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# Backbone colouring: tree backbones with small diameter in planar graphs

Victor Campos\*      Frédéric Havet†      Rudini Sampaio\*      Ana Silva\*

February 22, 2013

## Abstract

Given a graph  $G$  and a spanning subgraph  $T$  of  $G$ , a backbone  $k$ -colouring for  $(G, T)$  is a mapping  $c : V(G) \rightarrow \{1, \dots, k\}$  such that  $|c(u) - c(v)| \geq 2$  for every edge  $uv \in E(T)$  and  $|c(u) - c(v)| \geq 1$  for every edge  $uv \in E(G) \setminus E(T)$ . The backbone chromatic number  $BBC(G, T)$  is the smallest integer  $k$  such that there exists a backbone  $k$ -colouring of  $(G, T)$ . In 2007, Broersma et al. [2] conjectured that  $BBC(G, T) \leq 6$  for every planar graph  $G$  and every spanning tree  $T$  of  $G$ . In this paper, we prove this conjecture when  $T$  has diameter at most four.

**Keywords:** Backbone colouring, planar graphs, Broersma's conjecture.

## 1 Introduction

All the graphs considered in this paper are simple. Let  $G = (V, E)$  be a graph, and let  $H = (V, E(H))$  be a spanning subgraph of  $G$ . A  $k$ -colouring of  $G$  is a mapping  $f : V \rightarrow \{1, 2, \dots, k\}$ . Let  $f$  be a  $k$ -colouring of  $G$ . It is a *proper colouring* if  $|f(u) - f(v)| \geq 1$ . It is a *backbone colouring* for  $(G, H)$  if  $f$  is a proper colouring of  $G$  and  $|f(u) - f(v)| \geq 2$  for all edges  $uv \in E(H)$ . The *chromatic number*  $\chi(G)$  is the smallest integer  $k$  for which there exists a proper  $k$ -colouring of  $G$ . The *backbone colouring number*  $BBC(G, H)$  is the smallest integer  $k$  for which there exists a backbone  $k$ -colouring of  $(G, H)$ .

If  $f$  is a proper  $k$ -colouring of  $G$ , then  $g$  defined by  $g(v) = 2f(v) - 1$  is a backbone  $(2k - 1)$ -colouring of  $(G, H)$  for any spanning subgraph  $H$  of  $G$ . Hence,  $BBC(G, H) \leq 2\chi(G) - 1$ . In [1, 2], Broersma et al. showed that for any integer  $k$  there is a graph  $G$  with a spanning tree  $T$  such that  $BBC(G, T) = 2k - 1$ .

The above inequality and the Four Colour Theorem implies that for any planar graph  $G$  and spanning subgraph  $H$  then  $BBC(G, H) \leq 7$ . However Broersma et al. [2] conjectured that this is not best possible if  $T$  is a tree.

**Conjecture 1** *If  $G$  is a planar graph and  $T$  a spanning tree of  $G$ , then  $BBC(G, T) \leq 6$ .*

If true this conjecture would be best possible. Broersma et al. [2] gave an example of a graph  $G^*$  with a spanning tree  $T^*$  such that  $BBC(G, T) = 6$ . See Figure 1.

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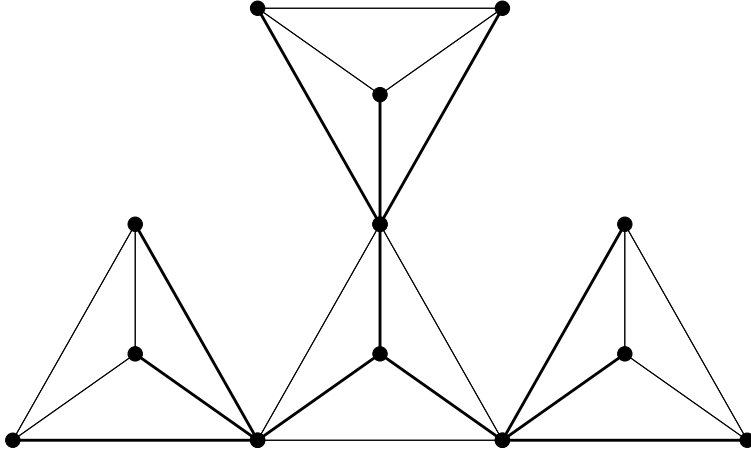


Figure 1: A planar graph  $G^*$  with a spanning tree  $T^*$  (bold edges) such that  $BBC(G^*, T^*) = 6$ .

Bu and Zhang [5] proved that, if  $G$  is a connected non-bipartite  $C_4$ -free planar graph, then there exists a spanning tree  $T$  of  $G$  such that  $BBC(G, T) = 4$ . On the other hand, Bu and Li [4] proved that, if  $G$  is a connected planar graph that is  $C_6$ -free or  $C_7$ -free and without adjacent triangles, then there exists a spanning tree  $T$  of  $G$  such that  $BBC(G, T) \leq 4$ . In [7], Wang et al. investigated backbone colouring for special graph classes such as Halin graphs, complete graphs, wheels, graphs with small maximum average degree and graphs with maximum degree 3.

The *diameter* of a graph is the maximum distance between two vertices in this graph. If  $T$  has diameter 2, then it is a *star*, that is a tree in which a vertex  $v$ , called the *center*, is adjacent to every other. If a planar graph  $G$  has a spanning star  $T$ , with center  $v$ , then  $G - v$  is an outerplanar graph which can be properly 3-coloured with  $\{1, 2, 3\}$ . Thus assigning the colour 5 to  $v$ , we obtain a backbone 5-colouring of  $(G, T)$ . This result may be extended if  $G$  has a spanning tree with diameter at most 3.

**Proposition 2** *Let  $G$  be a planar graph with a spanning tree  $T$ . If  $T$  has diameter at most three, then  $BBC(G, T) \leq 5$ .*

**Proof.** Free to add some edges, we may assume that  $G$  is triangulated. If  $T$  has diameter at most 3, then there exists two adjacent vertices  $x$  and  $y$  such that all edges of  $T$  are incident to  $x$  or  $y$ . Let  $z_1, \dots, z_p$  be the common neighbours of  $x$  and  $y$ , ordered in clockwise order around  $x$  (and so in anti-clockwise order around  $y$ ). We consider an embedding of  $G$  with outer face  $xyz_1$ .

For  $1 \leq i \leq p - 1$ , let  $G_i$  be the graph induced by the vertices in the cycle  $xz_iyz_{i+1}x$  and inside, and let  $H_i = G_i \setminus \{x, y\}$ . Since  $G$  is triangulated, all the vertices are in at least one  $G_i$ . Furthermore, every  $H_i$  is outerplanar, and every vertex in  $V(H_i) \setminus \{z_i, z_{i+1}\}$  is adjacent to exactly one of  $x, y$ .

We shall now define a backbone 5-colouring  $c$  of  $(G, T)$ .

First, we set  $c(x) = 1$ ,  $c(y) = 5$  and  $c(z_1) = 3$ . Next, we extend this colouring to the  $H_i$  one after another. Since  $H_i$  is outerplanar, it is 3-colourable. Let  $c_i$  be a proper 3-colouring of  $H_i$  in  $\{2, 3, 4\}$  such that  $c_i(z_i) = c(z_i)$  and  $c_i(z_{i+1}) \in \{3, 4\}$  if  $z_{i+1}x \in E(T)$  and  $c_i(z_{i+1}) \in \{2, 3\}$

if  $z_{i+1}y \in E(T)$ . We set  $c(z_{i+1}) = c_i(z_{i+1})$ , and for every vertex  $v$  of  $V(H_i) \setminus \{z_i, z_{i+1}\}$ , we define

- $c(v) = c_i(v)$ , if  $c_i(v) = 3$ , or  $c_i(v) = 2$  and  $vy \in E(T)$ , or  $c_i(v) = 4$  and  $vx \in E(T)$ ;
- $c(v) = 5$ , if  $c_i(v) = 2$  and  $vx \in E(T)$ ;
- $c(v) = 1$ , if  $c_i(v) = 4$  and  $vy \in E(T)$ .

It is easy to check that  $c$  is a backbone 5-colouring of  $(G, T)$ . □

**Remark 3** Notice that the proof of Proposition 2 contains an explicit polynomial time algorithm to obtain a backbone 5-colouring of  $(G, T)$  when  $G$  is planar and  $T$  has diameter at most 3, since 3-colourings of outerplanar graphs can be obtained in polynomial time [6]. Proposition 2 is best possible, because when  $G$  is a complete graph on four vertices and  $T$  a spanning star of  $G$ ,  $BBC(G, T) = 5$ .

In this paper, we settle Conjecture 1 for tree with diameter at most 4.

**Theorem 4** *Let  $G$  be a planar graph with a spanning tree  $T$ . If  $T$  has diameter at most 4, then  $BBC(G, T) \leq 6$ .*

Note that this result is best possible as the tree  $T^*$  in the above example has diameter 4.

In the next section, we outline the proof of Theorem 4 whose details are postponed to Section 3.

## 2 The proof

We denote by  $Z_6$  the set  $\{1, 2, 3, 4, 5, 6\}$  and, for any integer  $a \in Z_6$ , we denote by  $[a]$  the set  $\{a - 1, a, a + 1\} \cap Z_6$ .

Let  $G = (V, E)$  be a planar graph and  $T$  a spanning tree of  $G$  with diameter at most 4.  $T$  has a vertex  $r$  such that every vertex is at distance two from it in  $T$ . We call such a vertex the *root* of  $T$ . A vertex of  $V \setminus \{r\}$ , is a *twig* if it is adjacent to  $r$  in  $T$  and a *leaf* otherwise.

We shall prove a slightly stronger result than the one of Theorem 4.

**Theorem 5**  *$(G, T)$  admits a backbone colouring in  $Z_6$  such that the root is assigned 1.*

**Proof.** In the remaining, by  $(G, T)$ -colouring, one should understand a backbone 6-colouring of  $(G, T)$  such that  $r$  is assigned 1.

We will prove it by considering a minimum counterexample  $(G, T)$  with respect to its number of vertices. An edge of  $E \setminus E(T)$  is said to be *thin*. Free to add some more thin edges, we may assume that  $G$  is triangulated.

If  $T$  has a unique twig, then it has diameter 2, and we have the result by the proof of Proposition 2. (The root corresponds to  $x_1$  and the twig to  $x_2$ .) Hence  $T$  has at least two twigs. We consider an embedding of  $G$  in the plane such that the outer face contains  $r$  and a minimum number of thin edges.

The *interior* (resp. *exterior*) of a cycle  $C$ , denoted  $C^{int}$  (resp.  $C^{ext}$ ) is the subgraph of  $G$  induced by  $C$  and the vertices inside  $C$  (resp. outside  $C$ ).

Let  $e$  be a thin edge. The graph  $T \cup \{e\}$  has a unique cycle  $C_e$  (which contains  $e$ ). The edge  $e$  is *overstepping* if there is a vertex inside  $C_e$ . In other words,  $V(C_e^{int}) \neq \emptyset$ . Let  $O$  be the set of overstepping edges. There is a partial order  $\leq$  on  $O$  defined as follows:  $e_1 \leq e_2$  if  $e_1 = e_2$  or  $e_1$  is inside  $C_{e_2}$  (Lemma 6 proves that  $\leq$  is a partial order). Observe that the Hasse diagram of such a partial order is a set of at most two disjoint trees, each one rooted at an overstepping thin edge in the outer face. Indeed, it is easy to see that every overstepping edge  $e$  that is not maximal has a unique *successor* for  $\leq$  (i.e. overstepping edge  $f$  such that if  $e \leq e' \leq f$  then  $e' \in \{e, f\}$ ). This successor is one of the two edges of the face containing  $e$  contained in  $C_e^{ext}$ . Furthermore, every edge  $e$  has at most two *predecessors* for  $\leq$ : the two other edges of the face containing  $e$  contained in  $C_e^{int}$ .

The idea of the proof is to find a “good” overstepping edge  $e$ , such that a backbone 6-colouring of the graph induced by  $V(C_e^{ext})$  (which exists by minimality of  $(G, T)$ ) can be extended to  $V(C_e^{int})$  to obtain a  $(G, T)$ -colouring. This will be a contradiction.

Natural candidates for such a good edge are overstepping edges  $e$  which are *minimal* for  $\leq$  (i.e. such that  $e' \leq e$  implies  $e' = e$ ) or their successors. However we will need to consider a more precise partial ordering. If there are two overstepping edges  $e_3 = rv_1$  and  $e_4 = v_1v_2$  such that  $v_1$  and  $v_2$  are leaves and  $e_4 \not\leq e_3$ , (i.e.  $e_4$  is not inside  $C_{e_3}$ ), then we would like to have  $e_3$  smaller than  $e_4$  in the ordering.

This leads to the following binary relation  $\preceq$  between overstepping edges:  $e_1 \preceq e_2$  if  $e_1 \leq e_2$  or there exist two edges  $e_3 = rv_1$  and  $e_4 = v_1v_2$  such that  $v_1$  and  $v_2$  are leaves,  $e_4 \not\leq e_3$ ,  $e_1 \leq e_3$  and  $e_4 \leq e_2$ . In Lemma 6, we prove that  $\preceq$  is a partial order.

In the remainder of the paper, we will only consider the partial order  $\preceq$ . Hence the terms minimal, predecessor, successor, and so on refer to  $\preceq$ .

We first show some properties of minimal overstepping edges and deduce in Lemma 14 that if  $e$  is a minimal overstepping edge, then  $C_e^{int}$  is isomorphic to one of the graphs  $A_1$ ,  $A_2$  or  $A_3$ , depicted in Figure 2. In addition, if  $C_e^{int} = A_1$ , then  $rv_1 \in E(G)$ .

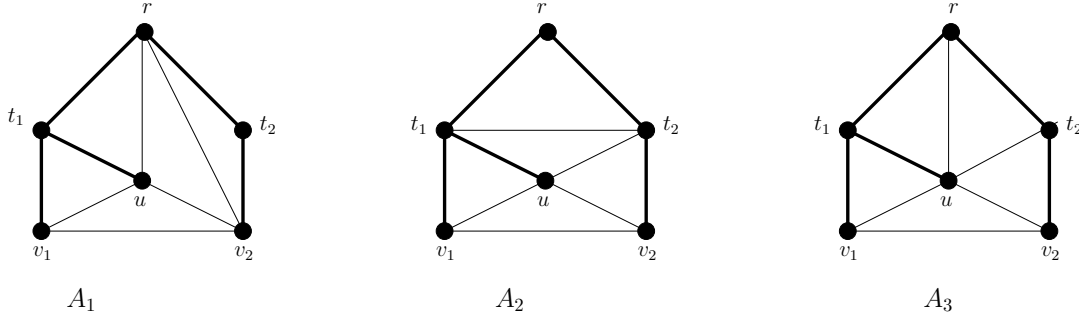


Figure 2: Configurations  $A_1$ ,  $A_2$  and  $A_3$

As any ordering,  $\preceq$  may be decomposed into levels. The first level  $L_1$  the *maximal* edges for  $\preceq$  (i.e. such that  $e \preceq e'$  implies  $e' = e$ ). This level contains at most two edges, depending on the number of thin overstepping edges in the outer face. Then, for every  $j \geq 1$ , the level  $L_{j+1}$  is the set of predecessors of elements of  $L_j$ . The *depth* of  $\preceq$ , denoted  $D$ , is the maximum  $j$  such that  $L_j$  is not empty. An overstepping edge of  $L_D$  is said to be *ultimate*. An edge of  $L_{D-1}$  having at least one (ultimate) predecessor is said to be *penultimate*. An edge of  $L_{D-2}$  having at least one penultimate predecessor is said to be *antepenultimate*.

If  $f$  is a penultimate edge, then it has one or two predecessors. Furthermore each of this predecessors  $e$  is ultimate and so minimal. Thus  $C_e^{int}$  is isomorphic to  $A_1$ ,  $A_2$  or  $A_3$ . Analyzing all possible cases, we show (Corollary 17) that, if  $f$  is a penultimate edge, then  $C_f^{int}$  is isomorphic to  $B_1$  or  $B_2$ , and that moreover  $rv_1 \in E(G)$  and, if  $C_f^{int} = B_2$ ,  $rv_3 \notin E(G)$ .

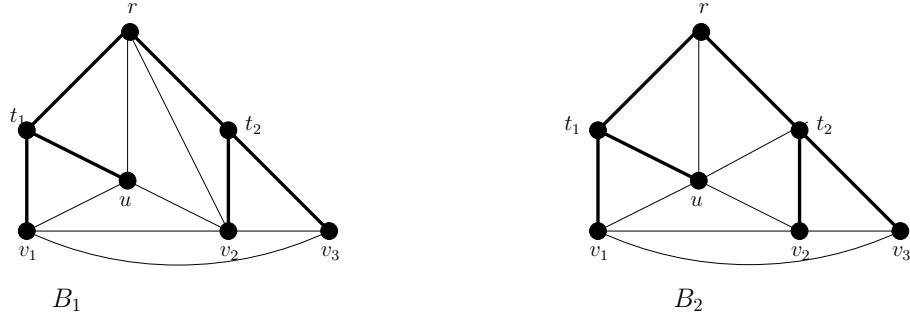


Figure 3: Configurations  $B_1$  and  $B_2$

Now if  $g$  is an antepenultimate edge, then it has one or two predecessors. Furthermore at least one of its predecessors  $f$  is penultimate (and so  $C_f^{int}$  is isomorphic to  $B_1$  or  $B_2$ ), and the other predecessor  $f'$  (if it exists) is either penultimate (so  $C_{f'}^{int}$  is isomorphic to  $B_1$  or  $B_2$ ) or ultimate (so  $C_{f'}^{int}$  is isomorphic to  $A_1$ ,  $A_2$  or  $A_3$ ). Analyzing all the possible cases again, we show that there are no antepenultimate edges (Corollary 24).

Now, suppose that  $G$  contains at least one overstepping edge. If  $e$  is a minimal edge, then  $C_e^{int}$  is isomorphic to some configuration  $A_i$ . In any of these cases, there is at least one face containing the root and only one thin edge. Therefore, the partial order considered contains a unique maximal overstepping edge  $e_0$ . Furthermore, since  $e_0$  is not antepenultimate,  $C_{e_0}^{int}$  must be isomorphic to one of the  $A_i$  or  $B_j$  configurations. We get a contradiction as the outer face contains  $r$  and the endpoints of  $e_0$  and  $e_0$  is the unique thin edge in this configuration and  $T$  would not be a tree.

We proved that  $G$  contains no overstepping edge. If the outer face of  $G$  contains only one thin edge, then  $G$  contains three vertices and the diameter of  $G$  is 2. If the outer face contains two thin edges  $e_1$  and  $e_2$ , then one thin edge (say  $e_1$ ) is adjacent to  $r$ , since  $r$  is on the outer face, and the other (say  $e_2$ ) is adjacent to a twig  $t$  while both are incident to a vertex  $v$  in the outer face. Now, both  $r$  and  $v$  have a twig  $v'$  as a common neighbour through edges of  $T$  as  $T$  is a spanning tree. Since neither  $e_1$  nor  $e_2$  are overstepping, then  $V(G) = \{r, t, v, v'\}$  and  $G$  has diameter 3. Both of these cases are solved using Proposition 2 and both can give colour 1 to the root, a contradiction.  $\square$

### 3 The details

**Lemma 6** *The binary relation  $\preceq$  is a partial order.*

**Proof.** Let  $e_0, e_1, e_2$  be overstepping edges. At first, we prove that  $\preceq$  is a partial order. By the definition, it is clearly reflexive. Now suppose that  $e_1 \preceq e_2$  and  $e_2 \preceq e_1$ . Then, by the

definition,  $e_1$  is inside  $C_{e_2}$  and  $e_2$  is inside  $C_{e_1}$ . Clearly, this is only possible if  $e_1 = e_2$ . Then,  $\leq$  is antisymmetric. Now suppose that  $e_0 \leq e_1 \leq e_2$ . Then  $e_0$  is inside  $C_{e_1}$  and  $e_1$  is inside  $C_{e_2}$ . This implies that  $C_{e_1}$  is inside  $C_{e_2}$ , and consequently  $e_0$  is inside  $C_{e_2}$ . Then  $e_0 \leq e_2$  and  $\leq$  is transitive.

Now we prove that  $\preceq$  is a partial order. Since  $e_1 \leq e_2$  implies that  $e_1 \preceq e_2$ , then  $\preceq$  is reflexive.

We claim that  $\preceq$  is antisymmetric. To prove this, suppose that  $e_1 \preceq e_2$  and  $e_2 \preceq e_1$ . If  $e_1 \leq e_2$  and  $e_2 \leq e_1$ , then  $e_1 = e_2$ , since  $\leq$  is antisymmetric. So, assume that  $e_1 \not\leq e_2$  and  $e_2 \leq e_1$ . Since  $e_1 \preceq e_2$ , we have by the definition of  $\preceq$  that there exist two overstepping edges  $e_3 = rv_1$  and  $e_4 = v_1v_2$  such that  $v_1$  and  $v_2$  are leaves,  $e_4 \not\leq e_3$ ,  $e_1 \leq e_3$  and  $e_4 \leq e_2$ . Then  $e_4 \leq e_2 \leq e_1 \leq e_3$  and, by transitivity,  $e_4 \leq e_3$ , a contradiction.

Now assume that  $e_1 \not\leq e_2$  and  $e_2 \not\leq e_1$ . Since  $e_1 \preceq e_2$  and  $e_2 \preceq e_1$ , we have by the definition of  $\preceq$  that there exist four overstepping edges  $e_3 = rv_1$ ,  $e_4 = v_1v_2$ ,  $e_5 = rw_1$  and  $e_6 = w_1w_2$  such that  $v_1, v_2, w_1, w_2$  are leaves,  $e_4 \not\leq e_3$ ,  $e_1 \leq e_3$ ,  $e_4 \leq e_2$ ,  $e_5 \not\leq e_6$ ,  $e_2 \leq e_5$  and  $e_6 \leq e_1$ . By transitivity,  $e_6 \leq e_3$  and  $e_4 \leq e_5$ . If  $v_1 = w_1$ , then  $e_3 = e_5$  and  $e_4 \leq e_5 = e_3$ , a contradiction. Then,  $v_1 \neq w_1$  and  $w_1$  is inside  $C_{e_3}$ , since  $e_6 \leq e_3$ . By planarity,  $rw_1 = e_5$  is also inside  $C_{e_3}$ . Then  $e_5 \leq e_3$  and then  $e_4 \leq e_5 \leq e_3$ , a contradiction since  $e_4 \not\leq e_3$ .

We then conclude that  $e_1 \preceq e_2$  and  $e_2 \preceq e_1$  implies that  $e_1 \leq e_2$  and  $e_2 \leq e_1$ , and consequently,  $e_1 = e_2$ , proving that  $\preceq$  is antisymmetric.

We claim that  $\preceq$  is transitive. To prove this, suppose that  $e_0 \preceq e_1$  and  $e_1 \preceq e_2$ . If  $e_0 \leq e_1$  and  $e_1 \leq e_2$ , then by transitivity  $e_0 \leq e_2$  and consequently  $e_0 \preceq e_2$ . So, assume that  $e_0 \leq e_1$  and  $e_1 \not\leq e_2$ . Since  $e_1 \preceq e_2$ , we have by the definition of  $\preceq$  that there exist two overstepping edges  $e_3 = rv_1$  and  $e_4 = v_1v_2$  such that  $v_1$  and  $v_2$  are leaves,  $e_4 \not\leq e_3$ ,  $e_1 \leq e_3$  and  $e_4 \leq e_2$ . By transitivity  $e_0 \leq e_3$  and then  $e_3$  and  $e_4$  also satisfy the condition to conclude that  $e_0 \preceq e_2$ .

Now assume that  $e_0 \not\leq e_1$  and  $e_1 \not\leq e_2$ . Since  $e_0 \preceq e_1$  and  $e_1 \preceq e_2$ , we have by the definition of  $\preceq$  that there exist four overstepping edges  $e_3 = rv_1$ ,  $e_4 = v_1v_2$ ,  $e_5 = rw_1$  and  $e_6 = w_1w_2$  such that  $v_1, v_2, w_1, w_2$  are leaves,  $e_4 \not\leq e_3$ ,  $e_0 \leq e_3$ ,  $e_4 \leq e_1$ ,  $e_5 \not\leq e_6$ ,  $e_1 \leq e_5$  and  $e_6 \leq e_2$ . By transitivity,  $e_4 \leq e_5$ . If  $v_1 = w_1$ , then  $e_3 = e_5$  and  $e_4 \leq e_5 = e_3$ , a contradiction. Then,  $v_1 \neq w_1$  and  $v_1$  is inside  $C_{e_5}$ , since  $e_4 \leq e_5$ . By planarity,  $rv_1 = e_3$  is also inside  $C_{e_5}$ . Then  $e_3 \leq e_5$  and then  $e_0 \leq e_5$ . Thus,  $e_5$  and  $e_6$  also satisfy the condition to conclude that  $e_0 \preceq e_2$ . In other words,  $\preceq$  is transitive.  $\square$

**Lemma 7** *Let  $x$  be a vertex of  $G$ . If  $d_T(x) = 1$ , then  $d_G(x) \geq 4$ .*

**Proof.** Suppose for a contradiction that  $d_T(x) = 1$  and  $d_G(x) \leq 3$ . By minimality of  $(G, T)$ , there is a  $(G - x, T - x)$ -colouring  $c$ . At  $x$ , at most 3 colours are forbidden by its neighbour in  $T$  and at most 2 colours are forbidden by its two other neighbours. So one colour of  $Z_6$  is still available to colour the vertex  $x$ . Hence one can extend  $c$  to  $(G, T)$ , a contradiction.  $\square$

### 3.1 Minimal overstepping edges

**Lemma 8** *Let  $e = uv$  be a minimal overstepping edge. Then there are at most two vertices inside  $C_e$ . Moreover if there are two, then they are adjacent in  $T$  and one of them is a twig and the other is a leaf.*

**Proof.** Since  $G$  is triangulated,  $uv$  is incident to two triangular faces, one of which, say  $F$ , is included in  $C_e^{int}$ . Let  $w$  be the third vertex incident to  $F$ . Let  $P$  be the path joining  $u$  to  $v$

in  $T$  and  $Q$  be the path joining  $w$  to  $P$  in  $T$ . Since  $T$  has diameter 4 and  $r$  is on the outer face, then  $Q$  has length at most 2.

Then  $C_e^{int}$  is divided into at most three regions:  $F$ ,  $C_{uw}^{int}$  and  $C_{vw}^{int}$  (the region  $C_{uw}^{int}$  or  $C_{vw}^{int}$  may not exist if  $uw \in E(T)$  or  $vw \in E(T)$  respectively). As  $F$  is a face, its interior is empty, and there are no vertices inside  $C_{uw}^{int}$  and  $C_{vw}^{int}$  because  $uw$  and  $vw$  are not overstepping since  $e$  is minimal. Hence the only possible vertices inside  $C_e$  are those of  $Q$ . Therefore there are at most two vertices inside  $C_e$  as  $Q$  has length at most 2.

Furthermore, if there are two vertices inside  $C_e$ , they must be adjacent as they are in  $Q$ . In addition, since  $r$  is on the outer face, none of these vertices is the root and thus one of them is a twig and the other is a leaf.  $\square$

**Lemma 9** *No minimal overstepping edge joins two leaves adjacent to a same twig.*

**Proof.** Suppose for a contradiction that an edge  $e = uv$  joins two leaves adjacent to a same twig  $t$ . Then  $C_e = tuvt$ . The root  $r$  is not in  $C_e^{int}$  as it is on the outer face. So by Lemma 8 and because  $G$  is triangulated,  $C_e^{int}$  is a  $K_4$  and there is a unique vertex  $x$  inside  $C_e$ . Hence,  $x$  contradicts Lemma 7.  $\square$

**Lemma 10** *No minimal overstepping edge joins two twigs.*

**Proof.** Suppose for a contradiction that two twigs  $s$  and  $t$  are joined by a minimal edge  $e$ . Then  $C_e = rstr$ . If there is a unique vertex  $u$  inside  $C_e^{int}$ , then  $u$  contradicts Lemma 7. So by Lemma 8, we may assume that the interior of  $C_e$  contains two adjacent vertices  $u_1$  and  $u_2$  and that  $u_1$  is a twig and  $u_2$  a leaf. By minimality of  $(G, T)$ , there is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$ . Set  $c(u_2) = 2$  and choose  $c(u_1)$  in  $Z_6 \setminus \{1, 2, 3, c(s), c(t)\}$ . This yields a  $(G, T)$ -colouring, a contradiction.  $\square$

**Lemma 11** *No minimal overstepping edge joins the root and a leaf.*

**Proof.** Suppose for a contradiction that a minimal edge  $e$  joins the root  $r$  and a leaf  $v$ . Let  $t$  be the twig adjacent to  $v$ .

Suppose there is a unique vertex  $u$  inside  $C_e$ . Then this vertex has only 3 neighbours, and  $d_T(u) = 1$ . This contradicts Lemma 7. Hence by Lemma 8, we may assume that there are two adjacent vertices  $u_1$  and  $u_2$  inside  $C_e$ . Without loss of generality,  $u_2$  is a leaf and  $u_1$  is a twig. By Lemma 7,  $d_G(u_2) \geq 4$ , so  $N_G(u_2) = \{u_1, r, v, t\}$ . By minimality of  $G$ , there is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$ . Let  $c(u_2)$  be a colour in  $\{2, 3\} \setminus \{c(v), c(t)\}$ . (Such a colour exists because  $|c(v) - c(t)| \geq 2$ .) Now by planarity,  $u_1$  has at most one neighbour  $x$  in  $\{v, t\}$  as  $ru_2$  is an edge. The set of forbidden colours in  $u_1$  is  $I = [1] \cup [c(u_2)] \cup \{c(x)\}$  which has cardinality at most 5 by the choice of  $c(u_2)$ . Hence assigning to  $u_1$  a colour  $c(u_1)$  in  $Z_6 \setminus I$ , we obtain a  $(G, T)$ -colouring, a contradiction.  $\square$

**Lemma 12** *No minimal overstepping edge joins a leaf and a twig.*

**Proof.** Suppose for a contradiction that a minimal overstepping edge  $e = sv$  joins a twig  $s$  and a leaf  $v$ . Then  $C_e = svtrs$ . By Lemma 8 there are at most two vertices inside  $C_e$ .



Suppose that there is a unique vertex  $u$  inside  $C_e$ . As  $d_T(u) = 1$ , by Lemma 7,  $d_G(u) \geq 4$ . So  $N_G(u) = \{r, s, t, v\}$ . Note that  $rv$  or  $st$  is not an edge, by planarity. Then, removing  $u$  and contracting  $rv$  or  $st$ , we find by the minimality of  $G$  a  $(G - u, T - u)$ -colouring  $c$  such that  $c(v) = 1$  or  $c(s) = c(t)$ . Since the set of forbidden colours for  $u$  has at most 5 colours, one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction.

Hence by Lemma 8, inside  $C_e$  there are a twig  $u_1$  and leaf  $u_2$  which are adjacent in  $T$ . As  $d_T(u_2) = 1$ ,  $d_G(u_2) \geq 4$  by Lemma 7.

- Suppose first that  $r$  is not adjacent to  $u_2$ . By Lemma 7,  $d_G(u_2) \geq 4$ . So  $N_G(u_2) = \{u_1, s, t, v\}$ .

Hence  $u_1$  is not adjacent to  $v$  by planarity. By minimality of  $(G, T)$ , there is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$ . Assign to  $u_2$  a colour  $c(u_2)$  in  $\{1, 2\} \setminus c(v)$ . Observe that it is valid since  $s$  and  $t$  are not coloured in  $\{1, 2\}$ . Then the set of forbidden colours in  $u_1$  is included in  $\{1, 2, 3, c(s), c(t)\}$  and so has cardinality at most 5. Hence one can extend  $c$  into a  $(G, T)$ -colouring a contradiction.

- Suppose now that  $r$  is adjacent to  $u_2$ .

By planarity,  $u_1$  is adjacent to at most one vertex  $w$  in  $\{s, t\}$ . By minimality of  $(G, T)$ , there is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$ .

If  $c(v) \neq 2$ , then set  $c(u_2) = 2$ . This it is valid since  $s$  and  $t$  are not coloured 2. Then the set of forbidden colours in  $u_1$  is included in  $\{1, 2, 3, c(v), c(w)\}$  and so has cardinality at most 5. Hence one can extend  $c$  into a  $(G, T)$ -colouring a contradiction. Hence we may assume that  $c(v) = 2$ .

If no neighbour of  $u_2$  is coloured 6, then set  $c(u_2) = 6$ . The set of forbidden colours in  $u_1$  is then  $\{1, 2, 5, 6, c(w)\}$  and so one can extend  $c$  into a  $(G, T)$ -colouring a contradiction. Hence we may assume that a neighbour  $y$  of  $u_2$  is coloured 6.

If no neighbour of  $u_2$  is coloured 3, then set  $c(u_2) = 3$ . The set of forbidden colours in  $u_1$  is then  $\{1, 2, 3, 4, c(w)\}$  and so one can extend  $c$  into a  $(G, T)$ -colouring a contradiction. Hence we may assume that a neighbour  $y$  of  $u_2$  is coloured 3. But this neighbour cannot be  $t$  since  $c(v) = 2$ . Thus  $c(s) = 3$  and  $c(t) = 6$ .

If  $w = s$ , that is if  $u_1$  is not adjacent to  $t$ , then setting  $c(u_1) = 6$  and  $c(u_2) = 4$  yields a  $(G, T)$ -colouring, a contradiction.

If  $w = t$ , then setting  $c(u_1) = 3$  and  $c(u_2) = 5$  yields a  $(G, T)$ -colouring, a contradiction.

□

**Lemma 13** *If  $e$  is a minimal overstepping edge joining two leaves, then there is one vertex inside  $C_e$ .*

**Proof.** Let  $e = v_1v_2$  and for  $i = 1, 2$ , let  $t_i$  be the twig adjacent to  $v_i$ . By Lemma 9,  $t_1 \neq t_2$ . Since  $e$  is minimal and  $G$  is triangulated,  $u_2v_1, u_2v_1 \in E(G)$ .

Suppose for a contradiction that more than one vertex is inside  $C_e$ . Then, by Lemma 8, inside  $C_e$ , there are a twig  $u_1$  and a leaf  $u_2$  which are adjacent in  $T$ . Moreover, by Lemma 7,  $d_G(u_2) \geq 4$  and so  $d_G(u_1) \leq 5$ .

Let us first suppose that  $ru_2$  is not an edge. By symmetry, we may assume that  $u_1v_1$  is not an edge. Set  $G' = (G - \{u_1, u_2\}) \cup \{rv_1, rv_2\}$ . By minimality of  $(G, T)$ , there is a  $(G', T - \{u_1, u_2\})$ -colouring, which is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$  such that  $c(v_1) \neq 1$  and  $c(v_2) \neq 1$ . Then setting  $c(u_2) = 1$  and colouring  $u_1$  with a colour in  $Z_6 \setminus \{1, 2, c(t_1), c(t_2), c(v_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence we may assume that  $ru_2 \in E(G)$ . Then, since  $e$  is minimal,  $u_1v_1$  is not an edge. By symmetry, we may assume that  $ru_2$  is inside the cycle  $rt_1v_1u_2u_1r$ . Thus  $N(u_1) \subset \{r, t_2, v_2, u_2\}$ .

Assume now that  $rv_1$  is not an edge. Let  $(G', T')$  be the graph pair obtained from  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$  by identifying  $r$  and  $v_1$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring which is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$  such that  $c(v_1) = c(r) = 1$ . If  $c(v_2) \neq 2$ , then setting  $c(u_2) = 2$  and colouring  $u_1$  with a colour in  $Z_6 \setminus \{1, 2, 3, c(t_2), c(v_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. If  $c(v_2) = 2$ , then  $c(t_2) \geq 4$ . If  $c(t_1) \neq 3$ , then colour  $u_2$  with 3 and  $u_1$  with some colour in  $\{5, 6\} \setminus \{c(t_2)\}$ ; otherwise, colour  $u_1$  with 3 and  $u_2$  with a colour in  $\{5, 6\} \setminus \{c(t_2)\}$ . In both cases, we obtain a  $(G, T)$ -colouring, a contradiction. Hence we may assume that  $rv_1 \in E(G)$ .

Assume that  $rv_2$  is not an edge. Let  $(G', T')$  be the graph pair obtained from  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$  by identifying  $r$  and  $v_2$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring which is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$  such that  $c(v_2) = c(r) = 1$ . If there is a colour  $\alpha \in \{2, 3, 6\}$  which does not appear on the neighbourhood of  $u_2$ , then setting  $c(u_2) = \alpha$  and colouring  $u_1$  with a colour in  $Z_6 \setminus (\{1, 2, c(t_2)\} \cup [\alpha])$ , we obtain a  $(G, T)$ -colouring, a contradiction. So all the colours of  $\{2, 3, 6\}$  appear on the neighbourhood of  $u_2$ . Necessarily, in this case,  $u_2$  is adjacent to  $v_1$ ,  $t_1$  and  $t_2$  and  $c(v_1) = 2$ ,  $c(t_1) = 6$  and  $c(t_2) = 3$ . Then setting  $c(u_2) = 4$  and  $c(u_1) = 6$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence we may assume that  $rv_2 \in E(G)$ .

We now distinguish several cases depending on the position of  $rv_1$  and  $rv_2$  regarding  $C_e$ .

1. Assume first that  $rv_1$  and  $rv_2$  are in  $C_e^{ext}$ . Then  $t_1t_2$  is not an edge by planarity.

Let  $(G', T')$  be the graph pair obtained from  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$  by identifying  $t_1$  and  $t_2$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring which is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$  such that  $c(t_1) = c(t_2) = \alpha$ . If  $2 \notin \{c(v_1), c(v_2)\}$ , then setting  $c(u_2) = 2$  and colouring  $u_1$  with a colour in  $Z_6 \setminus \{1, 2, 3, \alpha, c(v_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $2 \in \{c(v_1), c(v_2)\}$ , so  $\alpha \geq 4$ .

If  $\{c(v_1), c(v_2)\} \neq \{2, 3\}$ , then setting  $c(u_2) = 3$  and colouring  $u_1$  with a colour in  $\{5, 6\} \setminus \{\alpha, c(v_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $\{c(v_1), c(v_2)\} = \{2, 3\}$ , so  $\alpha \geq 5$ .

If  $c(v_2) \neq 3$  or  $u_1v_2 \notin E(G)$ , then setting  $c(u_1) = 3$  and colouring  $u_2$  with a colour in  $\{5, 6\} \setminus \{\alpha\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $c(v_2) = 3$  and  $u_1v_2 \in E(G)$ . By planarity, this implies that  $u_2t_2$  is not an edge.

Observe that at least one of the two edges  $rv_1$  and  $rv_2$  is not overstepping otherwise one of them would be smaller than  $e$  in the order  $\preceq$ .

If  $rv_1$  is not overstepping, then the interior of  $rt_1v_1$  is empty. Hence  $N_G(t_1) = \{r, v_1, u_2\}$ . Setting  $c(u_1) = 4$ ,  $c(u_2) = 6$  and recolouring  $t_1$  with 5, we obtain a  $(G, T)$ -colouring, a contradiction.

If  $rv_2$  is not overstepping, then the interior of  $rt_2v_2$  is empty. Hence  $N_G(t_2) = \{r, u_1, v_2\}$ . Setting  $c(u_1) = 6$ ,  $c(u_2) = 4$  and recolouring  $t_2$  with 5, we obtain a  $(G, T)$ -colouring, a

contradiction.

2. Assume that  $rv_1$  and  $rv_2$  are in  $C_e^{int}$ . Then  $N_G(u_1) = \{r, u_2, v_2\}$ . By minimality of  $(G, T)$ , there is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$ . Colour  $u_2$  with a colour  $c(u_2)$  in  $\{2, 3, 6\} \setminus \{c(v_1), c(v_2)\}$ . Then the set of forbidden colours in  $u_1$  is  $\{1, 2, c(v_2)\} \cup [c(u_2)]$  which has cardinality at most 5 because  $\{1, 2\} \cup [c(u_2)]$  has cardinality at most 4. Hence one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction.

3. Assume that  $rv_1$  is in  $C_e^{int}$  and  $rv_2$  is in  $C_e^{ext}$ .

Assume that  $d_G(u_2) = 5$ , so  $N_G(u_2) = \{r, u_1, v_1, v_2, t_2\}$  and  $N_G(u_1) = \{r, t_2, u_2\}$ . By minimality of  $(G, T)$ , there is a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$ . If one can colour  $u_2$  with a colour in  $\{2, 3, 6\}$ , then  $\{1, 2\} \cup [c(u_2)]$  has cardinality at most 4 and so at most 5 colours are forbidden for  $u_1$ . Hence one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction. So we may assume  $\{c(t_2), c(v_1), c(v_2)\} = \{2, 3, 6\}$ . If  $c(t_2) = 6$ , then setting  $c(u_1) = 3$  and  $c(u_2) = 5$ , we obtain a  $(G, T)$ -colouring, a contradiction. If  $c(t_2) \neq 6$ , then setting  $c(u_1) = 6$  and  $c(u_2) = 4$ , we obtain a  $(G, T)$ -colouring, a contradiction.

Henceforth we may assume that  $d_G(u_2) = 4$ , so  $N_G(u_2) = \{r, u_1, v_1, v_2\}$  and  $N_G(u_1) = \{r, t_2, v_2, u_2\}$ .

If  $\{c(v_1), c(v_2)\} \neq \{2, 3\}$ , then one can colour  $u_2$  with a colour in  $\{2, 3\}$  and  $u_1$  with a colour in  $\{5, 6\} \setminus \{c(t_2), c(v_2)\}$  to obtain a  $(G, T)$ -colouring, a contradiction.

If  $\{c(v_1), c(v_2)\} = \{2, 3\}$ , then colouring  $u_1$  with a colour  $c(u_1)$  in  $\{4, 6\} \setminus \{c(t_2)\}$  and  $u_2$  with the colour in  $\{4, 6\} \setminus \{c(u_1)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction.

4. Assume  $rv_2$  is in  $C_e^{int}$  and  $rv_1$  is in  $C_e^{ext}$ . Then  $N_G(u_1) = \{r, u_2, v_2\}$  and  $N_G(u_2) = \{r, u_1, t_1, v_1, v_2\}$ . By minimality of  $(G, T)$ , there exists a  $(G - \{u_1, u_2\}, T - \{u_1, u_2\})$ -colouring  $c$ .

If  $c(v_2) = 2$ , then colouring  $u_2$  with a colour  $c(u_2)$  in  $Z_6 \setminus \{1, 2, c(t_1), c(v_1)\}$  and  $u_1$  with a colour in  $\{3, 4, 5, 6\} \setminus [c(u_2)]$ , we obtain a  $(G, T)$ -colouring, a contradiction. So we may assume that  $c(v_2) \neq 2$ .

If one can colour  $u_2$  with a colour in  $\{2, 3, 6\}$ , then  $\{1, 2\} \cup [c(u_2)]$  has cardinality at most 4 and so at most 5 colours are forbidden in  $u_1$ . Hence one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction.

So we may assume  $\{c(t_1), c(v_1), c(v_2)\} = \{2, 3, 6\}$ . Necessarily,  $c(v_1) = 2$ ,  $c(v_2) = 3$  and  $c(t_1) = 6$ . Setting  $c(u_1) = 6$  and  $c(u_2) = 4$ , we obtain a  $(G, T)$ -colouring, a contradiction.

□

**Lemma 14** *If  $e$  is a minimal overstepping edge, then  $C_e^{int}$  is one of the graphs depicted in Figure 2. In addition, if  $C_e^{int} = A_1$ , then  $rv_1 \in E(G)$ .*

**Proof.** Let  $e$  be a minimal edge. According to the previous lemmas, it has to join two leaves  $v_1$  and  $v_2$  and there is a unique vertex  $u$  inside  $C_e$ . For  $i = 1, 2$ , let  $t_i$  be the twig adjacent to  $v_i$ . By Lemma 9,  $t_1 \neq t_2$ .

- Assume first that  $u$  is a twig.

If  $d_G(u) \leq 4$ , then consider a  $(G - u, T - u)$ -colouring  $c$ , which exists by minimality of  $(G, T)$ . In  $u$ , there are at most 5 colours forbidden as  $r$  is coloured 1, and thus forbids only two colours. Hence, one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction.

So we may assume that  $d_G(u) \geq 5$ , and thus  $N_G(u) = \{r, t_1, t_2, v_1, v_2\}$ .

If  $rv_1$  is not an edge, then let  $(G', T')$  be the pair obtained from  $(G - u, T - u)$  by identifying  $r$  and  $v_1$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which is a  $(G - u, T - u)$ -colouring such that  $c(v_1) = c(r) = 1$ . Then the set of forbidden colours in  $u$  is included in  $\{1, 2, c(t_1), c(t_2), c(v_2)\}$  and so has cardinality at most 5. Hence one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction.

Hence we may assume that  $rv_1$  is an edge. This edge must be in  $C_e^{ext}$  by planarity of  $G$ . Thus  $t_1t_2$  is not an edge of  $G$ . Let  $(G', T')$  be the pair obtained from  $(G - u, T - u)$  by identifying  $t_1$  and  $t_2$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring  $c$  which is a  $(G - u, T - u)$ -colouring such that  $c(t_1) = c(t_2)$ . Then the set of forbidden colours in  $u$  is included in  $\{1, 2, c(t_1), c(v_1), c(v_2)\}$  and so has cardinality at most 5. Hence one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction.

- Assume now that  $u$  is a leaf. By symmetry, we may assume that  $u$  is adjacent to  $t_1$ . By Lemma 7 and since  $G$  is triangulated,  $C_e^{int}$  is one of the graphs  $A_1, A_2$  or  $A_3$ .

Assume now that  $C_e^{int} = A_1$  and  $rv_1 \notin E(G)$ . Let  $(G', T')$  be the pair obtained from  $(G - u, T - u)$  by identifying  $r$  and  $v_1$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring which is a  $(G - u, T - u)$ -colouring  $c$  such that  $c(v_1) = c(r) = 1$ . Then the set of forbidden colours in  $u$  is included in  $\{1, c(v_2)\} \cup [c(t_1)]$  and so has cardinality at most 5. Hence one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction.  $\square$

### 3.2 Penultimate edges

**Lemma 15** *Let  $f$  be an edge which is the successor of a minimal edge  $e$ . If  $e$  is the unique predecessor of  $f$ , then  $C_f^{int}$  is one of the graphs depicted in Figure 3, and  $rv_1 \in E(G)$ . Moreover, if  $C_f^{int} = B_2$ ,  $rv_3 \notin E(G)$ .*

**Proof.** Let  $e'$  be the third edge of the triangle bounded by  $f$  and  $e$  in  $C_f^{int}$ . Suppose, by way of contradiction, that  $e$  is the unique predecessor of  $f$ . Then  $e'$  is not overstepping. So all the vertices inside  $C_f$  are in  $C_e^{int}$ . By Lemma 14,  $C_e^{int}$  is one of the graphs  $A_1, A_2$  or  $A_3$ .

One of the endvertices of  $f$  must be  $v_1$  and  $v_2$  (as defined for  $A_i$ ). We now distinguish many cases depending on  $C_e^{int}$  and the possible endvertices of  $f$ .

1. Assume that  $C_e^{int}$  is  $A_1$ .

- 1.1. Assume  $f = rv_1$ . Then the 4-cycle  $rt_2v_2v_1$  has no chord, because  $rv_2$  is in  $C_e^{int}$  and  $v_1t_2$  is not an edge since  $f$  is the successor of  $e$ . This contradicts the fact that  $G$  is triangulated.

- 1.2 Observe that  $f = t_1v_2$  is impossible since  $rv_1$  is an edge. Assume that  $f = t_2v_1$ . Let  $G' = (G - \{u, v_2\}) \cup t_1t_2$ . By minimality of  $(G, T)$ , there exists a  $(G', T - \{u, v_2\})$ -colouring

which is a  $(G - \{u, v_2\}, T - \{u, v_2\})$ -colouring  $c$  such that  $c(t_1) \neq c(t_2)$ . If  $c(t_1) = 6$ , then one can greedily extend  $c$  to  $v_2$  and then  $u$  to get a  $(G, T)$ -colouring, a contradiction. If  $c(t_1) \neq 6$ , then colouring  $v_2$  with a colour in  $\{c(t_1) - 1, c(t_1) + 1\} \setminus [c(t_2)]$  and  $u$  with a colour in  $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$  we obtain a  $(G, T)$ -colouring, a contradiction.

- 1.3. Assume that  $f = v_1 t_3$  with  $t_3$  a twig distinct from  $t_2$ . Since  $rv_1$  is an edge,  $t_1 t_3$  is not an edge. Let  $G'$  be the graph pair obtained from  $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ . If one can colour  $v_2$  with a colour  $c(v_2)$  in  $\{2, 3, 6\}$ , then  $\{1, 2\} \cup [c(v_2)]$  has cardinality at most 4 and so at most 5 colours are forbidden in  $t_2$ . Hence one can extend  $c$  into a  $(G, T)$ -colouring, a contradiction. So we may assume that  $\{c(u), c(v_1), c(t_3)\} = \{2, 3, 6\}$ . If  $c(t_3) = 3$ , set  $c(v_2) = 4$  and  $c(t_2) = 6$ . If  $c(t_3) = 6$ , set  $c(v_2) = 5$  and  $c(t_2) = 3$ . In both cases, we obtain a  $(G, T)$ -colouring, a contradiction.
- 1.4. Assume that  $f = v_2 t_3$  with  $t_3$  a twig distinct from  $t_1$ . By minimality of  $(G, T)$ , there exists a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$ . Setting  $c(t_1) = 6$  and choosing  $c(v_1)$  in  $Z_6 \setminus \{1, 5, 6, c(t_3), c(v_2)\}$  and  $c(u)$  in  $Z_6 \setminus \{1, 5, 6, c(v_1), c(v_2)\}$ , we get a  $(G, T)$ -colouring, a contradiction.
- 1.5.  $f$  cannot be  $v_2 v_3$  with  $v_3$  a leaf adjacent to  $t_1$  because  $rv_1$  is an edge.
- 1.6. Assume that  $f = v_1 v_3$  with  $v_3$  a leaf adjacent to  $t_2$ . Then  $C_e^{int} = B_1$ . By Lemma 14,  $rv_1 \in E(G)$ .

- 1.7. Assume  $f = v_2 v_3$  with  $v_3$  a leaf adjacent in  $T$  to a twig  $t_3$  not in  $\{t_1, t_2\}$ . Then  $v_1 v_3 \in E(G)$  and either  $rv_3 \in E(G)$  or  $t_3 v_1 \in E(G)$ . Since  $rv_1$  is an edge, we have that  $N(t_1) = \{r, u, v_1\}$ . By minimality of  $G$ , there exists a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$ .

If  $\{c(t_3), c(v_3), c(v_2)\} \neq \{2, 3, 4\}$ , then setting  $c(t_1) = 6$  and choosing  $c(v_1)$  in  $\{2, 3, 4\} \setminus \{c(t_3), c(v_3), c(v_2)\}$  and  $c(u)$  in  $\{2, 3, 4\} \setminus \{c(v_1), c(v_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $\{c(t_3), c(v_3), c(v_2)\} = \{2, 3, 4\}$ , and so  $c(t_3) = 4$ ,  $c(v_3) = 2$  and  $c(v_2) = 3$ . Then setting  $c(t_1) = 3$ ,  $c(u) = 5$  and  $c(v_1) = 6$  yields a  $(G, T)$ -colouring, a contradiction.

- 1.8. Assume  $f = v_1 v_3$  with  $v_3$  a leaf adjacent in  $T$  to a twig  $t_3$  not in  $\{t_1, t_2\}$ . Since  $rv_1 \in E(G)$ , then  $t_1 t_3 \notin E(G)$ .

Assume first that  $rv_3 \in C_f^{int}$ . By minimality of  $(G, T)$ , there exists a  $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ -colouring  $c$ . One can choose a colour  $c(v_2)$  in  $Z_6 \setminus \{1, c(u), c(v_1), c(v_3)\}$  such that  $I = [c(v_2)] \cup \{1, 2, c(v_3)\} \neq Z_6$ . Then choosing  $c(t_2) \in Z_6 \setminus I$ , we obtain a  $(G, T)$ -colouring, a contradiction.

Hence we may assume that  $rv_3$  is not in  $C_f^{int}$ . Let  $(G', T')$  be the graph obtained from  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$  by identifying  $t_1$  and  $t_3$ . By minimality of  $(G, T)$ , there exists a  $(G', T')$ -colouring which is a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$  such that  $c(t_1) = c(t_3)$ . If  $c(t_1) \neq 6$ , then one can choose a colour  $c(v_2) \in \{c(t_1) - 1, c(t_1) + 1\}$  such that  $I = [c(v_2)] \cup \{1, 2, c(v_3)\} \neq Z_6$ . Then choosing  $c(t_2) \in Z_6 \setminus I$  and  $c(u)$  in  $Z_6 \setminus ([c(t_1)] \cup \{1, c(v_1)\})$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence we may suppose that  $c(t_1) = 6$ . If  $v_2 t_3 \notin E(G)$ , then setting  $c(v_2) = 6$  and choosing  $c(t_2) \in \{3, 4\} \setminus c(v_3)$  and  $c(u)$  in  $Z_6 \setminus \{1, 5, 6, c(v_1)\}$  yields a  $(G, T)$ -colouring, a contradiction.

If  $v_2t_3 \in E(G)$ , then setting  $c(v_2) = 5$ ,  $c(t_2) = 3$  and choosing  $c(u)$  in  $Z_6 \setminus \{1, 5, 6, c(v_1)\}$  yields a  $(G, T)$ -colouring, a contradiction.

2. Assume that  $C_e^{int}$  is  $A_2$ .

2.1. Assume  $f = rv_1$ . Since  $f$  is the successor of  $e$ , then  $v_1t_2$  is not an edge and so  $rv_2 \in E(G)$  because  $G$  is triangulated. By minimality of  $G$ , there exists a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$ . Setting  $c(u) = 1$ , one can then extend  $c$  greedily to  $t_2$  and  $v_2$  to get a  $(G, T)$ -colouring, a contradiction.

2.2. Assume that  $f = rv_2$ . By minimality of  $(G, T)$ , there is a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring. Setting  $c(u) = 1$ , one can then extend  $c$  greedily to  $t_1$  and  $v_1$  to get a  $(G, T)$ -colouring, a contradiction.

2.3 Assume that  $f = t_1v_2$ . Since  $f$  is the successor of  $e$ , the cycle  $t_1v_1v_2$  is empty, and so  $v_1$  contradicts Lemma 7. Similarly, if  $f = t_2v_1$ , then  $v_2$  contradicts Lemma 7.

2.4. Assume that  $f = v_1t_3$  with  $t_3$  a twig distinct from  $t_2$ . Since  $f$  is the successor of  $e$ ,  $t_2v_1$  is not an edge. Then either  $rv_2$  is an edge or  $t_2t_3$  is an edge. Set  $G' = (G - \{u, t_2, v_2\}) \cup rv_1$ . By minimality of  $(G, T)$ , there is a  $(G', T - \{u, t_2, v_2\})$ -colouring  $c$  which is a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$  such that  $c(v_1) \neq c(r) = 1$ . Set  $c(u) = 1$ .

If  $c(v_1) \neq 2$ , then setting  $c(v_2) = 2$  and colouring  $t_2$  with a colour in  $Z_6 \setminus \{1, 2, 3, c(t_1), c(t_3)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. So  $c(v_1) = 2$  and thus  $c(t_1) \geq 4$ .

If  $c(t_3) \neq 3$ , then setting  $c(t_2) = 3$  and choosing  $c(v_2)$  in  $\{5, 6\} \setminus \{c(t_3)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. So  $c(t_3) = 3$ .

Choosing  $c(t_2)$  in  $\{4, 6\} \setminus \{c(t_1)\}$  and  $c(v_2)$  in  $\{4, 6\} \setminus \{c(t_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction.

2.5. Assume that  $f = v_2t_3$  with  $t_3$  a twig distinct from  $t_1$ . Then either  $rv_1$  is an edge or  $t_1t_3$  is an edge. Set  $G' = (G - \{u, t_1, v_1\}) \cup rv_2$ . By minimality of  $(G, T)$ , there exists a  $(G', T - \{u, t_1, v_1\})$ -colouring which is a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$  such that  $c(v_2) \neq c(r) = 1$ . Set  $c(u) = 1$ .

If  $c(v_2) \neq 2$ , then setting  $c(v_1) = 2$  and colouring  $t_1$  with a colour in  $Z_6 \setminus \{1, 2, 3, c(t_2), c(t_3)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. So  $c(v_2) = 2$  and thus  $c(t_2) \geq 4$ .

If  $c(t_3) \neq 3$ , then setting  $c(t_1) = 3$  and choosing  $c(v_1)$  in  $\{5, 6\} \setminus \{c(t_3)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. So  $c(t_3) = 3$ .

Choosing  $c(t_1)$  in  $\{4, 6\} \setminus \{c(t_2)\}$  and  $c(v_1)$  in  $\{4, 6\} \setminus \{c(t_1)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction.

2.6. Assume that  $f = v_2v_3$  with  $v_3$  a leaf adjacent to  $t_1$ . Since  $f$  is the successor of  $e$ , then  $t_1v_2$  is not inside  $v_3t_1v_1v_2$  and so  $v_1v_3 \in E(G)$ . Set  $G' = (G - \{u, v_1\}) \cup t_2v_3$ . By minimality of  $(G, T)$ , there is a  $(G', T - \{u, v_1\})$ -colouring which is a  $(G - \{u, v_1\}, T - \{u, v_1\})$ -colouring  $c$  such that  $c(t_2) \neq c(v_3)$ . Setting  $c(u) = c(v_3)$  and colouring  $v_1$  with a colour in  $Z_6 \setminus (\{c(u), c(v_2)\} \cup [c(t_1)])$ , we obtain a  $(G, T)$ -colouring, a contradiction.

2.7. Assume that  $f = v_1v_3$  with  $v_3$  a leaf adjacent to  $t_2$ . Since  $f$  is the successor of  $e$ , then  $t_2v_1$  is not inside  $v_2t_2v_3v_1$  and so  $v_2v_3 \in E(G)$ . Set  $G' = (G - \{u, v_2\}) \cup t_1v_3$ . By minimality of  $(G, T)$ , there is a  $(G', T - \{u, v_2\})$ -colouring, which is a  $(G - \{u, v_2\}, T - \{u, v_2\})$ -colouring  $c$  such that  $c(t_1) \neq c(v_3)$ . If  $c(t_2) \in [c(t_1)]$ , then one can also extend  $c$  greedily to  $v_2$  and then  $u$  to obtain a  $(G, T)$ -colouring, a contradiction. Hence  $|c(t_1) - c(t_2)| \geq 2$ . Thus one can colour  $v_2$  with  $c(t_1)$  and then colour  $u$  with a colour in  $Z_6 \setminus ([c(t_1)] \cup \{c(t_2), c(v_1)\})$ . This yields a  $(G, T)$ -colouring, a contradiction.

2.8. Assume  $f = v_2v_3$  with  $v_3$  a leaf adjacent in  $T$  to a twig  $t_3$  not in  $\{t_1, t_2\}$ .

Suppose first that  $rv_1 \notin E(G)$ . By minimality of  $(G, T)$ , there is a  $(G - \{u, t_1, v_1\} \cup \{rv_3, rv_2\}, T - \{u, t_1, v_1\})$ -colouring which is a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$  such that  $c(v_2) \neq 1$  and  $c(v_3) \neq 1$ . Colour  $v_1$  with 1 and let  $L(t_1) \supseteq Z_6 \setminus \{1, 2, c(v_3), c(t_3), c(t_2)\}$  and  $L(u) = Z_6 \setminus \{1, c(t_2), c(v_2)\}$  be the list of colours available for  $t_1$  and  $u$ , respectively. Note that there is at most one colour  $\alpha$  in  $Z_6$  such that  $L(u) \setminus [\alpha] = \emptyset$ . Thus, if there exists  $\beta$  in  $L(t_1) \setminus \{\alpha\}$  if such  $\alpha$  exists, or in  $L(t_1)$  otherwise, then we can colour  $t_1$  with  $\beta$  and  $u$  with a colour in  $L(u) \setminus [\beta]$  to obtain a  $(G, T)$ -colouring, a contradiction. So we may assume that no such  $\beta$  exists, that is  $L(t_1) = \{\alpha\}$  and  $L(u) \setminus [\alpha] = \emptyset$ . Since  $|c(v_2) - c(t_2)| \geq 2$ , necessarily  $\alpha = 4$ ,  $L(t_1) = \{4\}$ ,  $c(t_2) = 6$ ,  $c(v_2) = 2$ ,  $\{c(v_3), c(t_3)\} = \{3, 5\}$  and  $v_3, t_3 \in N(t_1)$ . Then, recolouring  $v_1$  with 6 and colouring  $t_1$  with 4 and  $u$  with 1 yields a  $(G, T)$ -colouring, a contradiction.

Suppose now that  $rv_1 \in E(G)$ . Then there is no vertex inside  $rt_1v_1r$ . By minimality of  $(G, T)$ , there is  $(G - u, T - u)$ -colouring  $c$ . If  $c(v_2) \neq 1$ , then we can colour  $u$  with 1; so, suppose otherwise. If there is no colour available for  $u$  to extend  $c$ , then  $F_c = \{1, c(t_2), c(v_1)\} \cup [c(t_1)]$  is equal to  $Z_6$ ; thus,  $c(t_1) \in \{3, 4, 5\}$ . If  $c(t_1) = 3$ , then  $\{c(v_1), c(t_2)\} = \{5, 6\}$ . If  $c(t_1) = 4$ , then  $\{c(v_1), c(t_2)\} = \{2, 6\}$ . If  $c(t_1) = 5$ , then  $\{c(v_1), c(t_2)\} = \{2, 3\}$ . If the colour of  $t_1$  can be changed, we obtain a  $(G - u, T - u)$ -colouring  $c'$  such that  $F_{c'} \neq Z_6$  which can be extended in a  $(G, T)$ -colouring, a contradiction. Hence,  $c(t_1) = i$  is the sole colour in  $Z_6 \setminus (\{1, 2, c(t_2)\} \cup [c(v_1)])$ . Thus,  $c(v_1) \neq 2$  and  $(c(v_1), c(t_2)) \neq (6, 5)$ . Then, necessarily (\*)  $c(v_1) = 5$ ,  $c(t_1) = 3$  and  $c(t_2) = 6$ . If  $c(t_3), c(v_3) \neq 3$ , then recolour  $t_1$  with 5 and  $v_1$  with 3. Otherwise, if  $c(t_3), c(v_3) \neq 6$ , then recolour  $v_1$  with 6. Otherwise (i.e.,  $\{c(t_3), c(v_3)\} = \{3, 6\}$ ), recolour  $v_1$  with 2. In any case, the resulting colouring  $c_1$  does not satisfy (\*). Hence, either  $F_{c_1} \neq Z_6$  or  $t_1$  can be recoloured to get a colouring  $c'_1$  such that  $F_{c'_1} \neq Z_6$ . Hence one of  $c_1, c'_1$  can be extended in a  $(G, T)$ -colouring, a contradiction.

2.9. Assume  $f = v_1v_3$  with  $v_3$  a leaf adjacent in  $T$  to a twig  $t_3$  not in  $\{t_1, t_2\}$ .

Suppose first that  $rv_2 \in E(G)$ . Set  $G' = (G - \{u, t_2, v_2\}) \cup \{t_1t_3, t_1v_3\}$ . By minimality of  $(G, T)$ , there is a  $(G', T - \{u, t_2, v_2\})$ -colouring, which is a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$  such that  $c(t_1) \neq c(t_3)$  and  $c(t_1) \neq c(v_3)$ . Set  $c(v_2) = c(t_1)$ . Then choosing  $c(t_2)$  in  $\{3, 4, 5, 6\} \setminus [c(t_1)]$  and  $c(u)$  in  $Z_6 \setminus ([c(t_1)] \cup \{c(t_2), c(v_1)\})$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $rv_2 \notin E(G)$ .

Suppose now that  $rv_3 \notin E(G)$ . Let  $(G', T')$  be the graph pair obtained from  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$  by identifying  $v_3$  and  $r$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which is a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$  such that  $c(v_3) = 1$ . Set  $c(u) = 1$ . For  $t_2$ , there at least two possible colours, namely the ones not in  $\{1, 2, c(t_1), c(t_3)\}$ . One of them, say  $\alpha$ , is such that  $I = [\alpha] \cup \{1, c(v_1), c(t_3)\}$

is not equal to  $Z_6$ . Thus, setting  $c(t_2) = \alpha$  and choosing  $c(v_2)$  in  $Z_6 \setminus I$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $rv_3 \in E(G)$ .

Assume that  $rv_3$  is inside  $C_f$ . Then  $t_2t_3 \notin E(G)$ . By minimality of  $(G, T)$ , there is a  $(G - \{u, t_2, v_2\} \cup rv_1, T - \{u, t_2, v_2\})$ -colouring, which is a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$  such that  $c(v_1) \neq 1$ . Thus, setting  $c(v_2) = 1$  and colouring  $u$  with a colour in  $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$  and  $t_1$  with a colour in  $Z_6 \setminus \{1, 2, c(t_1), c(u), c(v_3)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence we may assume that  $rv_3$  is outside  $C_f$ .

So, by planarity,  $t_1t_3 \notin E(G)$ . Let  $(G', T')$  be the graph pair obtained from  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$  by identifying  $t_1$  and  $t_3$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which is a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$  such that  $c(t_1) = c(t_3)$ . Set  $c(u) = c(v_3)$ . Let  $\alpha$  be a colour of  $Z_6 \setminus \{1, 2, c(t_1), c(v_3)\}$  such that  $I = [\alpha] \cup \{c(v_1), c(v_3), c(t_3)\}$  is not  $Z_6$ . Then setting  $c(t_2) = \alpha$  and choosing  $c(v_2)$  in  $Z_6 \setminus I$ , we obtain a  $(G, T)$ -colouring, a contradiction.

3. Assume that  $C_e^{int}$  is  $A_3$ .

- 3.1. Assume  $f = rv_1$ . Then  $rv_2$  is an edge. By minimality of  $G$ , there exists a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$ . Colour  $u$  with a colour  $c(u)$  in  $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ . Set  $c(t_2) = 6$  if  $c(u) \neq 6$  and  $c(t_2) = 5$  otherwise. In both cases, at most five colours are forbidden for  $v_2$ , and one can extend greedily the colouring into a  $(G, T)$ -colouring, a contradiction.
- 3.2. Assume that  $f = rv_2$ . Then  $rv_1$  is an edge. By minimality of  $G$ , there exists a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$ . Set  $c(t_1) = 6$ , then colour  $u$  with any colour in  $Z_6 \setminus \{1, 5, 6, c(t_2), c(v_2)\}$  and  $v_1$  with any colour in  $Z_6 \setminus \{1, 5, 6, c(v_2), c(u)\}$ . This yields a  $(G, T)$ -colouring, a contradiction.
- 3.3 Assume that  $f = t_1v_2$ . Since  $f$  is the successor of  $e$ , then the cycle  $t_1v_1v_2$  is empty, and so  $v_1$  contradicts Lemma 7. Similarly, if  $f = t_2v_1$ , then  $v_2$  contradicts Lemma 7.
- 3.4. Assume that  $f = v_1t_3$  with  $t_3$  a twig distinct from  $t_2$ .

Assume first that  $t_2t_3 \in E(G)$ . Then  $rv_2$  is not an edge. Set  $G' = (G - \{u, t_2, v_2\}) \cup rv_1$ . By minimality of  $(G, T)$ , there is a  $(G', T - \{u, t_2, v_2\})$ -colouring which is a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$  such that  $c(v_1) \neq c(r) = 1$ . Setting  $c(v_2) = 1$  and choosing  $c(u)$  in  $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$  and  $c(t_2)$  in  $Z_6 \setminus \{1, 2, c(u), c(t_3)\}$ , we get a  $(G, T)$ -colouring, a contradiction. So  $t_2t_3 \notin E(G)$  and thus  $rv_2 \in E(G)$ .

By minimality of  $G$ , there exists a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$ .

Assume that  $c(v_1) \neq 2$ . If  $c(t_1) = 3$ , then setting  $c(v_2) = 2$  and choosing  $c(u)$  in  $Z_6 \setminus \{1, 2, 3, 4, c(v_1)\}$  and  $c(t_2)$  in  $Z_6 \setminus \{1, 2, 3, c(u)\}$  yields a  $(G, T)$ -colouring, a contradiction. If  $c(t_1) \geq 4$ , then setting  $c(u) = 2$  and choosing  $c(v_2)$  in  $Z_6 \setminus \{1, 2, c(v_1), c(t_3)\}$  and  $c(t_2)$  in  $Z_6 \setminus (\{1, 2\} \cup [c(v_2)])$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $c(v_1) = 2$ .

If  $c(t_1) \neq 4$ , then colouring  $v_2$  with  $c(v_2) \in \{4, 6\} \setminus \{c(t_3)\}$ ,  $t_2$  with  $c(t_2) \in \{4, 6\} \setminus \{c(v_2)\}$  and  $u$  with  $c(u)$  in  $\{3, 5\} \setminus [c(t_1)]$ , we get a  $(G, T)$ -colouring, a contradiction. So  $c(t_1) = 4$ .

Colouring  $u$  with 6,  $v_2$  with  $c(v_2) \in \{3, 5\} \setminus \{c(t_3)\}$  and  $t_2$  with  $c(t_2)$  in  $\{3, 5\} \setminus [c(v_2)]$ , we get a  $(G, T)$ -colouring, a contradiction.



3.5. Assume that  $f = v_2t_3$  with  $t_3$  a twig distinct from  $t_1$ .

Assume first that  $t_1t_3$  is an edge. Set  $G' = (G - \{u, t_1, v_1\}) \cup rv_2$ . By minimality of  $(G, T)$ , there is a  $(G', T - \{u, t_1, v_1\})$ -colouring which is a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$  such that  $c(v_2) \neq c(r) = 1$ . Set  $c(v_1) = 1$ . If  $c(v_2) \neq 2$ , then setting  $c(u) = 2$  and assigning to  $t_1$  a colour in  $Z_6 \setminus \{1, 2, 3, c(t_3)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. So  $c(v_2) = 2$  and  $c(t_2) \geq 4$ . Setting  $c(u) = 3$  and assigning to  $t_1$  a colour in  $Z_6 \setminus \{1, 2, 3, 4, c(t_3)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $t_1t_3$  is not an edge.

So  $rv_1$  is an edge. Since  $e$  is minimal, then  $rv_1$  is not overstepping and  $C_{rv_1}^{int}$  is empty. Let  $G'$  be the graph from  $G - \{u, t_1, v_1\}$  by adding the edge  $t_2t_3$  if it does not exist. By minimality of  $(G, T)$ , there is a  $(G', T - \{u, t_1, v_1\})$ -colouring which is a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$  such that  $c(t_2) \neq c(t_3)$ . Set  $c(t_1) = 6$ . If  $c(t_2) \notin \{5, 6\}$ , then set  $c(v_1) = c(t_2)$  (this is possible because  $c(t_3) \neq c(t_2)$ ), otherwise colour  $v_1$  with any colour in  $Z_6 \setminus \{1, 5, 6, c(t_3), c(v_2)\}$ . Then colouring  $u$  with a colour in  $Z_6 \setminus \{1, 5, 6, c(v_1), c(v_2)\}$ , we get a  $(G, T)$ -colouring, a contradiction.

3.6. Assume that  $f = v_2v_3$  with  $v_3$  a leaf adjacent to  $t_1$ . Set  $G' = (G - \{u, v_1\}) \cup \{t_2v_3, rv_3\}$ . By minimality of  $(G, T)$ , there is a  $(G', T - \{u, v_1\})$ -colouring which is a  $(G - \{u, v_1\}, T - \{u, v_1\})$ -colouring  $c$  such that  $c(v_3) \notin \{c(r), c(t_2)\}$ . Setting  $c(u) = c(v_3)$  and colouring  $v_1$  with a colour in  $Z_6 \setminus (\{c(u), c(v_2)\} \cup [c(t_1)])$ , we obtain a  $(G, T)$ -colouring, a contradiction.

3.7. Assume that  $f = v_1v_3$  with  $v_3$  a leaf adjacent to  $t_2$ . Then  $C_f^{int} = B_2$ .

Assume first that  $rv_3 \in E(G)$ . Then  $t_1t_2 \notin E(G)$ . Let  $(G', T')$  be the graph pair obtained from  $(G - u, T - u)$  by identifying  $t_1$  and  $t_2$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which yields a  $(G - u, T - u)$ -colouring such that  $c(t_1) = c(t_2)$ . Then setting  $c(u) = c(v_3)$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $rv_3 \notin E(G)$ .

Now assume that  $rv_1 \notin E(G)$ . Let  $(G', T')$  be the graph pair obtained from  $(G - u, T - u)$  by identifying  $v_1$  and  $r$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which yields a  $(G - u, T - u)$ -colouring such that  $c(v_1) = 1$ . If  $c(t_1) = c(t_2)$ , then, setting  $c(u) = c(v_3)$ , we obtain a  $(G, T)$ -colouring, a contradiction. If  $c(t_1) = 6$  or  $c(t_2) \in [c(t_1)]$  or  $c(v_2) \in [c(t_1)]$ , then colouring  $u$  with a colour in  $Z_6 \setminus (\{1, c(t_2), c(v_2)\} \cup [c(t_1)])$ , we obtain a  $(G, T)$ -colouring, a contradiction. So, assume that  $c(t_1) \neq 6$  and  $c(t_2) \notin [c(t_1)]$  and  $c(v_2) \notin [c(t_1)]$ . If  $c(t_1) = 3$ , then  $c(t_2), c(v_2) \in \{5, 6\}$ , a contradiction. If  $c(t_1) = 5$ , then  $c(t_2), c(v_2) \in \{2, 3\}$ , a contradiction. Then,  $c(t_1) = 4$ ,  $c(t_2) = 6$  and  $c(v_2) = 2$ . Recolouring  $v_2$  with a colour in  $\{3, 4\} \setminus \{c(v_3)\}$  and setting  $c(u) = 2$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $rv_1 \in E(G)$ .

3.8. Assume  $f = v_2v_3$  with  $v_3$  a leaf adjacent in  $T$  to a twig  $t_3$  not in  $\{t_1, t_2\}$ .

Assume first that  $rv_1$  is not an edge. By minimality of  $(G, T)$ , there is a  $(G - \{u, t_1, v_1\}) \cup \{rv_2, rv_3\}, T - \{u, t_1, v_1\})$ -colouring which is a  $(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$  such that  $c(v_2) \neq 1$  and  $c(v_3) \neq 1$ . Colour  $v_1$  with 1. Colour  $t_1$  with a colour  $\alpha$  in  $A = Z_6 \setminus \{1, 2, c(t_3), c(v_3)\}$  such that  $[\alpha] \neq Z_6 \setminus \{1, c(t_2), c(v_2)\}$ . This is possible since  $|A| \geq 2$ . Then colouring  $u$  with a colour in  $Z_6 \setminus (\{1, c(t_2), c(v_2)\} \cup [\alpha])$ , we obtain a  $(G, T)$ -colouring, a contradiction.

Suppose now that  $rv_1$  is an edge. Then  $t_1t_3$  and  $t_1v_3$  are not edges. By minimality of  $(G, T)$ , there is a  $(G - \{u, t_1, v_1\}) \cup \{t_2t_3, t_2v_3\}, T - \{u, t_1, v_1\})$ -colouring which is a

$(G - \{u, t_1, v_1\}, T - \{u, t_1, v_1\})$ -colouring  $c$  such that  $c(t_3) \neq c(t_2)$  and  $c(v_3) \neq c(t_2)$ . Set  $c(t_1) = 6$ . Let  $L(u) = \{2, 3, 4\} \setminus \{t_2, v_2\}$  and let  $L(v_1) = \{2, 3, 4\} \setminus \{v_2, t_3, v_3\}$ . If  $L(v_1)$  is empty, then  $c(t_3) = 4$ ,  $c(v_3) = 2$  and  $c(v_2) = 3$ . In this case, recolouring  $t_1$  with 3, colouring  $u$  with a colour in  $\{5, 6\} \setminus \{c(t_2)\}$  and colouring  $v_1$  with a colour in  $\{5, 6\} \setminus \{c(u)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $L(v_1)$  is not empty. If  $|L(u)| > 1$ , we can colour  $v_1$  with a colour in  $L(v_1)$  and colour  $u$  with a colour in  $L(u) \setminus \{c(v_1)\}$  to obtain a  $(G, T)$ -colouring, a contradiction. Then  $|L(u)| = 1$  and consequently  $c(t_2) = 4$  and  $c(v_2) = 2$ . Then colouring  $u$  with 3 and colouring  $v_1$  with 4, we obtain a  $(G, T)$ -colouring, since  $c(t_3)$  and  $c(v_3)$  are distinct from  $c(t_2) = 4$ , a contradiction.

3.9. Assume  $f = v_1v_3$  with  $v_3$  a leaf adjacent in  $T$  to a twig  $t_3$  not in  $\{t_1, t_2\}$ .

Suppose first that  $rv_2 \notin E(G)$ . By minimality of  $(G, T)$ , there is a  $(G - \{t_2, v_2\} \cup \{rv_1, rv_3\}, T - \{t_2, v_2\})$ -colouring which is a  $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ -colouring  $c$  such that  $c(v_1) \neq 1$  and  $c(v_3) \neq 1$ . Setting  $c(v_2) = 1$  and choosing  $c(t_2)$  in  $Z_6 \setminus \{1, 2, c(u), c(t_3), c(v_3)\}$  yields a  $(G, T)$ -colouring, a contradiction. Hence  $rv_2 \in E(G)$ .

Assume that  $v_2t_3 \notin E(G)$ . By minimality of  $(G, T)$ , there is a  $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ -colouring. We can choose  $c(v_2)$  in  $Z_6 \setminus \{1, c(u), c(v_1), c(v_3)\}$  such that  $I = [c(v_2)] \cup \{1, 2, c(u)\} \neq Z_6$  and  $c(t_2) \in Z_6 \setminus I$  to obtain a  $(G, T)$ -colouring, a contradiction. Hence  $v_2t_3 \in E(G)$ .

Now assume that  $rv_3$  is not an edge. Let  $(G', T')$  be the graph pair obtained from  $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$  by identifying  $v_3$  and  $r$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which yields a  $(G - \{t_2, v_2\}, T - \{t_2, v_2\})$ -colouring such that  $c(v_3) = 1$ . Then colouring  $v_2$  with a colour  $\alpha \in Z_6 \setminus \{1, c(u), c(v_1), c(t_3)\}$  such that  $[\alpha] \cup \{1, 2, c(u)\} \neq Z_6$  and colouring  $t_2$  with a colour in  $Z_6 \setminus ([\alpha] \cup \{1, 2, c(u)\})$ , we obtain a  $(G, T)$ -colouring, a contradiction.

Hence,  $rv_3$  is an edge and, since  $v_2t_3$  is an edge,  $t_1t_3$  is not an edge by planarity. Let  $(G', T')$  be the graph pair obtained from  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$  by identifying  $t_1$  and  $t_3$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which yields a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring such that  $c(t_1) = c(t_3)$ . Then setting  $c(u) = c(v_3)$ , colouring  $v_2$  with a colour  $\alpha \in Z_6 \setminus \{1, c(v_1), c(t_3), c(v_3)\}$  such that  $[\alpha] \cup \{1, 2, c(u)\} \neq Z_6$  and colouring  $t_2$  with a colour in  $Z_6 \setminus ([\alpha] \cup \{1, 2, c(u)\})$ , we obtain a  $(G, T)$ -colouring, a contradiction.

□

**Lemma 16** *Every penultimate edge has a unique predecessor.*

**Proof.** By contradiction. Suppose that a penultimate edge  $f$  has two predecessors  $e$  and  $e'$ . Then  $e$  and  $e'$  are ultimate and so minimal. According to Lemma 14,  $C_e^{int}$  and  $C_{e'}^{int}$  are isomorphic to some of  $A_1$ ,  $A_2$  or  $A_3$ . Let us denote the vertices of  $C_e^{int}$  by their names in Figure 2 and the vertices of  $C_{e'}^{int}$  by their names in Figure 2 augmented with a prime.

Since  $f$ ,  $e$  and  $e'$  are bounding the face incident to  $f$  in  $C_f^{int}$ , the edge  $f$  is  $v_1v'_2$ ,  $v_1v'_1$ ,  $v_2v'_2$  or  $v_2v'_1$ . If  $f = v_2v'_1$ , then swapping the names of  $e$  and  $e'$ , we are left with  $f = v_1v'_2$ . Hence we may assume that  $f \in \{v_1v'_2, v_1v'_1, v_2v'_2\}$ . Note that if  $f = v_1v'_2$ , then  $t_2 = t'_1$  and  $v_2 = v'_1$ , if  $f = v_1v'_1$ , then  $t_2 = t'_2$  and  $v_2 = v'_2$ , and if  $f = v_2v'_2$ , then  $t_1 = t'_1$  and  $v_1 = v'_1$ .

Observe that if  $C_e^{int}$  is isomorphic to  $A_1$ , then  $f$  cannot be  $v_2v'_2$  because  $rv_1$  must be an edge that would cross  $f$ . Moreover if  $C_e^{int}$  and  $C_{e'}^{int}$  are both isomorphic to  $A_1$ , then  $f$  cannot be  $v_1v'_1$  since  $G$  has no multiple edges. Hence must be in one of the following cases:

- $C_e^{int}$  and  $C_{e'}^{int}$  are isomorphic to  $A_1$  and  $f = v_1v'_2$ .

By minimality of  $G$ , there is a  $(G - \{u', t_2, v_2\}, T - \{u', t_2, v_2\})$ -colouring  $c$ . Colour  $t_2$  with 6. If  $\{c(u), c(v_1), c(v'_2)\} \neq \{2, 3, 4\}$ , then colouring  $v_2$  with a colour in  $Z_6 \setminus \{1, 5, 6, c(u), c(v_1), c(v'_2)\}$  and colouring  $u'$  with a colour in  $\{2, 3, 4\} \setminus \{c(v_2), c(v'_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. If  $\{c(u), c(v_1), c(v'_2)\} = \{2, 3, 4\}$ , then re-colouring  $t_2$  with 3, and setting  $c(v_2) = 5$  and  $c(u') = 6$ , we obtain a  $(G, T)$ -colouring, a contradiction.

- $C_e^{int}$  is isomorphic to  $A_1$ ,  $C_{e'}^{int}$  is isomorphic to  $A_2$  and  $f = v_1v'_i$  for  $i \in \{1, 2\}$ .

By minimality of  $(G, T)$ , there is a  $(G - \{t_2, v_2, u'\} \cup rv'_i, T - \{t_2, v_2, u'\})$ -colouring which is a  $(G - \{t_2, v_2, u'\}, T - \{t_2, v_2, u'\})$ -colouring  $c$  such that  $c(v'_i) \neq 1$ . Colouring  $u'$  with 1, colouring  $v_2$  with a colour  $\alpha \in Z_6 \setminus \{1, c(u), c(v_1), c(v'_i)\}$  such that  $\{1, 2, c(t'_i)\} \cup [\alpha] \neq Z_6$  and colouring  $t_2$  with a colour in  $Z_6 \setminus (\{1, 2, c(t'_i)\} \cup [\alpha])$ , we obtain a  $(G, T)$ -colouring, a contradiction.

- $C_e^{int}$  is isomorphic to  $A_1$ ,  $C_{e'}^{int}$  is isomorphic to  $A_3$  and  $f = v_1v'_2$ .

By minimality of  $(G, T)$ , there is a  $(G - \{t_2, v_2, u'\} \cup \{ut'_2, v_1t'_2\}, T - \{t_2, v_2, u'\})$ -colouring which is a  $(G - \{t_2, v_2, u'\}, T - \{t_2, v_2, u'\})$ -colouring  $c$  such that  $c(u) \neq c(t'_2)$  and  $c(v_1) \neq c(t'_2)$ . If  $\{c(u), c(v_1), c(v'_2)\} = \{2, 3, 4\}$ , then colour  $t_2$  with 3, colour  $u'$  with a colour in  $\{5, 6\} \setminus \{c(t'_2)\}$  and colour  $v_2$  with a colour in  $\{5, 6\} \setminus \{c(u')\}$ . If  $2 \notin \{c(u), c(v_1), c(v'_2)\}$ , then set  $c(t_2) = 6$ ,  $c(v_2) = 2$  and colour  $u'$  with a colour in  $\{3, 4\} \setminus \{c(t'_2), c(v'_2)\}$ . If  $4 \notin \{c(u), c(v_1), c(v'_2)\}$ , then set  $c(t_2) = 6$ ,  $c(v_2) = 4$  and colour  $u'$  with a colour in  $\{2, 3\} \setminus \{c(t'_2), c(v'_2)\}$ . In any of these cases, we obtain a  $(G, T)$ -colouring, a contradiction. So  $2, 4 \in \{c(u), c(v_1), c(v'_2)\}$  and  $3 \notin \{c(u), c(v_1), c(v'_2)\}$ . Colour  $t_2$  with 6 and  $v_2$  with 3. Notice that  $\{c(t'_2), c(v'_2)\} \neq \{2, 4\}$ , since necessarily  $c(t'_2) = 4$  and  $c(v'_2) = 2$ , but  $c(u), c(v_1) \neq c(t'_2) = 4$  contradicts the fact that  $2, 4 \in \{c(u), c(v_1), c(v'_2)\}$ . Then colouring  $u'$  with  $\{2, 4\} \setminus \{c(t'_2), c(v'_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction.

- $C_e^{int}$  is isomorphic to  $A_1$ ,  $C_{e'}^{int}$  is isomorphic to  $A_3$  and  $f = v_1v'_1$ .

By minimality of  $(G, T)$ , there is a  $(G - \{u, t_2, v_2\} \cup \{t_1u', t_1v'_1\}, T - \{u, t_2, v_2\})$ -colouring which is a  $(G - \{u, t_2, v_2\}, T - \{u, t_2, v_2\})$ -colouring  $c$  such that  $c(u') \neq c(t_1)$  and  $c(v'_1) \neq c(t_1)$ . Colour  $u$  with a colour in  $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ . If  $c(v_1) = 1$  or  $c(v'_1) = 1$  or  $|\{c(u), c(u'), c(v_1), c(v'_1)\}| < 4$ , then colouring  $v_2$  with a colour  $\alpha \in Z_6 \setminus \{1, c(u), c(u'), c(v_1), c(v'_1)\}$  such that  $[\alpha] \cup \{1, c(u')\} \neq Z_6$  and colouring  $c(t_2)$  with a colour in  $Z_6 \setminus ([\alpha] \cup \{1, c(u')\})$ , we obtain a  $(G, T)$ -colouring, a contradiction.

Then  $\{c(u), c(v_1)\} \cap \{c(u'), c(v'_1)\} = \emptyset$ . This is only possible if  $\{c(t_1), c(t'_1)\} = \{3, 5\}$ ,  $\{c(t_1), c(t'_1)\} = \{3, 6\}$  or  $\{c(t_1), c(t'_1)\} = \{4, 6\}$ . If  $c(t'_1) = 3$ , then  $\{c(u'), c(v'_1)\} = \{5, 6\}$  and, since  $c(u') \neq c(t_1)$  and  $c(v'_1) \neq c(t_1)$ ,  $c(t_1) \notin \{5, 6\}$ . If  $c(t'_1) = 4$ , then  $\{c(u'), c(v'_1)\} = \{2, 6\}$  and consequently  $c(t_1) \notin \{2, 6\}$ . If  $c(t'_1) = 5$ , then  $\{c(u'), c(v'_1)\} = \{2, 3\}$  and consequently  $c(t_1) \notin \{2, 3\}$ . Then the only possibilities are  $(c(t_1), c(t'_1)) = (3, 6)$  or  $(c(t_1), c(t'_1)) = (4, 6)$ . In these cases,  $c(u') \neq 6$ .

If  $(c(t_1), c(t'_1)) = (3, 6)$ , then colouring  $t_2$  with 6 and  $v_2$  with a colour in  $\{2, 3, 4\} \setminus \{c(u'), c(v'_1)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction. Then  $(c(t_1), c(t'_1)) =$

(4, 6). Consequently,  $\{c(u), c(v_1)\} = \{2, 6\}$  and, since  $\{c(u), c(v_1)\} \cap \{c(u'), c(v'_1)\} = \emptyset$ ,  $\{c(u'), c(v'_1)\} = \{3, 4\}$ . This is a contradiction, since  $c(t_1) \neq c(u')$  and  $c(t_1) \neq c(v'_1)$ .

- $C_e^{int}$  is isomorphic to  $A_2$  or  $A_3$ ,  $C_{e'}^{int}$  is isomorphic to  $A_2$  or  $A_3$  and  $f = v_1 v'_1$ .

By minimality of  $(G, T)$ , there is a  $(G - \{u, u', t_2, v_2\} \cup \{rv_1, rv'_1\}, T - \{u, u', t_2, v_2\})$ -colouring  $c$  which is a  $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring  $c$  such that  $c(v_1), c(v'_1) \neq 1$ . Colour  $u$  with some colour in  $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$  and  $u'$  with some colour in  $Z_6 \setminus (\{1, c(v'_1)\} \cup [c(t'_1)])$ . Then, colour  $v_2$  with 1. If either  $t_2$  is adjacent to at most one in  $\{t_1, t'_1\}$  or  $\{c(t_1), c(u), c(t'_1), c(u')\} \neq \{3, 4, 5, 6\}$ , then we can assign to  $t_2$  a colour in  $\{3, 4, 5, 6\}$  not assigned to any of its neighbours to get a  $(G, T)$ -colouring, a contradiction.

So  $t_2 t_1$  and  $t_2 t'_1$  are edges and  $\{c(t_1), c(u), c(t'_1), c(u')\} = \{3, 4, 5, 6\}$ . By planarity,  $ru$  and  $ru'$  are not edges and we can recolour  $u$  and  $u'$  with 1. Then, colouring  $t_2$  with a colour  $\alpha \in Z_6 \setminus \{1, 2, c(t_1), c(t'_1)\}$  such that  $[\alpha] \cup \{1, c(v_1), c(v'_1)\} \neq Z_6$  and colouring  $v_2$  with a colour in  $Z_6 \setminus ([\alpha] \cup \{1, c(v_1), c(v'_1)\})$ , we obtain a  $(G, T)$ -colouring, a contradiction.

- $C_e^{int}$  is isomorphic to  $A_2$  or  $A_3$ ,  $C_{e'}^{int}$  is isomorphic to  $A_2$  or  $A_3$  and  $f = v_2 v'_2$ .

By minimality of  $(G, T)$ , there is a  $(G - \{u, u', t_2, v_2\} \cup \{rv_2, rv'_2\}, T - \{u, u', t_2, v_2\})$ -colouring which is a  $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring  $c$  such that  $c(v_1), c(v'_1) \neq 1$ . Choose  $c(u)$  in  $\{2, 3\} \setminus \{c(v_2), c(t_2)\}$  and  $c(u')$  in  $\{2, 3\} \setminus \{c(v'_2), c(t'_2)\}$  and set  $c(v_1) = 1$ . If  $t_1$  has at most one neighbour in  $\{t_2, t'_2\}$  or  $\{c(t_2), c(t'_2)\} \neq \{5, 6\}$ , then we can colour  $t_1$  with a colour in  $\{5, 6\}$  not appearing on any of its neighbours to get a  $(G, T)$ -colouring, a contradiction. Hence  $t_1$  is adjacent to  $t_2$  and  $t'_2$  (that is  $C_e^{int}$  and  $C_{e'}^{int}$ ) are isomorphic to  $A_2$  and  $\{c(t_2), c(t'_2)\} = \{5, 6\}$ . Recolouring  $u$  with  $c(t'_2)$  and  $u'$  with  $c(t_2)$  and colouring  $t_1$  with 3, we obtain a  $(G, T)$ -colouring, a contradiction.

- $C_e^{int}$  is isomorphic to  $A_2$  or  $A_3$ ,  $C_{e'}^{int}$  is isomorphic to  $A_2$  or  $A_3$  and  $f = v_1 v'_2$ .

By minimality of  $(G, T)$ , there exists a  $G - \{u, u', t_2, v_2\} \cup \{rv_1, rv'_2\}, T - \{u, u', t_2, v_2\}$ -colouring  $c$  which is a  $(G - \{u, u', t_2, v_2\}, T - \{u, u', t_2, v_2\})$ -colouring such that  $c(v_1) \neq 1$  and  $c(v'_2) \neq 1$ . Set  $c(v_2) = 1$  and colour  $u$  with some colour in  $Z_6 \setminus (\{1, c(v_1)\} \cup [c(t_1)])$ ,  $u'$  with some colour in  $\{2, 3\} \setminus \{c(t'_2), c(v'_2)\}$ . Note that the set  $F$  of forbidden colours for  $t_2$  is the union of  $\{1, 2, c(u)\} \cup [c(u')]$  and the set of colours of the neighbours of  $t_2$  in  $\{t_1, t'_2\}$ . Moreover  $F = Z_6$  for otherwise we could colour  $t_2$  with a colour in  $Z_6 \setminus F$  to obtain a  $(G, T)$ -colouring, a contradiction.

If  $c(u') = 2$ , then, since  $|F| = 6$ ,  $t_2 t_1$  and  $t_2 t'_2$  are edges and  $\{c(u), c(t_1), c(t'_2)\} = \{4, 5, 6\}$ . Since  $|c(u) - c(t_1)| \geq 2$ , necessarily  $\{c(t_1), c(u)\} = \{4, 6\}$  and  $c(t'_2) = 5$ . If  $c(t_1) = 6$ , then recolouring  $u$  with a colour in  $\{2, 3\} \setminus c(v_1)$  and assigning 4 to  $t_2$ , we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $c(t_1) = 4$  and  $c(u) = 6$ . So  $c(v_1) = 2$ , and thus  $c(v'_2) = 3$ . Then recolouring  $u$  and  $u'$  with 1 and  $v_2$  with 4 and setting  $c(t_2) = 6$ , we obtain a  $(G, T)$ -colouring, a contradiction.

Now, suppose that  $c(u') = 3$ . Then, since  $|F| = 6$ , two neighbours of  $t_2$  in  $\{u, t_1, t'_2\}$  are coloured 5 and 6. Assume that  $c(u) \notin \{5, 6\}$ , then  $t_2 t_1$  and  $t_2 t'_2$  are edges and  $\{c(t_1), c(t'_2)\} = \{5, 6\}$ . Note that, in this case,  $c(v_1) \leq 4$  and  $c(v'_2) \leq 4$ . Recolour  $u$  and  $u'$  with 1,  $v_2$  with 6 and colour  $t_2$  with 3 to get a  $(G, T)$ -colouring, a contradiction. Hence  $c(u) \in \{5, 6\}$ . Thus  $c(t_1) \leq 4$  and so  $c(t'_2) \in \{5, 6\}$  and  $c(v'_2) \leq 4$ . Thus,  $t_2 t'_2$  is an edge and  $c(t'_2) \in (\{5, 6\} \setminus \{c(u)\})$ . Recolour  $u'$  with  $c(u)$ . If  $t_1 \notin N(t_2)$  or  $c(t_1) \neq 3$ , colouring

$t_2$  with 3 yields a  $(G, T)$ -colouring, a contradiction. So  $t_1 \in N(t_2)$  and  $c(t_1) = 3$ . Then, recolour  $u$  and  $u'$  with 3 (note that  $c(v_1) \geq 5$  and  $c(v'_2) \neq 3$  as  $u'$  was coloured 3) and  $t_2$  with  $i \in \{5, 6\} \setminus \{c(t'_2)\}$ . This gives a  $(G, T)$ -colouring, a contradiction.

□

Lemmas 15 and 16 immediately imply the following.

**Corollary 17** *If  $f$  is a penultimate edge, then  $C_f^{int}$  is isomorphic to  $B_1$  or  $B_2$ , and  $rv_1 \in E(G)$ . Moreover, if  $C_f^{int} = B_2$ ,  $rv_3 \notin E(G)$ .*

### 3.3 Antepenultimate edges

We first prove that no antepenultimate edge  $g$  has two penultimate predecessors  $f$  and  $f'$ .

**Lemma 18** *Every antepenultimate edge has a unique penultimate predecessor.*

**Proof.** By contradiction. Suppose that an antepenultimate edge  $g$  has two penultimate predecessors  $f$  and  $f'$ .

According to Corollary 17,  $C_f^{int}$  and  $C_{f'}^{int}$  are isomorphic to one of the graphs  $B_1$  and  $B_2$ . Let us denote the vertices of  $C_f^{int}$  by their names in Figure 3 and the vertices of  $C_{f'}^{int}$  by their names in Figure 3 augmented with a prime.

Since  $g$ ,  $f$  and  $f'$  are bounding the face incident to  $g$  in  $C_g^{int}$ , the edge  $g$  is  $v_1v'_1$ ,  $v_1v'_3$ ,  $v_3v'_3$  or  $v_3v'_1$ . Since  $rv_1$  and  $rv'_1$  are edges, then  $g = v_1v'_1$ , for otherwise  $rv_1$  would cross  $g$ .

First, suppose that  $C_{f'}^{int}$  is isomorphic to  $B_1$ , i.e.,  $rv'_2 \in E$ . By minimality of  $(G, T)$ , there is a  $((G - \{v_2, v_3\}) \cup \{v'_2u, v'_2v_1\}, T - \{v_2, v_3\})$ -colouring, which is a  $(G - \{v_2, v_3\}, T - \{v_2, v_3\})$ -colouring such that  $c(v'_2) \notin \{c(u), c(v_1)\}$ . Setting  $c(v_2) = c(v'_2)$  and  $c(v_3) = 1$  gives a  $(G, T)$ -colouring, a contradiction.

The case  $C_f^{int}$  is isomorphic to  $B_1$  is symmetric, so we may assume that both are  $C_e^{int}$  and  $C_{e'}^{int}$  are isomorphic to  $B_2$ . By minimality of  $(G, T)$ , there exists a  $(G - \{v_2, v'_2, v_3, t_2\}, T - \{v_2, v'_2, v_3, t_2\})$ -colouring. Set  $c(v_2) = c(v'_2) = 1$ . Then, one can choose  $c(t_2)$  in  $L = Z_6 \setminus \{1, 2, c(u), c(u')\}$  such that  $I = [c(t_2)] \cup \{1, c(v_1), c(v'_1)\} \neq Z_6$  because  $|L| \geq 2$ . Hence colouring  $v_3$  with a colour in  $Z_6 \setminus I$ , we obtain a  $(G, T)$ -colouring, a contradiction. □

From Lemma 18, for every antepenultimate edge  $g$ ,  $g$  has only one predecessor  $f$  (which must be penultimate), or  $g$  has two predecessors: a penultimate edge  $f$  and an ultimate edge  $e'$ . From Lemmas 15 and 14,  $C_f^{int}$  is  $B_1$  or  $B_2$ , and  $C_{e'}^{int}$  is  $A_1$ ,  $A_2$  or  $A_3$ .

To deal with these cases, we need the following two auxiliary lemmas.

**Lemma 19** *Suppose that  $(G, T)$  contains a configuration isomorphic to  $B_1$  (see Figure 3). If there is a  $(G - \{u, v_2\}, T - \{u, v_2\})$ -colouring  $c$  satisfying one of the following conditions :*

- (a)  $c(t_2) = 6$  and  $(c(t_1), c(v_3)) \neq (5, 4)$ ;
- (b)  $c(v_3) = 1$  and  $c(t_1) \neq c(t_2)$ ;

*Then there is a  $(G, T)$ -colouring.*

**Proof.** Let  $L(u) = Z_6 \setminus ([c(t_1)] \cup \{1, c(v_1)\})$  and  $L(v_2) = Z_6 \setminus ([c(t_2)] \cup \{1, c(v_1), c(v_3)\})$  be the set of colours available for  $u$  and  $v_2$  respectively. Clearly  $L(u) \neq \emptyset$ . Observe that the conditions (a) and (b) also imply that  $L(v_2) \neq \emptyset$ . So, if  $|L(u)| \geq 2$ ,  $|L(v_2)| \geq 2$  or  $L(u) \neq L(v_2)$ , one can choose distinct colours  $c(u) \in L(u)$  and  $c(v_2) \in L(v_2)$  to obtain a  $(G, T)$ -colouring. It is a simple matter to check that in both cases these conditions are satisfied.  $\square$

**Lemma 20** *Suppose that  $(G, T)$  contains a configuration isomorphic to  $B_2$  (see Figure 3). If there is a  $(G - \{u, v_2\}, T - \{u, v_2\})$ -colouring  $c$  satisfying one of the following conditions :*

- (a)  $c(t_1) = c(t_2)$  and  $c(v_3) \neq 1$ ;
- (b)  $c(t_1) \neq c(t_2)$  and
  - (b1)  $c(t_1) = 6$ ; or
  - (b2)  $c(v_1) = c(t_2)$ ; or
  - (b3)  $c(t_2) \in [c(t_1)]$ .

*Then  $G$  has a  $(G, T)$ -colouring.*

**Proof.** Let  $L(u) = Z_6 \setminus \{1, [c(t_1)], c(t_2), c(v_1)\}$  and  $L(v_2) = Z_6 \setminus \{[c(t_2)], c(v_1), c(v_3)\}$  be the set of colours available for  $u$  and  $v_2$  respectively. Clearly  $L(v_2) \neq \emptyset$ . Observe that the conditions (a), (b1), (b2) and (b3) also imply that  $L(u) \neq \emptyset$ . So, if  $|L(u)| \geq 2$ ,  $|L(v_2)| \geq 2$  or  $L(u) \neq L(v_2)$ , one can choose distinct colours  $c(u) \in L(u)$  and  $c(v_2) \in L(v_2)$  to obtain a  $(G, T)$ -colouring. It is a simple matter to check that in each case these conditions are satisfied.  $\square$

Now we prove that the case of an antepenultimate edge  $g$  with a penultimate predecessor  $f$  and an ultimate predecessor  $e'$  is impossible.

**Lemma 21** *Every antepenultimate edge has a unique predecessor.*

**Proof.** By contradiction. Suppose that an antepenultimate edge  $g$  has two predecessors  $f$  and  $f'$ . By Lemma 18, one of those is not penultimate. So, without loss of generality,  $f$  is penultimate, and  $f'$  is not. Hence  $f'$  is minimal.

According to Corollary 17,  $C_f^{int}$  is isomorphic to  $B_1$  or  $B_2$ , and according to Lemma 14,  $C_{f'}^{int}$  is isomorphic to some of  $A_1$ ,  $A_2$  or  $A_3$ . Let us denote the vertices of  $C_f^{int}$  by their names in Figure 3 and the vertices of  $C_{f'}^{int}$  by their names in Figure 2 augmented with a prime.

Since  $g$ ,  $f$  and  $f'$  are bounding the face incident to  $g$  in  $C_g^{int}$ , the edge  $g$  is  $v_1v'_1$ ,  $v_1v'_2$ ,  $v_3v'_1$  or  $v_3v'_2$ . Moreover, since  $rv_1$  is an edge,  $rv_3$  is not an edge if  $C_f^{int}$  is isomorphic to  $B_2$ , and  $rv'_1$  is an edge if  $C_{f'}^{int}$  is isomorphic to  $A_1$ , we must be in one of the following cases:

- $C_f^{int}$  is isomorphic to  $B_1$ ,  $C_{f'}^{int}$  is isomorphic to  $A_2$  or  $A_3$  and  $g = v_1v'_2$ .

By minimality of  $(G, T)$ , there is a  $(G - \{v_2, t_2, u', v_3\} \cup \{v'_2r\}, T - \{v_2, t_2, u', v_3\})$ -colouring  $c$  which is a  $(G - \{v_2, t_2, u', v_3\}, T - \{v_2, t_2, u', v_3\})$ -colouring such that  $c(v'_2) \neq 1$ . Set  $c(v_3) = 1$ . Let  $L(t_2) \supseteq Z_6 \setminus \{1, 2, c(t'_2)\}$ ,  $L(v_2) = Z_6 \setminus \{1, c(u), c(v_1)\}$  and  $L(u') = Z_6 \setminus \{1, c(t'_2), c(v'_2)\}$ . Clearly, there exists at most one  $i \in Z_6$  such that  $L(u') = [i]$  and at most one  $j \in Z_6$  such that  $L(v_2) = [j]$ . Thus, as  $|L(t_2)| \geq 3$ , there exists  $k \in L(t_2)$

such that  $L(u') \setminus [k] \neq \emptyset$  and  $L(v_2) \setminus [k] \neq \emptyset$ . Setting  $c(t_2) = k$  and colouring  $u'$  and  $v_2$  by colours in  $L(u') \setminus [k]$  and  $L(v_2) \setminus [k]$ , respectively, we obtain a  $(G, T)$ -colouring, a contradiction.

- $C_f^{int}$  is isomorphic to  $B_2$ ,  $C_{f'}^{int}$  is isomorphic to  $A_2$  or  $A_3$  and  $g = v_1 v'_2$ .

By minimality of  $(G, T)$ , there is a  $((G - \{v_2, t_2, v_3, u'\}) \cup \{t'_2 u, t'_2 v_1\}, T - \{v_2, t_2, v_3, u'\})$ -colouring  $c$  which is a  $(G - \{v_2, t_2, v_3, u'\}, T - \{v_2, t_2, v_3, u'\})$ -colouring such that  $c(t'_2) \notin \{c(u), c(v_1)\}$ .

Suppose that  $t_2 t'_2 \in E(G)$ . If we can colour  $t_2$  with  $\beta \in [c(v_1)] \cup \{6\}$ , then we can colour  $u'$  with some colour in  $Z_6 \setminus (\{c(t'_2), c(v'_2)\} \cup [\beta])$ ,  $v_3$  with some colour in  $Z_6 \setminus (\{c(u'), c(v'_2), c(v_1)\} \cup [\beta])$  and  $v_2$  with some colour in  $Z_6 \setminus (\{c(v_3), c(u), c(v_1)\} \cup [\beta])$ , a contradiction. So, there is no available colour in  $[c(v_1)] \cup \{6\}$  for  $t_2$ ; that is,  $[c(v_1)] \cup \{6\} \subseteq \{1, 2, c(u), c(t'_2)\}$ . Since  $c(v_1) \notin \{1, c(u), c(t'_2)\}$ , we must have  $c(v_1) = 2$  and  $\{c(u), c(t'_2)\} = \{3, 6\}$ . colour  $u'$  with 2 (since  $v'_2 \in N(v_1)$  we know that  $c(v'_2) \neq c(v_1)$ ),  $v_2$  with 1 and  $v_3$  with  $c(t'_2)$ . Colour  $t_2$  with 4 if  $c(u) = 3$  and with 5 otherwise. This gives a  $(G, T)$ -colouring, a contradiction.

Now, suppose that  $ru' \in E(G)$ . If  $c(u) \neq 6$ , then we can colour  $t_2$  with 6 and  $u'$ ,  $v_3$  and  $v_2$  can be greedily coloured in this order, a contradiction; thus,  $c(u) = 6$ . Let  $L(u') = Z_6 \setminus \{1, c(t'_2), c(v'_2)\}$  be the colours available for  $u'$ ; note that if  $L(u') = [i]$  for some  $i \in Z_6$  then  $c(t'_2) = 6$  and  $c(v'_2) = 2$ , a contradiction since  $c(t'_2) \neq c(u)$ . Clearly, there exists  $\beta \in [c(v_1)] \setminus \{1, 2\}$  so we can colour  $t_2$  with  $\beta$ ,  $u'$  with any colour in  $L(u') \setminus [\beta]$  (recall that  $L(u') \neq [i]$  for all  $i \in Z_6$ ). Then colour  $v_3$  and  $v_2$  greedily gives a  $(G, T)$ -colouring, a contradiction.

- $C_f^{int}$  is isomorphic to  $B_1$  or  $B_2$ ,  $C_{f'}^{int}$  is isomorphic to  $A_2$  or  $A_3$  and  $g = v_1 v'_1$ .

By minimality of  $(G, T)$ , there is a  $((G - \{t_2, v_2, v_3\}) \cup \{uv'_1, rv'_1\}, T - \{t_2, v_2, v_3\})$ -colouring, which is a  $(G - \{t_2, v_2, v_3\}, T - \{t_2, v_2, v_3\})$ -colouring such that  $c(v'_1), c(u') \neq 1$  and  $c(v'_1) \neq c(u)$ .

Suppose that we can colour  $t_2$  with  $\beta \in [c(v_1)] \cup \{6\}$ . We know that there is at least one colour  $i \in Z_6 \setminus (\{1, c(u), c(v_1)\} \cup [\beta])$  available for  $v_2$  and at least one colour  $j \in Z_6 \setminus (\{c(v'_1), c(u'), c(v_1)\} \cup [\beta])$  available for  $v_3$ . Since  $c(v'_1) \notin \{1, c(u)\}$ , then  $i \neq j$  and we can colour  $v_2$  with  $i$  and  $v_3$  with  $j$  to obtain a  $(G, T)$ -colouring, a contradiction.

So, suppose that the colours of  $[c(v_1)] \cup \{6\}$  all appear in  $N(t_2)\{v_2, v_3\}$ ; since  $|([c(v_1)] \cup \{6\}) \setminus \{1, 2\}| \geq 2$ ,  $t_2$  must be adjacent to at least one of  $u$  and  $t'_1$ .

Assume first  $t_2 t'_1 \in E$ . Then recolour  $u'$  with 1. If  $ut_2 \notin E$  or  $[c(v_1)] \cup \{6\} \not\subseteq \{1, 2, c(u), c(t'_1)\}$ , note that we can apply the same argument as before since it holds even if  $c(u') = 1$ ; so suppose otherwise. In this case we must have: either (a)  $c(v_1) = 6$ ,  $c(u) = 5$  and  $c(t'_1) = 6$ ; or (b)  $c(v_1) = 2$  and  $\{c(u), c(t'_1)\} = \{3, 6\}$ . colour  $v_2$  with 1. If (a) occurs, then colour  $t_2$  with 3 and  $v_3$  with 5; if (b) occurs and  $c(t'_1) = 6$ , then colour  $t_2$  with 4 and  $v_3$  6; if (b) occurs and  $c(t'_1) = 3$ , then colour  $t_2$  with 5 and  $v_3$  with 3.

Hence  $t_2 t'_1 \notin E$ , and so  $t_2 u \in E$ . The possible situations are: (c)  $c(v_1) = 6$ ,  $c(u) = 5$  and  $c(u') = 6$ ; or (d)  $c(v_1) = 2$  and  $\{c(u), c(u')\} = \{3, 6\}$ . If (c) occurs, then colour  $v_3$  with  $\{2, 5\} \setminus \{c(v'_1)\}$  and  $t_2$  with  $\{3, 4\} \setminus [c(v_3)]$ . If (d) occurs, then colour  $v_3$  with  $c(u)$  (recall that  $c(v'_1) \neq c(u)$ ) and  $t_2$  with  $\{4, 5\} \setminus [c(v_3)]$ . In both cases we get a  $(G, T)$ -colouring, a contradiction.

- $C_f^{int}$  is isomorphic to  $B_1$ ,  $C_{f'}^{int}$  is isomorphic to  $A_1$  and  $g = v_1v'_1$ .  
 Let  $(G', T')$  be the graph pair obtained from  $(G - \{u, v_2, t_2, v_3, u'\}, T - \{u, v_2, t_2, v_3, u'\})$  by identifying  $t_1$  and  $t'_1$ . By minimality of  $(G, T)$ , there exists a  $(G', T')$ -colouring which is a  $(G - \{u, v_2, t_2, v_3, u'\}, T - \{u, v_2, t_2, v_3, u'\})$ -colouring such that  $c(t_1) = c(t'_1)$ . Set  $c(t_2) = 6$  and  $c(u') = c(v_1)$ . Let  $L = \{2, 3, 4\} \setminus \{c(v_1), c(v'_1)\}$ .  
 If  $c(t_1) \neq 5$ , then choosing  $c(v_3) \in L$ , and applying Lemma 19, we obtain a  $(G, T)$ -colouring, a contradiction. Hence  $c(t_1) = c(t'_1) = 5$ .  
 If  $L \neq \{4\}$ , then we can choose  $c(v_3) \in L \setminus \{4\}$ , and apply Lemma 19 to get a  $(G, T)$ -colouring, a contradiction. Hence  $\{c(v_1), c(v'_1)\} = \{2, 3\}$ .  
 Now setting  $c(v_3) = 5$ ,  $c(v_2) = 6$ ,  $c(u) = c(v'_1)$  and recolouring  $t_2$  with 3, we obtain a  $(G, T)$ -colouring, a contradiction.
- $C_f^{int}$  is isomorphic to  $B_1$ ,  $C_{f'}^{int}$  is isomorphic to  $A_1$  and  $g = v_3v'_1$ .  
 Let  $(G', T')$  be the graph pair obtained from  $(G - \{v_1, v_2, u, t_1, u'\}, T - \{v_1, v_2, u, t_1, u'\})$  by identifying  $t_2$  and  $t'_1$ . By minimality of  $(G, T)$ , there exists a  $(G', T')$ -colouring which is a  $(G - \{v_1, v_2, u, t_1, u'\}, T - \{v_1, v_2, u, t_1, u'\})$ -colouring such that  $c(t_2) = c(t'_1)$ . Set  $c(t_1) = 6$ .  
 If  $c(t_2) = c(t') = 6$ , then set  $c(u) = c(v_3)$ . One can then greedily extend the colouring to  $v_1, v_2$  and  $u'$  in this order, a contradiction.  
 If  $c(t_2) \neq 6$ , then one can choose  $c(v_1) \in [c(t_2)] \setminus \{5, 6\}$ . This is valid since  $c(v_3)$  and  $c(v'_1)$  are not in  $[c(t_2)]$ . One can then greedily extend the colouring to  $v_2, u$  and  $u'$  in this order, a contradiction.
- $C_f^{int}$  is isomorphic to  $B_2$ ,  $C_{f'}^{int}$  is isomorphic to  $A_1$  and  $g = v_3v'_1$ .  
 By minimality of  $(G, T)$ , there exists a  $(G - \{t_1, u, v_1, v_2\} \cup \{rv_3\}, T - \{t_1, u, v_1, v_2\})$ -colouring  $c$ . Set  $c(v_2) = 1$  and let  $L(v_1) = Z_6 \setminus \{1, c(v_3), c(u'), c(v'_1)\}$  be the colours available for  $v_1$ .  
 If  $L(v_1) \neq \{5, 6\}$ , then colouring  $t_1$  with 6,  $v_1$  with some colour in  $L(v_1) \setminus \{5, 6\}$  and  $u$  with some colour in  $Z_6 \setminus \{1, 5, 6, c(v_1), c(t_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction.  
 If  $L(v_1) = \{5, 6\}$ , then  $c(u'), c(v'_1) \in \{2, 3, 4\}$  and consequently  $c(t'_1) \in \{5, 6\}$ . We can suppose that  $c(t'_1) = 5$  and  $c(v_3) = 4$  for otherwise we can recolour  $u'$  with  $c(v_3)$  and fall in the case  $L(v_1) \neq \{5, 6\}$ . So,  $\{c(u'), c(v'_1)\} = \{2, 3\}$  and  $c(t_2) \geq 6$ . Setting  $c(t_1) = 3$ ,  $c(u) = 5$  and  $c(v_1) = 6$ , we obtain a  $(G, T)$ -colouring, a contradiction.

□

The next two lemmas prove that the case of an antepenultimate edge  $g$  with only one predecessor  $f$ , which must be penultimate, is also impossible. Lemma 22 prove for  $C_f^{int} = B_1$  and Lemma 23 prove for  $C_f^{int} = B_2$ .

**Lemma 22** *There is no antepenultimate edge  $g$  with only one penultimate predecessor  $f$  such that  $C_f^{int}$  is  $B_1$ .*

**Proof.** One of the endvertices of  $g$  must be  $v_1$  or  $v_3$  (see Figure 3). We now distinguish some cases depending on the possible endvertices of  $g$ .



- (a) Assume  $g = v'v_3$  with  $v'$  a leaf with twig  $t'$ . Since  $rv_1 \in E(G)$ , by planarity,  $t' \neq t_1$ . Let  $(G', T')$  be the graph pair obtained from  $(G - \{t_1, v_1, u, v_2\}, T - \{t_1, v_1, u, v_2\})$  by identifying  $t'$  and  $t_2$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring which is a  $(G - \{t_1, v_1, u, v_2\}, T - \{t_1, v_1, u, v_2\})$ -colouring such that  $c(t') = c(t_2)$ . Set  $c(v_2) = c(v')$  and  $c(t_1) = 6$ .

If  $c(t') \in \{5, 6\}$ , then setting  $c(u) = c(v_3)$  and choosing  $c(v_1)$  in  $\{2, 3, 4\} \setminus \{c(v'), c(v_3)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction.

If  $c(t') \in \{3, 4\}$ , then setting  $c(v_1) = c(t') - 1$  and choosing  $c(u)$  in  $\{2, 3, 4\} \setminus \{c(v_1), c(v_2)\}$ , we obtain a  $(G, T)$ -colouring, a contradiction.

- (b) Assume  $g = t'v_3$  with  $t'$  a twig. We can apply an argument similar to (a) choosing  $c(v_2) \in Z_6 \setminus (\{1, c(v_3)\} \cup [c(t_2)])$ .
- (c) Assume  $g = v_1r$ . Since  $G$  is triangulated, the edge  $v_1t_2$  must exist. This is a contradiction, since  $f$  is the successor of  $e$ .
- (d) Assume  $g = v_1t_2$ . Then  $v_3$  is a leaf of degree at most 3, a contradiction from Lemma 7.
- (e) Assume  $g = v_1v'$  with  $v'$  a leaf with twig  $t_2$ . Let  $(G', T')$  be the graph pair obtained from  $(G - \{v_3\}, T - \{v_3\})$  by identifying  $v_2$  and  $v'$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring which is a  $(G - \{v_3\}, T - \{v_3\})$ -colouring such that  $c(v_2) = c(v')$ . Hence one can colour  $v_3$  with a colour from  $Z_6 \setminus \{[c(t_2)], c(v_2), c(v_1)\}$  to obtain a  $(G, T)$ -colouring, a contradiction.
- (f) Assume  $g = v_1t'$  with  $t' \neq t_2$  a twig. Since  $g$  is the successor of  $f$ ,  $v_1t_2$  is not an edge and  $v_1r$  is not inside  $C_g$ , so  $v_3t' \in E$ .

Assume first that  $rv_3 \notin E(G)$ . By minimality of  $(G, T)$ , there is a  $((G - \{u, v_2, v_3\}) \cup t_1t_2, T - \{u, v_2, v_3\})$ -colouring which is a  $(G - \{u, v_2, v_3\}, T - \{u, v_2, v_3\})$ -colouring such that  $c(t_1) \neq c(t_2)$ . Since  $c(t'), c(v_1) \neq 1$ , we can colour  $v_3$  with 1. Then, by Lemma 19 (b), there is a  $(G, T)$ -colouring, a contradiction.

Assume now that  $rv_3 \in E(G)$ . Then  $t't_2 \notin E$  by planarity. Let  $(G', T')$  be the graph pair obtained from  $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$  by identifying  $t_1$  and  $t'$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which is a  $((G - \{u, t_2, v_2, v_3\}), T - \{u, t_2, v_2, v_3\})$ -colouring such that  $c(t_1) = c(t')$ . Set  $c(t_2) = 6$ . One can choose  $c(v_3) \in \{2, 3\} \setminus \{c(t'), c(v_1)\}$  because  $|c(v_1) - c(t')| \geq 2$ . Then by Lemma 19 (a), there is a  $(G, T)$ -colouring, a contradiction.

- (g) Assume  $g = v_1v'$  with  $v'$  a leaf with twig  $t' \neq t_2$ . Since  $g$  is the successor of  $f$ ,  $v_1t_2$  is not an edge and  $v_1r$  and  $v_1t'$  are not inside  $C_g$ , so  $v_3t' \in E$ .

Assume first that  $rv_3 \notin E(G)$ . By minimality of  $(G, T)$ , there is a  $((G - \{u, t_2, v_2, v_3\}) \cup rv', T - \{u, t_2, v_2, v_3\})$ -colouring which is a  $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ -colouring such that  $c(v') \neq 1$ . Since  $c(t'), c(v'), c(v_1) \neq 1$ , we can colour  $v_3$  with 1. Then, colouring  $t_2$  with a colour from  $Z_6 \setminus \{1, 2, c(t'), c(v'), c(t_1)\}$  and using Lemma 19 (b), we obtain a  $(G, T)$ -colouring, a contradiction.

Assume now that  $rv_3 \in E(G)$ . Let  $(G', T')$  be the graph pair obtained from  $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$  by identifying  $t_1$  and  $t'$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring, which is a  $((G - \{u, t_2, v_2, v_3\}), T - \{u, t_2, v_2, v_3\})$ -colouring such

that  $c(t_1) = c(t')$ . If  $\{c(v_1), c(v')\} \neq \{2, 3\}$ , then one can choose  $c(v_3)$  in  $\{2, 3\} \setminus \{c(t'), c(v'), c(v_1)\}$ . Then setting  $c(t_2) = 6$  and applying Lemma 19 (a), we obtain a  $(G, T)$ -colouring, a contradiction. Thus  $\{c(v_1), c(v')\} = \{2, 3\}$ , and so  $c(t_1) \geq 5$ . Setting  $c(u) = c(v')$ ,  $c(t_2) = 3$  and choosing  $c(v_3)$  in  $\{5, 6\} \setminus c(t')$  and  $c(v_2)$  in  $\{5, 6\} \setminus c(v_3)$  yields a  $(G, T)$ -colouring, a contradiction.

□

**Lemma 23** *There is no antepenultimate edge  $g$  with only one penultimate predecessor  $f$  such that  $C_f^{int}$  is  $B_2$ .*

**Proof.** One of the endvertices of  $g$  must be  $v_1$  or  $v_3$  (see Figure 3). We now distinguish some cases depending on the possible endvertices of  $g$ .

- (a) Assume  $g = v'v_3$  with  $v'$  a leaf with twig  $t'$ . Since  $rv_1 \in E(G)$ , by planarity,  $t' \neq t_1$ . By minimality of  $(G, T)$ , there is a  $((G - \{t_1, v_1, u\}) \cup \{t_2v', t_2t'\}, T - \{t_1, v_1, u\})(G', T')$ -colouring which is a  $(G - \{t_1, v_1, u\}, T - \{t_1, v_1, u\})$ -colouring such that  $c(v') \neq c(t_2)$  and  $c(t') \neq c(t_2)$ . Hence one can colour  $v_1$  with  $c(t_2)$  and colour  $t_1$  with a colour in  $Z_6 \setminus ([c(t_2)] \cup \{1, 2\})$ . From Lemma 20 (b2), we obtain a  $(G, T)$ -colouring, a contradiction.
- (b) Assume  $g = t'v_3$  with  $t'$  a twig. We can apply an argument similar to (a).
- (c) Assume  $g = v_1r$ . Since  $G$  is triangulated, the edge  $v_1t_2$  must exist. This is a contradiction, since  $f$  is the successor of  $e$ .
- (d) Assume  $g = v_1t_2$ . Then  $v_3$  is a leaf of degree at most 3, a contradiction from Lemma 7.
- (e) Assume  $g = v_1v'$  with  $v'$  a leaf adjacent to  $t_2$  in  $T$ . Since  $f$  is the successor of  $v_1v_3$ ,  $v_1t_2 \notin E(G)$  and so  $v_2v_3 \in E(G)$  because  $G$  is triangulated. Let  $(G', T')$  be the graph pair obtained from  $(G - \{v_3\}, T - \{v_3\})$  by identifying  $v_2$  and  $v'$ . By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring which is a  $(G - \{v_3\}, T - \{v_3\})$ -colouring such that  $c(v_2) = c(v')$ . Hence one can colour  $v_3$  with a colour from  $Z_6 \setminus \{[c(t_2)], c(v_2), c(v_1)\}$  to obtain a  $(G, T)$ -colouring, a contradiction.
- (f) Assume  $g = v_1t'$  with  $t' \neq t_2$  a twig. Let  $(G', T')$  be the graph pair obtained from  $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$  by identifying  $t_1$  and  $t'$ . This is possible since  $t_1t'$  is not an edge by planarity. By minimality of  $(G, T)$ , there is a  $(G', T')$ -colouring which is a  $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ -colouring such that  $c(t') = c(t_1)$ . Let  $L(t_2) = Z_6 \setminus \{1, 2, c(t')\}$ . If  $c(t_1) = 6$ , then, colouring  $v_3$  with 1 and  $t_2$  with a colour from  $L(t_2) \setminus \{6\}$ , and using Lemma 20 (b1), we obtain a  $(G, T)$ -colouring, a contradiction. So  $c(t_1) \neq 6$ , that is  $c(t_1) \in \{3, 4, 5\}$ . We can colour  $t_2$  with a colour from  $[c(t_1)] \setminus \{c(t'), c(t_1)\} \subseteq L(v_2)$ . By Lemma 20 (b3), there is a  $(G, T)$ -colouring, a contradiction.
- (g) Assume  $g = v_1v'$  with  $v'$  a leaf with twig  $t' \neq t_2$ . By minimality of  $(G, T)$ , there is a  $((G - \{u, t_2, v_2, v_3\}) \cup \{v_1t'\}, T - \{u, t_2, v_2, v_3\})(G', T')$ -colouring which is a  $(G - \{u, t_2, v_2, v_3\}, T - \{u, t_2, v_2, v_3\})$ -colouring such that  $c(v_1) \neq c(t')$ . If  $c(v_1) \neq 2$ , we can colour  $t_2$  with  $c(v_1)$ , since  $c(t'), c(v') \neq c(v_1)$ . Then, colouring  $v_3$  with a colour from  $Z_6 \setminus \{[c(v_1)], c(t'), c(v')\}$ , and applying Lemma 20 (b2), we obtain a  $(G, T)$ -colouring, a contradiction. So,  $c(v_1) = 2$ .

Suppose that  $c(v') \neq 1$ . Since  $c(t'), c(v') \notin \{1, 2\}$ , then  $\{c(t'), c(v')\} \in \{\{3, 5\}, \{3, 6\}, \{4, 6\}\}$ . Let  $L(v_3) = Z_6 \setminus \{2, c(t'), c(v')\}$  and let  $L(t_2) = Z_6 \setminus \{1, 2, c(t'), c(v')\}$ . If  $L(v_2) \cap ([c(t_1)] \setminus \{c(t_1)\}) \neq \emptyset$ , then choosing  $c(t_2)$  in  $[c(t_1)] \setminus \{c(t_1)\}$  and  $c(v_3)$  in  $L(v_3) \setminus [c(t_2)]$  (observe that  $|[c(t_2)] \cap L(v_3)| \leq 2$ , since  $L(v_3)$  has no three consecutive integers), and using Lemma 20 (b3), we obtain a  $(G, T)$ -colouring, a contradiction. Then  $L(v_2) \cap ([c(t_1)] \setminus \{c(t_1)\}) = \emptyset$ . If  $c(t_1) = 3$ , then  $\{c(t'), c(v')\} = \{4, 6\}$ . In this case, colouring  $t_2$  with 3,  $v_3$  and  $u$  with 5 and  $v_2$  with 6, we can obtain a  $(G, T)$ -colouring, a contradiction. If  $c(t_1) = 4$ , then  $\{c(t'), c(v')\} = \{3, 5\}$ , and if  $c(t_1) = 5$ , then  $\{c(t'), c(v')\} = \{4, 6\}$ . In both cases, setting  $c(t_2) = c(t_1)$ , choosing  $c(v_3)$  in  $Z_6 \setminus \{1, 2, [c(t_1)]\}$  and using Lemma 20 (a), we obtain a  $(G, T)$ -colouring, a contradiction.

Hence  $c(v') = 1$ . If  $c(t_1) \in \{3, 4\}$ , colour  $t_2$  with 3 (if  $c(t') \neq 3$ ) or 4 (otherwise). If  $c(t_1) = 5$ , colour  $t_2$  with 6 (if  $c(t') \neq 6$ ) or 5 (otherwise). These cases satisfy the conditions  $c(t_2) \in [c(t_1)]$  and  $Z_6 \setminus \{1, 2, c(t'), [c(t_2)]\} \neq \emptyset$ . Then, colouring  $v_3$  with a colour from  $Z_6 \setminus \{1, 2, c(t'), [c(t_2)]\}$ , and using Lemma 20 ((a) or (b3)), we obtain  $(G, T)$ -colouring, a contradiction.

□

Lemmas 18, 21, 22 and 23 directly imply the following.

**Corollary 24**  $(G, T)$  has no antepenultimate edges.

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