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Pseudo-holomorphic functions at the critical exponent

Laurent Baratchart*, Alexandre Borichev†, Slah Chaabi‡

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Abstract. We study Hardy classes on the disk for the equation $\bar{\partial}w = \alpha\bar{w}$ when $\alpha \in L^r$ with $2 \leq r < \infty$. The paper seems first to deal with $r = 2$. We prove an analog of the M. Riesz theorem and a strong topological converse to the similarity principle. Using the connection between pseudo-holomorphic functions and conjugate Beltrami equations, we deduce well-posedness on smooth domains of the Dirichlet problem with weighted L^p boundary data for 2-D isotropic conductivity equations whose coefficients have logarithm in $W^{1,2}$. In particular these are not strictly elliptic. Our results depend on a new multiplier theorem for $W_0^{1,2}$ -functions.

1 Introduction

Pseudoanalytic functions of one complex variable, *i.e.* solutions to a $\bar{\partial}$ equation whose right-hand side is a real linear function of the variable, are perhaps the simplest generalization of holomorphic functions. They received early attention in [38, 11] and extensive treatment in [6, 40] when the coefficients are L^r -summable, $r > 2$. While [6] takes on a function-theoretic viewpoint, [40] dwells on integral equations and leans on applications to geometry, elasticity and hydrodynamics. Recent further developments and applications to various boundary values problems can be found in [27, 42, 14]. The case of Besov-smooth coefficients is considered in [8]. Hardy classes for such functions were introduced in [32] and subsequently considered in [28, 29, 30, 5] in the range of exponents $1 < p < \infty$, see [15, 31, 16, 4] for further generalizations to multiply connected domains. The connection between pseudo holomorphic functions and conjugate Beltrami equations makes pseudo-holomorphic Hardy classes a convenient framework to solve Dirichlet problems with L^p boundary data for isotropic conductivity equations [5, 15, 4]. These are also instrumental in [17, 18, 19, 16] to approach certain inverse boundary problems.

The present paper seems first to deal with the critical exponent $r = 2$. We develop a theory of pseudo-holomorphic Hardy spaces on the disk in the range $1 < p < \infty$, prove existence of L^p boundary values, and show an analog of the M. Riesz theorem in this context. We also develop a topological parametrization by holomorphic Hardy functions which is new even when $r > 2$. We apply our result to well-posedness of the Dirichlet problem with weighted L^p boundary data for 2-D conductivity equations whose coefficients have logarithm in $W^{1,2}$. In particular these are not bounded away from zero nor infinity, and no strict ellipticity prevails. Accordingly, solutions may be locally unbounded.

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Like in previous work on pseudo-analytic functions, we make extensive use of the similarity principle, but in our case it requires a thorough analysis of smoothness and boundedness properties of exponential of $W^{1,2}$ functions which is carried out in a separate appendix. There we prove a theorem, one of the main technical results of the paper, asserting that a $W_0^{1,2}$ function in the disk is a multiplier from the space of functions with L^p maximal function on \mathbb{T} to the space of functions satisfying a Hardy condition of order p . This would have higher dimensional analogs, but we make no attempt at developing them and stick to dimension 2 throughout the paper.

2 Notations and preliminaries

Let $\mathbb{C} \sim \mathbb{R}^2$ be the complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. We designate by $\mathbb{T}_{\xi,\rho}$ and $\mathbb{D}_{\xi,\rho}$ respectively the circle and the open disk centered at ξ of radius ρ . We simply write \mathbb{T}_ρ , \mathbb{D}_ρ when $\xi = 0$, and if $\rho = 1$ we omit subscript. If f is a function on \mathbb{D}_ρ , we often put f_ρ for the function on \mathbb{D} defined by $f_\rho(\xi) := f(\rho\xi)$. To $\xi \in \mathbb{T}$ and $\gamma \in (0, \pi/2)$, we let $\Gamma_{\xi,\gamma}$ indicate the cone with vertex ξ and aperture 2γ , symmetric with respect to the line through 0, ξ . A complex-valued function f on \mathbb{D} has non-tangential limit ℓ at ξ if $f(z)$ tends to ℓ as $z \rightarrow \xi$ inside $\Gamma_{\xi,\gamma}$ for every γ . The non-tangential maximal function of f (with aperture 2γ) is the real valued map $\mathcal{M}_\gamma f$ on \mathbb{T} given by

$$\mathcal{M}_\gamma f(\xi) := \sup_{z \in \mathbb{D} \cap \Gamma_{\xi,\gamma}} |f(z)|, \quad \xi \in \mathbb{T}. \quad (1)$$

For $E \subset \mathbb{C}$ and f a function on a set containing E , we let $f|_E$ indicate the restriction of f to E . We put $|E|$ for the planar Lebesgue measure of E , as no confusion can arise with complex modulus. The differential of that measure is denoted interchangeably by

$$dm(z) = dxdy = (i/2) dz \wedge d\bar{z}, \quad z = x + iy.$$

When $\Omega \subset \mathbb{C}$ is an open set, we put $\mathcal{D}(\Omega)$ for the space of C^∞ -smooth complex-valued functions with compact support in Ω , equipped with the usual topology¹. Its dual $\mathcal{D}'(\Omega)$ is the space of distributions on Ω . For $p \in [1, \infty]$, we let $L^p(\Omega)$ and $W^{1,p}(\Omega)$ be the familiar Lebesgue and Sobolev spaces with respect to dm ; we sometimes write $L_{\mathbb{R}}^p(\Omega)$ or $W_{\mathbb{R}}^{1,p}(\Omega)$ to emphasize restriction to real-valued functions. Recall that $W^{1,p}(\Omega)$ consists of functions in $L^p(\Omega)$ whose first distributional derivatives lie in $L^p(\Omega)$, with norm:

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\partial f\|_{L^p(\Omega)} + \|\bar{\partial} f\|_{L^p(\Omega)}.$$

Here, ∂ and $\bar{\partial}$ stand for the usual complex derivatives:

$$\partial f := \partial_z f = \frac{1}{2}(\partial_x - i\partial_y)f \quad \text{and} \quad \bar{\partial} f := \partial_{\bar{z}} f = \frac{1}{2}(\partial_x + i\partial_y)f, \quad z = x + iy.$$

Note the identities $\overline{\partial f} = \bar{\partial} \bar{f}$ and $\Delta = 4\partial\bar{\partial}$, where Δ is the Euclidean Laplacian. By Weyl's lemma [20, thm 24.9], those $u \in \mathcal{D}'(\Omega)$ such that $\Delta u = 0$ are exactly the harmonic

¹*i.e.* the inductive topology of subspaces \mathcal{D}_K comprised of functions supported in the compact set K , each \mathcal{D}_K being topologized by uniform convergence of all derivatives [36, Ch. I, Sec.2].

functions on Ω . Subsequently, those $\psi \in \mathcal{D}'(\Omega)$ such that $\bar{\partial}\psi = 0$ are exactly the holomorphic functions on Ω . The space $\mathcal{D}(\mathbb{R}^2)$ is dense in $W^{1,p}(\mathbb{R}^2)$ for $p \in [1, \infty)$, and $W^{1,\infty}(\Omega)$ identifies with Lipschitz-continuous functions on Ω [37, ch.V prop. 1 & sec. 6.2].

We also introduce the spaces $L_{loc}^p(\Omega)$ and $W_{loc}^{1,p}(\Omega)$ of distributions whose restriction to any relatively compact open subset $\Omega_0 \subset \Omega$ lies in $L^p(\Omega_0)$ and $W^{1,p}(\Omega_0)$ respectively. These are topologized by the family of semi-norms $\|f_{\Omega_n}\|_{L^p(\Omega_n)}$ and $\|f_{\Omega_n}\|_{W^{1,p}(\Omega_n)}$ where $\{\Omega_n\}$ is a sequence of relatively compact open subsets exhausting Ω .

Below we indicate properties of Sobolev functions, most of which are standard but not all. They are valid on bounded Lipschitz domains (*i.e.* domains Ω whose boundary $\partial\Omega$ is locally isometric to the graph of a Lipschitz function).

- For $1 \leq p \leq \infty$, each $f \in W^{1,p}(\Omega)$ is the restriction to Ω of some $\tilde{f} \in W^{1,p}(\mathbb{R}^2)$. In fact, there is a continuous linear map $E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^2)$ such that $(Ef)|_{\Omega} = f$ (the extension theorem [12, prop. 2.70]). When $\Omega = \mathbb{D}_{\rho}$, we may simply put $(Ef)|_{\mathbb{C} \setminus \mathbb{D}_{\rho}}(z) = \varphi(z)f(\rho^2/\bar{z})$ where $\varphi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi|_{\mathbb{D}_{\rho}} \equiv 1$. The extension theorem entails that smooth functions on $\bar{\Omega}$ are dense in $W^{1,p}(\Omega)$ when $1 \leq p < \infty$.
- For $p > 2$, $W^{1,p}(\Omega)$ embeds continuously in the space of Hölder-smooth functions with exponent $1 - 2/p$ on Ω , in particular functions in $W^{1,p}(\Omega)$ extend continuously to $\bar{\Omega}$ and $W^{1,p}(\Omega)$ is an algebra where multiplication is continuous and derivatives can be computed by the chain rule. When $1 \leq p < 2$ the embedding is in $L^{p^*}(\Omega)$ with $p^* = 2p/(2 - p)$, while $W^{1,2}(\Omega)$ is embedded in all $L^{\ell}(\Omega)$, $\ell \in [1, \infty)$ (the Sobolev embedding theorem [1, thms. 4.12 & 4.39]).
- For $p \leq 2$ the embedding $W^{1,p}(\Omega) \rightarrow L^{\ell}(\Omega)$ is compact when $\ell \in [1, p^*)$ (the Rellich-Kontrachov theorem [1, thm. 6.3]).
- If $g \in \mathcal{D}'(\Omega)$ has derivatives in $L^p(\Omega)$ for some $p \in [1, \infty)$, then $g \in W^{1,p}(\Omega)$ [12, thm. 6.74]². Moreover there is a constant $C = C(\Omega, p)$ such that,

$$\|g - g_{\Omega}\|_{L^p(\Omega)} \leq C(\|\partial g\|_{L^p(\Omega)} + \|\bar{\partial} g\|_{L^p(\Omega)}) \quad \text{where} \quad g_{\Omega} := \frac{1}{|\Omega|} \int_{\Omega} g \, dm \quad (2)$$

(the Poincaré inequality [41, thm. 4.2.1]).

- $W^{1,p}(\Omega)$ -functions need not be continuous nor even locally bounded when $p \leq 2$, but if $p > 1$ their non-Lebesgue points form a set of Bessel $B_{1,p}$ -capacity zero [41, thms. 3.10.2]. Such sets are very thin: not only do they have measure zero but their Hausdorff $H^{2-p+\varepsilon}$ -dimension is zero for each $\varepsilon > 0$ [41, thm. 2.6.16]. When speaking of pointwise values of $f \in W^{1,p}(\Omega)$, we pick a representative such that $f(z) = \lim_{\varepsilon \rightarrow 0} f_{\mathbb{D}_{z,\varepsilon}}$ outside a set of $B_{1,p}$ -capacity zero. At such a z , f is said to be *strictly defined*.
- If $L^{\lambda}(\partial\Omega)$ is understood with respect to arclength, then $W^{1,\lambda}(\partial\Omega)$ is naturally defined using local coordinates, since Lipschitz-continuous changes of variable preserve Sobolev classes [41, thm. 2.2.2]. Each $f \in W^{1,p}(\Omega)$ with $1 < p \leq \infty$ has a trace on $\partial\Omega$, denoted again by f or sometimes by $\text{tr}_{\partial\Omega} f$ for emphasis, which lies in the

²The proof given there for bounded C^1 -smooth Ω carries over to the Lipschitz case.

Sobolev space $W^{1-1/p,p}(\partial\Omega)$ of non-integral order³, The latter is a real interpolation space between $L^p(\partial\Omega)$ and $W^{1,p}(\partial\Omega)$ of exponent $1 - 1/p$, a norm on which is given by [1, thm 7.47]:

$$\|g\|_{W^{1-1/p,p}(\partial\Omega)} := \|g\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|g(t) - g(t')|^p}{(\Lambda(t, t'))^p} d\Lambda(t) d\Lambda(t') \right)^{1/p} \quad (3)$$

where $\Lambda(t, t')$ indicates the length of the arc (t, t') on $\partial\Omega$. Note that $|t - t'| \sim \Lambda(t, t')$ since $\partial\Omega$ is Lipschitz. The trace operator defines a continuous surjection from $W^{1,p}(\Omega)$ onto $W^{1-1/p,p}(\partial\Omega)$ [24, thm. 1.5.1.3]. The pointwise definition of $\text{tr}_{\partial\Omega} f$ Λ -a.e. rests on the extension theorem and the fact that non-Lebesgue points of Ef have Hausdorff H^1 -measure zero [41, rmk. 4.4.5]. Of course $\text{tr}_{\partial\Omega} f$ coincides with the restriction $f|_{\partial\Omega}$ whenever f is smooth on $\bar{\Omega}$. The subspace $W_0^{1,p}(\Omega)$ of functions whose trace is zero coincides with the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$.

Because the integral in the right hand side of (3) does not change if we add a constant to g , it follows from (2) by continuity of the trace operator that

$$\left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|g(t) - g(t')|^p}{(\Lambda(t, t'))^p} d\Lambda(t) d\Lambda(t') \right)^{1/p} \leq C \left(\|\partial g\|_{L^p(\Omega)} + \|\bar{\partial} g\|_{L^p(\Omega)} \right) \quad (4)$$

where the constant C depends on Ω and p .

A variant of the Poincaré inequality involving the trace is: whenever $E \subset \partial\Omega$ has arclength $\Lambda(E) > 0$, there is $C > 0$ depending only p , Ω and E such that

$$\left\| g - \int_E \text{tr}_{\partial\Omega} g \right\|_{L^p(\Omega)} \leq C (\|\partial g\|_{L^p(\Omega)} + \|\bar{\partial} g\|_{L^p(\Omega)}). \quad (5)$$

This follows immediately from the continuity of the trace operator, the Rellich-Kondratchov theorem, and [41, lem. 4.1.3].

- For $p \in (1, \infty)$, the trace operator has a continuous section [24, thm. 1.5.1.3]; that is, to each $\psi \in W^{1-1/p,p}(\partial\Omega)$, there is $g \in W^{1,p}(\Omega)$ such that and

$$\|g\|_{W^{1,p}(\Omega)} \leq C \|\psi\|_{W^{1-1/p,p}(\partial\Omega)}, \quad \text{tr}_{\partial\Omega} g = \psi, \quad (6)$$

with $C = C(\Omega, p)$. If we assume that Ω is C^1 -smooth and not just Lipschitz, then the function g in (6) can be chosen so as to be harmonic in Ω (elliptic regularity theory [26, p. 165 & thm. 1.3])⁴.

- The non-integral version of the Sobolev embedding theorem [1, thm. 7.34] asserts that $W^{1-1/\beta,\beta}(\partial\Omega)$ injects continuously in $L^{\beta/(2-\beta)}(\partial\Omega)$ if $1 < \beta < 2$, while $W^{1/2,2}(\partial\Omega)$ embeds in $L^\ell(\partial\Omega)$ for all $\ell \in [1, \infty)$. The corresponding generalization of the Rellich-Kondratchov theorem [12, thm 4.54] is: if $1 < \beta \leq 2$, then $W^{1-1/\beta,\beta}(\partial\Omega)$ embeds compactly in $L^\ell(\partial\Omega)$ for $\ell < \beta/(2 - \beta)$.

³We leave out the case $p = 1$ where the trace is merely defined in $L^1(\partial\Omega)$. The space $W^{1-1/p,p}(\partial\Omega)$ coincides with the Besov space $B_p^{1-1/p,p}(\partial\Omega)$, but we need not introduce Besov spaces here.

⁴In fact, elliptic regularity holds for $1 < p < \infty$ as soon as $\partial\Omega$ is locally the graph of a VMO function [25, thm. 1.1]. When $\partial\Omega$ is only Lipschitz-smooth, then the range of p has to be restricted in a manner that depends on the Lipschitz constant, see [26, 25].

- When $p \in (2, \infty)$, the nonlinear map $f \mapsto e^f$ is bounded and continuous from $W^{1,p}(\Omega)$ into itself: this follows from the Taylor expansion of \exp because $W^{1,p}(\Omega)$ is an algebra. When $p = 2$ this property no longer holds, but still $f \mapsto e^f$ maps $W^{1,2}(\Omega)$ continuously into $W^{1,q}(\Omega)$ for all $q \in [1, 2)$, in particular $\text{tr}_{\partial\Omega} e^f = e^{\text{tr}_{\partial\Omega} f}$ exists in $W^{1-1/q,q}(\partial\Omega)$ for $1 < q < 2$. This is the content of Proposition 1 in section 8 that we could not locate in the literature⁵.

Though continuous when $q \in [1, 2)$, the nonlinear map $f \mapsto e^f$ is not bounded from $W^{1,2}(\Omega)$ to $W^{1,q}(\Omega)$. A sufficient condition for a bounded set $\mathcal{E} \subset W^{1,2}(\Omega)$ to be mapped onto a bounded subset of $W^{1,q}(\Omega)$ is the uniform absolute continuity of the measures $(|\partial f|^2 + |\bar{\partial} f|^2)dm$ for $f \in \mathcal{E}$, see Corollary 2 in Section 8.

- We use at some point the Sobolev space $W^{2,p}(\Omega)$ of functions in $L^p(\Omega)$ whose first distributional derivatives lie in $W^{1,p}(\Omega)$, equipped with the norm:

$$\|f\|_{W^{2,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\partial f\|_{W^{1,p}(\Omega)} + \|\bar{\partial} f\|_{W^{1,p}(\Omega)}.$$

When $p \leq 2$, the Rellich-Kondratchov theorem implies that $W^{2,p}(\Omega)$ is compactly embedded in $W^{1,\ell}(\Omega)$ for $\ell \in [1, p*)$.

For Ω bounded and $h \in L^p(\Omega)$, $1 < p < \infty$, let \tilde{h} denote the extension of h by 0 off Ω . The Cauchy integral operator applied to \tilde{h} defines a function $\mathcal{C}(h) \in W_{loc}^{1,p}(\mathbb{R}^2)$ given by

$$\mathcal{C}(h)(z) = \frac{1}{\pi} \int_{\Omega} \frac{h(t)}{z-t} dm(t) = \frac{1}{2\pi i} \int_{\Omega} \frac{h(\xi)}{\xi-z} d\xi \wedge \bar{d}\xi, \quad z \in \mathbb{C}. \quad (7)$$

Indeed $\mathcal{C}(h)$ lies in $L_{loc}^1(\mathbb{C})$ by Fubini's theorem, and since $z \mapsto 1/(\pi z)$ is a fundamental solution of the $\bar{\partial}$ operator it follows that $\bar{\partial} \mathcal{C}(h) = \tilde{h}$ in the distributional sense. In another connection (see [3, thm. 4.3.10] and the remark thereafter), the complex derivative $\partial \mathcal{C}(h)$ is given by the singular integral

$$\mathcal{B}(h)(z) := \lim_{\varepsilon \rightarrow 0} -\frac{1}{\pi} \int_{\Omega \setminus D(z, \varepsilon)} \frac{h(\xi)}{(z-\xi)^2} dm(\xi), \quad z \in \mathbb{C}, \quad (8)$$

which is the so-called Beurling transform of \tilde{h} . By a result of Calderón and Zygmund (see [3, thm 4.5.3]) this transform maps $L^p(\mathbb{C})$ continuously into itself, and altogether we conclude that $\mathcal{C}(h) \in W_{loc}^{1,p}(\mathbb{C})$, as announced.

The discussion above shows in particular that $\varphi := \mathcal{C}(h)|_{\Omega}$ lies in $W^{1,p}(\Omega)$, and that

$$\|\partial \varphi\|_{L^p(\Omega)} + \|\bar{\partial} \varphi\|_{L^p(\Omega)} = \|\mathcal{B}(h)|_{\Omega}\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)} \leq c \|h\|_{L^p(\Omega)},$$

where c depends only on p . In addition, it is a consequence of Fubini's theorem that $\|\varphi\|_{L^p(\Omega)} \leq 6 \text{diam } \Omega \|h\|_{L^p(\Omega)}$ [3, thm. 4.3.12]. Therefore

$$\|\mathcal{C}(h)\|_{W^{1,p}(\Omega)} \leq C \|h\|_{L^p(\Omega)} \quad (9)$$

where C depends only on p and Ω . Moreover, if R is large enough that $\Omega \subset B_R$, then $\mathcal{C}(h)$ coincides on Ω with the convolution of \tilde{h} with $z \mapsto \chi_{B_{2R}}(z)/z$, where χ_E denotes the

⁵By the Sobolev embedding theorem, Proposition 1 also implies that $e^f \in L^\ell(\Omega)$ for all $\ell \in [1, \infty)$ when $f \in W^{1,2}(\Omega)$. However, the classical Trudinger-Moser inequality already yields a stronger result *via* the extension theorem: $\exp \left\{ C|f|^2 / \|f\|_{W^{1,2}(\Omega)}^2 \right\}$ lies in $L^1(\Omega)$ for some constant C , see [23, thm. 7.15].

characteristic function of a set E . Therefore $\partial\mathcal{C}(\varphi)|_\Omega = \mathcal{C}(\partial\varphi)|_\Omega$ whenever $\varphi \in \mathcal{D}(\Omega)$, and by density it follows for $p \in (1, \infty)$ and some $C = C(p, \Omega)$ that

$$\|\mathcal{C}(h)\|_{W^{2,p}(\Omega)} \leq C\|h\|_{W^{1,p}(\Omega)}, \quad h \in W_0^{1,p}(\Omega). \quad (10)$$

Properties of the Cauchy transform make it a basic tool to integrate $\bar{\partial}$ -equations in Sobolev classes. In this connection, we record the following facts.

- For $\Omega \subset \mathbb{C}$ a bounded open set and $a \in L^p(\Omega)$ with $p \in (1, \infty)$, a distribution $A \in \mathcal{D}'(\Omega)$ satisfies $\bar{\partial}A = a$ if and only if $A = \mathcal{C}(a) + \Phi$ where Φ is holomorphic in Ω . This follows from the relation $\bar{\partial}\mathcal{C}(a) = a$ and Weyl's lemma. Clearly by (9), A belongs to $W^{1,p}(\Omega)$ if and only if Φ does. By localization, it follows that if $f \in L_{loc}^1(\Omega)$ satisfies $\bar{\partial}f \in L_{loc}^p(\Omega)$, then $f \in W_{loc}^{1,p}(\Omega)$.
- For $\Omega \subset \mathbb{C}$ a bounded C^1 -smooth simply connected domain and $a \in L^p(\Omega)$ with $p \in (1, \infty)$, then to each $\psi \in W^{1-1/p,p}(\partial\Omega)$, $\lambda \in \mathbb{R}$, $\theta_0 \in \mathbb{R}$, there is a unique $A \in W^{1,p}(\Omega)$ such that $\bar{\partial}A = a$ with $\text{tr}_{\partial\Omega}\text{Re}(e^{i\theta_0}A) = \psi$, and $\int_{\partial\Omega} \text{Im}(e^{i\theta_0}A) = \lambda$. Moreover, there are constants C, c depending only on p and Ω such that

$$\|A\|_{W^{1,p}(\Omega)} \leq C(\|a\|_{L^p(\Omega)} + \|\psi\|_{W^{1-1/p,p}(\partial\Omega)}) + c|\lambda|. \quad (11)$$

To see this it is enough, in view of (9) and the previous item, to consider the case $a = 0$. Clearly, we may also assume that $\theta_0 = 0$. By elliptic regularity, there is a unique $u \in W_{\mathbb{R}}^{1,p}(\Omega)$, harmonic in Ω and such that $\text{tr}_{\partial\Omega}u = \psi$. Moreover u satisfies $\|u\|_{W^{1,p}(\Omega)} \leq C\|\psi\|_{W^{1-1/p,p}(\partial\Omega)}$. As Ω is simply connected, integrating the conjugate differential yields a so-called harmonic conjugate to u , that is a real-valued harmonic function v , such that $A := u + iv$ is holomorphic in Ω . By realness of u, v and the Cauchy-Riemann equations, we get $|\partial v| = |\bar{\partial}v| = |\partial u|$ hence $v \in W_{\mathbb{R}}^{1,p}(\Omega)$. Clearly v is unique up to an additive constant, and if $\int_{\partial\Omega} v = \lambda$ we deduce from (5) that $\|v\|_{W^{1,p}(\Omega)} \leq C_1\|u\|_{W^{1,p}(\Omega)} + c_1|\lambda|$ so that (11) holds (with $a = 0$), as desired.

When $h \in L^2(\mathbb{C})$ has unbounded support, definition (7) of the Cauchy transform is no longer suitable. Instead, one renormalizes the kernel and puts

$$\mathcal{C}_2(h)(z) := \frac{1}{\pi} \int_{\mathbb{R}^2} \left(\frac{1}{z-t} + \frac{\chi_{\mathbb{C} \setminus \mathbb{D}}(t)}{t} \right) h(t) dm(t), \quad z \in \mathbb{C}. \quad (12)$$

Because $h \in L^2(\mathbb{C})$, the integral in (12) converges for a.e. $z \in \mathbb{C}$ by Fubini's theorem and the Schwarz inequality. In fact the function $\mathcal{C}_2(h)$ belongs to the space $VMO(\mathbb{C})$ [3, thm. 4.3.9], see definition in section 8. Further, it holds that $\bar{\partial}\mathcal{C}_2(h) = h$ and $\partial\mathcal{C}_2(h) = \mathcal{B}(h)$ [3, thm. 4.3.10]. In particular $\mathcal{C}_2(h)$ lies in $W_{loc}^{1,2}(\mathbb{C})$ and the map $h \mapsto \mathcal{C}_2(h)$ maps $L^2(\mathbb{C})$ continuously into $W_{loc}^{1,2}(\mathbb{C})$.

In section 8.1 we prove the following estimate, valid for some absolute constants C, C' :

$$\frac{\|\mathcal{C}_2(h)\|_{L^2(\mathbb{D}_R)}}{R} \leq (C + C'(\log R)^{1/2}) \|h\|_{L^2(\mathbb{D}_R)}, \quad R \geq 1. \quad (13)$$

Hereafter, all classes of functions we consider are embedded in $L_{loc}^p(\Omega)$ for some $p \in (1, +\infty)$, and solutions to differential equations are understood in the distributional sense. On the disk, we often use the elementary fact that if $f \in W^{1,p}(\mathbb{D})$ then f_ρ converges to f in $W^{1,p}(\mathbb{D})$ as $\rho \rightarrow 1^-$. This is clear by uniform continuity when f is smooth on $\bar{\mathbb{D}}$, and follows by density and a three- ε argument in the general case.

As transpired from the above already, the same symbol (*e.g.* “ C ”) is used many times to mean different constants.

3 Pseudo-holomorphic functions

Pseudo-holomorphic functions on an open set $\Omega \subset \mathbb{C}$ are those Φ satisfying an equation of the form

$$\bar{\partial}\Phi(z) = a(z)\overline{\Phi(z)} + b(z)\Phi(z), \quad z \in \Omega. \quad (14)$$

We restrict ourselves to the case where Ω is bounded and $a, b \in L^r(\Omega)$ for some $r \in [2, \infty)$. Accordingly, we only consider solutions Φ which belong to $L_{loc}^\gamma(\Omega)$ for some $\gamma > r/(r-1)$ so that, by Hölder's inequality, the right hand side of (14) defines a function in $L_{loc}^\lambda(\Omega)$ for some $\lambda > 1$. In particular, Φ has in fact to lie in $W_{loc}^{1,\lambda}(\Omega)$.

Letting $B \in W^{1,r}(\Omega)$ be such that $\bar{\partial}B = b$, a short computation (using Proposition 1 in Section 8 if $r = 2$) shows that Φ satisfies (14) if and only if $w := e^{-B}\Phi$ satisfies

$$\bar{\partial}w = \alpha\bar{w} \quad (15)$$

where $\alpha := ae^{-2i\text{Im}B}$ has the same modulus as a . Note (again from Proposition 1 if $r = 2$) that $w \in W_{loc}^{1,\lambda'}(\Omega)$ for some $\lambda' > 1$. Therefore, by the Sobolev embedding theorem, w lies in $L_{loc}^{\gamma'}(\Omega)$ for some $\gamma' > 2$, and so equation (15) is a simpler but equivalent form of (14) which is the one we shall really work with.

We need a factorization principle which goes back to [38], and was extensively used in all works mentioned above. We provide a proof because we allow for $r = 2$ and discuss normalization issues when Ω is smooth.

Lemma 1 (Similarity principle) *Let $\Omega \subset \mathbb{C}$ be a bounded domain, $\alpha \in L^r(\Omega)$ for some $r \in [2, \infty)$, and $w \in L_{loc}^\gamma(\Omega)$ be a solution to (15) with $\gamma > r/(r-1)$.*

(i) *The function w admits a factorization of the form*

$$w(z) = \exp(s(z))F(z), \quad z \in \Omega, \quad (16)$$

where F is holomorphic in Ω and $s \in W^{1,r}(\Omega)$ with

$$\|s\|_{W^{1,r}(\Omega)} \leq C\|\alpha\|_{L^r(\Omega)}, \quad (17)$$

C being a constant depending only on r and Ω .

(ii) *Assume in addition that Ω is C^1 -smooth. If $w \not\equiv 0$ and we pick $\psi \in W^{1-1/r,r}(\partial\Omega)$, $\lambda \in \mathbb{R}$, and $\theta_0 \in \mathbb{R}$, then s can be uniquely chosen in (16) so that $\text{tr}_{\partial\Omega} \text{Re}(e^{i\theta_0}s) = \psi$ and $\int_{\partial\Omega} \text{Im}(e^{i\theta_0}s) = \lambda$. In this case, there are constants C, c depending only on r and Ω such that*

$$\|s\|_{W^{1,r}(\Omega)} \leq C(\|\alpha\|_{L^r(\Omega)} + \|\psi\|_{W^{1-1/r,r}(\partial\Omega)}) + c|\lambda|. \quad (18)$$

(iii) *Either $w \equiv 0$ or $w \not\equiv 0$ a. e. on Ω^6 . Moreover $w \in W_{loc}^{1,r}(\Omega)$ if $r > 2$ and $w \in W_{loc}^{1,q}(\Omega)$ for all $q \in [1, 2)$ if $r = 2$.*

⁶More in fact is true: if $r > 2$ then e^s never vanishes and w has at most countably many zeros, namely those of F . If $r = 2$, w is strictly defined and nonzero outside a set of Bessel $B_{1,2}$ -capacity zero (containing the zeros of F and the non Lebesgue points of s).

Proof. We pointed out already that $w \in W_{loc}^{1,\ell}(\Omega)$ for some $\ell > 1$. Set by convention $\overline{w(\xi)}/w(\xi) = 0$ if $w(\xi) = 0$, and let $s := \mathcal{C}(\alpha\bar{w}/w)|_{\Omega}$. Then $s \in W^{1,r}(\Omega)$ with $\bar{\partial}s = \alpha\bar{w}/w$, and we get from (9) that (17) holds. To show that $F = e^{-s}w$ is in fact holomorphic, we compute

$$\bar{\partial}(e^{-s}w) = e^{-s}(-\bar{\partial}s w + \bar{\partial}w) = e^{-s}\left(-\frac{\alpha\bar{w}}{w}w + \alpha\bar{w}\right) = 0,$$

where the use of the Leibnitz and the chain rules is justified by Proposition 1 in Section 8 if $r = 2$. This proves (i).

Because s is finite a.e. on Ω (actually outside a set of $B_{1,2}$ -capacity zero), e^s is a.e. nonzero and so is w unless the holomorphic function F is identically zero. If $r > 2$ it holds that $e^s \in W^{1,r}(\mathbb{D})$, and since F is locally smooth we get that $w \in W_{loc}^{1,r}(\Omega)$; if $r = 2$, it follows from Proposition 1 that $e^s \in W^{1,q}(\Omega)$ for all $q \in [1, 2)$, and thus $w = e^s F$ lies in $W_{loc}^{1,q}(\Omega)$. This proves (iii).

Finally, when Ω is C^1 -smooth and $w \not\equiv 0$ (hence $w \neq 0$ a.e. by what precedes), there uniquely exists $s \in W^{1,r}(\Omega)$ so that $\bar{\partial}s = \alpha\bar{w}/w$, $\text{tr}_{\partial\Omega}\text{Re}(e^{i\theta_0}s) = \psi$, $\int_{\partial\Omega}\text{Im}(e^{i\theta_0}s) = \lambda$, and (11) shows that (18) holds. Moreover if (16) holds for some $s \in W^{1,r}(\Omega)$ and some holomorphic F , we find upon differentiating that $\bar{\partial}s = \alpha\bar{w}/w$, therefore factorization (16) is unique with the afore-mentioned conditions. This proves (ii). ■

A weak converse to the similarity principle is: if $s \in W^{1,r}(\Omega)$ and F is holomorphic on Ω , then $w = e^s F$ meets (15) with $\alpha := \bar{\partial}s e^s F / (e^s \bar{F}) \in L^r(\Omega)$. This remark shows that, in general, we cannot expect solutions of (15) to lie in $L_{loc}^\infty(\Omega)$ when $r = 2$.

4 Holomorphic parametrization

When $r > 2$, it follows from [40, thm. 3.13] that to each holomorphic F on Ω and each $\alpha \in L^r(\Omega)$, there is $\Phi \in W^{1,r}(\Omega)$ such that $w := \Phi F$ satisfies (15). In this section we strengthen this to a strong converse of the similarity principle, valid for $2 \leq r < \infty$, which leads to a parametrization of pseudo-holomorphic functions by holomorphic functions. We state the result on the disk, which is our focus in the present paper, but we mention that it carries over at once to Dini-smooth⁷ simply connected domains, granted the conformal invariance of equation (15) pointed out in [4, sec. 3.2].

Theorem 1 *Let $\alpha \in L^r(\mathbb{D})$ for some $r \in [2, \infty)$ and $F \not\equiv 0$ be holomorphic on \mathbb{D} . Pick further $\psi \in W_{\mathbb{R}}^{1-1/r,r}(\partial\Omega)$, and $\lambda \in \mathbb{R}$. Then, there is a unique $s \in W^{1,r}(\mathbb{D})$ such that $w = e^s F$ is a solution of (15) with $\text{tr}_{\mathbb{T}}\text{Im}s = \psi$ and $\int_{\mathbb{T}}\text{Re}s = \lambda$. Moreover (18) holds for some constants C, c depending only on r .*

From the proof of the theorem, we shall also obtain the following variant thereof.

Corollary 1 *Theorem 1 remains valid if, instead of $\text{tr}_{\mathbb{T}}\text{Im}s = \psi$ and $\int_{\mathbb{T}}\text{Re}s = \lambda$, we prescribe $\text{tr}_{\mathbb{T}}\text{Re}s = \psi$ and $\int_{\mathbb{T}}\text{Im}s = \lambda$.*

Before establishing Theorem 1, we need to take a closer look at pairs s, F for which (16) and (15) hold. This we do in the following subsection.

⁷A domain is Dini-smooth if its boundary has a parametrization with Dini-continuous derivative. Conformal maps between such domains have derivatives that extend continuously up to the boundary.

4.1 Arguments of pseudo-holomorphic functions

Let $w \in L_{loc}^\gamma(\mathbb{D})$ satisfy (15), $\gamma > r/(r-1)$, and consider factorization (16) provided by Lemma 1. Locally around points where F does not vanish, w has a Sobolev-smooth argument, unique modulo $2\pi\mathbb{Z}$, which is given by $\arg w = \arg F + \operatorname{Im} s$. Since $\log F$ is harmonic and $\bar{\partial}s = \alpha\bar{w}/w$, we deduce that around such points $\Delta \log w = 4\partial(\alpha e^{-2i\arg w})$. In particular $\arg w$ satisfies the nonlinear (yet quasilinear) equation $\Delta \arg w = 4\operatorname{Im}(\partial(\alpha e^{-2i\arg w}))$, and then $\log |w|$ is determined by $\arg w$ up to a harmonic function which turns out to be entirely specified by (15). The lemma below dwells on this observation but avoids speaking of $\arg F$ (which may not be globally defined if F has zeros).

Lemma 2 *Let $\alpha \in L^r(\mathbb{D})$ for some $r \in [2, \infty)$ and F be a non identically zero holomorphic function in \mathbb{D} . If we set $\beta := \alpha\bar{F}/F$, then a function $s \in W^{1,r}(\mathbb{D})$ is such that $w := e^s F$ satisfies (15) if and only if $\bar{\partial}s = \beta e^{-2i\operatorname{Im} s}$. This is equivalent to saying that $s = \varphi_1 + i\varphi_2$ where $\varphi_1, \varphi_2 \in W_{\mathbb{R}}^{1,r}(\mathbb{D})$ satisfy:*

$$\Delta\varphi_2 = 4\operatorname{Im}(\partial(\beta e^{-2i\varphi_2})), \quad (19)$$

$$\varphi_1 = \operatorname{Re}\mathcal{C}(\beta e^{-2i\varphi_2}) + v, \quad (20)$$

where v is a harmonic conjugate to the harmonic function $u \in W_{\mathbb{R}}^{1,r}(\mathbb{D})$ such that $\operatorname{tr}_{\mathbb{T}} u = \operatorname{tr}_{\mathbb{T}} \operatorname{Im}(\mathcal{C}(\beta e^{-2i\varphi_2})) - \operatorname{tr}_{\mathbb{T}} \varphi_2$.

Proof. Using Proposition 1 in Section 8 to justify the computation in case $r = 2$, we find that $s \in W^{1,r}(\mathbb{D})$ is such that $w = e^s F$ meets (15) if and only if $\bar{\partial}s - \beta e^{\bar{s}-s} = 0$. Putting $\varphi_1 := \operatorname{Re} s$ and $\varphi_2 := \operatorname{Im} s$, this is equivalent to

$$\bar{\partial}\varphi_1 = \beta \exp(-2i\varphi_2) - i\bar{\partial}\varphi_2, \quad \varphi_1, \varphi_2 \in W_{\mathbb{R}}^{1,r}(\mathbb{D}). \quad (21)$$

Solving this $\bar{\partial}$ -equation for φ_1 using the Cauchy operator, we may rewrite (21) as

$$\varphi_1 = \mathcal{C}(\beta e^{-2i\varphi_2}) - i\varphi_2 + A, \quad \varphi_1, \varphi_2 \in W_{\mathbb{R}}^{1,r}(\mathbb{D}), \quad (22)$$

where A is holomorphic in \mathbb{D} . Since $\beta e^{-2i\varphi_2} \in L^r(\mathbb{D})$ we get that $\mathcal{C}(\beta e^{-2i\varphi_2}) \in W^{1,r}(\mathbb{D})$, hence $\varphi_1, \varphi_2 \in W^{1,r}(\mathbb{D})$ if and only if A does. Therefore, given $\varphi_2 \in W_{\mathbb{R}}^{1,r}(\mathbb{D})$, equation (22) gives rise to a real valued φ_1 in $W^{1,r}(\mathbb{D})$ if and only if the holomorphic function A lies in $W^{1,r}(\mathbb{D})$ and satisfies

$$-\operatorname{Im}\mathcal{C}(\beta e^{-2i\varphi_2}) + \varphi_2 = \operatorname{Im}A. \quad (23)$$

By the discussion after (11) such an A exists if and only if the left hand side of (23) is harmonic, and since Δ commutes with taking imaginary part this condition amounts to

$$\Delta\varphi_2 - 4\operatorname{Im}(\partial\bar{\partial}\mathcal{C}(\beta e^{-2i\varphi_2})) = \Delta\varphi_2 - 4\operatorname{Im}(\partial(\beta e^{-2i\varphi_2})) = 0 \quad (24)$$

which is (19). Clearly then, by (23), $\operatorname{Im}A$ is the harmonic function $h \in W_{\mathbb{R}}^{1,r}(\mathbb{D})$ having trace $\operatorname{tr}_{\mathbb{T}} \varphi_2 - \operatorname{tr}_{\mathbb{T}} \operatorname{Im}\mathcal{C}(\beta e^{-2i\varphi_2}) \in W^{1-1/r,r}(\mathbb{T})$. Subsequently $\operatorname{Re}A = \operatorname{Im}(iA)$ must be a harmonic conjugate to $-h = u$, and taking real parts in (22) yields (20). ■

4.2 Proof of Theorem 1

4.2.1 Existence part.

In this subsection, we prove existence of s as in Theorem 1. Note that (18) will automatically hold by Lemma 1 point (iii) applied with $\theta_0 = -\pi/2$. Let $A \in W^{1,r}(\mathbb{D})$ be holomorphic in \mathbb{D} with $\text{tr}_{\mathbb{T}} \text{Re } A = \psi$ and $\int_{\mathbb{T}} \text{Im } A = -\lambda$. Writing $e^s F = e^{s-iA}(e^{iA}F)$, we may assume that $\psi = 0$ and $\lambda = 0$ upon replacing s by $s-iA$ and F by $e^{iA}F$. In addition, upon changing α into $\alpha \bar{F}/F$, we can suppose that $F \equiv 1$ thanks to (19) and (20).

We first deal with the case $r = 2$ and begin with fairly smooth α , say $\alpha \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$. Consider the following (non-linear) operator G_α acting on $\varphi \in W_{\mathbb{R}}^{1,2}(\mathbb{D})$:

$$G_\alpha(\varphi)(z) := -\frac{2}{\pi} \int_{\mathbb{D}} \log \left| \frac{1 - \bar{z}t}{z - t} \right| \text{Im} \left(\partial \left(\alpha(t) e^{-2i\varphi(t)} \right) \right) dm(t), \quad z \in \mathbb{D}. \quad (25)$$

Since $|e^{-2i\varphi}| = 1$ and $\alpha \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$, we get from (88) that $\partial(\alpha e^{-2i\varphi}) \in L^2(\mathbb{D})$, therefore the above integral exists for every $z \in \mathbb{C}$ by the Schwarz inequality. In fact, $G_\alpha(\varphi)$ is the Green potential of $4\text{Im} \left(\partial(\alpha e^{-2i\varphi}) \right)$ in \mathbb{D} , that is, its distributional Laplacian is $4\text{Im} \left(\partial(\alpha e^{-2i\varphi}) \right)$ and its value on \mathbb{T} is zero, compare [34, thm. 3.7.4]. In order that s exists subject to $\psi = 0$, $\lambda = 0$ and $F \equiv 1$, it suffices by Lemma 2 to verify that G_α has a fixed point in $W_{\mathbb{R}}^{1,2}(\mathbb{D})$. First, we check that G_α is compact from $W_{\mathbb{R}}^{1,2}(\mathbb{D})$ into itself, meaning that it is continuous and maps bounded sets to relatively compact ones.

Lemma 3 *If $\alpha \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$, then the operator G_α is bounded and continuous from $W_{\mathbb{R}}^{1,2}(\mathbb{D})$ into $W_{\mathbb{R}}^{2,2}(\mathbb{D})$ and it is compact from $W_{\mathbb{R}}^{1,2}(\mathbb{D})$ into itself.*

Proof. To prove boundedness and continuity of G_α from $W_{\mathbb{R}}^{1,2}(\mathbb{D})$ into $W_{\mathbb{R}}^{2,2}(\mathbb{D})$, observe from (88) and the dominated convergence theorem that the map $\varphi \mapsto \text{Im} \left(\partial(\alpha e^{-2i\varphi}) \right)$ is bounded and continuous from $W_{\mathbb{R}}^{1,2}(\mathbb{D})$ into $L_{\mathbb{R}}^2(\mathbb{D})$. Therefore it is enough to prove boundedness from $L_{\mathbb{R}}^2(\mathbb{D})$ into $W_{\mathbb{R}}^{2,2}(\mathbb{D})$ of the linear potential operator:

$$P(\psi) := -\frac{1}{2\pi} \int_{\mathbb{D}} \log \left| \frac{1 - \bar{z}t}{z - t} \right| \psi(t) dm(t).$$

The latter is a consequence of properties of the Cauchy and Beurling transforms recalled in section 2 [3, sec. 4.8.3]. Compactness of G_α from $L^2(\mathbb{D})$ into $W^{1,2}(\mathbb{D})$ now follows from compactness of the embedding of $W^{2,2}(\mathbb{D})$ into $W^{1,2}(\mathbb{D})$ asserted by the Rellich-Kondratchov theorem. ■

Knowing that G_α is compact on $W_{\mathbb{R}}^{1,2}(\mathbb{D})$, a sufficient condition that it has a fixed point is given by the Leray-Schauder theorem [23, thm. 11.3]: to find a constant M for which the *a priori* estimate $\|\varphi\|_{W^{1,2}(\mathbb{D})} \leq M$ holds whenever $\varphi \in W_{\mathbb{R}}^{1,2}(\mathbb{D})$ and $\varepsilon \in [0, 1]$ satisfy

$$\varphi = \varepsilon G_\alpha(\varphi). \quad (26)$$

Now, if (26) is true, then (19) is satisfied with $\beta = \varepsilon\alpha$ and φ instead of φ_2 . Therefore by Lemma 2, there exist $\varphi_{1,\varepsilon} \in W_{\mathbb{R}}^{1,2}(\mathbb{D})$ and $s_\varepsilon := \varphi_{1,\varepsilon} + i\varphi$ such that

$$\bar{\partial} e^{s_\varepsilon} = \varepsilon \alpha \overline{e^{s_\varepsilon}}.$$

Applying Lemma 1 point (ii) with $\Omega = \mathbb{D}$, $F \equiv 1$, $s = s_\varepsilon$, $\psi \equiv 0$, $\theta_0 = -\pi/2$ and $\lambda = 0$, we get from (18) that for some absolute constant C

$$\|\varphi\|_{W^{1,2}(\mathbb{D})} \leq \|s_\varepsilon\|_{W^{1,2}(\mathbb{D})} \leq \varepsilon C \|\alpha\|_{L^2(\mathbb{D})} \leq C \|\alpha\|_{L^2(\mathbb{D})} =: M.$$

Thus, G_α indeed has a fixed point, which settles the case $r = 2$ and $\alpha \in W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$. Next, we relax our restriction on α and assume only that it lies in $L^2(\mathbb{D})$. Let (α_n) be a sequence of smooth functions (for instance lying in $\mathcal{D}(\mathbb{D})$) that converges to α in $L^2(\mathbb{D})$. By the first part of the proof, there is a sequence $(s_n) \subset W^{1,2}(\mathbb{D})$ with $\text{Im tr}_{\mathbb{T}} s_n = 0$ as well as $\int_{\mathbb{T}} \text{Re tr } s_n = 0$, such that $\bar{\partial} e^{s_n} = \alpha_n \bar{e}^{s_n}$ and also, because of (18),

$$\|s_n\|_{W^{1,2}(\mathbb{D})} \leq C \|\alpha_n\|_{L^2(\mathbb{D})} \leq C'. \quad (27)$$

By the Rellich-Kondrachov theorem we can find a subsequence, again denoted by (s_n) , converging pointwise and in all $L^q(\mathbb{D})$, $1 \leq q < \infty$ to some function s . By dominated convergence, the functions $\bar{\partial} s_n = \alpha_n e^{-2i\text{Im } s_n}$ converge to $\alpha e^{-2i\text{Im } s}$ in $L^2(\mathbb{D})$. Thus, applying (11) with $A = s_n - s_m$, $a = \bar{\partial} s_n - \bar{\partial} s_m$, $\theta_0 = -\pi/2$, $\psi \equiv 0$, and $\lambda = 0$, we conclude that (s_n) is a Cauchy sequence in $W^{1,2}(\mathbb{D})$ which must therefore converge to s . Hence $s \in W^{1,2}(\mathbb{D})$, $\text{Im tr}_{\mathbb{T}} s = 0$, $\int_{\mathbb{T}} \text{Re } s = 0$, and $\bar{\partial} s = \alpha e^{-2i\text{Im } s}$. By Lemma 2, this establishes the existence of s when $r = 2$. Suppose finally that $\alpha \in L^r(\mathbb{D})$ for some $r > 2$. *A fortiori* $\alpha \in L^2(\mathbb{D})$, so by what precedes there is $s \in W^{1,2}(\mathbb{D})$ such that $\text{Im tr}_{\mathbb{T}} s = 0$, $\int_{\mathbb{T}} \text{Re } s = 0$, and $\bar{\partial} e^s = \alpha \bar{e}^s$. To see that in fact $s \in W^{1,r}(\mathbb{D})$, observe from Proposition 1 that $\bar{\partial} s = \alpha e^{-2i\text{Im } s} =: a \in L^r(\mathbb{D})$. Then, equation (11) implies that s is the unique function $A \in W^{1,r}(\mathbb{D})$ meeting $\text{Im tr}_{\mathbb{T}} A = 0$, $\int_{\mathbb{T}} \text{Re } A = 0$, and $\bar{\partial} A = a$. ■

4.2.2 Uniqueness part

In this subsection we establish uniqueness of s as in Theorem 1. Clearly, it is enough to consider $r = 2$. Consider two functions $w_1 = e^{s_1} F$ and $w_2 = e^{s_2} F$ meeting (15) on \mathbb{D} with $s_j \in W^{1,2}(\mathbb{D})$, $\text{tr}_{\mathbb{T}} \text{Im } s_j = \psi$ and $\int_{\mathbb{T}} \text{Re } s_j = \lambda$ for $j = 1, 2$. We define

$$s(z) := s_1(z) - s_2(z) \in W^{1,2}(\mathbb{D})$$

and we must prove that $s \equiv 0$. First we estimate the $\bar{\partial}$ -derivative of s :

Lemma 4 *There is a constant $C > 0$ such that, for a.e. $z \in \mathbb{D}$, we have*

$$|\bar{\partial} s(z)| \leq C |\text{Im } s(z)| |\alpha(z)|. \quad (28)$$

Proof. Setting $\beta := \alpha \bar{F}/F$ and computing as we did to obtain (21), we find that $\bar{\partial} s_j = \beta e^{-2i\text{Im } s_j}$, hence $\bar{\partial} s = \beta e^{-2i\text{Im } s_1} (1 - e^{2i\text{Im } s})$ from which (28) follows at once. ■

Next, we extend the function s outside of $\bar{\mathbb{D}}$ by reflection:

$$s(z) := \overline{s(1/\bar{z})}, \quad z \in \mathbb{C} \setminus \bar{\mathbb{D}}, \quad (29)$$

Observe since s is real-valued on \mathbb{T} that this extension makes $s \in W_{loc}^{1,2}(\mathbb{C})$.

Lemma 5 *There is a constant $C > 0$ such that, for a.e. $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$,*

$$|\bar{\partial}s(z)| \leq C \frac{|\operatorname{Im}s(z)|}{|z|^2} |\alpha(1/\bar{z})|. \quad (30)$$

Proof. Putting $\zeta = 1/\bar{z} =: f(z)$ and applying the chain rule, we get since $\partial f = 0$ that

$$\partial(s(1/\bar{z})) = ((\partial_{\bar{\zeta}} s) \circ f) \partial \bar{f} = -\frac{1}{z^2} \bar{\partial}s(1/\bar{z}).$$

Thus,

$$\bar{\partial}s(z) = \overline{\partial(s(1/\bar{z}))} = -\overline{\left(\frac{\bar{\partial}s(1/\bar{z})}{z^2}\right)}, \quad |z| > 1,$$

and applying Lemma 4 yields (30) in view of (29). ■

From the two previous lemmas we derive the inequality:

$$|\bar{\partial}s(z)| \leq C \frac{|\operatorname{Im}s(z)|}{1 + |z|^2} |\alpha(R(z))|, \quad a.e. \ z \in \mathbb{C}, \quad (31)$$

where $R(z)$ is equal to z if $|z| \leq 1$ and to $1/\bar{z}$ otherwise. Since $\alpha \in L^2(\mathbb{D})$, it follows from (31) and the change of variable formula that $\bar{\partial}s/s \in L^2(\mathbb{C})$.

Recall now definition (12). We introduce two auxiliary functions ψ, ϕ on \mathbb{C} :

$$\psi(z) := \mathcal{C}_2(\bar{\partial}s/s)(z), \quad \phi(z) := \exp(-\psi(z)). \quad (32)$$

Since $\bar{\partial}s/s \in L^2(\mathbb{C})$, we know that $\psi \in W_{loc}^{1,2}(\mathbb{C})$ with $\bar{\partial}\psi = \bar{\partial}s/s$. Consider the function $s\phi$ on \mathbb{C} . By Proposition 1, we compute from (32) using the Leibnitz rule that $\bar{\partial}(s\phi) = 0$, hence $s\phi$ is entire. We claim that

$$\liminf_{R \rightarrow +\infty} \left(\frac{1}{R} \int_{\mathbb{T}_R} \log^+ |s\phi(\xi)| |d\xi| - \frac{1}{2} \log R \right) < 0. \quad (33)$$

Indeed, taking into account (29) and the fact that $s|_{\mathbb{D}} \in L^\ell(\mathbb{D})$ for all $1 \leq \ell < \infty$ by the Sobolev embedding theorem, we get from Jensen's inequality upon choosing $\ell > 48\pi$ that

$$\begin{aligned} & \frac{1}{R^2} \int_{R < \rho < 2R} \int_{0 < \theta < 2\pi} \log^+ |s(\rho e^{i\theta})| \rho d\rho d\theta \\ & \leq \frac{12\pi}{\ell} \frac{4R^2}{3\pi} \int_{1/(2R) < \rho < 1/R} \int_{0 < \theta < 2\pi} \log^+ (|s(\rho e^{i\theta})|^\ell) \rho d\rho d\theta \\ & \leq \frac{12\pi}{\ell} \log \left[\frac{4R^2}{3\pi} \int_{1/(2R) < \rho < 1/R} \int_{0 < \theta < 2\pi} \max\{1, |s(\rho e^{i\theta})|^\ell\} \rho d\rho d\theta \right] \\ & \leq \frac{1}{\delta} \log R + C \end{aligned} \quad (34)$$

for some $\delta > 2$ and some $C > 0$, whenever $R \geq 1$.

In another connection, it follows from (13) and the Schwarz inequality that

$$\frac{1}{\pi R^2} \int_{R < \rho < 2R} \int_{0 < \theta < 2\pi} |\psi(\rho e^{i\theta})| \rho d\rho d\theta \leq \frac{\|\psi\|_{L^2(\mathbb{D}_{2R})}}{\sqrt{\pi}R} = O((\log R)^{1/2}), \quad R \rightarrow +\infty. \quad (35)$$

Since $\log^+ |s\phi| \leq \log^+ |s| + |\psi|$, claim (33) easily follows from (34) and (35).

As $\log |s\phi|$ is subharmonic on \mathbb{C} , it holds as soon as $|z| < R$ that

$$2\pi \log |s\phi(z)| \leq \frac{R+|z|}{R-|z|} \int_0^{2\pi} \log |s\phi(Re^{i\theta})| d\theta \leq \frac{R+|z|}{R-|z|} \frac{1}{R} \int_{\mathbb{T}_R} \log^+ |s\phi(\xi)| |d\xi|$$

[34, thm. 2.4.1], so by (33) there is a sequence $\rho_n \rightarrow +\infty$ for which $\sup_{\mathbb{T}_{\rho_n}} |s\phi| = O(\rho_n^{1/2})$.

Therefore, by an easy modification of Liouville's theorem, $s\phi$ must be a constant.

More generally, (31) remains valid if we replace s by $s - a$ for $a \in \mathbb{R}$, entailing that

$$\left| \frac{\bar{\partial}s(z)}{s(z) - a} \right| \leq C \left| \frac{\alpha(R(z))}{1 + |z|^2} \right| \in L^2(\mathbb{C}), \quad a.e. \ z \in \mathbb{C}, \quad a \in \mathbb{R}. \quad (36)$$

Thus, reasoning as before, we deduce there is a constant $b(a) \in \mathbb{C}$ such that

$$(s(z) - a)\phi_a(z) \equiv b(a), \quad a \in \mathbb{R},$$

with

$$\psi_a := \mathcal{C}_2(\bar{\partial}s/(s-a)), \quad \phi_a := \exp(-\psi_a).$$

Given $A, R > 0$, we get from (36), Corollary 3 and Corollary 2 that the sets $\{(\phi_a)|_{\mathbb{D}_R}\}_{|a| \leq A}$, $\{(\phi_a^{-1})|_{\mathbb{D}_R}\}_{|a| \leq A}$ are bounded in $W^{1,q}(\mathbb{D}_R)$ for $q \in [1, 2)$, hence also in $L^2(\mathbb{D}_R)$ by the Sobolev embedding theorem. Fix $A > 0$ such that $m(\{\xi \in \mathbb{D}_R : |s(\xi)| \leq A\}) = \lambda > 0$. For each $\delta > 0$ we can cover $[-A, A]$ by $N \leq 2A/\delta + 1$ open intervals, hence there exists $a = a(\delta) \in [-A, A]$ with $m(E_a) \geq \lambda\delta/(2A + \delta)$, where $E_a = \{\xi \in \mathbb{D}_R : |s(\xi) - a| \leq \delta\}$. Moreover, we observe from the Schwarz inequality that

$$|b(a)|m(E_a) = \int_{E_a} |s(\xi) - a||\phi_a(\xi)| dm(\xi) \leq \delta \|\phi_a\|_{L^2(\mathbb{D}_R)} m(E_a)^{1/2},$$

so that our lower bound on $\Lambda(E_a)$ now yields

$$|b(a)| \leq \delta^{1/2} \sqrt{(2A + \delta)/\lambda} \sup_{|a| \leq A} \|\phi_a\|_{L^2(\mathbb{D}_R)},$$

implying that $b(a(\delta)) \rightarrow 0$ as $\delta \rightarrow 0$. By compactness, we can pick a sequence $\delta_n \rightarrow 0$ such that $a_n := a(\delta_n) \rightarrow c \in [-A, A]$. Considering the equality $s - a_n = b(a_n)\phi_{a_n}^{-1}$ and the boundedness of $\{\phi_{a_n}^{-1}\}$ in $L^2(\mathbb{D})_R$, we find that $s \equiv c$ on \mathbb{D}_R . Since R was arbitrary s is constant on \mathbb{C} , and in fact $s \equiv 0$ because $\int_{\mathbb{T}} s = 0$. ■

Remark 1 *Incidentally, we showed in the course of the proof that if $s \in W_{loc}^{1,2}(\mathbb{C})$ satisfies*

$$|\bar{\partial}s(z)| \leq C |\operatorname{Im} s(z)| g(z)$$

for some non-negative function $g \in L^2(\mathbb{C})$, and if moreover

$$\int_{\mathbb{C} \setminus \mathbb{D}} \frac{|s(\xi)|^\ell d\xi \wedge d\bar{\xi}}{|\xi|^4} < \infty, \quad \text{for some } \ell > 48\pi,$$

then s is constant. It is interesting to compare this result to known Liouville type results like [2, Proposition 3.3] and [3, Theorem 8.5.1].

4.3 Proof of Corollary 1

Uniqueness of s is established as in Theorem 1, except that the right hand side of (29) now has a minus sign because s is pure imaginary on \mathbb{T} . Note also that (18) will hold by Lemma 1 point (ii) applied with $\theta_0 = 0$.

As regards existence of s , the arguments given early in subsection 4.2.1 apply with obvious modifications to show that we may assume $\psi = 0$, $\lambda = 0$ and $F \equiv 1$. Moreover, it is enough to prove the result when $r = 2$ and $\alpha \in \mathcal{D}(\mathbb{D})$, for then the passage to $\alpha \in L^2(\mathbb{D})$ and subsequently to $r > 2$ is as in the theorem.

So, let us put $r = 2$, fix $\alpha \in \mathcal{D}(\mathbb{D})$, and write $s(\psi, \lambda, F)$ to emphasize the dependance on ψ , λ and F of the function $s \in W^{1,2}(\mathbb{D})$ whose existence and uniqueness is asserted in Theorem 1. For $u \in W^{1/2,2}(\mathbb{T})$, we denote by $E(u) \in W_{\mathbb{R}}^{1,2}(\mathbb{D})$ the harmonic extension of u , i.e. $E(u)$ is harmonic and $\text{tr}_{\mathbb{T}} E(u) = u$. We put $H(u) \in W^{1,2}(\mathbb{D})$ for the holomorphic function such that $\text{Im } H(u) = E(u)$ and $\int_{\mathbb{T}} \text{Re } H(u) = 0$. Observe that $\alpha \exp(-2iE(u))$ lies in $W^{1,2}(\mathbb{D}) \cap L^\infty(\mathbb{D})$, therefore the operator $G_{\alpha e^{-2iE(u)}}$ defined through (25) is compact from $W_{\mathbb{R}}^{1,2}(\mathbb{D})$ into itself by Lemma 3. In the course of the proof of Theorem 1, we showed that it has a unique fixed point which is none but $\text{Im } s(0, 0, e^{H(u)}) =: \mathcal{F}(u)$. Furthermore, by (18), it holds that $\|\mathcal{F}(u)\|_{W^{1,2}(\mathbb{D})} \leq C\|\alpha\|_{L^2(\mathbb{D})}$ for some absolute constant C .

Lemma 6 *The (nonlinear) operator $u \mapsto \mathcal{F}(u)$ is compact from $W_R^{1/2,2}(\mathbb{T})$ into $W_R^{1,2}(\mathbb{D})$.*

Proof. Pick a sequence u_n converging to u in $W^{1/2,2}(\mathbb{T})$. By elliptic regularity, $H(u_n)$ converges to $H(u)$ in $W^{1,2}(\mathbb{D})$, in particular $\|H(u_n) + \mathcal{F}(u_n)\|_{W^{1,2}(\mathbb{D})}$ is bounded independently of n . Besides, from (25) and the definition of \mathcal{F} , we see that

$$\mathcal{F}(u_n) = G_{\alpha e^{-2iE(u_n)}}(\mathcal{F}(u_n)) = G_\alpha(E(u_n) + \mathcal{F}(u_n)), \quad (37)$$

hence Lemma 3 implies that $\{\mathcal{F}(u_n)\}_{n \in \mathbb{N}}$ is relatively compact in $W_{\mathbb{R}}^{1,2}(\mathbb{D})$. Let some subsequence, again denoted by $\mathcal{F}(u_n)$, converges to φ in $W_{\mathbb{R}}^{1,2}(\mathbb{D})$. Then $E(u_n) + \mathcal{F}(u_n)$ converges to $E(u) + \varphi$ in $W_{\mathbb{R}}^{1,2}(\mathbb{D})$, therefore by (37) and the continuity of G_α we get that $\varphi = G_{\alpha e^{-2iE(u)}}(\varphi)$. This means that $\varphi = \mathcal{F}(u)$, hence the latter is the only limit point of $\{\mathcal{F}(u_n)\}_{n \in \mathbb{N}}$, which proves continuity of \mathcal{F} .

If we only assume that $\|u_n\|_{W^{1/2,2}(\mathbb{T})}$ is bounded independently of n , elliptic regularity is still to the effect that $\|E(u_n)\|_{W^{1,2}(\mathbb{D})}$ is bounded, hence $\{E(u_n) + \mathcal{F}(u_n)\}_{n \in \mathbb{N}}$ is again bounded in $W_{\mathbb{R}}^{1,2}(\mathbb{D})$. As before it implies that $\{\mathcal{F}(u_n)\}_{n \in \mathbb{N}}$ is relatively compact in $W^{1,2}(\mathbb{D})$, as desired. ■

For $u \in W_{\mathbb{R}}^{1/2,2}(\mathbb{T})$, let $\tilde{u} := -\text{tr}_{\mathbb{T}} \text{Re } H(u)$ indicate the so-called conjugate function of u . That is, \tilde{u} is the trace of the harmonic conjugate of $E(u)$ that has zero mean on \mathbb{T} . Put $\mathcal{M} \subset W_{\mathbb{R}}^{1/2,2}(\mathbb{T})$ for the subspace of functions with zero mean. By (11) the map $u \mapsto \tilde{u}$ is continuous from $W_{\mathbb{R}}^{1/2,2}(\mathbb{T})$ into \mathcal{M} , and since $\tilde{\tilde{u}} = -u + \int_{\mathbb{T}} u$ it is a homeomorphism of \mathcal{M} . Pick $u \in \mathcal{M}$ and let $\varphi := \text{Im } s(u, 0, 1)$. Since $s(u, 0, 1) - H(u) = s(0, 0, e^{H(u)})$ we have that $\varphi = E(u) + \mathcal{F}(u)$. Set for simplicity $C(u) := \mathcal{C}(\alpha \exp\{-2i(E(u) + \mathcal{F}(u))\})$. Applying the trace and conjugate operators to (20), we see that $\text{tr}_{\mathbb{T}} \text{Re } s(u, 0, 1) = 0$ if and only if

$$u = \text{tr}_{\mathbb{T}} \text{Im} \left(C(u) - \int_{\mathbb{T}} C(u) \right) - \overbrace{\text{tr}_{\mathbb{T}} \text{Re} \left(C(u) - \int_{\mathbb{T}} C(u) \right)}. \quad (38)$$

Let $B(u)$ denote the right hand side of (38). To achieve the proof, we must show that the (non linear) operator B has a fixed point in \mathcal{M} . To prove this, we claim first that B is compact from \mathcal{M} into itself. Indeed, by elliptic regularity, E is linear and bounded from $W_{\mathbb{R}}^{1/2,2}(\mathbb{T})$ into $W_{\mathbb{R}}^{1,2}(\mathbb{D})$ while \mathcal{F} is compact by Lemma 6. A fortiori, $E + \mathcal{F}$ is thus bounded and continuous from \mathcal{M} into $W_{\mathbb{R}}^{1,2}(\mathbb{D})$. Moreover, as $\alpha \in \mathcal{D}(\mathbb{D})$, it follows from (88) and the dominated convergence theorem that $h \mapsto \alpha \exp(-2ih)$ is bounded and continuous from $W_{\mathbb{R}}^{1,2}(\mathbb{D})$ into $W_0^{1,2}(\mathbb{D})$. In addition, we get from (10) that \mathcal{C} is bounded linear from $W_0^{1,2}(\mathbb{D})$ into $W^{2,2}(\mathbb{D})$, hence compact into $W^{1,2}(\mathbb{D})$ by the Rellich-Kondratchov theorem. Finally, by the trace theorem, $g \mapsto \text{tr}_{\mathbb{T}} \text{Im}(g - \int_{\mathbb{T}} g)$ is linear and bounded from $W^{1,2}(\mathbb{D})$ into \mathcal{M} . Since the conjugate operator is linear and bounded on \mathcal{M} and composition with bounded continuous maps preserves compactness, the claim follows. Appealing now to the Leray-Schauder theorem, we know that B has a fixed point if we can find a constant M such that $\|u\|_{W^{1/2,2}(\mathbb{T})} \leq M$ holds whenever $u = \varepsilon B(u)$ for some $\varepsilon \in [0, 1]$. However, such a u must be $\text{Im tr}_T s_\varepsilon$, where $s_\varepsilon \in W^{1,2}(\mathbb{D})$ has pure imaginary trace with zero mean on \mathbb{T} and e^{s_ε} satisfies (15) with α replaced by $\varepsilon\alpha$. Thus, from Lemma 1 point (ii) applied with $\theta_0 = 0$, we conclude that $M = C\|\alpha\|_{L^2(\mathbb{D})}$ will do for some absolute constant C . ■

5 Hardy spaces on the disk

5.1 Holomorphic Hardy spaces

For $p \in [1, \infty)$, let $H^p = H^p(\mathbb{D})$ be the Hardy space of holomorphic functions f on \mathbb{D} with

$$\|f\|_{H^p} := \sup_{0 < \rho < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^p d\theta \right)^{1/p} < +\infty. \quad (39)$$

The space H^∞ consists of bounded holomorphic functions endowed with the *sup* norm. We refer to [13, 22] for the following standard facts on holomorphic Hardy spaces.

Each $f \in H^p$ has a non-tangential limit at a.e. $\xi \in \mathbb{T}$, which is also the $L^p(\mathbb{T})$ limit of $f_\rho(\xi) := f(\rho\xi)$ as $\rho \rightarrow 1^-$ and whose norm matches the supremum in (39). Actually $\|f_\rho\|_{L^p(\mathbb{T})}$ is non-decreasing with ρ , hence instead of (39) we could as well have set⁸

$$\|f\|_{H^p} := \sup_{0 < \rho < 1} \left(\int_{\mathbb{T}_\rho} |f(\xi)|^p |d\xi| \right)^{1/p} < +\infty \quad (40)$$

where the integral is now with respect to arclength. We customarily keep the same notation for f and its non-tangential limit when no confusion can arise, or write sometimes $f|_{\mathbb{T}}$ to emphasize that the non-tangential limit lives on \mathbb{T} . Note that $f|_{\mathbb{T}}$ coincides with $\text{tr} f$ when $f \in W^{1,p}(\mathbb{D})$ [5]. Each function in H^p is both the Cauchy and the Poisson integral of its non-tangential limit. As regards the non-tangential maximal function, it holds when $1 \leq p < \infty$ and $f \in H^p$ that

$$\|\mathcal{M}_\gamma f\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})} \quad (41)$$

⁸In fact (39) expresses that $|f|^p$ has a harmonic majorant whereas (40) bounds the L^p -norm of f on curves tending to the boundary; the first condition defines the Hardy space and the second the so-called Smirnov space. These coincide when harmonic measure and arclength are comparable on the boundary, [13, Ch. 10], [39], which is the case for smooth domains. The name ‘‘Hardy space’’ is then more common.

where the constant C depends only on γ and p [22, ch. II, thm. 3.1].

Traces of H^p -functions on \mathbb{T} are exactly those functions in $L^p(\mathbb{T})$ whose Fourier coefficients of negative index do vanish. In particular if $f \in H^p$ and $f|_{\mathbb{T}} \in L^q(\mathbb{T})$, then $f \in H^q$. It is obvious from Fubini's theorem that $H^p \subset L^p(\mathbb{D})$, but more in fact is true [5, lem. 5.2.1]:

$$\|f\|_{L^\lambda(\mathbb{D})} \leq C\|f\|_{H^p}, \quad p \leq \lambda < 2p, \quad (42)$$

where $C = C(p, \lambda)$. A sequence $(z_l) \subset \mathbb{D}$ is the zero set of a nonzero H^p function, counting multiplicities with repetitions, if and only if it meets the *Blaschke condition*:

$$\sum_l (1 - |z_l|) < \infty. \quad (43)$$

A non-negative function $h \in L^p(\mathbb{T})$ is such that $h = |f|$ for some nonzero $f \in H^p$ if and only if $\log h \in L^1(\mathbb{T})$. This entails that a nonzero H^p function cannot vanish on a subset of strictly positive Lebesgue measure on \mathbb{T} .

For $1 < p < \infty$, then to each $\psi \in L^p_{\mathbb{R}}(\mathbb{T})$ there is $g \in H^p$ such that $\operatorname{Re} g = \psi$ on \mathbb{T} [22, ch. III]. Such a g is unique up to an additive pure imaginary constant, and if we normalize it so that $\int_{\mathbb{T}} \operatorname{Im} g = 0$ then $\|g\|_{H^p} \leq C\|\psi\|_{L^p(\mathbb{T})}$ with $C = C(p)$. In fact $g = u + iv$ on \mathbb{D} , where u is the Poisson integral of ψ and v is the Poisson integral of

$$\tilde{\psi}(e^{i\theta}) := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{\varepsilon < |\theta - t| < \pi} \frac{\psi(e^{it})}{\tan(\frac{\theta - t}{2})} dt \quad (44)$$

which is the so-called *conjugate function* of ψ . This definition carries over to $L^p(\mathbb{T})$ the conjugation operator $\psi \mapsto \tilde{\psi}$ already introduced on $W^{1/2,2}(\mathbb{T})$ after the proof of Lemma 6 in section 4.3. It is a theorem of M. Riesz that the conjugation operator maps $L^p(\mathbb{T})$ continuously into itself. By elliptic regularity, it is also continuous from $W^{1-1/p,p}(\mathbb{T})$ into itself.

When $\psi \in L^1(\mathbb{T})$, the conjugate function $\tilde{\psi}$ is still defined pointwise almost everywhere *via* (44) but it needs not be in $L^1(\mathbb{T})$.

For $p \in (1, \infty)$, a non-negative function $\mathfrak{w} \in L^1(\mathbb{T})$ is said to satisfy condition A_p if

$$\{\mathfrak{w}\}_{A_p} := \sup_I \left(\frac{1}{\Lambda(I)} \int_I \mathfrak{w} d\Lambda \right) \left(\frac{1}{\Lambda(I)} \int_I \mathfrak{w}^{-1/(p-1)} d\Lambda \right)^{p-1} < +\infty \quad (45)$$

where the supremum is taken over all arcs $I \subset \mathbb{T}$. A theorem of Hunt, Muckenhoupt and Wheeden [22, ch. VI, thm. 6.2] asserts that if \mathfrak{w} meets condition A_p , then

$$\int_{\mathbb{T}} |\tilde{\phi}|^p \mathfrak{w} d\Lambda \leq C \int_{\mathbb{T}} |\phi|^p \mathfrak{w} d\Lambda, \quad \phi \in L^1(\mathbb{T}), \quad (46)$$

where C depends only on $\{\mathfrak{w}\}_{A_p}$; moreover, an inequality of the form (46) is possible only if \mathfrak{w} satisfies A_p . In (46), the assumption $\phi \in L^1(\mathbb{T})$ is just a means to ensure that $\tilde{\phi}$ is well defined.

5.2 Pseudo-holomorphic Hardy spaces

Given $\alpha \in L^r(\mathbb{D})$ for some $r \in [2, \infty)$ and $p \in (1, \infty)$, we define the Hardy space $G_\alpha^p(\mathbb{D})$ of those $w \in L_{loc}^\gamma(\mathbb{D})$ with $\gamma > r/(r-1)$ that satisfy (15) and are such that

$$\|w\|_{G_\alpha^p(\mathbb{D})} := \sup_{0 < \rho < 1} \left(\int_{\mathbb{T}_\rho} |w(\xi)|^p |d\xi| \right)^{1/p} < +\infty. \quad (47)$$

If we denote with \mathcal{H}^p the Banach space of complex measurable functions f on \mathbb{D} such that $\text{ess. sup}_{0 < \rho < 1} \rho \|f_\rho\|_{L^p(\mathbb{T})} < +\infty$, then $G_\alpha^p(\mathbb{D})$ identifies to a real subspace of \mathcal{H}^p . The fact that this subspace is closed, hence a Banach space in its own right, is part of Theorem 2 to come. Note if $w \in L_{loc}^\gamma(\mathbb{D})$ satisfies (15) that $w \in W_{loc}^{1,q}(\mathbb{D})$ for $q \in [1, 2)$ by Lemma 1; hence the integral in (47) is indeed finite for *each* ρ by the trace theorem. Clearly $G_0^p(\mathbb{D}) = H^p$, but $G_\alpha^p(\mathbb{D})$ is not a complex vector space when $\alpha \neq 0$. Spaces $G_\alpha^1(\mathbb{D})$ and $G_\alpha^\infty(\mathbb{D})$ could be defined similarly, but we shall not consider them.

For $r > 2$, such classes of functions were apparently introduced in [32] and subsequently considered in [28, 29, 30, 5, 15, 16, 4]. Unlike these references, our definition is modelled after (40) rather than (39), that is, integral means in (47) are with respect to arclength⁹ and *not* normalized arclength. This is unimportant when $r > 2$, but becomes essential¹⁰ if $r = 2$.

Below, we do consider the case $r = 2$ and stress topological connections with holomorphic Hardy spaces which are new even when $r > 2$, see Theorem 1 point (iii).

By Lemma 1, each solution to (15) in $L_{loc}^\gamma(\mathbb{D})$, $\gamma > r/(r-1)$, factors as $w = e^s F$ where

$$\|s\|_{W^{1,r}(\mathbb{D})} \leq C(r) \|\alpha\|_{L^r(\mathbb{D})} \quad (48)$$

and F is holomorphic in \mathbb{D} . Moreover, if $w \neq 0$, one can uniquely impose $\text{Im tr}_{\mathbb{T}} s = 0$ and $\int_{\mathbb{T}} \text{Res} = 0$ or $\text{Re tr}_{\mathbb{T}} s = 0$ and $\int_{\mathbb{T}} \text{Im} s = 0$. To distinguish these two factorizations, we write $w = e^{s^r} F^r$ in the first case, and $w = e^{s^i} F^i$ in the second; that is, s^r is real on \mathbb{T} and s^i is pure imaginary there. If $w \equiv 0$, we put $F^r = F^i = 0$ and do not define s^r and s^i . From the proof Lemma 1, it follows when $w \neq 0$ (hence w is a.e. nonzero) that if we let

$$\mathcal{R}(\beta)(z) := -\overline{\mathcal{C}(\beta)(1/\bar{z})} = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{z \bar{\beta}(\xi)}{1 - \xi z} d\xi \wedge \overline{d\xi}, \quad \beta \in L^r(\mathbb{D}), \quad z \in \mathbb{D}, \quad (49)$$

then s^r is given by

$$s^r(z) = \mathcal{C}(\alpha \bar{w}/w)(z) - \mathcal{R}(\alpha \bar{w}/w)(z) - \frac{1}{2\pi} \int_{\mathbb{T}} \text{Re } \mathcal{C}(\alpha \bar{w}/w)(\xi) |d\xi| \quad (50)$$

while s^i is given by

$$s^i(z) = \mathcal{C}(\alpha \bar{w}/w) + \mathcal{R}(\alpha \bar{w}/w) - \frac{1}{2\pi} \int_{\mathbb{T}} \text{Im } \mathcal{C}(\alpha \bar{w}/w)(\xi) |d\xi|. \quad (51)$$

Indeed, it is easy to check that $\mathcal{R}(\alpha \bar{w}/w)$ is a holomorphic function in $W^{1,r}(\mathbb{D})$ having zero mean on \mathbb{T} and assuming conjugate values to $-\mathcal{C}(\alpha \bar{w}/w)$ there.

⁹Thus, it would be more appropriate to call $G_\alpha^p(\mathbb{D})$ a pseudo-holomorphic Smirnov space.

¹⁰When $r = 2$, w may fail to satisfy condition (39) even though it meets (15) and (47). The problem lies with *small* values of r , as w needs not be locally bounded on \mathbb{D} .

From (48) it follows that if $r > 2$, then

$$\|e^{\pm s^r}\|_{W^{1,r}(\mathbb{D})} \leq C(r, \|\alpha\|_{L^r(\mathbb{D})}) \quad \text{and} \quad \|e^{\pm s^i}\|_{W^{1,r}(\mathbb{D})} \leq C(r, \|\alpha\|_{L^r(\mathbb{D})}). \quad (52)$$

When $r = 2$, we only deduce from (48), (50), (51), Proposition 2 and Corollary 2 that

$$\|e^{\pm s^r}\|_{W^{1,q}(\mathbb{D})} \leq C(q, \omega) \quad \text{and} \quad \|e^{\pm s^i}\|_{W^{1,q}(\mathbb{D})} \leq C(q, \omega), \quad 1 < q < 2, \quad (53)$$

where ω is a strictly positive function on $(0, +\infty)$ such that $\|\alpha\|_{L^2(Q_{\omega(\varepsilon)} \cap \mathbb{D})} < \varepsilon$ as soon as $Q_{\omega(\varepsilon)}$ is a cube of sidelength $\omega(\varepsilon)$.

- When $r > 2$, we conclude from (52) and the Sobolev embedding theorem that $e^{\pm s^r}$ and $e^{\pm s^i}$ are continuous and bounded independently of w on $\overline{\mathbb{D}}$. Hence w belongs to $G_\alpha^p(\mathbb{D})$ if and only if F^r or F^i lies in H^p (in which case both do). This way $G_\alpha^p(\mathbb{D})$ inherits many properties of H^p . In particular, each $w \in G_\alpha^p(\mathbb{D})$ has a nontangential limit a.e. on \mathbb{T} , denoted again by w or $w_{\mathbb{T}}$ for emphasis, which is also the limit of w_ρ as $\rho \rightarrow 1^-$ in $L^p(\mathbb{T})$. Moreover, $\|w_{\mathbb{T}}\|_{L^p(\mathbb{T})}$ is a norm equivalent to (40) on $G_\alpha^p(\mathbb{D})$, and we may as well have used (39) to define the latter. Also, from Theorem 1, we infer that condition (43) characterizes the zeros of non identically vanishing functions in $G_\alpha^p(\mathbb{D})$ ¹¹.
- When $r = 2$, all we conclude *a priori* from (53), Lemma 11, and Hölder's inequality is that F^r and F^i are in $\cup_{1 < \ell < p} H^\ell$ if $w \in G_\alpha^p(\mathbb{D})$. Conversely $w \in \cup_{1 < \ell < p} G_\alpha^p(\mathbb{D})$ if F^r or F^i lies in H^p . To clarify the matter, one should realize that factorizations $w = e^{s^r} F^r$ and $w = e^{s^i} F^i$ no longer play equivalent roles. For it may happen that $w \in G_\alpha^p(\mathbb{D})$ and $F^r \notin H^p$. In fact, if we let

$$w(z) := \frac{1}{\log(3/|z-1|)(z-1)^{1/p}}, \quad z \in \mathbb{D}, \quad (54)$$

we get that

$$\left| \frac{\bar{\partial} w(z)}{w(z)} \right| = \left(2|z-1| \log(3/|z-1|) \right)^{-1}$$

hence $w \in G_\alpha^p(\mathbb{D})$ with $\alpha := \bar{\partial} w / \bar{w} \in L^2(\mathbb{D})$, but the factorization

$$w(z) = e^{\log \log(3/|z-1|) - a} e^a (z-1)^{-1/p}, \quad a := \int_{\mathbb{T}} e^{\log \log(3/|z-1|)},$$

is such that $F^r = e^a (z-1)^{-1/p} \notin H^p$.

In contrast, it holds that $w \in G_\alpha^p(\mathbb{D})$ if and only if $F^i \in H^p$. Assume indeed that $0 \not\equiv w \in G_\alpha^p$. Since $F^i \in H^\ell$ for $1 \leq \ell < p$ and $e^{s_\rho^i}$ converges to e^{s^i} in $W^{1,q}(\mathbb{D})$ for all $q \in [1, 2)$ by Proposition 1, it follows from Lemma 11 and Hölder's inequality that $\text{tr}_{\mathbb{T}} w_\rho$ converges as $\rho \rightarrow 1^-$ to $e^{\text{tr}_{\mathbb{T}} s^i} F_{|\mathbb{T}}^i$ in $L^\lambda(\mathbb{T})$, for every $\lambda \in [1, p)$. Moreover, as $\text{tr}_{\mathbb{T}} w_\rho$ remains bounded in $L^p(\mathbb{T})$ by (47), it converges weakly there to $e^{\text{tr}_{\mathbb{T}} s^i} F_{|\mathbb{T}}^i$ when $\rho \rightarrow 1^-$, since this is the only weak limit possible granted the convergence of $\text{tr}_{\mathbb{T}} w_\rho$ in $L^\lambda(\mathbb{T})$. In particular $e^{\text{tr}_{\mathbb{T}} s^i} F_{|\mathbb{T}}^i \in L^p(\mathbb{T})$, and since $|e^{\text{tr}_{\mathbb{T}} s^i}| \equiv 1$ we conclude that $F_{|\mathbb{T}}^i \in L^p(\mathbb{T})$ hence $F^i \in H^p$. Conversely if $F^i \in H^p$, then w satisfies (47) by Corollary 4 Theorem 6 in section 8.

¹¹When $r = 2$, this property has no simple analog for w is only defined $B_{1,2}$ quasi-everywhere.

That $\text{tr}_{\mathbb{T}} w_{\rho}$ converges strongly in $L^p(\mathbb{T})$ as $\rho \rightarrow 1^-$, and not just weakly as we showed, is part of the next Theorem of which point (iii) only is new when $r > 2$ [32, 28, 5, 4].

Theorem 2 *Let $\alpha \in L^r(\mathbb{D})$ with $2 \leq r < \infty$ and fix $p \in (1, \infty)$.*

(i) *Each $w \in G_{\alpha}^p(\mathbb{D})$ has a trace $w_{\mathbb{T}}$ on \mathbb{T} given by*

$$w_{\mathbb{T}} := \lim_{\rho \rightarrow 1^-} \text{tr}_{\mathbb{T}} w_{\rho} \quad \text{in } L^p(\mathbb{T}). \quad (55)$$

When $r > 2$, the function $w_{\mathbb{T}}$ is also the non-tangential limit of w a.e. on \mathbb{T} .

(ii) *It holds for some strictly positive constant C depending only on $|\alpha|$ and p that*

$$\|w_{\mathbb{T}}\|_{L^p(\mathbb{T})} \leq \|w\|_{G_{\alpha}^p(\mathbb{D})} \leq C \|w_{\mathbb{T}}\|_{L^p(\mathbb{T})}, \quad (56)$$

and $G_{\alpha}^p(\mathbb{D})$ is a real Banach space on which $\|w_{\mathbb{T}}\|_{L^p(\mathbb{T})}$ is a norm equivalent to (47).

(iii) *The map $w \mapsto F^{\mathfrak{i}}$ is a homeomorphism from $G_{\alpha}^p(\mathbb{D})$ onto H^p . When $r > 2$, the map $w \mapsto F^{\mathfrak{r}}$ is also such a homeomorphism.*

(iv) *If $w \in G_{\alpha}^p(\mathbb{D})$ and $w_{\mathbb{T}} \in L^q(\mathbb{T})$ for some $q \in (1, \infty)$, then $w \in G_{\alpha}^q(\mathbb{D})$. A non-negative function $h \in L^p(\mathbb{T})$ is such that $h = |w_{\mathbb{T}}|$ for some nonzero $w \in G_{\alpha}^p(\mathbb{D})$ if and only if $\log h \in L^1(\mathbb{T})$.*

Proof. When $r > 2$, all properties except (iii) follow from their H^p -analog via the continuity and uniform boundedness of $e^{\pm s^{\mathfrak{r}}}$ or $e^{\pm s^{\mathfrak{i}}}$ discussed earlier in this section, see also [32, 28, 5, 4].

We postpone the proof of (iii) and assume for now that $r = 2$. Let $0 \neq w \in G_{\alpha}^p(\mathbb{D})$ and put $s = s^{\mathfrak{i}}$, $F = F^{\mathfrak{i}}$ to alleviate notation. For $(\rho_n) \subset (0, 1)$ a sequence tending to 1, we will show one can extract a subsequence (ρ_{n_k}) such that $\text{tr}_{\mathbb{T}} w_{\rho_{n_k}}$ converges to $e^{\text{tr}_{\mathbb{T}} s} F|_{\mathbb{T}}$ in $L^p(\mathbb{T})$, which will give us (55).

Since s_{ρ} converges to s in $W^{1,2}(\mathbb{D})$, we get from Lemma 11 that $\text{tr}_{\mathbb{T}} s_{\rho}$ converges to $\text{tr}_{\mathbb{T}} s$ in $L^{\ell}(\mathbb{T})$, as $\rho \rightarrow 1^-$, for all $\ell \in [1, \infty)$. Moreover, we pointed out before the theorem that $F \in H^p$, so $(F_{\rho})|_{\mathbb{T}}$ converges to $F|_{\mathbb{T}}$ in $L^p(\mathbb{T})$. Extracting a subsequence from ρ_n if necessary (still denoted by ρ_n), we can assume that $\text{tr}_{\mathbb{T}} s_{\rho_n}$ (resp. $(F_{\rho_n})|_{\mathbb{T}}$) also converges pointwise a.e. on \mathbb{T} to $\text{tr}_{\mathbb{T}} s$ (resp. $F|_{\mathbb{T}}$). Now, Corollary 4 to Theorem 6, applied with ps in place of s , implies that $\|e^{p \text{Re } s_{\rho}} F_{\rho}\|_{L^p(\mathbb{T})}$ is uniformly bounded as $\rho \rightarrow 1^-$. Therefore, as the weak limit coincides with the pointwise limit when both exist by Egoroff's theorem, there is a subsequence (ρ_{n_k}) such that $(e^{p \text{Re } s_{\rho_{n_k}}} |F_{\rho_{n_k}}|)$ converges weakly to $|F|$ in $L^p(\mathbb{T})$. Letting $1/p + 1/p' = 1$, this means that for each test function $\Theta \in L^{p'}(\mathbb{T})$ we have:

$$\lim_{k \rightarrow \infty} \left| \int_{\mathbb{T}} e^{p \text{Re } s_{\rho_{n_k}}(z)} |F_{\rho_{n_k}}(z)| \Theta(z) |dz| - \int_{\mathbb{T}} |F(z)| \Theta(z) |dz| \right| = 0. \quad (57)$$

Pick $\Theta_k = |F_{\rho_{n_k}}|^{p-1} \in L^{p'}(\mathbb{T})$. Convergence of $(F_{\rho})|_{\mathbb{T}}$ to $F|_{\mathbb{T}}$ in $L^p(\mathbb{T})$ easily implies that Θ_k converges to $|F|^{p-1}$ in $L^{p'}(\mathbb{T})$. In view of (57), this yields

$$\lim_{k \rightarrow \infty} \left| \int_{\mathbb{T}} e^{p \text{Re } s_{\rho_{n_k}}(z)} |F_{\rho_{n_k}}(z)|^p |dz| - \int_{\mathbb{T}} |F(z)|^p |dz| \right| = 0,$$

implying that $\|\mathrm{tr}_{\mathbb{T}} w_{\rho_{n_k}}\|_{L^p(\mathbb{T})} = \|e^{\mathrm{Re} s_{\rho_{n_k}}} F_{\rho_{n_k}}\|_{L^p(\mathbb{T})}$ tends to $\|e^{\mathrm{tr}_{\mathbb{T}} s} F_{\mathbb{T}}\|_{L^p(\mathbb{T})} = \|F_{\mathbb{T}}\|_{L^p(\mathbb{T})}$ when $k \rightarrow \infty$. However, from the discussion before the theorem, we know that $\mathrm{tr}_{\mathbb{T}} w_{\rho_{n_k}}$ converges weakly to $e^{\mathrm{tr}_{\mathbb{T}} s} F_{\mathbb{T}}$ in $L^p(\mathbb{T})$, so by uniform convexity of $L^p(\mathbb{T})$ the convergence must in fact be strong because we just showed the norm of the weak limit is the limit of the norms [9, thm. III 27]. This proves (i).

Next, we observe by absolute continuity of $|\alpha|^2 dm$ that to each $\varepsilon > 0$ there is $\omega(\varepsilon) > 0$ for which $\|\alpha\|_{L^2(Q_{\omega(\varepsilon)} \cap \mathbb{D})} < \varepsilon$ as soon as $Q_{\omega(\varepsilon)}$ is a cube of sidelength $\omega(\varepsilon)$. Thus, in view of (51), we can if $w \not\equiv 0$ apply Proposition 2 to $\beta := \alpha \bar{w}/w$ and obtain a strictly positive function $\tilde{\omega}$ on \mathbb{R} , *depending only on $|\alpha|$* , such that

$$\|\partial s\|_{L^2(Q_{\tilde{\omega}(\eta)} \cap \mathbb{D})} + \|\bar{\partial} s\|_{L^2(Q_{\tilde{\omega}(\eta)} \cap \mathbb{D})} < \eta \quad (58)$$

as soon as $Q_{\tilde{\omega}(\eta)}$ is a cube of sidelength $\tilde{\omega}(\eta)$. A fortiori, (58) holds with $\mathrm{Re} s$ instead of s . Now, picking any $\gamma \in (0, \pi/2)$ and recalling that $|w_{\mathbb{T}}| = |F_{\mathbb{T}}|$ because $\mathrm{Res} \in W_{0,\mathbb{R}}^{1,2}(\mathbb{D})$, we deduce from (41) and Theorem 6 applied to $f = e^{\mathrm{Re} s}$ and $g = e^{i\mathrm{Im} s} F$ that the right inequality in (56) holds. In another connection, the left inequality is obvious from (55). To show that $G_{\alpha}^p(\mathbb{D})$ is a Banach space, consider a sequence $(w_n) \subset G_{\alpha}^p(\mathbb{D})$ converging in \mathcal{H}^p to some function w . We must prove that $w \in G_{\alpha}^p$. We can assume $w \not\equiv 0$, therefore $w_n \not\equiv 0$ for n large enough. Convergence in \mathcal{H}^p being stronger than in $L^p(\mathbb{D})$, *a fortiori* w_n converges to w in $\mathcal{D}'(\mathbb{D})$ and some subsequence, again denoted by w_n , converges pointwise a.e. to w . Besides, if we write $w_n = e^{s_n} F_n$ where we mean as before that $s_n = s_n^i$ and $F_n = F_n^i$, we get from (53) that e^{s_n} is uniformly bounded in $W^{1,q}(\mathbb{D})$ for each $q \in (1, 2)$. Therefore, by the Sobolev embedding theorem, e^{s_n} is bounded in $L^{\ell}(\mathbb{D})$, independently of n , for each $\ell \in [1, \ell)$. In addition, since $|e^{\mathrm{tr}_{\mathbb{T}} s_n}| \equiv 1$, it follows from (56) that F_n is bounded in H^p hence also in $L^{\ell}(\mathbb{D})$ for each $\ell \in (1, 2p)$ by (42), independently of n . Altogether, by Hölder's inequality, w_n remains bounded in $L^{\gamma}(\mathbb{D})$ for some $\gamma > 2$. Consequently some subsequence converges weakly in $L^{\gamma}(\mathbb{D})$, and since the weak limit coincides with the pointwise limit if it exists we conclude that the weak limit is w . In particular $w \in L^{\gamma}(\mathbb{D})$. Moreover, by Hölder's inequality, $\alpha \bar{w}_n$ is bounded in $L^t(\mathbb{D})$ for some $t > 1$, and arguing as before we get that some subsequence (again denoted by $\alpha \bar{w}_n$) converges weakly to $\alpha \bar{w}$ there. Thus, passing to the distributional limit in the relation $\bar{\partial} w_n = \alpha \bar{w}_n$, we obtain (15) so that $w \in G_{\alpha}^p(\mathbb{D})$. This proves (ii).

We already know from Theorem 1 and the discussion before the statement of Theorem 2 that the map $w \mapsto F^i$ is injective from $G_{\alpha}^p(\mathbb{D})$ onto H^p . Since $|w_{\mathbb{T}}| = |F_{\mathbb{T}}^i|$, it is clear from (56) that this map and its inverse are continuous at 0. Let now w_n converge to $w \not\equiv 0$ in $G_{\alpha}^p(\mathbb{D})$ and write $w_n = e^{s_n^i} F_n^i$, $w = e^{s^i} F^i$. We claim that some subsequence of F_n^i converges to F^i in H^p and this will establish continuity of the map at every point. As $F_{n|\mathbb{T}}^i$ is bounded in $L^p(\mathbb{T})$ by (56), some subsequence converges weakly there to $\Phi_{\mathbb{T}}$ for some $\Phi \in H^p$. Thus, replacing w_n by a subsequence (again denoted as w_n), we may assume by the Cauchy formula that F_n^i converges locally uniformly to Φ on \mathbb{D} . Note that $\Phi \not\equiv 0$ for otherwise, in view of (53), we would have that w_n converges to the zero distribution, contradicting that $w \not\equiv 0$. In particular $\alpha \bar{F}_n^i / F_n^i$ converges in $L^2(\mathbb{D})$ to $\alpha \bar{\Phi} / \Phi$ by dominated convergence. Since $\bar{\partial} s_n^i = \alpha \bar{F}_n^i / F_n^i \exp(-2i\mathrm{Im} s_n^i)$ by Lemma 2 and $\|s_n^i\|_{W^{1,2}(\mathbb{D})}$ is uniformly bounded by (48), we can argue as we did after (27) (put $\alpha_n \equiv \alpha \bar{F}_n^i / F_n^i$ and $\theta_0 = 0$ in the discussion there) to the effect that some subsequence, again denoted by s_n^i , converges to some $\sigma \in W^{1,2}(\mathbb{D})$ such that $\mathrm{Re} \mathrm{tr}_{\mathbb{T}} \sigma = 0$ and $\int_{\mathbb{T}} \sigma = 0$, both pointwise a.e. and in $W^{1,2}(\mathbb{D})$. Refining the sequence if necessary, we can further assume that w_n converges

pointwise a.e. to w . Taking pointwise limits we get $w = e^\sigma \Phi$, hence $\sigma = s^i$ and $\Phi = F^i$ by the uniqueness part of Corollary 1. Thus, $F^i_{|\mathbb{T}}$ is the weak limit of $F^i_{n|\mathbb{T}}$, and since $\|F^i\|_{L^p(\mathbb{T})} = \|w\|_{L^p(\mathbb{T})}$ is the limit of $\|F^i_{n|\mathbb{T}}\|_{L^p(\mathbb{T})} = \|w_n\|_{L^p(\mathbb{T})}$, the convergence in fact takes place in $L^p(\mathbb{T})$, thereby proving the claim. Conversely, let $w_n = e^{s_n^i} F^i_n$ be a sequence in $G^p_\alpha(\mathbb{D})$ such that F^i_n converges to $\Phi \neq 0$ in H^p . By Corollary 1, $\|s_n^i\|_{W^{1,2}(\mathbb{D})}$ is bounded independently of n , and as before some subsequence, again denoted by s_n^i , converges in $W^{1,2}(\mathbb{D})$ to σ such that $\text{Re tr}_\mathbb{T} \sigma = 0$ and $\int_\mathbb{T} \sigma = 0$. Refining the sequence if necessary, we can assume in view of the trace theorem that $\text{tr}_\mathbb{T} s_n^i$ converges pointwise a.e. on \mathbb{T} to $\text{tr}_\mathbb{T} \sigma$, so that by dominated convergence $(w_n)_\mathbb{T}$ tends to $e^{\text{tr}_\mathbb{T} \sigma} \Phi_{|\mathbb{T}}$ in $L^p(\mathbb{T})$. Using (56) it implies that w_n converges in $G^p_\alpha(\mathbb{D})$ to some $w = e^{s^i} F^i$, and by the continuity proven before we conclude that $\Phi = F^i$. This proves (iii) when $r = 2$. That both $w \mapsto F^i$ and $w \mapsto F^r$ are homeomorphism when $r > 2$ is similar but easier because then $s \rightarrow e^s$ is bounded and continuous from $W^{1,r}(\mathbb{D})$ into $W^{1,r}(\mathbb{D}) \subset L^\infty(\mathbb{D})$. Finally (iv) follows from corresponding properties of H^p functions, the fact that $w \in G^p_\alpha$ if and only if $F^i \in H^p$, and the equality $|w_\mathbb{T}| = |F^i_{|\mathbb{T}}|$. ■

Remark 2 When $r = 2$, $w_\mathbb{T}$ in Theorem 2 needs not be the non-tangential limit of w . Indeed, if $(z_n) \subset \mathbb{D}$ is nontangentially dense on \mathbb{T} , then $s(z) := \sum_n 2^{-n} \log \log 2/|z - z_n|$ lies in $W^{1,2}(\mathbb{D})$ so that $e^s \in G^p_\alpha(\mathbb{D})$ for all $p \in (1, \infty)$ with $\alpha := \partial s$ by Lemma 11, yet e^s is not even nontangentially bounded at a single $\xi \in \mathbb{T}$.

6 The generalized conjugation operator

The M. Riesz theorem may be rephrased as follows. Given $\psi \in L^p_\mathbb{R}(\mathbb{T})$ with $p \in (1, \infty)$, the problem of finding a holomorphic function f in \mathbb{D} such that $\text{Re tr}_\mathbb{T} f_\rho$ tends to ψ in $L^p(\mathbb{T})$ has a solution in H^p which is unique up to an additive imaginary constant. In fact, if we normalize it to have mean $\int_\mathbb{T} \psi / 2\pi + ic$ on \mathbb{T} , then $f_{|\mathbb{T}} = \psi + i\tilde{\psi} + ic$ and it holds that $\|f\|_{H^p} \leq C\|\psi\|_{L^p(\mathbb{T})} + C'|c|$ for some constants C, C' depending only on p .

The corresponding problem for pseudo-holomorphic functions, *i.e.* for solutions to (15) when $\alpha \neq 0$, turns out to have a similar answer in G^p_α as long as $\alpha \in L^r(\mathbb{D})$ for some $r \geq 2$. When $r > 2$ this was essentially proven in [28], see also [5] and [4]. More precisely:

Theorem 3 ([28],[5],[4]) *Let $\alpha \in L^r(\mathbb{D})$ with $2 < r \leq \infty$ and $1 < p < \infty$. To each $\psi \in L^p_\mathbb{R}(\mathbb{T})$ and $c \in \mathbb{R}$, there is a unique $w \in G^p_\alpha(\mathbb{D})$ such that $\text{Re } w_\mathbb{T} = \psi$ and $\int_\mathbb{T} \text{Im } w_\mathbb{T} = c$. Moreover $\|w\|_{G^p_\alpha(\mathbb{D})} \leq C\|\psi\|_{L^p(\mathbb{T})} + C'|c|$, where C, C' depend only on p and r .*

Theorem 3 generalizes the M. Riesz theorem: to each $\psi \in L^p_\mathbb{R}(\mathbb{T})$ and $c \in \mathbb{R}$, there is a unique $\psi_c^\sharp \in L^p_\mathbb{R}(\mathbb{T})$ (a generalized conjugate of ψ) such that $\int_\mathbb{T} \psi_c^\sharp = c$ and $\psi + i\psi_c^\sharp = w_\mathbb{T}$ for some $w \in G^p_\alpha(\mathbb{D})$. Moreover $\|\psi_c^\sharp\|_{L^p(\mathbb{T})} \leq C\|\psi\|_{L^p(\mathbb{T})} + C'|c|$. The theorem below extends the result to the case $r = 2$ where solutions to (15) may be locally unbounded.

Theorem 4 *Let $\alpha \in L^2(\mathbb{D})$ and $1 < p < \infty$. To each $\psi \in L^p_\mathbb{R}(\mathbb{T})$ and $c \in \mathbb{R}$, there is a unique $w \in G^p_\alpha(\mathbb{D})$ such that $\text{Re } w_\mathbb{T} = \psi$ and $\int_\mathbb{T} \text{Im } w_\mathbb{T} = c$. Moreover*

$$\|w\|_{G^p_\alpha(\mathbb{D})} \leq C\|\psi\|_{L^p(\mathbb{T})} + C'|c|, \quad (59)$$

where C, C' depend only on p and $|\alpha|$.

Proof. We first show existence. Assume ψ, c are not both zero otherwise $w \equiv 0$ will do. Let α_n be a sequence of functions in $L^\infty(\mathbb{D})$, converging to α in $L^2(\mathbb{D})$. By Theorem 3, to each n there is $w_n \in G_{\alpha_n}^p(\mathbb{D})$ such that $\operatorname{Re} w_n|_{\mathbb{T}} = \psi$ and $\int_{\mathbb{T}} \operatorname{Im} w_n = c$. Notations being as in Section 5.2, let us write $w_n = e^{s_n^\tau} F_n^\tau$ where $s_n^\tau \in W^{1,2}(\mathbb{D})$ is real with zero mean on \mathbb{T} while $F_n^\tau \in H^p$. Below, we drop the superscript τ for simplicity.

It follows from (48) that $\|s_n\|_{W^{1,2}(\mathbb{D})} \leq C_0 \|\alpha_n\|_{L^2(\mathbb{D})}$ for some absolute constant C_0 , hence $\|s_n\|_{W^{1,2}(\mathbb{D})}$ is bounded independently of n . In view of the Rellich-Kondratchov theorem, we can find a subsequence, again denoted with s_n , converging to some function s both pointwise on \mathbb{D} and in $L^\ell(\mathbb{D})$ for all $\ell \in [1, \infty)$. By the trace theorem and the non integral version of the Rellich-Kondratchov theorem, we may further assume that $\operatorname{tr}_{\mathbb{T}} s_n$ converges to some function h both pointwise a.e. on \mathbb{T} and in $L^\ell(\mathbb{T})$. Moreover, convergence of α_n to α in $L^2(\mathbb{D})$ entails that a single function ω in (53) can be chosen to accomodate α and all the α_n . Therefore $e^{\pm s_n}$ is bounded in $W^{1,q}(\mathbb{D})$, independently of n and ψ , for each $q \in [1, 2)$. So, invoking again the trace and the Rellich-Kondratchov theorems, we may assume upon refining s_n further $e^{\pm \operatorname{tr}_{\mathbb{T}} s_n}$ converges to its pointwise limit $e^{\pm h}$ in $L^\ell(\mathbb{T})$, for all $\ell \in [1, \infty)$.

Thus, by Hölder's inequality, $\operatorname{Re}(F_n)|_{\mathbb{T}} = e^{-\operatorname{tr}_{\mathbb{T}} s_n} \psi$ converges to $e^{-h} \psi$ in $L^\lambda(\mathbb{T})$ for fixed but arbitrary $\lambda \in [1, p)$. Continuity of the conjugate operator now implies that $\widetilde{\operatorname{Re}(F_n)|_{\mathbb{T}}}$ in turn converges to $\widetilde{e^{-h} \psi}$ in $L^\lambda(\mathbb{T})$. Since $\int_{\mathbb{T}} \operatorname{Im} w_n = c$, it holds by inspection that $\operatorname{Im}(F_n)|_{\mathbb{T}} = \widetilde{\operatorname{Re}(F_n)|_{\mathbb{T}}} + c_n$ where the constant c_n is such that

$$c_n \int_{\mathbb{T}} e^{\operatorname{tr}_{\mathbb{T}} s_n} + \int_{\mathbb{T}} e^{\operatorname{tr}_{\mathbb{T}} s_n} \widetilde{\operatorname{Re}(F_n)|_{\mathbb{T}}} = c. \quad (60)$$

The first integral in (60) converges to $\int_{\mathbb{T}} e^h > 0$, and the second integral converges to $\int_{\mathbb{T}} e^h \widetilde{e^{-h} \psi}$ by Hölder's inequality. Therefore c_n converges to

$$c_0 := \left(c - \int_{\mathbb{T}} e^h \widetilde{e^{-h} \psi} \right) / \int_{\mathbb{T}} e^h, \quad (61)$$

and subsequently $(F_n)|_{\mathbb{T}}$ converges to

$$F_{\mathbb{T}} := e^{-h} \psi + i \widetilde{e^{-h} \psi} + i c_0 \quad (62)$$

in $L^\lambda(\mathbb{T})$, for all $\lambda \in [1, p)$. Thus, F_n converges in H^λ to F , the Poisson integral of $F_{\mathbb{T}}$. Note that F is not identically zero, for this would imply $\psi \equiv 0$ and $c = 0$, contrary to our initial assumption.

From what precedes and the dominated convergence theorem, we get that $\alpha_n e^{-2i \operatorname{Im} s_n} \bar{F}_n / F_n$ converges to $\alpha e^{-2i \operatorname{Im} s} \bar{F} / F$ in $L^2(\mathbb{D})$. As $\bar{\partial} s_n = \alpha_n e^{-2i \operatorname{Im} s_n} \bar{F}_n / F_n$ by Lemma 2, applying (11) with $A = s_n - s_m$, $a = \bar{\partial} s_n - \bar{\partial} s_m$, $\theta_0 = -\pi/2$, $\psi \equiv 0$, and $\lambda = 0$, we conclude that (s_n) is a Cauchy sequence in $W^{1,2}(\mathbb{D})$ which must therefore converge to s . Hence $s \in W^{1,2}(\mathbb{D})$ and $h = \operatorname{tr}_{\mathbb{T}} s$. Besides, since we get in the limit that $\bar{\partial} s = (\alpha \bar{F} / F) e^{-2i \operatorname{Im} s}$ we see from Lemma 2 that $w := e^s F$ satisfies (15). Moreover if we write $w = e^{s^\tau} F^\tau$ with the notation of section 5.2, we find that $s^\tau = s$ and $F^\tau = F$ because s inherits from s_n the properties $\operatorname{Im} \operatorname{tr}_{\mathbb{T}} s = 0$ and $\int_{\mathbb{T}} \operatorname{Re} s = 0$. As $F \in H^\lambda$ for all $\lambda \in [1, p)$, we further deduce from the discussion before (54) that $w \in G_\alpha^\lambda$ for all such λ . By inspection of (62), we get

$$w_{\mathbb{T}} = e^{\operatorname{tr}_{\mathbb{T}} s} F|_{\mathbb{T}} = e^{\operatorname{tr}_{\mathbb{T}} s} (e^{-\operatorname{tr}_{\mathbb{T}} s} \psi + i \widetilde{e^{-\operatorname{tr}_{\mathbb{T}} s} \psi} + i c_0) = \psi + i \left(e^{\operatorname{tr}_{\mathbb{T}} s} \widetilde{e^{-\operatorname{tr}_{\mathbb{T}} s} \psi} + e^{\operatorname{tr}_{\mathbb{T}} s} c_0 \right) \quad (63)$$

where we used that $h = \text{tr}_{\mathbb{T}} s$ is real valued. In particular, (63) entails that $\text{Re } w_{\mathbb{T}} = \psi$. To show that in fact $w \in G_{\alpha}^p(\mathbb{D})$, we must prove in view of Theorem 2 that $w_{\mathbb{T}} \in L^p(\mathbb{T})$. For this, note that $\psi \in L^p(\mathbb{T})$ by assumption and that $e^{\text{tr}_{\mathbb{T}} s} c_0 \in L^p(\mathbb{T})$ by the trace and Sobolev embedding theorems. In another connection, $p \text{tr}_{\mathbb{T}} s \in W^{1/2,2}(\mathbb{T}) \subset VMO(\mathbb{T})$ after (92) hence $e^{p \text{tr}_{\mathbb{T}} s}$ satisfies condition A_p , see Lemma 9. Thus, using (46),

$$\|e^{\text{tr}_{\mathbb{T}} s} \widetilde{e^{-\text{tr}_{\mathbb{T}} s} \psi}\|_p^p \leq C'' \|\psi\|_p^p, \quad C'' = C''(\{e^{p \text{tr}_{\mathbb{T}} s}\}_{A_p}), \quad (64)$$

implying in view of (63) that $w_{\mathbb{T}} \in L^p(\mathbb{T})$. This yields the existence part of Theorem 4. As for uniqueness, let $w_1, w_2 \in G_{\alpha}^p(\mathbb{D})$ be two solutions. Set $v := w_1 - w_2 \in G_{\alpha}^p(\mathbb{D})$, so that $\text{Re } v_{\mathbb{T}} = 0$, $\int_{\mathbb{T}} \text{Im } v_{\mathbb{T}} = 0$. If we write $v = e^{\sigma^{\mathfrak{r}}} \Phi^{\mathfrak{r}}$, we observe that $\text{Re } (\Phi^{\mathfrak{r}})_{|\mathbb{T}} \equiv 0$ hence the H^{λ} function $\Phi^{\mathfrak{r}}$, $1 \leq \lambda < p$, is a pure imaginary constant, say ζ . Thus, $v = \zeta e^{\sigma}$ and the relations $\int_{\mathbb{T}} \text{Im } v_{\mathbb{T}} = 0$, $\int_{\mathbb{T}} e^{\text{tr}_{\mathbb{T}} \sigma} > 0$ give us $\zeta = 0$ so that $v = 0$, as desired. Finally, we show that (59) holds. For this, in view of (56), it is enough to prove that

$$\|w_{\mathbb{T}}\|_{L^p(\mathbb{T})} \leq C \|\psi\|_{L^p(\mathbb{T})} + C' |c|, \quad (65)$$

where C, C' depend only on p . By (64), (63), (61) and Hölder's inequality, we need only establish that $\|e^{\text{tr}_{\mathbb{T}} s}\|_{L^p(\mathbb{T})}$, $\{e^{p \text{tr}_{\mathbb{T}} s}\}_{A_p}$, and $1/\int_{\mathbb{T}} e^{\text{tr}_{\mathbb{T}} s}$ are bounded from above independently of ψ . We pointed out earlier in the proof that e^{s_n} is bounded in $W^{1,q}(\mathbb{D})$, independently of n and ψ , for each $q \in [1, 2)$. Since s_n tends to s in $W^{1,2}(\mathbb{D})$, boundedness of $\|e^{\text{tr}_{\mathbb{T}} s}\|_{L^p(\mathbb{T})}$ follows from Proposition 1 and the (non-integral version of) the Sobolev embedding theorem. Next, we saw that $\|s_n\|_{W^{1,2}(\mathbb{D})} \leq C_0 \|\alpha_n\|_{L^2(\mathbb{D})}$ for some absolute constant C_0 , hence also $\|s\|_{W^{1,2}(\mathbb{D})} \leq C_0 \|\alpha\|_{L^2(\mathbb{D})}$. Thus, using concavity of \log , the Schwarz inequality and the trace theorem, we get for some absolute constant C'' that

$$\log \left(\frac{1}{2\pi} \int_{\mathbb{T}} e^{s(\xi)} |d\xi| \right) \geq \frac{1}{2\pi} \int_{\mathbb{T}} s(\xi) |d\xi| \geq -\|s\|_{L^2(\mathbb{T})} \geq -C'' \|s\|_{W^{1,2}(\mathbb{D})} \geq -C_0 C'' \|\alpha\|_{L^2(\mathbb{D})},$$

showing that $\int_{\mathbb{T}} e^{\text{tr}_{\mathbb{T}} s} \geq \exp\{-C_0 C'' \|\alpha\|_{L^2(\mathbb{D})}\}$. To majorize $\{e^{p \text{tr}_{\mathbb{T}} s}\}_{A_p}$ independently of ψ , it is enough by Lemma 9 to prove that $M_{\text{tr}_{\mathbb{T}} s}(J)$ (see definition (81)) can be made arbitrary small with $\Lambda(J)$, uniformly with respect to ψ , as J ranges over open arcs on \mathbb{T} . Let ω be as in (53). By (50) and Proposition 2, there is a strictly positive function $\tilde{\omega}$ on $(0, +\infty)$, depending only on ω , such that (58) holds. Now, if $\Lambda(J) < 1$, it is elementary to check that $R(J, \Lambda(J))$ (cf. definition (105)) is contained in a square of sidelength $\Lambda(J)$. Therefore if we pick $\Lambda(J) < \min\{1/2, \tilde{\omega}(\eta)\}$ we deduce from (92) and Lemma 13 that $M_{\text{tr}_{\mathbb{T}} s}(J) \leq C_1 \eta$ where C_1 is an absolute constant. This achieves the proof of Theorem 4. \blacksquare

7 Dirichlet problem for $\exp - W^{1,2}$ conductivity

The following connection between pseudo-holomorphic functions and conductivity equations is instrumental in [2] and was investigated in the context of pseudo-holomorphic Hardy spaces in [5, 4] when $r > 2$. We start from a 2-d isotropic conductivity equation with exp-Sobolev smooth coefficient:

$$\text{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega, \quad \sigma \geq 0, \quad \log \sigma \in W^{1,r}(\Omega), \quad r \in [2, \infty). \quad (66)$$

When $r > 2$, the assumption that $\log \sigma \in W^{1,r}(\Omega)$ simply means that $\sigma \in W^{1,r}(\Omega)$ and that $0 < c < \sigma$ (strict ellipticity). If $r = 2$, then σ lies in $W^{1,q}(\Omega)$ for all $q \in [1, 2)$ by Proposition 1, but it needs not be bounded away from zero nor infinity which makes this case particularly interesting because (66) may no longer be strictly elliptic. Put $\nu := (1 - \sigma)/(1 + \sigma)$ and consider the conjugate Beltrami equation:

$$\bar{\partial}f = \nu \bar{\partial}f \quad \text{in } \Omega, \quad -1 \leq \nu \leq 1, \quad \operatorname{artanh} \nu \in W^{1,r}(\Omega), \quad r \in [2, \infty), \quad (67)$$

where the assumptions on ν translate those on σ shown in (66). The fact that $\sigma \in W^{1,q}(\Omega)$ for all $q \in [1, 2)$ implies easily the same holds for ν . If we restrict to solutions $f \in L_{loc}^\gamma(\Omega)$ for some $\gamma > r/(r-1)$ and write $f = u + iv$ to separate real and imaginary parts, we find that (67) is equivalent to the generalized Cauchy-Riemann system:

$$\begin{cases} \partial_x v = -\sigma \partial_y u, \\ \partial_y v = \sigma \partial_x u, \end{cases} \quad (68)$$

whose compatibility condition is the conductivity equation (66). Hence (67) is a means to rewrite (66) as a complex equation of the first order. Now, if we set

$$w := \frac{f - \nu \bar{f}}{\sqrt{1 - \nu^2}} = \sigma^{1/2} u + i \sigma^{-1/2} v, \quad \alpha = \bar{\partial} \log \sigma^{1/2} \in L^r, \quad (69)$$

a straightforward computation using (68) shows that (15) holds. Note that any constant c solves (67), the corresponding solution in (15) being $\sigma^{1/2} \operatorname{Re} c + i \sigma^{-1/2} \operatorname{Im} c$. The preceding discussion makes the study of (67) essentially equivalent to that of (15), (66). In particular, Theorem 4 translates into the following result that seems first to describe a class of non strictly elliptic equations with unbounded coefficients for which the Dirichlet problem is well-posed with (weighted) L^p -boundary data.

Theorem 5 *Let $\sigma \geq 0$ be such that $\log \sigma \in W^{1,2}(\mathbb{D})$, and fix $p \in (1, \infty)$. To each ψ such that $\psi \operatorname{tr}_{\mathbb{T}} \sigma^{1/2} \in L^p(\mathbb{T})$, there is a unique solution u to (66) such that*

$$\sup_{0 < \rho < 1} \left(\int_{\mathbb{T}_\rho} |u(\xi)|^p \sigma^{p/2}(\xi) |d\xi| \right)^{1/p} < +\infty \quad (70)$$

and $\lim_{\rho \rightarrow 1} \operatorname{tr}_{\mathbb{T}}(u_\rho \sigma_\rho^{1/2}) = \psi \operatorname{tr}_{\mathbb{T}} \sigma^{1/2}$ in $L^p(\mathbb{T})$. Moreover, the supremum in (70) is less than $C \|\psi \sigma^{1/2}\|_{L^p(\mathbb{T})}$ for some $C = C(p, \sigma)$.

8 Appendix

8.1 Mean growth of Cauchy transforms

In this subsection we prove estimate (13). First, we evaluate $\mathcal{C}_2(h)_{\mathbb{D}_R}$, the mean of $\mathcal{C}_2(h)$ on \mathbb{D}_R , for $h \in L^2(\mathbb{C})$ and $R \geq 1$. To this end, we use the identity (see [3, sec. 4.3.2]):

$$\mathcal{C}(\chi_{\mathbb{D}_R})(t) = \begin{cases} \bar{t} & \text{if } |t| \leq R, \\ R^2/t & \text{if } |t| > R. \end{cases} \quad (71)$$

When h has compact support, we deduce from (12), (71) and Fubini's theorem that

$$\begin{aligned}
\mathcal{C}_2(h)_{\mathbb{D}_R} &= \frac{1}{\pi R^2} \int_{\mathbb{D}_R} \left(\frac{1}{\pi} \int_{\mathbb{C}} \frac{h(t)}{z-t} dm(t) + \frac{1}{\pi} \int_{\mathbb{C} \setminus \mathbb{D}} \frac{h(t)}{t} dm(t) \right) dm(z) \\
&= -\frac{1}{\pi R^2} \int_{\mathbb{D}_R} h(t) \bar{t} dm(t) - \frac{1}{\pi} \int_{\mathbb{C} \setminus \mathbb{D}_R} \frac{h(t)}{t} dm(t) + \frac{1}{\pi} \int_{\mathbb{C} \setminus \mathbb{D}} \frac{h(t)}{t} dm(t) \\
&= -\frac{1}{\pi R^2} \int_{\mathbb{D}_R} h(t) \bar{t} dm(t) + \frac{1}{\pi} \int_{1 \leq |t| \leq R} \frac{h(t)}{t} dm(t), \tag{72}
\end{aligned}$$

so by density (72) holds for all $h \in L^2(\mathbb{C})$. From (72) and the Schwarz inequality, we get

$$|\mathcal{C}_2(h)_{\mathbb{D}_R}| \leq \frac{\|h\|_{L^2(\mathbb{C})}}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} \|h\|_{L^2(\mathbb{C})} (\log R)^{1/2}, \quad R \geq 1. \tag{73}$$

Next, we observe from the Poincaré inequality that

$$\|\mathcal{C}_2(h) - \mathcal{C}_2(h)_{\mathbb{D}_R}\|_{L^2(\mathbb{D}_R)} \leq C_R (\|h\|_{L^2(\mathbb{D}_R)} + \|\mathcal{B}(h)\|_{L^2(\mathbb{D}_R)}) \leq 2C_R \|h\|_{L^2(\mathbb{C})}, \tag{74}$$

where C_R is the best constant in (2) when $p = 2$ and $\Omega = \mathbb{D}_R$. Since

$$\frac{\|\mathcal{C}_2(h)\|_{L^2(\mathbb{D}_R)}}{\sqrt{\pi}R} \leq \frac{\|\mathcal{C}_2(h) - \mathcal{C}_2(h)_{\mathbb{D}_R}\|_{L^2(\mathbb{D}_R)}}{\sqrt{\pi}R} + |\mathcal{C}_2(h)_{\mathbb{D}_R}|,$$

(13) now follows from (73), (74) and the fact that $C_R = RC_1$ by homogeneity.

8.2 Functions of vanishing mean oscillation

The space $BMO(\mathbb{R}^2)$ of functions with bounded mean oscillation consists of those $h \in L^1_{loc}(\mathbb{R}^2)$ such that

$$\|h\|_{BMO(\mathbb{R}^2)} := \sup_Q \frac{1}{|Q|} \int_Q |h(\xi) - h_Q| dm(\xi) < \infty, \quad h_Q := \frac{1}{|Q|} \int_Q h dm, \tag{75}$$

where the supremum is taken over all squares $Q \subset \mathbb{C}$. Clearly squares could be replaced by any other substantial family, for instance by balls. Notice that $\|\cdot\|_{BMO(\mathbb{R}^2)}$ is a genuine norm modulo additive constants only.

The John-Nirenberg theorem [21, thm. 7.1.6] asserts there are absolute constants C, c such that, for each $h \in BMO(\mathbb{R}^2)$, every square Q , and any $\lambda > 0$,

$$\frac{|\{\xi \in Q : |h(\xi) - h_Q| > \lambda\}|}{|Q|} \leq C \exp\left(\frac{-c\lambda}{\|h\|_{BMO(\mathbb{R}^2)}}\right); \tag{76}$$

in fact, $C = e$ and $c = 1/4e$ will do.

The space $VMO(\mathbb{R}^2)$ of functions with vanishing mean oscillation is the subspace of $BMO(\mathbb{R}^2)$ comprised of those h for which

$$\lim_{\varepsilon \rightarrow 0} \sup_{|Q| < \varepsilon} \frac{1}{|Q|} \int_Q |h(\xi) - h_Q| dm(\xi) = 0. \tag{77}$$

In fact, $VMO(\mathbb{R}^2)$ is the closure in $BMO(\mathbb{R}^2)$ of continuous functions with compact support [10]. For $h \in L^1_{loc}(\mathbb{R}^2)$ and Q a square, let us define

$$M_h(Q) := \sup_{Q' \subset Q} \frac{1}{|Q'|} \int_{Q'} |h - h_{Q'}| dm$$

where the supremum is taken over all squares $Q' \subset Q$.

Lemma 7 *For any $h \in BMO(\mathbb{R}^2)$ and every square Q , it holds that:*

$$\int_Q e^{|h|/(8eM_h(Q))} dm \leq (1+e)|Q|e^{|h_Q|/(8eM_h(Q))}. \quad (78)$$

Proof. Inspecting the standard proof of the John-Nirenberg inequality which uses recursively the Calderón-Zygmund decomposition in dyadic subcubes of Q [21, thm. 7.1.6], one can check that (76) remains valid if we replace $\|h\|_{BMO(\mathbb{R}^2)}$ by $M_h(Q)$:

$$\frac{|\{\xi \in Q : |h(\xi) - h_Q| > \lambda\}|}{|Q|} \leq C \exp\left(\frac{-c\lambda}{M_h(Q)}\right). \quad (79)$$

Now, if $h|_Q \equiv 0$ there is nothing to prove, otherwise $M_h(Q) \neq 0$ and we pick $C' \in (0, c)$ with c as in (79). We set $g := C'|h - h_Q|/M_h(Q)$ and compute as in [21, cor. 7.1.7]:

$$\frac{1}{|Q|} \int_Q e^g dm = 1 + \frac{1}{|Q|} \int_Q (e^g - 1) dm = 1 + \frac{1}{|Q|} \int_0^\infty e^\lambda |\{\xi \in Q : g(\xi) > \lambda\}| d\lambda$$

where the second equality uses Fubini's theorem. Then, using (79) to estimate the distribution function of g , we find that

$$\frac{1}{|Q|} \int_Q e^{C'|h-h_Q|/M_h(Q)} dm = \frac{1}{|Q|} \int_Q e^g dm \leq 1 + C \int_0^\infty e^\lambda e^{-c\lambda/C'} d\lambda = 1 + \frac{C}{c/C' - 1}. \quad (80)$$

Multiplying by $|Q|e^{C'|h_Q|/M_h(Q)}$ and choosing for definiteness $C = e$, $c = 1/(4e)$, and $C' = 1/8e$ yields (78). \blacksquare

Since by definition $M_h(Q)$ goes to zero uniformly with $|Q|$ when $h \in VMO(\mathbb{R}^2)$, Lemma 7 makes it clear in this case that $e^h \in L^p_{loc}(\mathbb{C})$ for all $p \in [1, \infty)$.

The spaces $BMO(\Gamma)$ and $VMO(\Gamma)$, for Γ the graph of a Lipschitz function over \mathbb{R} or a Lipschitz Jordan curve, are defined similarly:

$$\|h\|_{BMO(\Gamma)} := \sup_I \frac{1}{\Lambda(I)} \int_I |h(t) - h_I| d\Lambda(t), \quad h_I := \frac{1}{\Lambda(I)} \int_I h(t) d\Lambda(t),$$

where Λ indicates arclength on Γ and I ranges over all arcs such that $\Lambda(I) < \infty$. Setting

$$M_h(I) := \sup_{I' \subset I} \frac{1}{\Lambda(I')} \int_{I'} |h - h_{I'}| d\Lambda, \quad (81)$$

the analog of Lemma 7 goes as follows.

Lemma 8 *Let Γ be the graph of a Lipschitz function over \mathbb{R} or a Lipschitz Jordan curve. and Λ indicate arclength on Γ . For any $h \in BMO(\Gamma)$ and every arc $I \subset \Gamma$, it holds that*

$$\int_I e^{|h|/(4eM_h(I))} d\Lambda \leq (1+e)\Lambda(I) e^{|h_I|/(4eM_h(I))}. \quad (82)$$

Proof. In dimension one, the John-Nirenberg inequality (76) is valid with $C = e$ and $c = 1/2e$. Thus, arguing as we did to obtain (80) while choosing $C' = 1/4e$, we find that

$$\frac{1}{\Lambda(I)} \int_I e^{|h-h_I|/(4eM_h(I))} d\Lambda \leq 1 + e \quad (83)$$

from which (82) follows. ■

We shall mainly be concerned with $\Gamma = \mathbb{T}$. It is well-known that exponentials of real valued $VMO(\mathbb{T})$ -functions satisfy A_p condition (45), for all $p \in (1, \infty)$. Below, we record for later use an estimate showing the dependence of the A_p norm on the function.

Lemma 9 *Assume $h \in VMO_{\mathbb{R}}(\mathbb{T})$ and $p \in (1, \infty)$. Let $\eta = \eta(h, p) > 0$ be so small that $\max(4eM_h(I), 4eM_h(I)/(p-1)) \leq 1$ for all arcs $I \subset \mathbb{T}$ satisfying $\Lambda(I) < \eta$. Then*

$$\{e^h\}_{A_p} := \sup_I \left(\frac{1}{\Lambda(I)} \int_I e^h d\Lambda \right) \left(\frac{1}{\Lambda(I)} \int_I e^{-h/(p-1)} d\Lambda \right)^{p-1} \leq C \quad (84)$$

where C depends only on η , p , and $\|e^h\|_{L^1(\mathbb{T})}$.

Proof. If we put $p' = p/(p-1)$, then $1/(p-1) = p' - 1$ and it follows easily from the definition that $\{e^h\}_{A_p} = \{e^{-h/(p-1)}\}_{A_{p'}}^{1/(p-1)}$. Therefore we may assume that $p \geq 2$.

Now, the left hand side of (84) can be rewritten as

$$\sup_I \left(\frac{1}{\Lambda(I)} \int_I e^{h-h_I} d\Lambda \right) \left(\frac{1}{\Lambda(I)} \int_I e^{-(h-h_I)/(p-1)} d\Lambda \right)^{p-1}.$$

If $\Lambda(I) < \eta$ then $4eM_h(I)$ and $4eM_h(I)/(p-1) < 1$, thus by (83) and Hölder's inequality:

$$\frac{1}{\Lambda(I)} \int_I e^{h-h_I} d\Lambda \leq \left(\frac{1}{\Lambda(I)} \int_I e^{|h-h_I|/(4eM_h(I))} d\Lambda \right)^{4eM_h(I)} \leq (1+e)^{4eM_h(I)} \leq (1+e),$$

$$\left(\frac{1}{\Lambda(I)} \int_I e^{-(h-h_I)/(p-1)} d\Lambda \right)^{p-1} \leq \left(\frac{1}{\Lambda(I)} \int_I e^{|h-h_I|/(4eM_h(I))} d\Lambda \right)^{4eM_h(I)} \leq (1+e).$$

This shows that (84) holds with $C = (1+e)^2$ when the supremum is restricted to those I of length less than η . In another connection, if $\Lambda(I) \geq \eta$, then obviously

$$\frac{1}{\Lambda(I)} \int_I e^h d\Lambda \leq \eta^{-1} \|e^h\|_{L^1(\mathbb{T})}$$

and likewise, taking into account that $p \geq 2$ and using Hölder's inequality, we obtain

$$\left(\frac{1}{\Lambda(I)} \int_I e^{-h/(p-1)} d\Lambda \right)^{p-1} \leq \frac{1}{\Lambda(I)} \int_I e^{|h|} d\Lambda \leq \eta^{-1} \|e^{|h|}\|_{L^1(\mathbb{T})}.$$

Thus, (84) holds with $C = (\|e^{|h|}\|_{L^1(\mathbb{T})}/\eta)^2$ in this case. ■

8.3 Exp-summability of Sobolev functions at critical exponents

When Ω is the square Q_ρ centered at 0 of sidelength ρ , it is easily seen by homogeneity that the best constant in (2) is $O(\rho)$. Hence it holds for $g \in W^{1,p}(\mathbb{R}^2)$ that

$$\left(\frac{1}{|Q_\rho|} \int_{Q_\rho} |g - g_{Q_\rho}|^p dm \right)^{1/p} \leq C \rho^{1-2/p} (\|\partial g\|_{L^p(Q_\rho)} + \|\bar{\partial} g\|_{L^p(Q_\rho)}) \quad (85)$$

for some constant $C = C(p)$. Consequently, by Hölder's inequality:

$$\frac{1}{|Q_\rho|} \int_{Q_\rho} |g - g_{Q_\rho}| dm \leq C \rho^{1-2/p} (\|\partial g\|_{L^p(Q_\rho)} + \|\bar{\partial} g\|_{L^p(Q_\rho)}). \quad (86)$$

When $p = 2$ the right hand side of (86) is uniformly small with ρ for fixed g , by absolute continuity of $|\partial g|^2 dm$, $|\bar{\partial} g|^2 dm$ and since $1 - 2/p = 0$. Therefore $W^{1,2}(\mathbb{R}^2)$ embeds continuously in $VMO(\mathbb{R}^2)$ (see also [10]), and the results in the previous section show that $e^f \in L_{loc}^\ell(\mathbb{R}^2)$ for all $\ell \in [1, \infty)$ when $f \in W^{1,2}(\mathbb{R}^2)$. Actually much more is true, as shown in Proposition 1 below. First, we derive a specific estimate on a square:

Lemma 10 *Let Q be a square and $f \in W_{\mathbb{R}}^{1,2}(Q)$. For $\ell \in [1, \infty)$, let $\varepsilon = \varepsilon(f, \ell) > 0$ be so small that $\|\partial f\|_{L^2(Q_\varepsilon)} < \pi/(16\sqrt{2}\ell)$ whenever $Q_\varepsilon \subset Q$ is a square of sidelength ε . Then*

$$\|e^{|f|}\|_{L^\ell(Q)} \leq |Q|(1+e)^{1/\ell} \exp \left(\|f\|_{L^2(Q)} \frac{|Q|^{1/2} + \varepsilon}{|Q|^{1/2}\varepsilon} \right). \quad (87)$$

Proof. Let a be the sidelength of Q and $N = [a/\varepsilon]$ where brackets indicate integer part. Divide Q into subsquares $Q^{(1)}, \dots, Q^{(N+1)^2}$ of sidelength $a/(N+1) < \varepsilon$ with non-overlapping interiors. When $p = 2$ and g is real-valued, inequality (85) holds with $C = \sqrt{2}/\pi$ by convexity of Q_ρ [33]. Therefore, since $\bar{\partial} f = \overline{\partial f}$ because f is real, inequality (86) implies by definition of ε that $M_f(Q^{(j)}) \leq 1/(8\ell)$. Hence, by Lemma 7, and Hölder's inequality, we have for each $j \in \{1, 2, \dots, (N+1)^2\}$ that

$$\left(\frac{1}{|Q^{(j)}|} \int_{Q^{(j)}} e^{\ell|f|} dm \right)^{1/\ell} \leq \left(\frac{1}{|Q^{(j)}|} \int_{Q^{(j)}} e^{|f|/(8\ell M_f(Q^{(j)})a)} dm \right)^{8\ell M_f(Q^{(j)})} \leq (1+e)^{1/\ell} e^{|h_{Q^{(j)}}|}.$$

By the Schwarz inequality $|h_{Q^{(j)}}| \leq \|f\|_{L^2(Q)} / |Q^{(j)}|^{1/2} = \|f\|_{L^2(Q)}(N+1)/a$, so the previous equation yields

$$\|e^{|f|}\|_{L^\ell(Q^{(j)})} \leq |Q^{(j)}|^{1/\ell} (1+e)^{1/\ell} e^{(N+1)\|f\|_{L^2(Q)}/a} \leq |Q^{(j)}|^{1/\ell} (1+e)^{1/\ell} e^{\frac{a+\varepsilon}{\varepsilon a} \|f\|_{L^2(Q)}},$$

and since $\sum_j |Q^{(j)}|^{1/\ell} \leq |Q|$ by concavity of $x \mapsto x^{1/\ell}$ for $x > 0$, summing over j and using the triangle inequality gives us

$$\|e^{|f|}\|_{L^\ell(Q)} \leq \sum_{j=1}^{(N+1)^2} \|e^{|f|}\|_{L^\ell(Q^{(j)})} \leq |Q|(1+e)^{1/\ell} e^{\frac{a+\varepsilon}{\varepsilon a} \|f\|_{L^2(Q)}}$$

which is the desired estimate. ■

The point in Lemma 10 is that the right hand side of (87) depends on f only through $\|f\|_{L^2(Q)}$ and ε , which measures how small the squares on which the derivative of f is L^2 -small. We now prove that e^f has in fact quite a bit of smoothness when $f \in W^{1,2}(\Omega)$.

Proposition 1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Fix $p \in (1, \infty)$ and pick $\ell \in [1, \min(p, 2))$. Then the map $(g, f) \mapsto ge^f$ is continuous from $W^{1,p}(\Omega) \times W^{1,2}(\Omega)$ into $W^{1,\ell}(\Omega)$, and the derivative is computed using the Leibnitz and the chain rules:*

$$\partial(ge^f) = e^f \partial g + ge^f \partial f, \quad \bar{\partial}(ge^f) = e^f \bar{\partial} g + ge^f \bar{\partial} f. \quad (88)$$

In particular, for every $q \in [1, 2)$, the map $f \mapsto e^f$ is continuous from $W^{1,2}(\Omega)$ into $W^{1,q}(\Omega)$ and the map $f \mapsto e^{tr_{\partial\Omega} f}$ is continuous from $W^{1,2}(\Omega)$ into $W^{1-1/q,q}(\partial\Omega)$.

Proof. By the extension theorem, we may assume that Ω is a square. Let $g \in W^{1,p}(\Omega)$ and $f \in W^{1,2}(\Omega)$. Consider two sequences (f_n) and (g_n) of smooth functions on Ω , converging respectively to f and g in $W^{1,2}(\Omega)$ and $W^{1,p}(\Omega)$. We claim that e^{f_n} converges to e^f in $L^\ell(\Omega)$ for all $\ell \in [1, \infty)$. To see this, consider first the case of real-valued functions. By the mean-value theorem and convexity of $t \mapsto e^t$, we have that

$$\int_{\Omega} |e^f - e^{f_n}|^\ell dm \leq \int_{\Omega} |f - f_n|^\ell |e^f + e^{f_n}|^\ell dm \leq \|f - f_n\|_{L^{2\ell}(\Omega)}^\ell \|e^f + e^{f_n}\|_{L^{2\ell}(\Omega)}^\ell$$

where we used the Schwarz inequality. By the Sobolev embedding theorem $\|f - f_n\|_{L^{2\ell}(\Omega)}$ tends to zero as $n \rightarrow \infty$. Moreover $\|f_n\|_{L^{2\ell}(\Omega)}$ tends to $\|f\|_{L^{2\ell}(\Omega)}$, and we may apply Lemma 10 to f and f_n with ℓ replaced by 2ℓ and the same ε when n is large enough, since $\|\partial f_n - \partial f\|_{L^2(\Omega)} \rightarrow 0$. Hence $\|e^f + e^{f_n}\|_{L^{2\ell}(\Omega)}$ is uniformly bounded with n , thus the right hand side of the above equation indeed goes to zero as $n \rightarrow \infty$.

If now f, f_n are complex-valued, say $f = u + iv$, $f_n = u_n + iv_n$, we write

$$\|e^f - e^{f_n}\|_{L^\ell(\Omega)} \leq \|e^u(e^{iv} - e^{iv_n})\|_{L^\ell(\Omega)} + \|e^{iv_n}(e^u - e^{u_n})\|_{L^\ell(\Omega)}.$$

By what precedes the last term in the right hand side tends to zero when $n \rightarrow \infty$, and so does the first since we can extract pointwise convergent subsequences from any subsequence of v_n and apply the dominated convergence theorem. *This proves the claim.* Next, we observe that $g_n e^{f_n}$ is smooth on Ω and that

$$\partial(g_n e^{f_n}) = e^{f_n} \partial g_n + g_n e^{f_n} \partial f_n. \quad (89)$$

Assume for instance that $p < 2$. Then, by the Sobolev embedding theorem, g_n converges to g in $L^{p*}(\Omega)$ where $p* = 2p/(2-p) > 2$. From this and the previous claim, we deduce by Hölder's inequality that $g_n e^{f_n}$ converges to ge^f in $L^\ell(\Omega)$ for $\ell \in [1, p*)$, hence also as a distribution. By the same token, the right hand side of (89) converges to $e^f \partial g + ge^f \partial f$ in $L^\ell(\Omega)$ for each $\ell \in [1, p)$. The case $p = 2$ is similar except that $p*$ can be taken arbitrarily large, hence the convergence in the right hand-side of (89) takes place in $L^\ell(\Omega)$ for all $\ell < 2$. When $p > 2$ then g is even bounded, but this does not improve the estimate. Repeating the argument for $\bar{\partial}(ge^f)$ proves that $g_n e^{f_n}$ converges to ge^f in $W^{1,\ell}$ for $\ell \in [1, \min(p, 2))$ and that (88) holds. Hence $(g, f) \mapsto ge^f$ is defined from $W^{1,p}(\Omega) \times W^{1,2}(\Omega)$ into $W^{1,\ell}(\Omega)$ and (88) is valid. Relaxing the smoothness assumption on f_n, g_n and arguing as before shows this map is actually continuous. This proves the first assertion.

Setting $g \equiv 1$, the second assertion follows by Sobolev embedding and the trace theorem. ■

Although we just showed it is continuous for $q \in [1, 2)$, the map $f \mapsto e^f$ from $W^{1,2}(\Omega)$ into $W^{1,q}(\Omega)$ is not bounded. Below, we derive further conditions on a bounded subset of $W^{1,2}(\Omega)$ to be mapped by the exponential onto a bounded subset of $W^{1,q}(\Omega)$.

Corollary 2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and \mathcal{E} a bounded subset of $W^{1,2}(\Omega)$ such that, to each $\eta > 0$, there is $\varepsilon > 0$ for which*

$$\|\partial f\|_{L^2(Q_\varepsilon \cap \Omega)} + \|\bar{\partial} f\|_{L^2(Q_\varepsilon \cap \Omega)} < \eta \quad \forall f \in \mathcal{E}, \quad (90)$$

as soon as Q_ε is a square of sidelength ε . Then $\{e^f : f \in \mathcal{E}\}$ is a bounded subset of $W^{1,q}(\Omega)$ for each $q \in (1, 2)$. The bound depends on q , on $\sup_{f \in \mathcal{E}} \|f\|_{W^{1,2}(\Omega)}$, and on the function $\eta \mapsto \varepsilon(\eta)$.

Proof. By Proposition 1 and Hölder's inequality, it is enough to show for each $\ell \in [1, \infty)$ that e^f is uniformly bounded in $L^\ell(\Omega)$ as f ranges over \mathcal{E} . For this we can replace \mathcal{E} by $\mathcal{E}' := \{\operatorname{Re} f, f \in \mathcal{E}\}$, observing that property (90) holds for \mathcal{E}' as well. Thus, we may assume that \mathcal{E} consists of real-valued functions (in which case one of the derivatives is redundant in (90) for $\bar{\partial} f = \overline{\partial f}$). When Ω is a square, the result follows from Lemma 10. To reduce the general case to this one, we need a refinement of the extension theorem where the extension operator $E : W^{1,2}(\Omega) \rightarrow W^{1,2}(\mathbb{R}^2)$ satisfies

$$\text{to each } \eta > 0, \quad \text{there is } \varepsilon > 0 \text{ such that } \|\partial E(f)\|_{L^2(Q_\varepsilon)} < \eta, \quad \forall f \in \mathcal{E}, \quad \forall Q_\varepsilon, \quad (91)$$

where Q_ε indicates as usual a generic square of sidelength ε . Indeed, if (91) holds, we can pick a square $Q \supset \Omega$ and apply the previous case to $E(\mathcal{E})|_Q \subset W^{1,2}(Q)$. This *a fortiori* yields the desired boundedness of e^f in $W^{1,q}(\Omega)$.

When Ω is disk, reflection followed by multiplication with a $\mathcal{D}(\mathbb{R}^2)$ -function which is 1 on Ω does the job. In the general case, it is lengthy but straightforward to check that the standard extension process [12, prop. 2.70] by local reflection across $\partial\Omega$ satisfies (91). ■

That $\phi \mapsto e^\phi$ is continuous from $W^{1/2,2}(\Gamma)$ into $L^\ell(\Gamma)$ when $\ell \in [1, \infty)$, if Γ is a Lipschitz Jordan curve, may be proven much like Lemma 10, because $W^{1/2,2}(\Gamma) \subset VMO(\Gamma)$ [10]. This last fact depends on the estimate:

$$\begin{aligned} \frac{1}{\Lambda(I)} \int_I |h - h_I| d\Lambda &\leq \frac{1}{(\Lambda(I))^2} \int_{I \times I} |h(t) - h(t')| d\Lambda(t) d\Lambda(t') \\ &\leq \frac{1}{\Lambda(I)} \int_{I \times I} \frac{|h(t) - h(t')|}{\Lambda(t, t')} d\Lambda(t) d\Lambda(t') \\ &\leq \left(\int_{I \times I} \frac{|h(t) - h(t')|^2}{(\Lambda(t, t'))^2} d\Lambda(t) d\Lambda(t') \right)^{1/2} \\ &\leq \|h\|_{W^{1/2,2}(I)} \leq \|h\|_{W^{1/2,2}(\Gamma)}, \end{aligned} \quad (92)$$

where the next to last step used the Schwarz inequality.

We turn to summability properties that depend more specifically on the geometry of the domain. We state and prove them only for the case $\Omega = \mathbb{D}$, which is easier to address and sufficient for our purposes.

Proposition 2 *Let $\beta \in L^2(\mathbb{D})$ and ω be a strictly positive function on $(0, +\infty)$ such that $\|\beta\|_{L^2(Q_{\omega(\varepsilon)} \cap \mathbb{D})} < \varepsilon$ as soon as $Q_{\omega(\varepsilon)}$ is a cube of sidelength $\omega(\varepsilon)$.*

(i) *If we let (cf. (7))*

$$\mathcal{C}(\beta)(z) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{\beta(\xi)}{\xi - z} d\xi \wedge \overline{d\xi}, \quad z \in \mathbb{C},$$

there is a strictly positive function ω_1 on $(0, +\infty)$, depending only on ω , such that

$$\|\partial \mathcal{C}(\beta)\|_{L^2(Q_{\omega_1(\eta)})} + \|\bar{\partial} \mathcal{C}(\beta)\|_{L^2(Q_{\omega_1(\eta)})} < \eta \quad (93)$$

as soon as $Q_{\omega_1(\eta)}$ is a cube of sidelength $\omega_1(\eta)$.

(ii) If we let (cf. (49))

$$\mathcal{R}(\beta)(z) := \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{z\bar{\beta}(\xi)}{1-\xi z} d\xi \wedge \overline{d\xi}, \quad z \in \mathbb{D},$$

then $\mathcal{R}(\beta) \in W^{1,2}(\mathbb{D})$ is holomorphic in \mathbb{D} and there exists a strictly positive function ω_2 on $(0, +\infty)$, depending only on ω , such that

$$\|\partial\mathcal{R}(\beta)\|_{L^2(Q_{\omega_2(\eta)} \cap \mathbb{D})} < \eta \quad (94)$$

as soon as $Q_{\omega_2(\eta)}$ is a cube of sidelength $\omega_2(\eta)$.

Proof. Since $\beta \in L^2(\mathbb{D})$, we know that $\mathcal{C}(\beta) \in W_{loc}^{1,2}(\mathbb{C})$. Set $\delta(\eta) := \eta/(3\|\beta\|_{L^2(\mathbb{D})} + \eta)$ and $\omega_1(\eta) := \min\{1/\sqrt{2}, \omega^2(\eta/3), \delta^2(\eta)\}$. For any cube $Q_{\omega_1(\eta)}$, we have a nested triple of concentric cubes with parallel sides: $Q_{\omega_1(\eta)} \subset Q_{\omega_1^{1/2}(\eta)} \subset Q_{\omega(\eta/3)}$. Thus, letting $\tilde{\beta}$ be the extension of β by 0 off \mathbb{D} , we deduce since $\bar{\partial}\mathcal{C}(\beta) = \tilde{\beta}$ that

$$\|\bar{\partial}\mathcal{C}(\beta)\|_{L^2(Q_{\omega_1(\eta)})} \leq \|\beta\|_{L^2(Q_{\omega(\eta/3)} \cap \mathbb{D})} < \eta/3. \quad (95)$$

Next, we write

$$\partial\mathcal{C}(\beta) = \mathcal{B}(\tilde{\beta}) = \mathcal{B}(\chi_{Q_{\omega_1^{1/2}(\eta)}}\tilde{\beta}) + \mathcal{B}(\chi_{\mathbb{C} \setminus Q_{\omega_1^{1/2}(\eta)}}\tilde{\beta}) \quad (96)$$

where \mathcal{B} indicates the Beurling transform, cf. (8). As \mathcal{B} is an isometry of $L^2(\mathbb{C})$, we get

$$\|\mathcal{B}(\chi_{Q_{\omega_1^{1/2}(\eta)}}\tilde{\beta})\|_{L^2(\mathbb{C})} = \|\beta\|_{L^2(Q_{\omega_1^{1/2}(\eta)} \cap \mathbb{D})} \leq \|\beta\|_{L^2(Q_{\omega(\eta/3)} \cap \mathbb{D})} < \eta/3. \quad (97)$$

Moreover, formula (8) and the Cauchy-Schwarz inequality give us the pointwise estimate:

$$\mathcal{B}(\chi_{\mathbb{C} \setminus Q_{\omega_1^{1/2}(\eta)}}\tilde{\beta})(z) \leq \frac{\|\beta\|_{L^2(\mathbb{D})}}{\omega_1^{1/2}(\eta) - \omega_1(\eta)}, \quad z \in Q_{\omega_1(\eta)},$$

and integrating over $Q_{\omega_1(\eta)}$ yields

$$\|\mathcal{B}(\chi_{\mathbb{C} \setminus Q_{\omega_1^{1/2}(\eta)}}\tilde{\beta})\|_{L^2(Q_{\omega_1(\eta)})} \leq \frac{\|\beta\|_{L^2(\mathbb{D})} \omega_1^{1/2}(\eta)}{1 - \omega_1^{1/2}(\eta)} \leq \frac{\|\beta\|_{L^2(\mathbb{D})} \delta(\eta)}{1 - \delta(\eta)} = \eta/3. \quad (98)$$

Inequality (93) now follows from (95), (96), (97) and (98), thereby proving (i).

Consider now $\mathcal{R}(\beta)$. Clearly it is holomorphic in \mathbb{D} and vanishes at 0. In addition $\overline{\mathcal{R}(\beta)(z)} = -\mathcal{C}(\beta)(1/\bar{z})$ and since $\mathcal{C}(\beta) \in W_{loc}^{1,2}(\mathbb{C})$ we get that $\mathcal{R}(\beta) \in W^{1,2}(\mathbb{D})$.

Let us show that putting $\omega_2(\eta) := \min\{\omega_1(\eta)/4, (\eta\sqrt{\pi}/(16\|\beta\|_{L^2(\mathbb{D})}))^{1/2}\}$ ensures (94) to hold, with ω_1 defined as before. First, as $\omega_1 \leq 1/\sqrt{2}$, each cube $Q_{\omega_2(\eta)}$ has diameter at most $1/4$, hence is disjoint from $\mathbb{D}_{1/2}$ if it meets $\mathcal{A}_{3/4} := \{z : 1 \geq |z| \geq 3/4\}$. In this case the reflection of $Q_{\omega_2(\eta)} \cap \mathbb{D}$ is contained in a cube of sidelength $4\omega_2(\eta) \leq \omega_1(\eta)$, and since

$$\partial\mathcal{R}(\beta)(z) = \frac{\overline{\partial\mathcal{C}(\beta)(1/\bar{z})}}{z^2}, \quad z \neq 0,$$

we deduce from (93) and the change of variable formula that $\|\partial\mathcal{R}(\beta)\|_{L^2(Q_{\omega_2(\eta)})} \leq \eta$.

Assume now that $Q_{\omega_2(\eta)} \subset \mathbb{D}_{3/4}$. Differentiating under the integral sign, we obtain

$$\partial \mathcal{R}(\beta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{\bar{\beta}(\xi)}{1 - \xi z} d\xi \wedge \overline{d\xi} + \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{z \bar{\xi} \bar{\beta}(\xi)}{(1 - \xi z)^2} d\xi \wedge \overline{d\xi},$$

so if $z \in \mathbb{D}_{3/4}$ we get by the Schwarz inequality that $|\partial \mathcal{R}(\beta)(z)| \leq 16 \|\beta\|_{L^2(\mathbb{D})} / \sqrt{\pi}$. Integrating over $Q_{\omega_2(\eta)}$ yields

$$\|\partial \mathcal{R}(\beta)\|_{L^2(Q_{\omega_2(\eta)})} \leq \omega_2^2 16 \|\beta\|_{L^2(\mathbb{D})} / \sqrt{\pi} \leq \eta,$$

as desired. ■

Corollary 3 *Let $\beta \in L^2(\mathbb{C})$ and ω be a strictly positive function on $(0, +\infty)$ such that $\|\beta\|_{L^2(Q_{\omega(\varepsilon)})} < \varepsilon$ as soon as $Q_{\omega(\varepsilon)}$ is a cube of sidelength $\omega(\varepsilon)$. If we let (cf. (12))*

$$\mathcal{C}_2(\beta)(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \left(\frac{1}{z - t} + \frac{\chi_{\mathbb{C} \setminus \mathbb{D}}(t)}{t} \right) \beta(t) dm(t), \quad z \in \mathbb{C},$$

there is a strictly positive function ω_1 on $(0, +\infty)$, depending only on ω , such that

$$\|\partial \mathcal{C}_2(\beta)\|_{L^2(Q_{\omega_1(\eta)})} + \|\bar{\partial} \mathcal{C}_2(\beta)\|_{L^2(Q_{\omega_1(\eta)})} < \eta \quad (99)$$

as soon as $Q_{\omega_1(\eta)}$ is a cube of sidelength $\omega_1(\eta)$.

Proof. This is proved in the same way as (93), replacing $\tilde{\beta}$ by β . ■

Lemma 11 *If $f \in W^{1,q}(\mathbb{D})$ for some $q \in (1, 2)$ and $\ell := q/(2 - q)$, then*

$$\sup_{\rho \in (0, 1]} \left(\int_{\mathbb{T}_\rho} |f(\xi)|^\ell |d\xi| \right)^{1/\ell} \leq C \|f\|_{W^{1,q}(\mathbb{D})} \quad (100)$$

where $C = C(q)$ is a constant.

Proof. Set $f_\rho(\xi) := f(\rho\xi)$ so that

$$\left(\int_{\mathbb{T}_\rho} |f(\xi)|^\ell |d\xi| \right)^{1/\ell} = \rho^{1/\ell} \left(\int_{\mathbb{T}} |f_\rho(\xi)|^\ell |d\xi| \right)^{1/\ell}. \quad (101)$$

By the trace theorem and (the non-integral version of) the Sobolev embedding theorem

$$\left(\int_{\mathbb{T}} |f_\rho(\xi)|^\ell |d\xi| \right)^{1/\ell} \leq C \|f_\rho\|_{W^{1,q}(\mathbb{D})} \quad (102)$$

with $C = C(q)$, and from the change of variable formula we get for $\rho > 0$ that

$$\|f_\rho\|_{W^{1,q}(\mathbb{D})} = \rho^{-2/q} \|f\|_{L^q(\mathbb{D}_\rho)} + \rho^{1-2/q} (\|\partial f\|_{L^q(\mathbb{D}_\rho)} + \|\bar{\partial} f\|_{L^q(\mathbb{D}_\rho)}). \quad (103)$$

Since $1/\ell - 2/q = -1$, we are left in view of (101), (102), and (103) to bound $\rho^{-1}\|f\|_{L^q(\mathbb{D}_\rho)}$ by $C\|f\|_{W^{1,q}(\mathbb{D})}$ for some $C = C(q)$. From (85) (which holds for disks as well as squares only with a different constant), we see that it is equivalent to prove the estimate:

$$\rho^{2/q-1} |f_{\mathbb{D}_\rho}| = \left| \frac{1}{\pi \rho^{3-2/q}} \int_{\mathbb{D}_\rho} f \, dm \right| \leq C \|f\|_{W^{1,q}(\mathbb{D})}, \quad 0 < \rho \leq 1, \quad (104)$$

with $C = C(q)$. Now, the Sobolev embedding theorem implies there is $C = C(q)$ such that $\|f\|_{L^{2q/(2-q)}(\mathbb{D})} \leq C\|f\|_{W^{1,q}(\mathbb{D})}$, so by Hölder's inequality

$$\left| \int_{\mathbb{D}_\rho} f \, dm \right| \leq C \rho^{3-2/q} \|f\|_{W^{1,q}(\mathbb{D})}$$

which is exactly (104) with C replaced by C/π . ■

For $J \subset \mathbb{T}$ an open arc and $\delta \in (0, 1)$ a real number, we denote by $R(J, \delta)$ the open curvilinear rectangle in \mathbb{D} (an annulus if $J = \mathbb{T}$) defined by

$$R(J, \delta) = \{z : z = \rho\xi, \xi \in J, 1 - \delta < \rho < 1\}. \quad (105)$$

Lemma 12 *For $f \in W_0^{1,2}(\mathbb{D})$ and $\rho \in (0, 1]$, it holds for every arc $I \subset \mathbb{T}_\rho$ that*

$$\left| \frac{1}{\Lambda(I)} \int_I f(\zeta) |d\zeta| \right| \leq \frac{(1-\rho)^{1/2}}{(\Lambda(I))^{1/2}} \left(\|\partial f\|_{L^2(R(J, 1-\rho))} + \|\bar{\partial} f\|_{L^2(R(J, 1-\rho))} \right), \quad (106)$$

where $J \subset \mathbb{T}$ is the arc such that $\rho J = I$.

Proof. By density it is enough to prove (106) when $f \in \mathcal{D}(\mathbb{D})$. If we write $\zeta \in I$ as $\zeta = \rho\xi$ with $\xi \in J$, we get

$$f(\zeta) = - \int_\rho^1 (\partial f(t\xi)\xi + \bar{\partial} f(t\xi)\bar{\xi}) \, dt$$

and integrating with respect to $|d\zeta| = \rho|d\xi|$ yields

$$\left| \int_I f(\zeta) |d\zeta| \right| = \left| \rho \int_J \int_\rho^1 (\partial f(t\xi)\xi + \bar{\partial} f(t\xi)\bar{\xi}) \, dt |d\xi| \right| \leq \int_{R(J, 1-\rho)} (|\partial f(t\xi)| + |\bar{\partial} f(t\xi)|) \, t dt |d\xi|.$$

Since $m(R(J, 1-\rho)) = \Lambda(I)(1-\rho^2)/2$, estimate (106) follows from the Schwarz inequality. ■

Lemma 13 *Let J be a proper open subarc of \mathbb{T} and $\delta_0 \in (0, 1)$ a real number. To each $\delta \in (0, \delta_0]$, there is $C > 0$ depending only on δ_0 and $\Lambda(J)/\delta$ such that, for all $f \in W^{1,2}(R(J, \delta))$ (cf. definition (105)),*

$$\left(\int_{\partial R(J, \delta) \times \partial R(J, \delta)} \frac{|f(t) - f(t')|^2}{(\Lambda(t, t'))^2} d\Lambda(t) d\Lambda(t') \right)^{1/2} \leq C (\|\partial f\|_{L^2(R(J, \delta))} + \|\bar{\partial} f\|_{L^2(R(J, \delta))}). \quad (107)$$

Proof. Pick $\delta \in (0, \delta_0]$, and write e^{ia}, e^{ib} for the endpoints of J with $a < b$ and $|a-b| < 2\pi$. The map $\varphi(\rho, \theta) := (\rho \cos \theta, \rho \sin \theta)$ is a diffeomorphism from $R := (1 - \delta, 1) \times (a, b)$ onto $R(J, \delta)$ satisfying $|||D\varphi||| \leq 1$ and $|||(D\varphi)^{-1}||| \leq 1/(1 - \delta) \leq 1/(1 - \delta_0)$, where $D\varphi$ indicates the derivative and $|||\cdot|||$ the operator norm. In particular φ^{-1} extends to a Lipschitz homeomorphism from $\partial R(J, \delta)$ onto ∂R with Lipschitz constant depending only on δ_0 , and by the change of variable formula it is enough to show putting $h := f \circ \varphi$ that

$$\left(\int_{\partial R \times \partial R} \frac{|h(t) - h(t')|^2}{(\Lambda(t, t'))^2} d\Lambda(t) d\Lambda(t') \right)^{1/2} \leq C(\|\partial h\|_{L^2(R)} + \|\bar{\partial} h\|_{L^2(R)}) \quad (108)$$

where the constant C depends only on $\Lambda(J)/\delta = 2\pi(b - a)/\delta$. The result now follows from the fact that when $p = 2$ and Ω is a rectangle, then the constant in (4) depends only on the ratio of sidelengths, a fact which is obvious by homogeneity. \blacksquare

The next theorem is basic to our study of G_α^p when $\alpha \in L^2(\mathbb{D})$ in section 5 but is of independent interest. It is best stated in terms of multipliers. Recall from (1) the definition of the non-tangential maximal function $\mathcal{M}_\gamma f$. We let $\mathfrak{M}^{\gamma, p}$ be the Banach space of complex valued functions on \mathbb{D} such that $\|\mathcal{M}_\gamma f\|_{L^p(\mathbb{T})} < \infty$. Remember also from Section 5.2 the Banach space \mathcal{H}^p of functions satisfying a Hardy condition.

Theorem 6 *Let $\gamma \in (0, \pi/2)$ and $p \in [1, \infty)$. If $f \in W_{0, \mathbb{R}}^{1,2}(\mathbb{D})$, multiplication by f is continuous from $\mathfrak{M}^{\gamma, p}$ into \mathcal{H}^p . More precisely, for any function g on \mathbb{D} , we have that*

$$\sup_{0 < \rho < 1} \left(\int_{\mathbb{T}_\rho} e^{pf(\xi)} |g(\xi)|^p |d\xi| \right)^{1/p} < C \|\mathcal{M}_\gamma g\|_{L^p(\mathbb{T})} \quad (109)$$

where C depends on p, γ , and on $\varepsilon > 0$ small enough that $\|\partial f\|_{L^2(Q_\varepsilon \cap \mathbb{D})} < C'/p$ whenever Q_ε is a square of sidelength ε , with C' a constant that depends only on γ .

Proof. Pick $\rho \in (0, 1)$ and $\zeta \in \mathbb{T}$. A little geometry shows that $\Gamma(\zeta, \gamma)$ contains \mathbb{T}_ρ if $\rho < \sin \gamma$, in which case we obviously have:

$$\int_{\mathbb{T}_\rho} e^{pf(\xi)} |g(\xi)|^p |d\xi| \leq \mathcal{M}_\gamma^p g(\zeta) \int_{\mathbb{T}_\rho} e^{p\ell f(\xi)} |d\xi|,$$

and averaging over $\zeta \in \mathbb{T}$ yields

$$\int_{\mathbb{T}_\rho} e^{pf(\xi)} |g(\xi)|^p |d\xi| \leq \frac{1}{2\pi} \left(\int_{\mathbb{T}} \mathcal{M}_\gamma^p g(\zeta) |d\zeta| \right) \left(\int_{\mathbb{T}_\rho} e^{p\ell f(\xi)} |d\xi| \right). \quad (110)$$

By Lemma 11 applied to e^f rather than f , with $\ell = p$ and $q := 2p/(p+1)$, we get

$$\left(\int_{\mathbb{T}_\rho} e^{p\ell f(\xi)} |d\xi| \right)^{1/p} \leq c_0 \|e^f\|_{W^{1,q}(\mathbb{D})} \quad (111)$$

for some $c_0 = C_0(p)$. Moreover, if $\tilde{f} \in W_{\mathbb{R}}^{1,2}(\mathbb{R}^2)$ indicates the extension of f by 0 off \mathbb{D} , then Proposition 1 together with Hölder's inequality imply since f is real valued:

$$\begin{aligned} \|e^f\|_{W^{1,q}(\mathbb{D})} &= \|e^f\|_{L^q(\mathbb{D})} + 2\|\partial f e^f\|_{L^q(\mathbb{D})} \leq \|e^f\|_{L^{2p}(\mathbb{D})} \left(\sqrt{\pi} + 2\|\partial f\|_{L^2(\mathbb{D})} \right) \\ &\leq \|e^{\tilde{f}}\|_{L^{2p}(Q)} \left(\sqrt{\pi} + 2\|\partial \tilde{f}\|_{L^2(Q)} \right), \end{aligned} \quad (112)$$

where $Q \supset \mathbb{D}$ is the square centered at 0 with sidelength 2. Letting $\varepsilon_0 > 0$ be so small that $\|\partial \tilde{f}\|_{L^2(Q_{\varepsilon_0})} < \pi/(32\sqrt{2}ep)$ whenever Q_{ε_0} is a square of sidelength ε_0 , we deduce from Lemma 10 that

$$\|e^{\tilde{f}}\|_{L^{2p}(Q)} \leq 4(1+e)^{1/(2p)} \exp\left(\|\tilde{f}\|_{L^2(Q)} \frac{2+\varepsilon_0}{2\varepsilon_0}\right). \quad (113)$$

Next, from Poincaré's inequality (cf. (5) with $\Omega = Q$ and $E = \partial Q$), we know there is an absolute constant c such that $\|\tilde{f}\|_{L^2(Q)} \leq c\|\partial \tilde{f}\|_{L^2(Q)}$; in fact, using that $\tilde{f} = 0$ in $Q \setminus \mathbb{D}$, a computation similar to the proof of Lemma 12 shows that $c = \sqrt{2}$ will do. Writing Q as a union of $(1+2/\varepsilon_0)^2$ cubes with disjoint interiors and sidelength less than ε_0 , we get

$$\|\tilde{f}\|_{L^2(Q)} \leq \sqrt{2}\|\partial \tilde{f}\|_{L^2(Q)} \leq \frac{\pi(1+2/\varepsilon_0)^2}{32ep} \quad (114)$$

and we conclude from (110), (111), (112), (113), and (114) that

$$\sup_{0 < \rho < \sin \gamma} \left(\int_{\mathbb{T}_\rho} e^{pf(\xi)} |g(\xi)|^p |d\xi| \right)^{1/p} \leq C_0 \|\mathcal{M}_\gamma g\|_{L^p(\mathbb{T})}$$

with

$$C_0 := \frac{c_0}{(2\pi)^{1/p}} 4(1+e)^{1/(2p)} \left(\sqrt{\pi} + \frac{\pi(1+2/\varepsilon_0)^2}{16\sqrt{2}ep} \right) \exp\left(\frac{\pi(2+\varepsilon_0)^3}{64ep\varepsilon_0^3}\right), \quad (115)$$

which is (109) for those $\rho < \sin \gamma$.

Assume next that $\rho \geq \sin \gamma$. Now $\Gamma(\zeta, \gamma)$ cuts out two disjoint open arcs on \mathbb{T}_ρ one of which is centered at $\xi = \rho\zeta$. Denote this arc by A_ξ . Its length $\Lambda(A_\xi)$ is independent of ζ and it is easy to check that $C_1(1-\rho) \leq \Lambda(A_\xi) \leq C_2(1-\rho)$ for strictly positive constants C_1, C_2 depending¹² only on γ . Put $N_\rho := 2[2\pi\rho/\Lambda(A_\xi)] + 2$, where the brackets indicate integer part, and partition \mathbb{T}_ρ into N_ρ semi open arcs of equal length, say $I_{\xi_1}, \dots, I_{\xi_{N_\rho}}$, centered at equidistant points $\xi_1, \dots, \xi_{N_\rho} \in \mathbb{T}_\rho$. Put $\zeta_j = \xi_j/\rho \in \mathbb{T}$, and consider the partition of \mathbb{T} made of the N_ρ semi open arcs $J_{\zeta_j} := I_{\xi_j}/\rho$ centered at ζ_j . By construction, we have whenever $\zeta \in J_{\zeta_j}$ that $I_{\xi_j} \subset \Gamma_{\zeta, \gamma}$. Consequently

$$\int_{I_{\xi_j}} e^{pf(\xi)} |g(\xi)|^p |d\xi| \leq \mathcal{M}_\gamma^p g(\zeta) \int_{I_{\xi_j}} e^{pf(\xi)} |d\xi|,$$

and averaging over $\zeta \in J_{\zeta_j}$ gives us

$$\int_{I_{\xi_j}} e^{pf(\xi)} |g(\xi)|^p |d\xi| \leq \frac{1}{\Lambda(J_{\zeta_j})} \left(\int_{J_{\zeta_j}} \mathcal{M}_\gamma^p g(\zeta) |d\zeta| \right) \left(\int_{I_{\xi_j}} e^{pf(\xi)} |d\xi| \right).$$

Since $\Lambda(J_{\zeta_j}) = \Lambda(I_{\xi_j})/\rho$ we deduce upon summing over j that

$$\int_{\mathbb{T}_\rho} e^{pf(\xi)} |g(\xi)|^p |d\xi| \leq \rho \left(\int_{\mathbb{T}} \mathcal{M}_\gamma^p g(\zeta) |d\zeta| \right) \sup_{1 \leq j \leq N_\rho} \left(\frac{1}{\Lambda(I_{\xi_j})} \int_{I_{\xi_j}} e^{pf(\xi)} |d\xi| \right). \quad (116)$$

¹²In fact, using the sagitta formula, a small computation yields $|A_\xi| = 2\rho\lambda$ where $\lambda \in (0, \pi/2)$ meets $\tan \lambda = (1/\kappa - 1) \tan \gamma$ with $\kappa := \left(1 - (1 - \rho^2)(1 + \tan^2 \gamma)\right)^{1/2}$. It is then easy to see from the mean value theorem that we can take $C_1 = 2 \sin(2\gamma)$ and $C_2 = 2\pi(1 + \tan^2 \gamma)^3 / \min(\tan^3 \gamma, 1 + 2 \tan^2 \gamma)$.

Recall from (105) the notation $R(J, \delta)$ and let C be the constant in Lemma 13 where $\delta_0 = 1 - \sin \gamma$ and $\Lambda(J)/\delta = C_2/(2 \sin \gamma)$; note that C depends only on γ . Because $\Lambda(J_{\zeta_j})/(1 - \rho) \leq C_2/(2 \sin \gamma)$, the arc $J'_{\zeta_j} \subset \mathbb{T}$ of length $(1 - \rho)C_2/(2 \sin \gamma)$ centered at ζ_j does contain J_{ζ_j} . Therefore $R(J_{\zeta_j}, 1 - \rho)$ is contained in $R(J'_{\zeta_j}, 1 - \rho)$ and I_{ξ_j} is contained in $I'_{\xi_j} := J'_{\zeta_j}/\rho$, hence (107) *a fortiori* implies

$$\left(\int_{I_{\xi_j} \times I_{\xi_j}} \frac{|f(t) - f(t')|^2}{(\Lambda(t, t'))^2} d\Lambda(t) d\Lambda(t') \right)^{1/2} \leq C(\|\partial f\|_{L^2(R(J'_{\zeta_j}, 1 - \rho))} + \|\bar{\partial} f\|_{L^2(R(J'_{\zeta_j}, 1 - \rho))}). \quad (117)$$

Now, it is elementary to check that $R(J'_{\zeta_j}, 1 - \rho)$ is contained in a rectangle, one side of which is tangent to \mathbb{T} at ζ_j and has length $\Lambda(J'_{\zeta_j})$ while those perpendicular to that one have length $1 - \rho + \rho\Lambda^2(J'_{\zeta_j})/8$. Thus, $R(J'_{\zeta_j}, 1 - \rho)$ is contained in a square of sidelength $C_3(1 - \rho)$ where C_3 depends only¹³ on γ . So, if we let ε_1 be so small that $\|\partial f\|_{L^2(Q_{\varepsilon_1})} < 1/(8Cep)$ whenever Q_{ε_1} is a square of sidelength ε_1 , we get since f is real valued that

$$\|\partial f\|_{L^2(R(J'_{\zeta_j}, 1 - \rho))} + \|\bar{\partial} f\|_{L^2(R(J'_{\zeta_j}, 1 - \rho))} \leq \frac{1}{4Cep}, \quad \max(\sin \gamma, 1 - \varepsilon_1/C_3) \leq \rho < 1. \quad (118)$$

Set for simplicity $\rho_0 := \max(\sin \gamma, 1 - \varepsilon_1/C_3)$. Then, from (92), (117), and (118), we see that for all subarcs $I \subset I_{\xi_j}$

$$\begin{aligned} \frac{1}{\Lambda(I)} \int_I |f - f_I| d\Lambda &\leq \left(\int_{I \times I} \frac{|f(t) - f(t')|^2}{(\Lambda(t, t'))^2} d\Lambda(t) d\Lambda(t') \right)^{1/2} \\ &\leq \left(\int_{I_{\xi_j} \times I_{\xi_j}} \frac{|f(t) - f(t')|^2}{(\Lambda(t, t'))^2} d\Lambda(t) d\Lambda(t') \right)^{1/2} \\ &\leq 1/4ep \quad \text{if } \rho_0 \leq \rho < 1. \end{aligned} \quad (119)$$

If we let $h := \text{tr}_{\mathbb{T}_\rho} f$, inequality (119) asserts that

$$M_h(I_{\xi_j}) \leq 1/4ep, \quad \rho_0 \leq \rho < 1, \quad (120)$$

with $M_h(I_{\xi_j})$ defined by (81) where we set Γ to be \mathbb{T}_ρ . By (120), (82), and Hölder's inequality, it follows that if $\rho_0 \leq \rho < 1$ then

$$\left(\frac{1}{\Lambda(I_{\xi_j})} \int_{I_{\xi_j}} e^{pf(\xi)} |d\xi| \right)^{1/p} \leq (1 + e)^{1/p} \exp \left(\left| \frac{1}{\Lambda(I_{\xi_j})} \int_{I_{\xi_j}} f(\zeta) |d\zeta| \right| \right). \quad (121)$$

In another connection, keeping in mind (118) and the inclusion $R(J_{\zeta_j}, 1 - \rho) \subset R(J'_{\zeta_j}, 1 - \rho)$, an application of (106) yields

$$\left| \frac{1}{\Lambda(I_{\xi_j})} \int_{I_{\xi_j}} f(\zeta) |d\zeta| \right| \leq \frac{(1 - \rho)^{1/2}}{(\Lambda(I_{\xi_j}))^{1/2}} \frac{1}{4Cep}. \quad (122)$$

Put $\rho_1 := \max(\rho_0, C_1/(C_1 + \pi))$, and assume for a while that $\rho \geq \rho_1$; in particular $2\pi\rho/(C_1(1 - \rho)) > 2$. Then

$$\Lambda(I_{\xi_j}) = \frac{2\pi\rho}{2 + 2[2\pi\rho/|A_\xi|]} \geq \frac{\pi\rho}{1 + 2\pi\rho/(C_1(1 - \rho))} \geq \frac{C_1(1 - \rho)}{3}, \quad (123)$$

¹³Since $\Lambda(J'_{\zeta_j}) = C_2(1 - \rho)/(2\rho)$, we can take for instance $C_3 = \max\{1 + C_2^2/(128 \sin^2 \gamma), C_2/(2 \sin \gamma)\}$, where C_2 was estimated in the previous footnote.

and using (123) in (122) we obtain

$$\left| \frac{1}{\Lambda(I_{\xi_j})} \int_{I_{\xi_j}} f(\zeta) |d\zeta| \right| \leq \frac{\sqrt{3}}{4\sqrt{C_1}Cep}, \quad \rho_1 \leq \rho < 1. \quad (124)$$

Plugging (124) in the right hand side of (121) and using (116) now gives us

$$\sup_{\rho_1 \leq \rho < 1} \left(\int_{\mathbb{T}_\rho} e^{pf(\xi)} |g(\xi)|^p |d\xi| \right)^{1/p} \leq (1+e)^{1/p} \exp \left(\frac{\sqrt{3}}{4\sqrt{C_1}Cep} \right) \|\mathcal{M}_\gamma g\|_{L^p(\mathbb{T})}$$

which is another part of (109).

It remains for us to dispose of the case $\rho \in [\sin \gamma, \rho_1)$ when the latter interval is nonempty. On the one hand, in this range of ρ , the first two inequalities in (123) imply that

$$\Lambda(I_{\xi_j}) \geq \frac{\pi \sin \gamma}{1 + 2\pi\rho_1/(C_2(1 - \rho_1))}. \quad (125)$$

On the other hand, (111), (112), (113), and (114) (which required no assumption on ρ) provide us with the estimate:

$$\left(\int_{I_{\xi_j}} e^{pf(\xi)} |d\xi| \right)^{1/p} \leq \left(\int_{\mathbb{T}_\rho} e^{pf(\xi)} |d\xi| \right)^{1/p} \leq (2\pi)^{1/p} C_0 \quad (126)$$

where C_0 is defined by (115). Therefore by (125) and (126) we have that

$$\left(\frac{1}{\Lambda(I_{\xi_j})} \int_{I_{\xi_j}} e^{pf(\xi)} |d\xi| \right)^{1/p} \leq \left(\frac{1 + 2\pi\rho_1/(C_2(1 - \rho_1))}{\pi \sin \gamma} \right)^{1/p} (2\pi)^{1/p} C_0, \quad \sin \gamma \leq \rho < \rho_1,$$

and using this in (116) achieves the proof. ■

Corollary 4 *If $w = e^s F$, where $s \in W^{1,2}(\mathbb{D})$ with $\operatorname{Re} \operatorname{tr}_{\mathbb{T}} s \equiv 0$ and $F \in H^p$, then*

$$\sup_{0 < \rho < 1} \left(\frac{1}{2\pi} \int_{\mathbb{T}_\rho} |w(\xi)|^p |d\xi| \right)^{1/p} < +\infty.$$

Proof. This follows from (41) and Theorem 6 applied with $f = e^{\operatorname{Re} s}$ and $g = e^{i\operatorname{Im} s} F$. ■

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