Algebraic parameter estimation of a multi-sinusoidal waveform signal from noisy data

Rosane Ushirobira, Wilfrid Perruquetti, Mamadou Mboup, Michel Fliess

Algebraic parameter estimation of a multi-sinusoidal waveform signal from noisy data*

Rosane Ushirobira[1], Wilfrid Perruquetti[2], Mamadou Mboup[3] and Michel Fliess[4]

***Abstract*— In this paper, we apply an algebraic method to estimate the amplitudes, phases and frequencies of a biased and noisy sum of complex exponential sinusoidal signals. Let us stress that the obtained estimates are integrals of the noisy measured signal: these integrals act as time varying filters. Compared to usual approaches, our algebraic method provides a more robust estimation of these parameters within a fraction of the signal's period. We provide some computer simulations to demonstrate the efficiency of our method.**

I. Introduction

Numerous practical engineering problems involve the estimation of the frequencies of a biased and noisy sum of complex exponential sinusoidal signals, e.g. signal demodulation in communications, regulation of electronic converters power, the circadian rhythm of biological cells and the modal identification for flexible structures (see [51]). A very motivating example, developed in [42], is the position reconstruction of a human body in the sagittal plane using only accelerometer measurements.

Several different methods have been elaborated to solve this particular estimation problem, (see [25], [49] for surveys), such as linear regression [5], [41], adaptive least square method [47], subspace methods (high resolution) [12], [22], [46], [24], the extended Kalman filter introduced in [26], [27], [28] and refined in [2] where a simple tuning rule is given, the notches filter introduced simultaneously in [21] and [44] providing biased estimates of the frequency for standard notch (see [48]) with a first improvement obtained in [1] and an adaptive version in [9] (see also [45]), adaptive sogi-filters [13], techniques borrowed from adaptive nonlinear control [23], [38] or alternatively [29], [30] and more recently [3], [6], [7], [56]. Let us stress that almost all the above mentioned results (except [25], [5], [13] that needs half of the period to recover the parameters and [51] that uses also algebraic techniques for a single sinusoidal) deal only with the frequency estimation problem: here our method can be extended to estimate all the parameters including amplitudes and phases (see example in sections IV and V). Nevertheless, obtaining a robust estimation in the presence of noise and an unknown constant bias, continues to be an issue not quite solved.

Other interesting feature of our algebraic approach is that it is fast and online. Therefore, a comparison procedure with offline methods, such as the maximum likelihood estimation (see [11]), does not apply.

The algebraic methods used in this paper are inspired by the fundamental work of M. Fliess et al. [20], [18], [17], [19], [15], [35]. For more results in practical examples, we refer to [40], [50], [51], [53].

The parameter estimation problem for a finite sum of sinusoidal functions was notably studied by G. Riche de Prony in his 1795 seminal paper [43] (see also [24], [41]). In this paper, we are interested in Prony's problem for the estimation of frequencies in a sum of complex sinusoidal functions. In other words, our first goal is to estimate the frequencies of the signal

$$x(t) = \sum_{k=1}^{n} \alpha_k \exp\left(i\left(\omega_k t + \phi_k\right)\right) \tag{1}$$

from the biased and noisy output measure

$$y(t) = x(t) + \beta + \varpi$$

where β is an unknown constant bias and ϖ is a noise.[1]

Let us remark that, in its generality, the problem of parameter estimation problem for $x(t)$ consists on the estimation of the triplet $(\alpha_k, \omega_k, \phi_k)$ for all k. For the sake of brevity, we present here only the frequencies estimation in the general case. However, an example in the case $n = 3$ is treated at the end of this paper, where we also indicate how to proceed for the calculation of the triplets (amplitude, frequencies, phases).

II. Problem formulation

To formulate the problem of the parametric estimation, we start with a signal depending on a set of parameters. We wish to estimate some of these parameters. They form a vector that denoted by Θ. Based on the observed noisy signal, our aim is to obtain a "good" approximation of Θ.

*This work was supported by Inria Lille - Nord Europe, France. It was partially supported by the European INTERREG IV A 2 Mers Seas Zeen Cross-border Cooperation Program 2007-2013 under SYSIASS 6-20 project. It was also partially supported by Ministry of Higher Education and Research, Nord-Pas de Calais Regional Council and FEDER through the Contrat de Projets Etat Region (CPER) CIA 2007-2013.

[1]Rosane Ushirobira is with Non-A, Inria & Institut de Mathématiques de Bourgogne - Université de Bourgogne, France (e-mail: Rosane.Ushirobira@inria.fr).

[2] Wilfrid Perruquetti is with Université Lille Nord de France & École Centrale de Lille & Non-A, Inria & LAGIS (CNRS), France (e-mail: Wilfrid.Perruquetti@inria.fr).

[3] Mamadou Mboup is with CReSTIC - Université de Reims Champagne Ardenne & Non-A, Inria, France (e-mail: Mamadou.Mboup@univ-reims.fr).

[4] Michel Fliess is with LIX (CNRS), École polytechnique (e-mail: Michel.Fliess@polytechnique.edu) & AL.I.E.N., Nancy, France, www.alien-sas.com

[1]Here, the noise is interpreted as a *fast oscillation* and it does not depend on any probabilistic modeling, as in [14], [15].

Let us denote by θ_k $(1 \leq k \leq n)$ a multiple of the elementary symmetric polynomial in n variables $\omega_1, \ldots, \omega_n$:

$$\theta_k := (-i)^k \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \omega_{j_1} \omega_{j_2} \ldots \omega_{j_k}. \tag{2}$$

That means that $\theta_1, \ldots, \theta_n$ can be obtained as the coefficients of the polynomial in the variable X given by

$$\prod_{\ell=1}^{n} (X - i\omega_\ell) = X^n + \theta_1 X^{n-1} + \theta_2 X^{n-2} + \cdots + \theta_n.$$

It is easy to see that the signal $z(t) = x(t) + \beta$ and the vector $\Theta = \{\theta_1, \ldots, \theta_n\}$ satisfy a linear differential algebraic relation provided by the differential equation:

$$z^{(n)}(t) + \sum_{k=0}^{n-1} \theta_{k+1} z^{(k)}(t) - \theta_n \beta = 0. \tag{3}$$

The Laplace transform applied on the equation (3) gives the following relation in the operational domain:

$$\begin{aligned} &s\left(s^n + \sum_{k=0}^{n-1} \theta_{k+1} s^k\right) Z(s) - s^2 z^{(n-1)}(0) \\ &- \sum_{j=0}^{n-2} \left(s^{n-j} + \sum_{k=j+1}^{n-1} \theta_{k+1} s^{k-j}\right) z^{(j)}(0) - \theta_n \beta = 0. \end{aligned} \tag{4}$$

We use the notation $\Theta_{\text{est}} := \{\theta_1, \theta_2, \ldots, \theta_n\}$, rather than Θ for the desired parameters. For $1 \leq \ell \leq n$, let us set $\theta_{n+\ell} := -x^{(\ell-1)}(0)$. Since the bias $\theta_{2n+1} := \beta$ is not of interest, we also define $\Theta_{\overline{\text{est}}} := \{\theta_{n+1}, \ldots, \theta_{2n}, \theta_{2n+1}\}$, the set of undesired parameters. The frequencies ω_k $(1 \leq k \leq n)$ can be deduced from Θ_{est} from a straightforward computation.

Now, consider the algebraic extensions $\mathbb{C}_{\Theta_{\text{est}}} := \mathbb{C}(\Theta_{\text{est}})$ and $\mathbb{C}_{\Theta_{\overline{\text{est}}}} := \mathbb{C}(\Theta_{\overline{\text{est}}})$ and denote by $\mathbb{C}_{\Theta_{\text{est}}}[s]$ (respectively $\mathbb{C}_{\Theta_{\overline{\text{est}}}}[s]$) the polynomial ring in the variable s with coefficients in $\mathbb{C}_{\Theta_{\text{est}}}$ (respectively in $\mathbb{C}_{\Theta_{\overline{\text{est}}}}$). We set

$$T(s) := s^n + \sum_{k=0}^{n-1} \theta_{k+1} s^k = \prod_{\ell=1}^{n} (s - i\omega_\ell) \in \mathbb{C}_{\Theta_{\text{est}}}[s].$$

The relation below arises naturally from equation (4):

$$\mathcal{R}\left(s, Z(s), \Theta_{\text{est}}, \Theta_{\overline{\text{est}}}\right) := P(s)\, Z(s) + \overline{Q}(s) = 0 \tag{5}$$

where $P(s) = s\, T(s) \in \mathbb{C}_{\Theta_{\text{est}}}[s]$ and

$$\begin{aligned} \overline{Q}(s) = s \sum_{j=0}^{n-1} \left(s^{n-j-1} + \sum_{k=j+1}^{n-1} \theta_{n-k} s^{k-j-1}\right) \theta_{n+j+1} \\ - T(s)\theta_{2n+1} \in \mathbb{C}_{\Theta_{\text{est}}}[\Theta_{\overline{\text{est}}}][s] \end{aligned} \tag{6}$$

We start by eliminating $\Theta_{\overline{\text{est}}}$ in equation (5). In other words, the polynomial $\overline{Q}(s)$ must be annihilated. Then we shall obtain a system of equations depending uniquely on Θ_{est}. For that purpose, we proceed in three steps:

1) Algebraic elimination of $\Theta_{\overline{\text{est}}}$: we use the canonical form of the **minimal $\overline{Q}$-annihilator**, the operator in $\mathbb{C}_{\Theta_{\text{est}}}(s)\left[\frac{d}{ds}\right]$ [2] that generates all differential operators annihilating $\overline{Q}$.
2) Obtaining a system of equations on Θ_{est}: the canonical forms of the differential operators generated by the minimal $\overline{Q}$-annihilator provide a system of equations with good numerical properties in the time domain.
3) Resolution of the resulting system: to bring the equations back to the time domain, we use the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{s^m}\frac{d^p Z(s)}{ds^p}\right) = \frac{(-1)^p}{(m-1)!}\int_0^t v_{m-1,p}(\tau) z(\tau) d\tau \tag{7}$$

with $v_{m,p}(\tau) = (t-\tau)^m \tau^p, \forall\ p, m \in \mathbb{N}, m \geq 1$. To reduce the noisy influence in our estimation, we choose the integers m and p as small as possible.

The algebraic framework for our method is described in Section III, where we also define the minimal annihilators mentioned in the first point. In subsection III-A, we detail the canonical form of the annihilators and recall some well-known properties of the Weyl algebra. In Section IV, we give the frequencies estimation. Numerical simulations are given in Section V to illustrate the efficiency of our algebraic method, using a comparison with the modified Prony's method.

III. Annihilators via the Weyl algebra

As we have seen in the preceding Section, our goal is to annihilate the polynomial $\overline{Q}$, see (6). For that, we use differential operators, that is, polynomials in the variable $\frac{d}{ds}$ with polynomial coefficients in the variable s. The polynomial $\overline{Q}$ has degree n, therefore it is clear that any differential operator of lowest degree, with respect to the variable $\frac{d}{ds}$, greater than n annihilates $\overline{Q}$, for example $\Pi_1 = \left(s\frac{d}{ds} - n\right) \circ \cdots \circ \left(s\frac{d}{ds} - 1\right) \circ \left(s\frac{d}{ds}\right)$ and $\Pi_2 = \frac{d^n}{ds^n}$. Some natural questions arise such as whether these annihilators are the same or if there exists a lower order [3] annihilator. The structure of the Weyl algebra $\mathbb{C}_\Theta(s)\left[\frac{d}{ds}\right]$ helps answering these questions.

Let us stress that this algebraic point of view is inspired by the work of M. Fliess et al. [17], [18], [19], [15], [35] [4]. The algebraic notions defined below are detailed in [10] and [37].

A. The Weyl Algebra

Definition 1: Let $\mathbb{K}$ be a field of characteristic zero. Let $k \in \mathbb{N} \setminus \{0\}$. The **Weyl algebra** $\mathsf{A}_k(\mathbb{K})$ is the $\mathbb{C}$-algebra generated by $p_1, q_1, \ldots, p_k, q_k$ satisfying the relations

$$[p_i, q_j] = \delta_{ij},\ [p_i, p_j] = [q_i, q_j] = 0, \forall\ 1 \leq i, j \leq k$$

where $[\cdot, \cdot]$ is the commutator defined by $[u, v] := uv - vu$, for all $u, v \in \mathsf{A}_k(\mathbb{K})$. Sometimes we write simply A_k.

[2] The polynomial ring in $\frac{d}{ds}$ with coefficients in $\mathbb{C}_{\Theta_{\text{est}}}[s]$

[3] The order of an operator $\Pi \in \mathbb{C}_\Theta(s)\left[\frac{d}{ds}\right]$ is its degree as a polynomial in the variable $\frac{d}{ds}$.

[4] Similar tools were used for numerical differentiation of noisy signal [36], [33] and spike detection [16].

A very useful realization of the Weyl algebra A_k is to consider it as the algebra of polynomial differential operators on $\mathbb{K}[s_1,\ldots,s_k]$ such that

$$p_i = \frac{\partial}{\partial s_i} \quad \text{and} \quad q_i = s_i \times \cdot \ , \forall\ 1 \leq i \leq k.$$

As a consequence, we can write

$$\mathsf{A}_k = \mathbb{K}[q_1,\ldots,q_k][p_1,\ldots,p_k] = \mathbb{K}[s_1,\ldots,s_k]\left[\frac{\partial}{\partial s_1},\ldots,\frac{\partial}{\partial s_k}\right]$$

(remark that the same notation is used for the variable s_i and for the operator "multiplication by s_i").

A closely related algebra to $\mathsf{A}_k(\mathbb{K})$ is defined as the differential operators on $\mathbb{K}[s_1,\ldots,s_k]$ with coefficients in the rational functions field $\mathbb{K}(s_1,\ldots,s_k)$. We denote it by $\mathsf{B}_k(\mathbb{K})$, or B_k for short. We can write

$$\mathsf{B}_k := \mathbb{K}(q_1,\ldots,q_k)[p_1,\ldots,p_k] = \mathbb{K}(s_1,\ldots,s_k)\left[\frac{\partial}{\partial s_1},\ldots,\frac{\partial}{\partial s_k}\right].$$

In the case $k=1$ for instance, we have

$$\mathsf{A}_1 = \langle p,q \mid pq-qp=1\rangle = \mathbb{K}[s]\left[\frac{d}{ds}\right] \quad \text{and} \quad \mathsf{B}_1 = \mathbb{K}(s)\left[\frac{d}{ds}\right]$$

Proposition 2: A basis for A_k is given by $\left\{q^I p^J \mid I,J \in \mathbb{N}^k\right\}$ where $q^I := q_1^{i_1}\ldots q_k^{i_k}$ and $p^J := p_1^{j_1}\ldots p_k^{j_k}$ if $I=(i_1,\ldots,i_k)$ and $J=(j_1,\ldots,j_k)$. Therefore an operator $F \in \mathsf{A}_k$ can be written in a *canonical form*,

$$F = \sum_{I,J} \lambda_{IJ} q^I p^J \quad \text{with} \quad \lambda_{IJ} \in \mathbb{K}.$$

Example 3: We need later the fallowing useful identity:

$$p^n q^m \quad = \quad q^m p^n + \sum_{k=1}^{n} \binom{n}{i}\binom{m}{i} i! q^{m-i} p^{n-i}$$

An element $F \in \mathsf{B}_k$ can be similarly written as

$$F = \sum_I \lambda_I g_I(s) p^I, \quad \text{where} \quad g_I(s) \in \mathbb{K}(s_1,\ldots,s_k).$$

The **order** of an element $F \in \mathsf{B}_k$, $F = \sum_I \lambda_I g_I(s) p^I$ is defined as $\mathrm{ord}(F) := \max\{|I| \mid g_I(s) \neq 0\}$. Notice that the same definition holds for the Weyl algebra A_k since $\mathsf{A}_k \subset \mathsf{B}_k$. Some important properties of A_k and B_k are given by the following propositions:

Proposition 4: A_k is a domain. Moreover, A_k is simple and Noetherian.

These properties are shared by B_k. Furthermore, A_k is neither a principal right domain, nor a principal left domain, while this is true for B_k:

Proposition 5: B_1 admits a left division algorithm, that is, if F, $G \in \mathsf{B}_1$, then there exists Q, $R \in \mathsf{B}_1$ such that $F = QG + R$ and $\mathrm{ord}(R) < \mathrm{ord}(G)$. Consequently, B_1 is a principal left domain.

Lastly, it follows from the fact that $\frac{d}{ds}$ is a derivation:

Proposition 6 (Derivation): Given $P_1,\ P_2 \in \mathbb{C}[s]$, we have (Leibniz rule):

$$\frac{d^n}{ds^n}(P_1\ P_2) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k P_1}{ds^k}\ \frac{d^{n-k}P_2}{ds^{n-k}}$$

B. Annihilator

We set $\mathsf{B} := \mathsf{B}_1(\mathbb{C}) = \mathbb{C}(s)\left[\frac{d}{ds}\right]$.

Definition 7: Let $Q \in \mathbb{C}_{\overline{\Theta_{\text{est}}}}[s]$. A *$Q$-annihilator w.r.t.* B is an element of $\mathrm{Ann}_{\mathsf{B}}(Q) = \left\{F \in \mathsf{B} \mid F(Q) = 0\right\}$.

Since B is a left principal domain (see Proposition 5), then $\mathrm{Ann}_{\mathsf{B}}(Q)$ is a left principal ideal, i.e. it is generated by a unique $\Pi_{\min} \in \mathsf{B}$, up to multiplication by a polynomial in B. That means $\mathrm{Ann}_{\mathsf{B}}(Q) = \mathsf{B}\ \Pi_{\min}$. We call $\Pi_{\min}$ a **minimal Q-annihilator w.r.t.** B. Remark that $\mathrm{Ann}_{\mathsf{B}}(Q)$ contains annihilators in finite integral form, *i.e.* operators with coefficients in $\mathbb{C}\left[\frac{1}{s}\right]$. We have the following lemmas:

Lemma 8: Consider $Q(s) = s^n$, $n \in \mathbb{N}$. A minimal Q-annihilator is given by

$$\Pi_n = s\frac{d}{ds} - n.$$

For m, $n \in \mathbb{N}$, the operators Π_m and Π_n commute. Thus, one can use the following Lemma

Lemma 9: Let $P_1, P_2 \in \mathbb{C}_{\overline{\Theta_{\text{est}}}}[s]$. Let Π_i be a P_i-annihilator $(i=1,2)$ such that $\Pi_1\Pi_2 = \Pi_2\Pi_1$. Then $\Pi_1\Pi_2$ is a $(\mu P_1 + \eta P_2)$-annihilator for all $\mu,\eta \in \mathbb{C}_{\overline{\Theta_{\text{est}}}}$.

Now, recall that

$$\overline{Q}(s) = s\sum_{j=0}^{n-1}\left(s^{n-j-1} + \sum_{k=j+1}^{n-1}\theta_{n-k}s^{k-j-1}\right)\theta_{n+j+1} \\ -T(s)\theta_{2n+1}$$

So, the above Lemma provides a minimal $\overline{Q}$-annihilator w.r.t. B:

$$\Pi_{\min} = \left(s\frac{d}{ds} - n\right) \circ \cdots \circ \left(s\frac{d}{ds} - 1\right) \circ \left(s\frac{d}{ds}\right). \quad (8)$$

From the identity in Example 3, it results:

$$\Pi_{\min} = s^n \frac{d^n}{ds^n}. \quad (9)$$

Lemma 10: Let $Q \in \mathbb{C}_{\Theta_{\text{est}}}[s]$. Then a minimal Q-annihilator w.r.t $\mathsf{B}_{\Theta_{\text{est}}}$ is $\Pi_{\min} = Q\frac{d}{ds} - \frac{dQ}{ds}$.

IV. Parameter Estimation

The first step of the estimation is to determine a minimal $\overline{Q}$-annihilator $\Pi_{\min}$. Then, we choose a suitable family of annihilators $\mathcal{F} = (\Pi_i)_{i=1}^r$ in $\mathbb{C}(s)\left[\frac{d}{ds}\right]$ generated by $\Pi_{\min}$ so that $\mathcal{F}$ applied to (5) provides a system of equations in Θ_{est}. Finally, the frequencies are the solutions of this system in the time domain. A similar procedure can be also applied to estimate the remaining parameters (phases and amplitudes).

The order of the operators is one of the factors that must be taken in account when choosing the family $\mathcal{F}$: it must be minimal to reduce noise sensitivity. Also, the use of finite-integral form annihilators is justified by (7). In addition, the choice of a well-balanced system of equations provides "good" numerical properties.

Since the family $\mathcal{F}$ is generated by the minimal $\overline{Q}$-annihilator that $\Pi_{\min}$, its elements are of the form:

$$\Pi = \left(\sum_{i=0}^{\ell} f_i(s)\frac{d^i}{ds^i}\right)\Pi_{\min}(s), \quad (10)$$

with $f_i(s) \in \mathbb{C}(s), \forall\ 1 \leq i \leq \ell$.

We have seen that a minimal $\overline{Q}$-annihilator w.r.t. B is $\Pi_{\min} = s^n \frac{d^n}{ds^n}$, that applied on the relation (5) gives

$$\Pi_{\min}\left(P(s)\,Z(s)\right) = \sum_{j=0}^{n} P_j(s) \frac{d^j Z(s)}{ds^j}$$

where for all $1 \leq j \leq n$,

$$P_j(s) = \frac{(n+1)!}{(n-j)!} s^{n+j+1} + \frac{1}{(n-j)!} \sum_{k=n-1-j}^{n-1} \theta_{k+1}\,(k+1)! s^{k+1+j}$$

Given that $\Pi_{\min}$ annihilates $\overline{Q}$, from the relation (5) follows an algebraic relation involving $\theta_1, \ldots, \theta_n$:

$$\sum_{j=0}^{n} P_j(s) \frac{d^j Z(s)}{ds^j} = 0.$$

However, we need n independent equations to linearly identify Θ_{est}. One can show that this cannot be done in the operational domain, see for instance [55] for a similar proof in a low-dimensional case.

Therefore, the idea is to write the annihilator in (10) in a canonical form:

$$\Pi = \sum_{i=n}^{\ell} g_i(s) \frac{d^i}{ds^i}, \quad \text{with } g_i(s) \in \mathbb{C}(s), \forall\ n \leq i \leq \ell.$$

A suitable choice of the rational functions g_i brings a consistent system of equations in the time domain : set $\ell = 2n-1$ and for $1 \leq i \leq 2n-1$, set $g_i(s) = 1$ and $g_k(s) = 0$, if $k \neq i$. Then, it suffices to solve the system. In the sequel, a very interesting example in the case of a sum of three sinusoidal waveform signals illustrates the usefulness of our algebraic method.

Remark 11: Let us note that in the case of a similar parameter estimation problem of a single noisy sinusoidal waveform signal, a consistent system can be found in the operational domain (see [54]).

Example 12: We apply our method in the case of a sum of three sinusoidal waveform signal, so $n = 3$. Since $\overline{Q}$-annihilators are of the form (10), we choose $\ell = 2$ to obtain a system with three equations. So the canonical form of the annihilator of order 6 is:

$$\Pi = g_0(s)\frac{d^4}{ds^4} + g_1(s)\frac{d^5}{ds^5} + g_2(s)\frac{d^6}{ds^6},$$

where $g_0(s), g_1(s), g_2(s) \in \mathbb{C}(s)$. Choosing $g_0(s) = 1$, $g_1(s) = 0$, $g_2(s)$; $g_0(s) = 0$, $g_1(s) = 1$, $g_2(s) = 0$ and $g_0(s) = 0$, $g_1(s) = 0$, $g_2(s) = 1$ gives three equations in the operational domain leading to the following system in the time domain:

$$\begin{pmatrix} \frac{1}{6}J_1 & \frac{1}{6}J_2 & J_3 \\ -\frac{1}{24}J_4 & -\frac{1}{6}J_5 & -\frac{1}{2}J_6 \\ \frac{1}{12}J_7 & \frac{1}{4}J_8 & J_9 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = - \begin{pmatrix} J_{10} \\ J_{11} \\ J_{12} \end{pmatrix}$$

where we set $v_{m,n} = v_{m,n}(u) = (t-u)^m u^n$, for $m, n \in \mathbb{N}$ and $J_i = \int_0^t I_i\ z(u)\ du$, for $1 \leq i \leq 12$,

$$\begin{aligned}
I_1 &= 2v_{3,4} - t\ v_{3,3} \\
I_2 &= 14\,v_{2,4} - 14\,tv_{2,3} + 3\,t^2\,v_{2,2} \\
I_3 &= 14\,v_{1,4} - 21\,t\ v_{1,3} + 9t^2\ v_{1,2} - t^3\ v_{1,1} \\
I_4 &= 9\,v_{3,5} - 5t\,v_{3,4} \\
I_5 &= 18v_{2,5} - 20t\,v_{2,4} + 5t^2\,v_{2,3} \\
I_6 &= 42\,v_{1,5} - 70\ t\,v_{1,4} + 35t^2\,v_{1,3} - 5t^3\,v_{1,2} \\
I_7 &= 5\,v_{3,6} - 3t\,v_{3,5} \\
I_8 &= 15\,v_{2,6} - 18t\,v_{2,5} + 5\ t^2\,v_{2,4} \\
I_9 &= 30v_{1,6} - 54t\,v_{1,5} + 30t^2\,v_{1,4} - 5t^3\,v_{1,3} \\
I_{10} &= -16\,v_{1,3} + 36\,v_{2,2} - 16v_{3,1} + v_{0,4} + v_{4,0} \\
I_{11} &= 20v_{1,4} - 60\,v_{2,3} + 40\,v_{3,2} - 5\,v_{4,1} - v_{0,5} \\
I_{12} &= -24\,v_{1,5} + 15\,v_{4,2} + 90\,v_{2,4} - 80\,v_{3,3} + v_{0,6}
\end{aligned}$$

Finally, the expressions for θ_1, θ_2 and θ_3 are:

$$\theta_1 = 12\,\frac{2\ a\ J_2 - 2\ b\ J_3 + d\ J_{10}}{\Lambda}$$

$$\theta_2 = 12\frac{-2\ a\ J_1 + c\ J_3 + e\ J_{10}}{\Lambda}$$

$$\theta_3 = \frac{4\ b\ J_1 - 2\ c\ J_2 + f\ J_{10}}{\Lambda}$$

where
$$\begin{aligned}
&a = 2J_9J_{11} + J_6J_{12},\ b = 3J_8J_{11} + 2J_5J_{12}, \\
&c = 2J_7J_{11} + J_4J_{12},\ d = 4J_5J_9 - 3J_6J_8, \\
&e = J_6J_7 - J_4J_9,\ f = 3J_4J_8 - 4J_5J_7, \\
&\Lambda = 2\ d\ J_1 + 2\ e\ J_2 + f\ J_3
\end{aligned}$$

In the general parameter estimation problem, we wish to estimate all parameters giving the amplitudes, the frequencies and the phases, that is the triplets $(\alpha_k, \omega_k, \phi_k)$, $1 \leq k \leq 3$ in (1). As mentioned in the Introduction, our algebraic method works very properly in this case. We give an idea of how to proceed:

- define two new vectors $\Theta_{\text{est}} = \{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6\}$ and $\Theta_{\overline{\text{est}}} = \{\theta_7\}$ where θ_1, θ_2, θ_3 are as in (2), $\theta_4 = \beta - z(0)$, $\theta_5 = -\dot{z}(0)$, $\theta_6 = -\ddot{z}(0)$ and $\theta_7 = -\beta$.
- according with the new Θ_{est} and $\Theta_{\overline{\text{est}}}$, we obtain a new polynomial $\overline{Q}$ in the relation $\mathcal{R}$ (5):

  $$\overline{Q}(s) = T(s)\ \theta_7 \in \mathbb{C}_{\Theta_{\overline{\text{est}}}}[s]$$

  and polynomials in $\mathbb{C}_{\Theta_{\text{est}}}[s]$:

  $$\begin{aligned}
  &T(s) = s^3 + \theta_3 s^2 + \theta_2 s + \theta_1 \\
  &Q(s) = s^3\theta_4 + s^2(\theta_5 + \theta_3\theta_4) + s(\theta_6 + \theta_2\theta_4 + \theta_3\theta_5).
  \end{aligned}$$

- choose a new minimal $\overline{Q}$-annihilator and estimate the frequencies $\{\theta_1, \theta_2, \theta_3\}$ by a similar reasoning as above.
- to linearly identify the remaining θ_i define a minimal $\overline{Q}$-annihilator with coefficients in $\mathbb{C}_{\Theta_{\text{est}}}(s)$, that means depending on the parameters θ_1, θ_2, θ_3 that we just calculated. In this case, the minimal $\overline{Q}$-annihilator is

  $$\Pi_{\min}^{\Theta_{\text{est}}} = T\frac{d}{ds} - T'$$

that generates a family of annihilators of the form

$$\Pi = \sum_{i=0}^{\ell} g_i(s) \frac{d}{ds} \circ \Pi_{\min}^{\Theta_{\text{est}}}$$

Once more, we apply the algebraic procedure and by a suitable choice of annihilators, we obtain a system of equations with coefficients depending on θ_1, θ_2, θ_3 that allows us to determine θ_4, θ_5 and θ_6.

For more details in an encouraging example on the position reconstruction of a human body in the sagittal plane using only accelerometer measurements, we refer to [42].

V. Simulations

The following figures show the estimation of parameters θ_1, θ_2 and θ_3 concerning the results concerning the normalized mean values and variances. More precisely, the "true" parameters are denoted by θ_1, θ_2 and θ_3 and $\widehat{\theta}_{i,k}$ denotes the estimation of θ_i obtained at the k-th run. The modified Prony's method (PM) is used as a reference. Each point is obtained by averaging the results over 100 trials.

Dotted line curves represent exact values, while solid line curves show the estimations by our algebraic method and dashed line curves, the results by the modified Prony's method.

Figure 1 shows the simulation results for the estimation of the parameters θ_1, θ_2 and θ_3 versus the estimation time.

In figure 2, we plot $\widehat{\theta}_i = \frac{1}{100}\sum_k \widehat{\theta}_{i,k}$ and $\frac{var(\widehat{\theta}_i)}{\theta_i^2}$ versus the estimation time. More simulation experiments will be presented in the final version.

VI. Conclusion

In this paper, we study the parameter estimation of a multi-sinusoidal waveform signal from noisy data. The methods used in this paper are of algebraic flavor. They allow a robust estimation, and very fast as well, within a fraction of the signal's period.

We emphasize an essential point: the estimation obtained are based on integrals of measured signals. These particular integrals play the role of time-varying filters.

The efficiency of our algebraic method is illustrated by computer simulations.

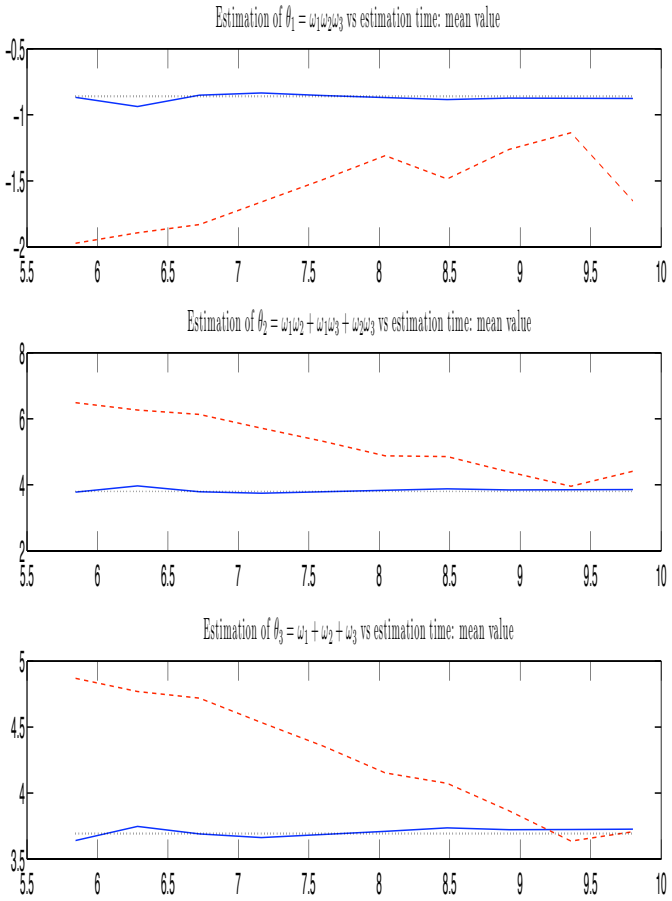

Fig. 1. Comparison with the modified Prony's method

References

[1] S. Bittanti, M. Campi, and S. Savaresi, Unbiased estimation of a sinusoid in colored noise via adapted notch filters. *Automatica*, **33**, 209–215, 1997.

[2] S. Bittanti and S. Savaresi, On the parametrization and design of an extended Kalman filter frequency tracker, *IEEE Trans. Automat. Control*, **45**, 1718–1724, 2000.

[3] A.A. Bobtsov, D. Efimov, A.A. Pyrkin, and A. Zolghadri, Switched algorithm for frequency estimation with noise rejection. *IEEE Trans. Automat. Control*, **57** (9), 2400–2404, 2012.

[4] M. Bodson and S. Douglas, Adaptive algorithms for the rejection of sinusoidal disturbances with unknown frequency. In *Proc. 13th IFAC World Conf.*, San Francisco, CA, USA, 1996.

[5] Y. Bresler and A. Macovski, Exact maximum likelihood parameter estimation of superimposed exponential signals in noise. *IEEE Trans. Acoust., Speech, Signal Process.*, **34** (5), 1081 – 1089, 1986.

[6] D. Carnevale and A. Astolfi, Hybrid observer for global frequency estimation of saturated signals. *IEEE Trans. Automat. Control*, **54** (10), 2461 –2464, 2009.

[7] D. Carnevale and A. Astolfi, A hybrid observer for frequency estimation of saturated multi-frequency signals. *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, 2577–2582, 2011.

[8] P. Cartier and Y. Perrin, *Integration over finite sets*, Nonstandard Analysis in Practice, F. & M. Diener (Eds), Springer, 195–204, 1995.

[9] M. H. Cheng and J. L. Tsai , A new IIR adaptive notch filter, *Signal Process.*, **86**, 1648–1655, 2006.

[10] J. Dixmier,*Algèbres enveloppantes*, Gauthier-Villars, 1974.

[11] W. Edmonson, W. Lee and J. Anderson, Maximum likehood estimation of sinusoidal parameteres using a gloabal optimisation algorithm, *Ninth Asilomar Conference on Signals, Systems and Computers*, **2**, 1167 – 1171, 1995.

[12] M. Favaro and G. Picci, Consistency of subspace methods with almost periodic components. *Automatica*, **48**, 514 – 520, 2012.

[13] G. Fedele, A. Ferrise, and D. Frascino, Multi-sinusoidal signal estimation by an adaptive sogi-filters bank. *15th IFAC Symposium on System Identification*, 2009.

[14] M. Fliess, Analyse non standard du bruit, *C.R. Acad. Sci. Paris, ser. I*, **342**, 797–802, 2006.

[15] M. Fliess, Critique du rapport signal à bruit en communications numériques, *ARIMA*, **9**, 419–429, 2008.

[16] M. Fliess, C. Join and M. Mboup, Algebraic change-point detection, *Applicable Algebra Engin. Communic. Comput.*, **21**, 131–143, 2010.

[17] M. Fliess, M. Mboup, H. Mounier and H. Sira-Ramírez, Questioning some paradigms of signal processing via concrete examples, in H. Sira-Ramírez, G. Silva-Navarro (Eds.): *Algebraic Methods in Flatness, Signal Processing and State Estimation*, Editorial Lagares, 1–21, 2003.

[18] M. Fliess and H. Sira-Ramírez, An algebraic framework for linear identification. *ESAIM Control Optim. Calc. Variat.*, **9**, 151–168, 2003.

[19] M. Fliess and H. Sira-Ramírez, Closed-loop parametric identification for continuous-time linear systems via new algebraic techniques, in H.

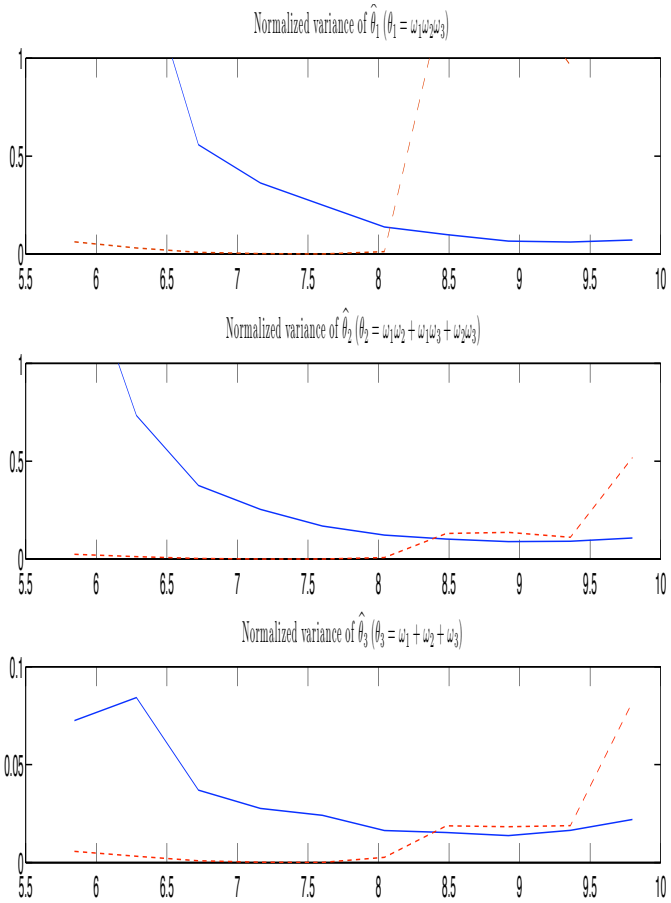

Fig. 2. Variances

Garnier & L. Wang (Eds): *Identification of Continuous-time Models from Sampled Data*, Springer, 362–391, 2008.

[20] M. Fliess, C. Join and H. Sira-Ramírez, Non-linear estimation is easy. *Int. J. Modeling Identification and Control*, **4** (1), 12–27, 2008.

[21] B. Friedlander and J. Smith, Analysis and performance evaluation of an adaptive notch filter. *IEEE Trans. Inf. Theory*, **30** (2), 283–295, 1984.

[22] S. Haykin, *Adaptive Filter Theory* (2^{nd} ed.), Englewood Cliffs, 1991.

[23] L. Hsu, R. Ortega and G. Damm, A globally convergent frequency estimator, *IEEE Trans. Automat. Control*, **44**, 698–713, 1999.

[24] M. Kahn, M. Mackisack, M. Osborne and G. K. Smyth, On the consistency of Prony's method and related algorithms, *J. Comput. Graph. Statist.*, **1**, 329–349, 1992.

[25] S. M. Kay and Jr S. L. Marple, Spectrum analysis-a modern perspective. *Proc. IEEE*, **69** (11), 1380–1419, 1981.

[26] B.F. La Scala, R.R. Bitmead, and M.R. James, Conditions for stability of the extended Kalman filter and their application to the frequency tracking problem. *Mathematics of Control, Signals, and Systems (MCSS)*, **8**, 1–26, 1995.

[27] B.F. La Scala and R.R. Bitmead. Design of an extended Kalman filter frequency tracker. *IEEE Trans. Signal Process.*, **44** (3), 739–742, 1996.

[28] B.F. La Scala, R.R. Bitmead, and B.G. Quinn. An extended Kalman filter frequency tracker for high-noise environments. *IEEE Trans. Signal Process.*, **44** (2), 431–434, 1996.

[29] T. H. Li and B. Kedem, Strong consistency of the contraction mapping method for frequency estimation, *IEEE Trans. Inform. Theory*, **39**, 989–998, 1993.

[30] D-J. Liu and C-T. Lin, Fundamental frequency estimation based on the joint time-frequency analysis of harmonic spectral structure. *IEEE Trans. Acoust., Speech, Signal Process.*, **9** (6), pp. 609–621, sep 2001.

[31] D. Y. Liu, O. Gibaru, W. Perruquetti, M. Fliess and M. Mboup, An error analysis in the algebraic estimation of a noisy sinusoidal signal, *16th Mediterranean conference on Control and automation* (MED' 2008), Ajaccio, 2008.

[32] D. Y. Liu, O. Gibaru and W. Perruquetti, Differentiation by integration with Jacobi polynomials, *Journal of Computational and Applied Mathematics*, **235** (9), 3015–3032, 2011.

[33] D. Y. Liu, O. Gibaru and W. Perruquetti, Error analysis of a Jacobi derivative estimators for noisy signals, *Numerical Algo.*, **58** (1), 53–83, 2011.

[34] M. Mboup, C. Join and M. Fliess, A revised look at numerical differentiation with an application to nonlinear feedback control, *Proc. 15^{th} Medit. Conf. Control Automation - MED'2007*, Athens, 2007.

[35] M. Mboup, Parameter estimation for signals described by differential equations, *Applicable Analysis*, **88**, 29–52, 2009.

[36] M. Mboup, C. Join and M. Fliess, Numerical differentiation with annihilators in noisy environment, *Numerical Algo.*, **50**, 439–467, 2009.

[37] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Amer. Math. Soc., 2000.

[38] M. Mojiri and A. R. Bakhsahi, An adaptive notch filter for frequency estimation of a periodic signal, *IEEE Trans. Automat. Control*, **49** (2), 314–318, 2004.

[39] J.I. Montojo and L.B. Milstein, Channel estimation for non-ideal ofdm systems. *IEEE Trans. Commun.*, **58** (1), 146 –156, 2010.

[40] A. Neves, M. Mboup and M. Fliess, An Algebraic Receiver for Full Response CPM Demodulation, *VI Int. Telecom. Symp. (ITS2006)*, Fortaleza , CE, Brazil, 2006.

[41] M. R. Osborne and G.K. Smyth, A modified Prony algorithm for exponential function fitting, *SIAM Journal on Scientific Computing*, **16** (1), 119—138, 1995.

[42] W. Perruquetti, V. Bonnet, M. Mboup, R. Ushirobira and P. Fraisse, An algebraic approach for human posture estimation in the sagittal plane using accelerometer noisy signal, in proc. IEEE CDC 2012, 2012.

[43] G. M. Riche de Prony, Essai exprimental et analytique : sur les lois de la dilatabilité de fluides élastiques et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alcool à différentes températures, *Journal de l'école polytechnique*, **1** (22), 24–76, 1795.

[44] D.B. Rao and Sun-Yuan Kung, Adaptive notch filtering for the retrieval of sinusoids in noise. *IEEE Trans. Acoust., Speech, Signal Process.*, **32** (4), 791–802, 1984.

[45] P. A. Regalia, *Adaptive IIR Filtering in Signal Processing and Control*. Marcel Dekker, 1995.

[46] R. Roy and T. Kailath, ESPRIT-estimation of signal parameters via rotational invariance techniques, *IEEE Trans. Signal Process.*, **37**, 984–995, 1989.

[47] H.C. So and F.K.W. Chang, A generalized weighted linear predictor frequency estimation approach for a complex sinusoid. *IEEE Trans. Signal Process.*, **54** (4), 1304–1315, 2006.

[48] P. Stoica and A. Nehorai, Performance analysis of an adaptive notch filter with constrained poles and zeros. *IEEE Trans. Acoust., Speech, Signal Process.*, **36** (6), 911–919, 1988.

[49] P. Stoica. List of references on spectral line analysis. *Signal Processing*, **31** (3), 329 – 340, 1993.

[50] J. R. Trapero, H. Sira-Ramírez and V. F. Battle, An algebraic frequency estimator for a biased and noisy sinusoidal signal, *Signal Processing*, **87**, 1188–1201, 2007.

[51] J. R. Trapero, H. Sira-Ramírez and V. F. Battle, A fast on-line frequency estimator of lightly damped vibrations in flexible structures, *J. Sound Vibration*, **307**, 365–378, 2007.

[52] J. R. Trapero, M. Mboup, E. Pereira-Gonzalez and V. F. Battle, On-line frequency and damping estimation in a single-link flexible manipulator based on algebraic identification, *MED*, 2008.

[53] J. R. Trapero, H. Sira-Ramírez and V. F. Battle, On the algebraic identification of the frequencies, amplitudes and phases of two sinusoidal signals from their noisy sum, *Int. J. Control*, **81**, 507–518, 2008.

[54] R. Ushirobira, W. Perruquetti, M. Mboup and M. Fliess, Estimation algébrique des paramètres intrinsèques d'un signal sinusoïdal biaisé en environnement bruité, *Proc. Gretsi*, 5-9 septembre, Bordeaux, France, 2011.

[55] R. Ushirobira, W. Perruquetti, M. Mboup and M. Fliess, Algebraic parameter estimation of a biased sinusoidal waveform signal from noisy data, *16th IFAC Symposium on System Identification*, 2012.

[56] X. Xia, Global frequency estimation using adaptive identifiers. *IEEE Trans. Autom. Control*, **47** (7), 1188–1193, 2002.