamma_{\mathbf{t},s}^{\epsilon,i}\|_p \leq C_{pp'}(\|\sigma\|_{0,i})\|\vec{\nabla}^i \beta_w^\epsilon\|_{pp_1} + C_{pp'}(\|\sigma\|_{0,i+1})\|\vec{\nabla}^{i+1} \beta_\epsilon\|_{L^{2p_1,p_1}[w,T];L^{pp_1}(\Omega)}, \quad (31)$$

$$\|\tilde{\gamma}_{\mathbf{t},s}^{\epsilon,i}\|_p \leq C_{pp'}(\|\sigma\|_{0,i})\|\vec{\nabla}^i \tilde{\beta}_w^\epsilon\|_{pp_1} + C_{pp'}(\|\sigma\|_{0,i+1})\|\vec{\nabla}^{i+1} \tilde{\beta}_\epsilon\|_{L^{2p_1,p_1}[w,T];L^{pp_1}(\Omega)},$$

$$\|\hat{\gamma}_{\mathbf{t},s}^{\epsilon,i}\|_p \leq C_{pp'}(\|\sigma\|_{0,i})\|\vec{\nabla}^i \hat{\beta}_w^\epsilon\|_{pp_1} + C_{pp'}(\|\sigma\|_{0,i+1})\|\vec{\nabla}^{i+1} \hat{\beta}_\epsilon\|_{L^{2p_1,p_1}[w,T];L^{pp_1}(\Omega)}. \quad (32)$$

From here one obtains the following result

Lemma 3. *Define for each fixed $r > 0$, a subset $A_i(r) \subseteq \{\mathbf{t} = (t_1, \dots, t_i) \in [0, r]^i; t_1 < \dots < t_i < r\}$ such that $A_i(r) \subseteq A_i(r')$ for $r < r'$ then we have that the following inequalities are satisfied for any $p > 0$ and $1/p_1 + 1/p'_1 = 1$ with $p_1 \geq 1$,*

$$\begin{aligned} \int_0^T \int_{A_i(s)} \|\gamma_{\mathbf{t},s}^{\epsilon,i}\|_p dt ds &\leq |A_i(T)| \left\{ C_{pp'}(\|\sigma\|_{0,i})\|\vec{\nabla}^i \beta^\epsilon\|_{L^{pp_1,pp_1}[0,T];L^{pp_1}(\Omega)} + C_{pp'}(\|\sigma\|_{0,i+1})\|\vec{\nabla}^{i+1} \beta^\epsilon\|_{L^{2p_1,p_1}[0,T];L^{pp_1}(\Omega)} \right\}, \\ \int_0^T \int_{A_i(s)} \|\tilde{\gamma}_{\mathbf{t},s}^{\epsilon,i}\|_p dt ds &\leq |A_i(T)| \left\{ C_{pp'}(\|\sigma\|_{0,i})\|\vec{\nabla}^i \tilde{\beta}^\epsilon\|_{L^{pp_1,pp_1}[0,T];L^{pp_1}(\Omega)} + C_{pp'}(\|\sigma\|_{0,i+1})\|\vec{\nabla}^{i+1} \tilde{\beta}^\epsilon\|_{L^{2p_1,p_1}[0,T];L^{pp_1}(\Omega)} \right\}, \\ \int_0^T \int_{A_i(s)} \|\hat{\gamma}_{\mathbf{t},s}^{\epsilon,i}\|_p dt ds &\leq |A_i(T)| \left\{ C_{pp'}(\|\sigma\|_{0,i})\|\vec{\nabla}^i \hat{\beta}^\epsilon\|_{L^{pp_1,pp_1}[0,T];L^{pp_1}(\Omega)} + C_{pp'}(\|\sigma\|_{0,i+1})\|\vec{\nabla}^{i+1} \hat{\beta}^\epsilon\|_{L^{2p_1,p_1}[0,T];L^{pp_1}(\Omega)} \right\}. \end{aligned}$$

Let us see how to deal with the norm estimate of (22) in the case that $i = j = k = 1$. In fact, this terms corresponds to (11). After applying the duality formula of Malliavin Calculus one has to bound

$$\int_0^T \left[\int_0^t \left(\int_{\phi(s)}^s \|(D_u((D_s((D_t c_{i,j,k}^0(T))c_{i,j,k}^1(t)))c_{i,j,k}^2(t,s)))c_{i,j,k}^3(t,s,u)\|_1 du \right) ds \right] dt. \quad (33)$$

The above iterated derivative can be written as sums of terms of the type

$$D_{\mathbf{t}_0,t}^{k_0} c_{i,j,k}^0(T) D_{\mathbf{t}_1}^{k_1} c_{i,j,k}^1(t) D_{\mathbf{t}_2}^{k_2} c_{i,j,k}^2(t,s) c_{i,j,k}^3(t,s,u),$$

where $k_0 = 1, 2, 3$, $k_1 = 0, 1, 2$, $k_2 = 0, 1$, $k_0 + k_1 + k_2 = 3$ and $\mathbf{t}_0 \cup \mathbf{t}_1 \cup \mathbf{t}_2 = \{u, s\}$ with $u < s < t$.

Note that

$$\begin{aligned} c_{1,1,1}^0(T) &= F_1, \\ c_{1,1,1}^1(t) &= \nabla \hat{\Gamma}_t^\epsilon U_t, \\ c_{1,1,1}^2(t,s) &= U_s^{-1} \left(\nabla \sigma(s, \bar{Y}_s, \underline{\theta}) \right) \sigma(\phi(s), \bar{Y}_{\phi(s)}) = U_s^{-1} N_s^1, \\ c_{1,1,1}^3(t,s,u) &= 1. \end{aligned} \quad (34)$$

Now note that for any $p \geq 1$

$$\begin{aligned} \|U_t\|_p + \|U_t^{-1}\|_p &\leq C \exp(C\|\sigma\|_{0,1}), \\ \|D_s U_t\|_p + \|D_s U_t^{-1}\|_p &\leq C(1 + \|\sigma\|_{0,2}) \exp(C\|\sigma\|_{0,1}). \end{aligned}$$

Therefore we obtain that

$$\begin{aligned} &\|(D_u((D_s((D_t c_{1,1,1}^0(T))c_{1,1,1}^1(t)))c_{1,1,1}^2(t,s)))c_{1,1,1}^3(t,s,u)\|_1, \\ &\leq C \sum \|D_{\mathbf{t}_0,t}^{k_0} c_{1,1,1}^0(T)\|_{\ell_0} \|\vec{\nabla}^{1+k_1} \hat{\Gamma}_t^\epsilon\|_{\ell_1} (1 + \|\sigma\|_{0,1+k_1})(1 + \|\sigma\|_{0,1+k_2}) \exp(C\|\sigma\|_{0,1}). \end{aligned} \quad (35)$$

Here the sum is with respect to the indices restrictions on k_0, k_1 and k_2 stated above and $\ell_0, \ell_1 > 1$ with $\ell_0^{-1} + \ell_1^{-1} < 1$

Note that in general for $\ell, \ell_i \geq 1$ such that $\ell_1^{-1} + \dots + \ell_{k_0+1}^{-1} = \ell^{-1}$

$$\|D_{\mathbf{t}_0, t}^{k_0} c_{1,1,1}^0(T)\|_p \leq C \|\hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j_1}\|_{\ell_1} \dots \|\hat{\gamma}_{\mathbf{t}_{k_0}}^{\epsilon, j_{k_0}}\|_{\ell_{k_0}} \|\hat{N}_{\mathbf{t}_{k_0+1}, T}^{j_{k_0+1}}\|_{\ell_{k_0+1}}.$$

The restrictions on the time variables and the indices are given in the corresponding sums. Furthermore note that in (35), the variable u may only enter on the right hand side through the time vector \mathbf{t}_0, t . We have that if u is within this vector and as $u < s < t < T$, then they will not influence the upper bound as given by (31) unless we have to consider the case of $\|\hat{\gamma}_u^\epsilon\|_p$. In that case, the upper bound is the one given in Lemma 2 which gives a bound independent of u . Therefore the upper bound for (33) is given by

$$\begin{aligned} & \prod_{l=1}^{k_0} \left\{ C_{p'_l \ell_l} (\|\sigma\|_{0, j_l}) \|(\vec{\nabla}^{j_l} \hat{\beta}_w^\epsilon)\|_{L^{\ell_l p_l, \ell_l p_l} [0, T]; L^{\ell_l p_l}(\Omega)} + C_{p'_l \ell_l} (\|\sigma\|_{0, j_l+1}) \|\vec{\nabla}^{j_l+1} \hat{\beta}^\epsilon\|_{L^{2 p_l p_l} [w, T]; L^{\ell_l p_l}(\Omega)} \right\} \\ & \times C_{\ell_{k_0+1}} (\|\sigma\|_{0, j_{k_0+1}}) \exp(C_{\ell_{k_0+1}} (\|\sigma\|_{0,0}) \|b_\epsilon\|_\infty) \|\vec{\nabla}^{1+k_1} \hat{\Gamma}_t^\epsilon\|_{\ell_1} (1 + \|\sigma\|_{0,1+k_1}) (1 + \|\sigma\|_{0,1+k_2}) \exp(C \|\sigma\|_{0,1}). \end{aligned} \quad (36)$$

Here $1/p_l + 1/p'_l = 1$. Furthermore, the indices satisfy the following restrictions

$$\begin{aligned} j_1 + \dots + j_{k_0} + j_{k_0+1} &= 0, \\ j_1, \dots, j_{k_0} &\geq -1, j_{k_0+1} \geq 0. \end{aligned}$$

Also in the case that $j_l = 0$ then $\|\vec{\nabla}^{j_l} \hat{\beta}_w^\epsilon\|_{L^{\ell_l p_l, \ell_l p_l} [0, T]; L^{\ell_l p_l}(\Omega)}$ is replaced by the sup norm of b_ϵ . Finally we obtain the convergence order $1/n$ with respect to the term (33) and the bounds stated in Proposition 3 of the main paper in the case that $k_0 + 1 + k_1 \leq 3$. Now we need to repeat this idea for all the other terms because they will involve only a total number of three derivatives of \mathbf{b}_ϵ . This is done in the next section in more detail than the above general idea for most of the remaining terms.

Instead, we now deal with the particular case, when 4 derivatives of \mathbf{b}_ϵ appear where we need a better estimate. For this, we return to (33). In order to obtain 4 derivatives of \mathbf{b}_ϵ , we need to have $k_2 = 0$ therefore the indices are reduced to $k_0 = 1, 2, 3$ and $k_1 = 3 - k_0$ and $\mathbf{t}_0 \cup \mathbf{t}_1 = \{u, s\}$. Therefore, following (27) through (28) one has to bound terms of the type

$$|(D_u((D_s((D_t c_{1,1,1}^0(T)) c_{1,1,1}^1(t)))) c_{1,1,1}^2(s)) c_{1,1,1}^3| = |(D_u(D_s((D_t F_1) \nabla \Gamma_t^\epsilon U_t))) U_s^{-1} N_s^1| \quad (37)$$

Note that we have simplified the dependence of the functions $c_{1,1,1}^i$ on the time variables as given in (34).

Now the goal is to find the terms that generate 4 derivatives of \mathbf{b}_ϵ . Also we want to study in all these terms the dependence on t .

Now we try to follow carefully the dependence on t in (37). In fact, using (25)

$$\begin{aligned} D_t c_{1,1,1}^0(T) &= D_t F_1 = N_t^1 \hat{N}_T^1 + \hat{\gamma}_t^\epsilon \hat{N}_T \\ &= N_t^1 \hat{N}_T^1 + \hat{N}_T \left(\hat{\Gamma}_t^\epsilon + N_t^1 \int_t^T N_r^1 \nabla \hat{\Gamma}_r^\epsilon dW_r - \frac{N_t^1}{2} \int_t^T \nabla(\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right). \end{aligned} \quad (38)$$

Writing things explicitly one finds that ${}^2 \mathbb{E}[\sup_s |N_s^1|^p] \leq C_p$ for the term that appears above. Also note that if any of the two derivatives $D_u D_s$ is applied to U_t the terms that will appear will only

²Recall that we have stated this property right after (5).

have at most three derivatives of \mathbf{b}_ϵ and therefore can be treated as before. Therefore we only need to concentrate on $D_u(D_s((D_t F_1)\nabla\hat{\Gamma}_t^\epsilon))$.

In equation (38) only the last two terms within the parenthesis will produce a first order derivative. Therefore the other two terms can be dealt with the technique previously explained as the number of derivatives of \mathbf{b}_ϵ that they will generate are at most three.

Therefore when applying the operator $D_u D_s$ and with the objective of generating the higher number of derivatives of $\hat{\Gamma}^\epsilon$ one will have to consider

$$N_t^1 \int_t^T N_r^1 D_u D_s (\nabla\hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T D_u D_s (\nabla(\hat{\Gamma}_r^\epsilon)^2) N_r^1 dr.$$

Therefore the only terms that will provide derivatives of order four are

$$\begin{aligned} & \nabla\hat{\Gamma}_t^\epsilon \hat{N}_T \left(N_t^1 \int_t^T N_r^1 D_u D_s (\nabla\hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T D_u D_s \nabla(\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right), \\ & (D_s \nabla\hat{\Gamma}_t^\epsilon) \hat{N}_T \left(N_t^1 \int_t^T N_r^1 D_u (\nabla\hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T D_u \nabla(\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right), \\ & (D_u D_s \nabla\hat{\Gamma}_t^\epsilon) \hat{N}_T \left(N_t^1 \int_t^T N_r^1 (\nabla\hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T \nabla(\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right). \end{aligned}$$

In either of the three cases, one can bound each term as follows:

$$|\nabla\hat{\Gamma}_t^\epsilon| \|\hat{N}_T\| \sup_{t \in [0, T]} \left| N_t^1 \int_t^T N_r^1 D_u D_s (\nabla\hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T D_u D_s \nabla(\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right|, \quad (39)$$

$$|D_s \nabla\hat{\Gamma}_t^\epsilon| \|\hat{N}_T\| \sup_{t \in [0, T]} \left| N_t^1 \int_t^T N_r^1 D_u (\nabla\hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T D_u \nabla(\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right|, \quad (40)$$

$$|D_u D_s \nabla\hat{\Gamma}_t^\epsilon| \|\hat{N}_T\| \sup_{t \in [0, T]} \left| N_t^1 \int_t^T N_r^1 (\nabla\hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T \nabla(\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right|. \quad (41)$$

From these bounds, one sees that only the first term on all three expressions depends on t . Furthermore, the other terms can be bounded in $L^p(\Omega)$ using the technique described previously³. For the first expression above, i.e. (39), if one integrates with respect to t , one finds that the $L^1[0, T]$ norm is applied to $|\nabla\hat{\Gamma}_t^\epsilon|$ and therefore one can use the analogous inequality to (30). That is, for any $p_1, p \geq 1$, one has

$$\|\vec{\nabla}^i \hat{\Gamma}^\epsilon\|_{L^{p_1}[w, T]; L^p(\Omega)} \leq C(\|\sigma\|_{0, i}) \|\vec{\nabla}^i \beta^\epsilon\|_{L^{p_1}[w, T]; L^p(\Omega)}.$$

The first term on the second and third expressions (i.e. (40) and (41)) are dealt similarly. In fact,

$$|D_s(\nabla\hat{\Gamma}_t^\epsilon)| \leq |\nabla^2 \hat{\Gamma}_t^\epsilon| \|D_s Y_t\| \leq |\nabla^2 \hat{\Gamma}_t^\epsilon| N_s^1 \sup_{t \in [0, T]} |N_t^1|.$$

Similarly,

$$|D_u D_s(\nabla\hat{\Gamma}_t^\epsilon)| \leq \sum_{j=2}^3 |\nabla^j \hat{\Gamma}_t^\epsilon| N_s^1 \sup_{t \in [0, T]} |N_t^1|. \quad (42)$$

³That is, using Holder's and Burkholder-Davis-Gundy inequalities.

In any case a similar conclusion is achieved. The above arguments lead to a bound of the following type:

$$|D_u(D_s(D_t c_{1,1,1}^0(T)c_{1,1,1}^1(t)))c_{1,1,1}^2(s)c_{1,1,1}^3| \leq \sum_{k_0=1}^3 C(\|\sigma\|_{0,3})|\hat{N}_{u,s,T}\|_{\xi_{u,s}^{\epsilon,4-k_0}}\|\nabla^{4-k_0}\hat{\Gamma}_t^\epsilon|. \quad (43)$$

The definition of $\xi_{u,s}^{\epsilon,4-k_0}$ is clear from the above calculation and they are given by

$$\begin{aligned} \xi_{u,s}^{\epsilon,3} &:= 1 + \sup_{t \in [0,T]} \left| N_t^1 \int_t^T N_r^1 (\nabla \hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T \nabla (\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right|, \\ \xi_{u,s}^{\epsilon,2} &:= \xi_{u,s}^{\epsilon,3} + \sup_{t \in [0,T]} \left| N_t^1 \int_t^T N_r^1 D_u (\nabla \hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T D_u \nabla (\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right| \\ &\quad + \sup_{t \in [0,T]} \left| N_t^1 \int_t^T N_r^1 D_s (\nabla \hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T D_s \nabla (\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right|, \\ \xi_{u,s}^{\epsilon,1} &:= \xi_{u,s}^{\epsilon,2} + \sup_{t \in [0,T]} \left| N_t^1 \int_t^T N_r^1 D_u D_s (\nabla \hat{\Gamma}_r^\epsilon) dW_r - \frac{N_t^1}{2} \int_t^T D_u D_s \nabla (\hat{\Gamma}_r^\epsilon)^2 N_r^1 dr \right|. \end{aligned}$$

Notice in particular that the fact that $\xi^{\epsilon,3}$ appears in $\xi^{\epsilon,2}$ is due to (42).

Therefore one has that (33) is bounded above by

$$\frac{1}{n} \sum_{k_0=1}^3 C(\|\sigma\|_{0,3}) \sup_{u,s} \mathbb{E} \left[|\hat{N}_{u,s,T}\|_{\xi_{s,u}^{\epsilon,4-k_0}} \|\vec{\nabla}^{4-k_0} \hat{\Gamma}^\epsilon\|_{L^1[0,T]} \right].$$

Now we apply Hölder's inequality to obtain that

$$\mathbb{E} \left[|\hat{N}_{u,s,T}\|_{\xi_{s,u}^{\epsilon,4-k_0}} \|\vec{\nabla}^{4-k_0} \hat{\Gamma}^\epsilon\|_{L^1[0,T]} \right] \leq C \|\hat{N}_{u,s,T}\|_{q'} \|\xi_{s,u}^{\epsilon,4-k_0}\|_{q'} \|\vec{\nabla}^{4-k_0} \hat{\Gamma}^\epsilon\|_{L^1[0,T]} \|q.$$

Here $1/q + 1/q' + 1/q'' = 1$ and note that the second term can be dealt using BDG inequalities. In fact, for example, we deal with

$$\begin{aligned} \|\xi_{u,s}^{\epsilon,1} - \xi_{u,s}^{\epsilon,2}\|_{q'}^p &\leq C(q'p) \mathbb{E} \left[\sup_{t \in [0,T]} \left| \int_t^T N_r^1 D_u D_s (\nabla \hat{\Gamma}_r^\epsilon) dW_r \right|^{q'p} + \left(\int_0^T |D_u D_s \nabla (\hat{\Gamma}_r^\epsilon)^2| dr \right)^{q'p} \right] \\ &\leq C(q'p) \mathbb{E} \left[\left(\int_0^T |N_r^1 D_u D_s (\nabla \hat{\Gamma}_r^\epsilon)|^2 dr \right)^{q'p/2} + \left(\int_0^T |D_u D_s \nabla (\hat{\Gamma}_r^\epsilon)^2| dr \right)^{q'p} \right] \\ &\leq C(q'p) \mathbb{E} \left[\left(\int_0^T |\mathbf{N}_{r,s,u}^2 \cdot \vec{\nabla}^2 \hat{\Gamma}_r^\epsilon|^2 dr \right)^{q'p/2} + \left(\int_0^T |\vec{\nabla}^2 (\hat{\Gamma}_r^\epsilon)^2| dr \right)^{q'p} \right]. \end{aligned}$$

From here we can now apply Lemmas 2 and 3 as before. In particular, the same argument after Lemma 2 applies here leading to the estimates of Lemma 3 for this term. The other terms follow a similar pattern.

3.2 Appendix: Norm upper estimates for terms with less than three derivatives

As stated previously in inequality (36) all terms which has a total of three derivatives or less of \mathbf{b}_ϵ can be treated with that general method through the use of Lemmas 2 and 3. Here we will show this by dealing with some of these terms. Their final order are written in blue and they can be checked to coincide with the orders in the statement of Proposition 3 once one applies Lemmas 1 and 2 in the main paper.

3.2.1 On $|\mathbb{E}[F\mathcal{E}_T]|$

Here used the norm simplified notation $|X|_{p,q} = \left(\mathbb{E} \left[\left(\int_0^T |X_s|^p ds \right)^{q/p} \right] \right)^{1/q}$. That is for example,

$$\|\vec{\nabla}^i \beta^\epsilon\|_{L^p[0,T];L^q(\Omega)} \equiv |\vec{\nabla}^i \beta^\epsilon|_{p,q}.$$

Equations (6)-(9) are rewritten as

$$\begin{aligned} F\mathcal{E}_T &= \bar{N}_T^{(0,1)} \int_0^T N_s^{(0,1)} \int_{\phi(s)}^s N_{r,s}^{(1,0)} dr dW_s + \bar{N}_T^{(0,1)} \int_0^T N_s^{(0,1)} \int_{\phi(s)}^s dW_r dW_s \\ &+ \bar{N}_T^{(0,1)} \int_0^T N_s^{(0,1)} \int_{\phi(s)}^s N_{r,s}^{(1,0)} dr ds + \bar{N}_T^{(0,1)} \int_0^T N_s^{(0,1)} \int_{\phi(s)}^s dW_r ds. \end{aligned}$$

Then from the Malliavin duality formula and an estimation (23) we have

$$|\mathbb{E}[F\mathcal{E}_T]| \leq \frac{1}{n} \int_0^T \sup_r \mathbb{E} \left[\left| (D_s \bar{N}_T^{(0,1)}) N_s^{(0,1)} N_{r,s}^{(1,0)} \right| \right] ds \quad (44)$$

$$+ \int_0^T \int_{\phi(s)}^s \mathbb{E} \left[\left| D_r \left((D_s \bar{N}_T^{(0,1)}) N_s^{(0,1)} \right) \right| \right] dr ds \quad (45)$$

$$+ \frac{1}{n} \int_0^T \sup_r \mathbb{E} \left[\left| \bar{N}_T^{(0,1)} N_s^{(0,1)} N_{r,s}^{(1,0)} \right| \right] ds \quad (46)$$

$$+ \int_0^T \int_{\phi(s)}^s \mathbb{E} \left[\left| D_r \left(\bar{N}_T^{(0,1)} N_s^{(0,1)} \right) \right| \right] dr ds. \quad (47)$$

And for $1/p + 1/p' = 1$ with $p \geq 1$,

$$\begin{aligned} (44) &\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \int_0^T \sup_r \mathbb{E} \left[\left| \gamma_s^{\epsilon,j} \bar{N}_{s,T}^{(0,k)} N_{r,s}^{(1,0)} \right| \right] ds \\ &\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \int_0^T \|\gamma_s^{\epsilon,j}\|_p \sup_r \|\bar{N}_{r,s,T}^{(1,k)}\|_{p'} ds. \end{aligned}$$

Using (31), for $1/p_1 + 1/p'_1 = 1$ with $p_1 \geq 1$,

$$\begin{aligned} \|\gamma_s^\epsilon\|_p &\leq C_{pp'_1}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_{pp_1} + C_{pp'_1}(\|\sigma\|_{0,1}) \|\vec{\nabla}^1 \beta^\epsilon\|_{L^{2p_1,p_1}[0,T];L^{pp_1}(\Omega)} \\ &\leq C_{pp'_1}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_{pp_1} + C_{pp'_1}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \beta^\epsilon|_{2p_1,pp_1}, \end{aligned}$$

we have

$$\begin{aligned} &\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \left(C_{pp'_1}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{pp'_1}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \beta^\epsilon|_{2p_1,pp_1} \right)^{1+j} |\bar{N}_{\cdot,T}^{(1,k)}|_{\infty,p'} \\ &\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \left(C_{pp'_1}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{pp'_1}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \beta^\epsilon|_{2p_1,pp_1} \right)^{1+j} C_{p'}(\|\sigma\|_{1,k}). \end{aligned}$$

Next, for $1/q + 1/q' + 1/q'' = 1$ with $q, q' \geq 1$, we similarly have

$$\begin{aligned}
(45) &\leq \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3 \\ (j+1)+(k+1)+l=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \int_0^T \int_{\phi(s)}^s \mathbb{E} \left[\left| \gamma_{\mathbf{t}_1}^{\epsilon, j} \gamma_{\mathbf{t}_2}^{\epsilon, k} \bar{N}_{\mathbf{t}_3, s, T}^{(0, l)} \right| \right] dr ds \\
&\leq \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3 \\ (j+1)+(k+1)+l=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \int_0^T \int_{\phi(s)}^s \|\gamma_{\mathbf{t}_1}^{\epsilon, j}\|_q \|\gamma_{\mathbf{t}_2}^{\epsilon, k}\|_{q'} \|\bar{N}_{\mathbf{t}_3, s, T}^{(0, l)}\|_{q''} dr ds \\
&= \int_0^T \int_{\phi(s)}^s \|\gamma_{r, s}^{\epsilon, 1}\|_q \|\bar{N}_{\mathbf{t}_3, s, T}^{(0, 1)}\|_{q''} dr ds + \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 3 \\ (j+1)+(k+1)+l=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^0 \int_0^T \int_{\phi(s)}^s \|\gamma_{\mathbf{t}_1}^{\epsilon, j}\|_q \|\gamma_{\mathbf{t}_2}^{\epsilon, k}\|_{q'} \|\bar{N}_{\mathbf{t}_3, s, T}^{(0, l)}\|_{q''} dr ds \\
&\leq \frac{1}{n} C_{q''}(\|\sigma\|_{0,1}) \left\{ C_{qp'}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \beta^\epsilon|_{qp_1, qp_1} + C_{qp'}(\|\sigma\|_{0,2}) |\vec{\nabla}^2 \beta^\epsilon|_{2p_1, qp_1} \right\} \\
&+ \frac{1}{n} \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 3 \\ (j+1)+(k+1)+l=3 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^0 C_{q''}(\|\sigma\|_{0,l}) \left\{ C_{qp'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{qp_1}(\|\sigma\|_{0,1}) \|\vec{\nabla}^1 \beta^\epsilon\|_{2p_1, q} \right\}^{1+j} \\
&\quad \times \left\{ C_{q'p'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{q'p_1}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \beta^\epsilon|_{2p_1, q'} \right\}^{1+k}.
\end{aligned}$$

We have

$$(46) \leq \frac{1}{n} \int_0^T \sup_r \mathbb{E} \left[\left| \bar{N}_{r, s, T}^{(1, 1)} \right| \right] ds \leq \frac{1}{n} C(\|\sigma\|_{1,1}).$$

Similarly, we have

$$\begin{aligned}
(47) &\leq \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \int_0^T \int_{\phi(s)}^s \mathbb{E} \left[\left| \gamma_r^{\epsilon, j} \bar{N}_{r, s, T}^{(0, k)} \right| \right] dr ds \\
&\leq \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \int_0^T \int_{\phi(s)}^s \|\gamma_r^{\epsilon, j}\|_p \|\bar{N}_{r, s, T}^{(0, k)}\|_{p'} dr ds \\
&\leq \frac{T}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \left(C_{pp'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{pp_1}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \beta^\epsilon|_{2p_1, pp_1} \right)^{1+j} C_{p'}(\|\sigma\|_{0,k}).
\end{aligned}$$

3.2.2 On $|\mathbb{E}[F_1 A_1]|$

Equations (10)-(13) are rewritten as

$$\begin{aligned}
F_1 A_1 &= \hat{N}_T \int_0^T \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \int_0^s N_r^{(0,1)} \int_{\phi(r)}^r N_{u,r}^{(1,0)} du dW_r dW_s \\
&+ \hat{N}_T \int_0^T \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \int_0^s N_r^{(0,1)} \int_{\phi(r)}^r dW_u dW_r dW_s \\
&+ \hat{N}_T \int_0^T \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \int_0^s N_r^{(0,1)} \int_{\phi(r)}^r N_{u,r}^{(1,0)} du dr dW_s \\
&+ \hat{N}_T \int_0^T \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \int_0^s N_r^{(0,1)} \int_{\phi(r)}^r dW_u dr dW_s.
\end{aligned}$$

Then from the Malliavin duality formula and an estimation (23) we have

$$|\mathbb{E}[F_1 A_1]| \leq \frac{1}{n} \int_0^T \int_0^s \sup_u \mathbb{E} \left[\left| (D_r \left((D_s \hat{N}_T) \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \right)) N_r^{(0,1)} N_{u,r}^{(1,0)} \right| \right] dr ds \quad (48)$$

$$+ \int_0^T \int_0^s \int_{\phi(r)}^r \mathbb{E} \left[\left| D_u \left((D_r \left((D_s \hat{N}_T) \vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1 \right)) N_r^{(0,1)} \right) \right| \right] du dr ds \quad (49)$$

$$+ \frac{1}{n} \int_0^T \int_0^s \sup_u \mathbb{E} \left[\left| (D_s \hat{N}_T) (\vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1) N_r^{(0,1)} N_{u,r}^{(1,0)} \right| \right] dr ds \quad (50)$$

$$+ \int_0^T \int_0^s \int_{\phi(r)}^r \mathbb{E} \left[\left| D_u \left((D_s \hat{N}_T) (\vec{\nabla}^1 \bar{\Gamma}_s^\epsilon \cdot \mathbf{N}_s^1) N_r^{(0,1)} \right) \right| \right] du dr ds. \quad (51)$$

And for $1/s^{(1)} + \dots + 1/s^{(5)} = 1$ with $s^{(1)}, \dots, s^{(4)} \geq 1$, we have

$$\begin{aligned}
&(48) \\
&\leq \frac{1}{n} \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}} \int_0^T \int_0^s \sup_u \mathbb{E} \left[\left| \hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \hat{N}_{\mathbf{t}_3, s, T}^{(0, l)} \nabla^i \bar{\Gamma}_s^\epsilon N_{u, r}^{(1, 1)} \right| \right] dr ds \\
&\leq \frac{1}{n} \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}} \int_0^T \int_0^s \left\| \hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \right\|_{s^{(1)}} \left\| \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \right\|_{s^{(2)}} \left\| \hat{N}_{\mathbf{t}_3, s, T}^{(0, l)} \right\|_{s^{(5)}} \left\| \nabla^i \bar{\Gamma}_s^\epsilon \right\|_{s^{(3)}} \sup_u \left\| N_{u, r}^{(1, 1)} \right\|_{s^{(4)}} dr ds \\
&\leq \frac{1}{n} C_{s^{(4)}, s^{(5)}} (\|\sigma\|_{1,1}) \int_0^T \int_0^s \left\| \hat{\gamma}_{r, s}^{\epsilon, 1} \right\|_{s^{(1)}} \left\| \nabla^1 \bar{\Gamma}_s^\epsilon \right\|_{s^{(3)}} dr ds \\
&+ \frac{1}{n} \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}} C_{s^{(4)}, s^{(5)}} (\|\sigma\|_{1,1}) \int_0^T \int_0^s \left\| \hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \right\|_{s^{(1)}} \left\| \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \right\|_{s^{(2)}} \left\| \nabla^i \bar{\Gamma}_s^\epsilon \right\|_{s^{(3)}} dr ds \\
&\leq \frac{1}{n} C_{s^{(4)}, s^{(5)}} (\|\sigma\|_{1,1}) \left\{ \int_0^T \int_0^s \left\| \hat{\gamma}_{r, s}^{\epsilon, 1} \right\|_{s^{(1)}}^{p_1} dr ds \right\}^{1/p_1} \left\{ \int_0^T \left\| \nabla^1 \bar{\mathbf{b}}_\epsilon(s) \right\|_{s^{(3)}}^{p_1'} ds \right\}^{1/p_1'} \\
&+ \frac{1}{n} \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}} C_{s^{(4)}, s^{(5)}} (\|\sigma\|_{1,1}) \left\{ C_{s^{(1)} p_1'} (\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{s^{(1)} p_1} (\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1, s^{(1)} p_1} \right\}^{1+j} \\
&\times \left\{ C_{s^{(2)} p_1'} (\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{s^{(2)} p_1} (\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1, s^{(2)} p_1} \right\}^{1+k} \int_0^T \left\| \nabla^i \bar{\mathbf{b}}_\epsilon(s) \right\|_{s^{(3)}} ds,
\end{aligned}$$

where $\nabla^i \bar{\mathbf{b}}_\epsilon(s) := \sum_{j=0}^i \nabla^j b_\epsilon(s, \theta' Y_s + (1 - \theta') \bar{Y}_s)$.

From inequality (32), where $w = s$ for $r < s$, and Minkowski's inequality, the term $\left\{ \int_0^T \int_0^T \|\hat{\gamma}_{r,s}^{\epsilon,1}\|_{s^{(1)}}^{p_1} dr ds \right\}^{1/p_1}$ can be estimated as, for $1/p_2 + 1/p_2' = 1$ with $p_2 \geq 1$ and $1 \leq p_1 \leq s^{(1)} p_2$,

$$\begin{aligned} & \left\{ \int_0^T \int_0^T \|\hat{\gamma}_{r,s}^{\epsilon,1}\|_{s^{(1)}}^{p_1} dr ds \right\}^{1/p_1} \\ & \leq \left\{ T \int_0^T \left\{ C_{s^{(1)} p_2'}(\|\sigma\|_{0,1}) \|\bar{\nabla}^1 \hat{\beta}_s^\epsilon\|_{s^{(1)} p_2} + C_{s^{(1)} p_2'}(\|\sigma\|_{0,2}) \|\bar{\nabla}^2 \hat{\beta}_s^\epsilon\|_{2 p_2, s^{(1)} p_2} \right\}^{p_1} ds \right\}^{1/p_1} \\ & \leq C_{s^{(1)} p_2', p_1}(\|\sigma\|_{0,1}) \left\{ \int_0^T \|\bar{\nabla}^1 \hat{\beta}_s^\epsilon\|_{s^{(1)} p_2}^{p_1} ds \right\}^{\frac{1}{p_1}} + C_{s^{(1)} p_2'}(\|\sigma\|_{0,2}) \|\bar{\nabla}^2 \hat{\beta}_s^\epsilon\|_{2 p_2, s^{(1)} p_2} \\ & \leq C_{s^{(1)} p_2', p_1}(\|\sigma\|_{0,1}) \|\bar{\nabla}^1 \hat{\beta}_s^\epsilon\|_{s^{(1)} p_2, s^{(1)} p_2} + C_{s^{(1)} p_2'}(\|\sigma\|_{0,2}) \|\bar{\nabla}^2 \hat{\beta}_s^\epsilon\|_{2 p_2, s^{(1)} p_2}. \end{aligned}$$

Then we have, for $1/q_1 + 1/q_1' + 1/q_1'' = 1$ with $q_1, q_1' \geq 1, q_1' \leq s^{(3)}$ and $q_1'' \leq s^{(4)}$,

$$\begin{aligned} (48) & \leq \frac{1}{n} C_{s^{(4)}, s^{(5)}}(\|\sigma\|_{1,1}) \left\{ C_{s^{(1)} p_2', p_1}(\|\sigma\|_{0,1}) \|\bar{\nabla}^1 \hat{\beta}_s^\epsilon\|_{s^{(1)} p_2, s^{(1)} p_2} + C_{s^{(1)} p_2', p_1}(\|\sigma\|_{0,2}) \|\bar{\nabla}^2 \hat{\beta}_s^\epsilon\|_{2 p_2, s^{(1)} p_2} \right\} \|\nabla^1 \bar{\mathbf{b}}_\epsilon\|_{s^{(3)}, s^{(3)}} \\ & + \frac{1}{n} \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^0 C_{s^{(4)}, s^{(5)}}(\|\sigma\|_{1,i}) \left\{ C_{s^{(1)} p_1'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{s^{(1)} p_1}(\|\sigma\|_{0,1}) \|\bar{\nabla}^1 \hat{\beta}_s^\epsilon\|_{2 p_1, s^{(1)} p_1} \right\}^{1+j} \\ & \times \left\{ C_{s^{(2)} p_1'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{s^{(2)} p_1}(\|\sigma\|_{0,1}) \|\bar{\nabla}^1 \hat{\beta}_s^\epsilon\|_{2 p_1, s^{(2)} p_1} \right\}^{1+k} \|\nabla^i \bar{\mathbf{b}}_\epsilon\|_{s^{(3)}, s^{(3)}}. \end{aligned}$$

The main term (Eq. (49)) has been estimated in Section 3.1.

Using similar inequality of (30), we have

$$\begin{aligned} (50) & \leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \int_0^T \int_0^s \sup_u \mathbb{E} \left[\|\hat{\gamma}_s^{\epsilon, j} \hat{N}_{u, r, s, T}^{(1, k)} \nabla \bar{\Gamma}_s^\epsilon\| \right] dr ds \\ & \leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \int_0^T \int_0^s \|\hat{\gamma}_s^{\epsilon, j}\|_q \sup_u \|\hat{N}_{u, r, s, T}^{(1, k)}\|_{q''} \|\nabla \bar{\Gamma}_s^\epsilon\|_{q'} dr ds \\ & \leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \left(C_{q p_1'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{q p_1'}(\|\sigma\|_{0,1}) \|\bar{\nabla}^1 \hat{\beta}_s^\epsilon\|_{2 p_1, q} \right)^{1+j} \\ & \quad \times C_{q''}(\|\sigma\|_{1, k}) C_{q' p_1'}(\|\sigma\|_{0,1}) \|\nabla^1 \bar{\mathbf{b}}_\epsilon\|_{q' p_1, q' p_1}. \end{aligned}$$

The last term in $|\mathbb{E}[F_1 A_1]|$ is estimated as follow. For $1/r^{(1)} + \dots + 1/r^{(4)} = 1$ with

$$r^{(1)}, r^{(2)}, r^{(3)} \geq 1,$$

$$\begin{aligned}
(51) &\leq \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^1 \int_0^T \int_0^s \int_{\phi(r)}^r \mathbb{E} \left[\|\hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \hat{N}_{\mathbf{t}_3, r, T}^{(0, l)} \nabla^i \bar{\Gamma}_s^\epsilon\| \right] du dr ds \\
&\leq \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^1 \int_0^T \int_0^s \int_{\phi(r)}^r \|\hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j}\|_{r^{(1)}} \|\hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k}\|_{r^{(2)}} \|\hat{N}_{\mathbf{t}_3, r, T}^{(0, l)}\|_{r^{(4)}} \|\nabla^i \bar{\Gamma}_s^\epsilon\|_{r^{(3)}} du dr ds \\
&\leq \int_0^T \int_0^s \int_{\phi(r)}^r \|\hat{\gamma}_{u, s}^{\epsilon, 1}\|_{r^{(1)}} \|\hat{N}_{\mathbf{t}_3, r, T}^{(0, 1)}\|_{r^{(4)}} \|\nabla \bar{\Gamma}_s^\epsilon\|_{r^{(3)}} du dr ds \\
&+ \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^0 \int_0^T \int_0^s \int_{\phi(r)}^r \|\hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j}\|_{r^{(1)}} \|\hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k}\|_{r^{(2)}} \|\hat{N}_{\mathbf{t}_3, r, T}^{(0, l)}\|_{r^{(4)}} \|\nabla^i \bar{\Gamma}_s^\epsilon\|_{r^{(3)}} du dr ds \\
&\leq C_{r^{(4)}} (\|\sigma\|_{0,1}) \left\{ \int_0^T \left(\int_0^T \int_{\phi(r)}^r \|\hat{\gamma}_{u, s}^{\epsilon, 1}\|_{r^{(1)}} du dr \right)^{p_1} ds \right\}^{1/p_1} \left\{ \int_0^T \|\nabla \bar{\Gamma}_s^\epsilon\|_{r^{(3)}}^{p'_1} ds \right\}^{1/p'_1} \\
&+ \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^0 C_{r^{(4)}} (\|\sigma\|_{0, l}) \int_0^T \int_0^s \int_{\phi(r)}^r \|\hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j}\|_{r^{(1)}} \|\hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k}\|_{r^{(2)}} \|\nabla^i \bar{\Gamma}_s^\epsilon\|_{r^{(3)}} du dr ds \\
&\leq \frac{C_{r^{(3)}, r^{(4)}, q_1} (\|\sigma\|_{0,1})}{n} \left\{ C_{p_2' r^{(1)}} (\|\sigma\|_{0,1}) |\vec{\nabla} \hat{\beta}^\epsilon|_{r^{(1)} p_2, r^{(1)} p_2} + C_{p_2' r^{(1)}} (\|\sigma\|_{0,2}) |\vec{\nabla}^2 \hat{\beta}^\epsilon|_{2 p_2, r^{(1)}} \right\} \|\nabla \bar{\mathbf{b}}_\epsilon\|_{r^{(3)}, r^{(3)}} \\
&+ \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^0 \frac{C_{r^{(4)}} (\|\sigma\|_{0, l})}{n} \left\{ C_{r^{(1)} p_1'} (\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{r^{(1)} p_1'} (\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2 p_1, r^{(1)}} \right\}^{1+j} \\
&\quad \times \left\{ C_{r^{(2)} p_1'} (\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{r^{(2)} p_1'} (\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2 p_1, r^{(2)}} \right\}^{1+k} \|\nabla^i \bar{\mathbf{b}}_\epsilon\|_{r^{(3)}, r^{(3)}},
\end{aligned}$$

where assume that $1 \leq q'_1 \leq r^{(3)}$.

3.2.3 On $\mathbb{E}[F_1 A_2]$

Equations (14)-(15) are rewritten as

$$F_1 A_2 = \hat{N}_T \int_0^T \int_{\phi(s)}^s \partial_t \Gamma^\epsilon(r, \bar{Y}_{\phi(s)}) dr dW_s + \hat{N}_T \int_0^T \vec{\nabla}^1 \bar{\Gamma}_{\phi(s)}^\epsilon \cdot \mathbf{N}_s^1 \int_{\phi(s)}^s dW_r dW_s.$$

Then from the Malliavin duality formula and an estimation (23) we have

$$|\mathbb{E}[F_1 A_2]| \leq \frac{1}{n} \int_0^T \mathbb{E} \left[|D_s \hat{N}_T| \mathcal{S}(\Gamma^\epsilon(\cdot, \bar{Y}_{\phi(s)}))(s) \right] ds \quad (52)$$

$$+ \int_0^T \int_{\phi(s)}^s \mathbb{E} \left[\left| D_r \left((D_s \hat{N}_T) \vec{\nabla}^1 \bar{\Gamma}_{\phi(s)}^\epsilon \cdot \mathbf{N}_s^1 \right) \right| \right] dr ds. \quad (53)$$

And we have

$$\begin{aligned}
(52) &\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \int_0^T \mathbb{E} \left[\|\hat{\gamma}_s^{\epsilon, j} \hat{N}_{s, T}^{(0, k)}\| \mathcal{S} \left(\Gamma^\epsilon(\cdot, \bar{Y}_{\phi(s)}) \right) (s) \right] ds \\
&\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \int_0^T \|\hat{\gamma}_s^{\epsilon, j}\|_q \|\hat{N}_{s, T}^{(0, k)}\|_{q'} \|\mathcal{S} \left(\Gamma^\epsilon(\cdot, \bar{Y}_{\phi(s)}) \right) (s)\|_{q'} ds, \\
&\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ k \geq 1, (j+1)+k=2}}^0 \left(C_{qp'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{qp'}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1, q} \right)^{1+j} \\
&\quad \times C_{q'}(\|\sigma\|_{0, k}) \|\mathcal{S}(b_\epsilon)(\cdot)\|_{q', q'}.
\end{aligned}$$

By using the same argument as Eq. (51), we have

$$\begin{aligned}
(53) &\leq \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \int_0^T \int_{\phi(s)}^s \mathbb{E} \left[\|\hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \hat{N}_{\mathbf{t}_3, T}^{(0, l)} \nabla^i \bar{\Gamma}_{\phi(s)}^\epsilon\| \right] dr ds \\
&\leq \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{r, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{r\} + \#\mathbf{t}_2 \cap \{r\} + \#\mathbf{t}_3 \cap \{r\} = 1}}^1 \int_0^T \int_{\phi(s)}^s \|\hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j}\|_{r^{(1)}} \|\hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k}\|_{r^{(2)}} \|\hat{N}_{\mathbf{t}_3, T}^{(0, l)}\|_{r^{(4)}} \|\nabla^i \bar{\Gamma}_{\phi(s)}^\epsilon\|_{r^{(3)}} dr ds \\
&\leq \frac{C_{r^{(3)}, r^{(4)}, q_1}(\|\sigma\|_{0,1})}{n} \left\{ C_{p_2' r^{(1)}}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{r^{(1)} p_2, r^{(1)} p_2} + C_{p_2' r^{(1)}}(\|\sigma\|_{0,2}) |\vec{\nabla}^2 \hat{\beta}^\epsilon|_{2p_2, r^{(1)}} \right\} \left| \vec{\nabla} \bar{\mathbf{b}}_{\epsilon, \phi} \right|_{r^{(3)}, r^{(3)}} \\
&+ \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 2, 1 \leq i \leq 2 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, s\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^0 \frac{C_{r^{(4)}}(\|\sigma\|_{0, i})}{n} \left\{ C_{r^{(1)} p_1'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{r^{(1)} p_1'}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1, r^{(1)}} \right\}^{1+j} \\
&\quad \times \left\{ C_{r^{(2)} p_1'}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{r^{(2)} p_1'}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1, r^{(2)}} \right\}^{1+k} \left| \nabla^i \bar{\mathbf{b}}_{\epsilon, \phi} \right|_{r^{(3)}, r^{(3)}},
\end{aligned}$$

where $\nabla^i \bar{\mathbf{b}}_{\epsilon, \phi}(s) := \sum_{j=0}^i \nabla^j b_\epsilon(s, \theta' \bar{Y}_s + (1 - \theta') \bar{Y}_{\phi(s)})$.

3.2.4 On $|\mathbb{E}[F_1 A_3]|$

Equations (16)-(19) are rewritten as

$$\begin{aligned}
F_1 A_3 &= \hat{N}_T \int_0^T \vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1 \int_0^s N_r^{(0,1)} \int_{\phi(r)}^r N_{u,r}^{(1,0)} du dW_r ds \\
&+ \hat{N}_T \int_0^T \vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1 \int_0^s N_r^{(0,1)} \int_{\phi(r)}^r dW_u dW_r ds \\
&+ \hat{N}_T \int_0^T \vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1 \int_0^s N_r^{(0,1)} \int_{\phi(r)}^r N_{u,r}^{(1,0)} du dr ds \\
&+ \hat{N}_T \int_0^T \vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1 \int_0^s N_r^{(0,1)} \int_{\phi(r)}^r dW_u dr ds.
\end{aligned}$$

Then from the Malliavin duality formula and an estimation (23) we have

$$|\mathbb{E}[F_1 A_3]| \leq \frac{1}{n} \int_0^T \int_0^s \sup_u \mathbb{E} \left[\left| (D_r (\hat{N}_T \vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1)) N_{u,r}^{(1,1)} \right| \right] dr ds \quad (54)$$

$$+ \int_0^T \int_0^s \int_{\phi(r)}^r \mathbb{E} \left[\left| D_u \left((D_r (\hat{N}_T \vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1)) N_r^{(0,1)} \right) \right| \right] du dr ds \quad (55)$$

$$+ \frac{1}{n} \int_0^T \int_0^s \sup_u \mathbb{E} \left[\left| \hat{N}_T (\vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1) N_{u,r}^{(1,1)} \right| \right] dr ds \quad (56)$$

$$+ \int_0^T \int_0^s \int_{\phi(r)}^r \mathbb{E} \left[\left| D_u \left(\hat{N}_T (\vec{\nabla}(\bar{\Gamma}_s^\epsilon)^2 \cdot \mathbf{N}_s^1) N_r^{(0,1)} \right) \right| \right] du dr ds. \quad (57)$$

And we have, by using the similar argument to equation (50),

$$\begin{aligned} (54) &\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \int_0^T \int_0^s \mathbb{E} \left[\left| \hat{\gamma}_r^{\epsilon, j} \hat{N}_{r,s,T}^{(0,k)} \nabla^i (\bar{\Gamma}_s^\epsilon)^2 N_{u,r}^{(1,1)} \right| \right] dr ds \\ &\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \int_0^T \int_0^s \left\| \hat{\gamma}_r^{\epsilon, j} \right\|_q \sup_u \left\| \hat{N}_{u,r,s,T}^{(1,k)} \right\|_{q'} \left\| \nabla^i (\bar{\Gamma}_s^\epsilon)^2 \right\|_{q'} dr ds, \\ &\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \left(C_{qp'_1}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{qp'_1}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1, q} \right)^{1+j} \\ &\quad \times C_{q''}(\|\sigma\|_{1,k}) C_{q'p'_1}(\|\sigma\|_{0,1}) |\nabla^i (\bar{\mathbf{b}}_\epsilon)^2|_{q'p_1, q'p_1}, \end{aligned}$$

where $\nabla^i (\bar{\mathbf{b}}_\epsilon)^2(s) := \sum_{j=0}^i \nabla^j (b_\epsilon)^2(s, \theta Y_s + (1-\theta)\bar{Y}_s)$.

Next we have, by using equation (51),

$$\begin{aligned} (55) &\leq \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 3 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, r\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^1 \int_0^T \int_0^s \int_{\phi(r)}^r \mathbb{E} \left[\left| \hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \hat{N}_{\mathbf{t}_3, s, T}^{(0,l)} \nabla^i (\bar{\Gamma}_s^\epsilon)^2 \right| \right] du dr ds \\ &\leq \sum_{\substack{-1 \leq j \leq 1, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 3 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, r\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^1 \int_0^T \int_0^s \int_{\phi(r)}^r \left\| \hat{\gamma}_{\mathbf{t}_1}^{\epsilon, j} \right\|_{r^{(1)}} \left\| \hat{\gamma}_{\mathbf{t}_2}^{\epsilon, k} \right\|_{r^{(2)}} \left\| \hat{N}_{\mathbf{t}_3, s, T}^{(0,l)} \right\|_{r^{(4)}} \left\| \nabla^i (\bar{\Gamma}_s^\epsilon)^2 \right\|_{r^{(3)}} du dr ds, \\ &\leq \frac{C_{r^{(3)}, r^{(4)}, q_1}(\|\sigma\|_{0,1})}{n} \left\{ C_{p_2', r^{(1)}}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{r^{(1)} p_2, r^{(1)} p_2} + C_{p_2', r^{(1)}}(\|\sigma\|_{0,2}) |\vec{\nabla}^2 \hat{\beta}^\epsilon|_{2p_2, r^{(1)}} \right\} |\nabla^1 (\bar{\mathbf{b}}_\epsilon)^2|_{r^{(3)}, r^{(3)}} \\ &+ \sum_{\substack{-1 \leq j \leq 0, -1 \leq k \leq 0, 1 \leq l \leq 3, 1 \leq i \leq 3 \\ i+(j+1)+(k+1)+l=4 \\ \mathbf{t}_1 \cup \mathbf{t}_2 \cup \mathbf{t}_3 = \{u, r\}, \#\mathbf{t}_1 = j+1, \#\mathbf{t}_2 = k+1 \\ \#\mathbf{t}_1 \cap \{u\} + \#\mathbf{t}_2 \cap \{u\} + \#\mathbf{t}_3 \cap \{u\} = 1}}^0 \frac{C_{r^{(4)}}(\|\sigma\|_{0,l})}{n} \left\{ C_{r^{(1)} p'_1}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{r^{(1)} p'_1}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1, r^{(1)}} \right\}^{1+j} \\ &\quad \times \left\{ C_{r^{(2)} p'_1}(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_{r^{(2)} p'_1}(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1, r^{(2)}} \right\}^{1+k} |\nabla^i (\bar{\mathbf{b}}_\epsilon)^2|_{r^{(3)}, r^{(3)}}, \end{aligned}$$

where we assume that $1 \leq q'_1 \leq r^{(3)}$.

Similarly, we have

$$\begin{aligned}
(56) &\leq \frac{1}{n} \int_0^T \int_0^s \sup_u \mathbb{E} \left[|\hat{N}_{u,r,s,T}^{(1,1)} \nabla(\bar{\Gamma}_s^\epsilon)^2| \right] dr ds \\
&\leq \frac{1}{n} \int_0^T \int_0^s \sup_u \|\hat{N}_{u,r,s,T}^{(1,1)}\|_{p'} \|\nabla(\bar{\Gamma}_s^\epsilon)^2\|_p dr ds \\
&\leq \frac{C_{p'}(\|\sigma\|_{1,1})}{n} \int_0^T \|\nabla(\bar{\Gamma}_s^\epsilon)^2\|_p ds \\
&\leq \frac{C_{p'}(\|\sigma\|_{1,1})}{n} |\nabla^1(\bar{\mathbf{b}}_\epsilon)^2|_{p,p}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(57) &\leq \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \int_0^T \int_0^s \int_{\phi(r)}^r \mathbb{E} \left[|\hat{\gamma}_u^{\epsilon,j} \hat{N}_{u,r,s,T}^{(0,k)} \nabla^i(\bar{\Gamma}_s^\epsilon)^2| \right] du dr ds \\
&\leq \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \int_0^T \int_0^s \int_{\phi(r)}^r \|\hat{\gamma}_u^{\epsilon,j}\|_q \|\hat{N}_{u,r,s,T}^{(0,k)}\|_{q'} \|\nabla^i(\bar{\Gamma}_s^\epsilon)^2\|_{q'} du dr ds \\
&\leq \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 C_{q'}(\|\sigma\|_{0,k}) \int_0^T \int_{\phi(r)}^r \|\hat{\gamma}_u^{\epsilon,j}\|_q du dr \int_0^T \|\nabla^i(\bar{\Gamma}_s^\epsilon)^2\|_{q'} ds \\
&\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 C_{q'}(\|\sigma\|_{0,k}) \left\{ C_q(\|\sigma\|_{0,0}) \|\mathbf{b}_\epsilon\|_\infty + C_q(\|\sigma\|_{0,1}) |\vec{\nabla}^1 \hat{\beta}^\epsilon|_{2p_1,q} \right\}^{1+j} |\nabla^i(\bar{\mathbf{b}}_\epsilon)^2|_{q',q'}.
\end{aligned}$$

3.2.5 On $|\mathbb{E}[F_1 A_4]|$

Equations (20)-(21) are rewritten as

$$F_1 A_4 = \hat{N}_T \int_0^T \int_{\phi(s)}^s \partial_t(\Gamma^\epsilon(r, \bar{Y}_{\phi(s)}))^2 dr ds + \hat{N}_T \int_0^T \vec{\nabla}^1(\bar{\Gamma}_{\phi(s)}^\epsilon)^2 \cdot \mathbf{N}_s \int_{\phi(s)}^s dW_r ds.$$

Then from the Malliavin duality formula and an estimation (23),

$$|\mathbb{E}[F_1 A_4]| \leq \frac{1}{n} \int_0^T \mathbb{E} \left[|\hat{N}_T| \mathcal{S}((\Gamma^\epsilon(\cdot, \bar{Y}_{\phi(s)}))^2)(s) \right] ds \quad (58)$$

$$+ \int_0^T \int_{\phi(s)}^s \mathbb{E} \left[\left| D_r \left(\hat{N}_T \vec{\nabla}^1(\bar{\Gamma}_{\phi(s)}^\epsilon)^2 \cdot \mathbf{N}_s \right) \right| \right] dr ds. \quad (59)$$

And we have

$$(58) \leq \frac{1}{n} \int_0^T \|\hat{N}_T\|_{p'} \left\| \mathcal{S}((\Gamma^\epsilon(\cdot, \bar{Y}_{\phi(s)}))^2)(s) \right\|_p ds \leq \frac{1}{n} C_{p'}(\|\sigma\|_{0,0}) \left\| \mathcal{S}((b_\epsilon)^2)(\cdot) \right\|_{p,p}.$$

Using similar arguments of (57),

$$\begin{aligned}
(59) &\leq \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \int_0^T \int_{\phi(s)}^s \mathbb{E} \left[\left| \hat{\gamma}_r^{\varepsilon, j} \hat{N}_{r, s, T}^{(0, k)} \nabla^i (\bar{\Gamma}_{\phi(s)}^\varepsilon)^2 \right| \right] dr ds \\
&\leq \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 \int_0^T \int_{\phi(s)}^s \left\| \hat{\gamma}_r^{\varepsilon, j} \right\|_q \left\| \hat{N}_{r, s, T}^{(0, k)} \right\|_{q''} \left\| \nabla^i (\bar{\Gamma}_{\phi(s)}^\varepsilon)^2 \right\|_{q'} dr ds \\
&\leq \frac{1}{n} \sum_{\substack{j \geq -1 \\ 2 \geq i, k \geq 1, i+(j+1)+k=3}}^0 C_{q''}(\|\sigma\|_{0, k}) \left\{ C_q(\|\sigma\|_{0, 0}) \|\mathbf{b}_\varepsilon\|_\infty + C_q(\|\sigma\|_{0, 1}) |\vec{\nabla}^1 \hat{\mathbf{b}}^\varepsilon|_{2p_1, q} \right\}^{1+j} \left\| \nabla^i (\bar{\mathbf{b}}_{\varepsilon, \phi})^2 \right\|_{q', q'}.
\end{aligned}$$

Finally we have that all the terms which are parts of the main term (36) and [blue parts in this subsection](#) satisfy the conditions stated in Proposition 3 after application of Lemmas 1 and 2 in the main paper.

References

- [1] V. Bally and D. Talay, *The law of the Euler scheme for stochastic differential equations. I. Convergence rate of the distribution function*, Probab. Theory Related Fields **104** (1996), no. 1, 43–60, DOI 10.1007/BF01303802.
- [2] ———, *The law of the Euler scheme for stochastic differential equations. II. Convergence rate of the density*, Monte Carlo Methods Appl. **2** (1996), no. 2, 93–128, DOI 10.1515/mcma.1996.2.2.93.
- [3] E. Gobet and R. Munos, *Sensitivity analysis using Itô-Malliavin calculus and Martingales, and application to stochastic optimal control*, SIAM J. Control Optim. **43** (2005), no. 5, 1676–1713, DOI 10.1137/S0363012902419059.
- [4] A. Kohatsu-Higa, *Weak Approximations: A Malliavin Calculus approach*, Mathematics of Computation **70** (2001), 135–172.
- [5] A. Kohatsu-Higa, A. Lejay, and K. Yasuda, *Weak rate of convergence of the Euler-Maruyama scheme for stochastic differential equations with non-regular drift* (2016).
- [6] D. Nualart, *The Malliavin calculus and related topics*, 2nd ed., Probability and its Applications (New York), Springer-Verlag, 2006.