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Lagrangian Curves in a 4-dimensional affine symplectic space

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Abstract

Lagrangian curves in $\mathbb{R}^4$ entertain intriguing relationships with second order deformation of plane curves under the special affine group and null curves in a 3-dimensional Lorentzian space form. We provide a natural affine symplectic frame for Lagrangian curves. It allows us to classify Lagrangrian curves with constant symplectic curvatures, to construct a class of Lagrangian tori in $\mathbb{R}^4$ and determine Lagrangian geodesics.

Keywords: Symplectic geometry; Lagrangian planes; Differential invariants; Moving frame; Lie group actions.

Mathematics Subject Classification: 53A55; 53A15; 53D12 .

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Introduction

The study of submanifolds in an affine symplectic space originated in the work of S.S. Chern and H.C. Wang [6]. The issue has however remained silent for many years, before being taken up on several occasions in recent literature [1, 7, 24, 33, 30, 44]. The renewed interest in this topic raises in connection with the modern approach to the moving frame method [24] and in investigations on integrable evolutions of curves in affine spaces or in the Grassmannians of the Lagrangian linear subspaces of $\mathbb{R}^{2n}$ [28, 29, 44]. The specific nature of the geometry of an affine symplectic space stems from the fact that the linear symplectic group does not act transitively on the Grassmannians of the linear subspaces of $\mathbb{R}^{2n}$. Already in the case of curves, the phenomenology is rather varied and depends on the typology of the osculating spaces along the curve which can be symplectic, isotropic, coisotropic or Lagrangian. This makes a non-trivial task the construction of a moving frame that works effectively in all possible cases, or say for all linearly full curve. So far, in the literature only the generic case (i.e. curves whose osculating spaces of even order are symplectic) has been investigated, while those cases which are more specific to the symplectic setting have not been examined.

In the present paper we focus on Lagrangian curves. They are the curves whose osculating spaces of order $\leq n$ are isotropic. First we provide an appropriate moving frame for those curves. It was claimed in [24] that Lagrangian curves were singular and required higher order moving frame. Such is not the case. We exhibit a frame that is of minimal order for generic curves but specializes well for Lagrangian curves. Second, with the moving frame at hand, we investigate the geometry of some remarkable Lagrangian curves. We first provide a classification of Lagrangian curves with constant symplectic curvatures. We then examine the notion of geodesics for Lagrangian curves. It turns out to be a subclass of the Lagrangian curves with constant curvatures.

In our treatment we only consider the four dimensional case. This choice is due to two main reasons: minimize the computational complications when constructing the moving frame and exploit the specificity of the four-dimensional case. We shall indeed disclose some interesting interconnections with the deformation problem [5, 10, 23] for plane curves and with the conformal geometry of null curves in a 3-dimensional pseudo-Riemannian space form (Minkowski, de Sitter or anti-de Sitter). This latter topic has its own significance in mathematics, theoretical physics and biophysics [2, 4, 21, 26, 32, 35, 34, 37, 39, 38, 9, 36].

The material is organized into five sections. In Section 1 we recall basic facts on the affine symplectic geometry of $\mathbb{R}^4$ and define the symplectic curvatures. In Section 2 we study Lagrangian curves. They are characterized by the fact that their osculating planes are Lagrangian subspaces of $\mathbb{R}^4$ and demonstrate intriguing connections to conformal and affine geometry. Lagrangian curves have a natural parametrization through which a notion of symplectic length can be defined. We then examine their link to the conformal geometry of the

Grassmannian of the Lagrangian vector subspaces of $\mathbb{R}^4$. In Section 3 we face the construction of a moving frame for linearly full curves in $\mathbb{R}^4$. We briefly outline the moving frame technology for parametrized submanifolds following [8, 16, 17, 18]. It is then applied to build the moving frame associated to a minimal-order section for the action of the affine symplectic group on the fourth-order jet space of linearly full parameterized curves of $\mathbb{R}^4$, regardless of the specific properties of the osculating spaces. We will also examine the cross-section corresponding to a Gram-Schmid process. Such a construction can be extended to provide appropriate moving frame in any dimension. In Section 4 we classify Lagrangian curves with constant symplectic curvatures and we show that, up to affine symplectic transformations, closed Lagrangian curves with constant curvatures depend on a rational parameter. Based on those closed curve we show how to construct Lagrangian tori. Finally, in Section 5 we study the symplectic arc-length functional for Lagrangian curves and we prove that its critical points form a subset of the Lagrangian curves with constant symplectic curvatures.

1 Curves in symplectic affine geometry

1.1 Symplectic structure

On $\mathbb{R}^4$ we consider the *standard symplectic form*

$$\Lambda(X,Y) = {}^tX \cdot J \cdot Y = \sum_{a,b=1}^{4} x_a\, J_{ab}\, y_b, \quad \forall X,Y \in \mathbb{R}^4,$$

where

$$J = (J_{ab}) = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

The *linear symplectic group* $\mathrm{Sp}(4,\mathbb{R})$ is the group of matrices that preserve Λ:

$$\mathrm{Sp}(4,\mathbb{R}) = \{\, A \in \mathrm{GL}(4,\mathbb{R}) \mid {}^tA\, J\, A = J\}.$$

It is a 10 dimensional Lie group. The semi-direct product $\mathbb{R}^4 \rtimes \mathrm{Sp}(4,\mathbb{R})$ is the *affine symplectic group*. Its action on $\mathbb{R}^4$ is given by $(a,A) \star x = Ax + a$. It admits the matrix representation:

$$\mathcal{S}(4,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ a & A \end{pmatrix} \in \mathrm{GL}(5,\mathbb{R}) \;\middle|\; a \in \mathbb{R}^4, A \in \mathrm{Sp}(4,\mathbb{R}) \right\}. \tag{1.1}$$

The Lie algebras $\mathfrak{sp}(4,\mathbb{R})$ and $\mathfrak{s}(4,\mathbb{R})$ of $\mathrm{Sp}(4,\mathbb{R})$ and $\mathcal{S}(4,\mathbb{R})$ are respectively

$$\mathfrak{sp}(4,\mathbb{R}) = \left\{ A \in \mathfrak{gl}(4,\mathbb{R}) \mid {}^tA \cdot J + J \cdot A = 0 \right\}$$

and

$$\mathfrak{s}(4,\mathbb{R}) = \left\{ \begin{pmatrix} 0 & 0 \\ a & A \end{pmatrix} \;\middle|\; a \in \mathbb{R}^4,\ A \in \mathfrak{sp}(4,\mathbb{R}) \right\}.$$

We can deduce that

$$\mathfrak{sp}(4,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \;\middle|\; A, B, C \in \mathfrak{gl}(2,\mathbb{R}) \text{ and } {}^tB = B, {}^tC = C \right\}$$

so that a basis for $\mathfrak{sp}(4,\mathbb{R})$ is provided by the matrices

$$\bar{A}^i_j = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}, \quad i,j = 1,2,$$

$$\bar{B}^i_i = \begin{pmatrix} 0 & E_{ii} \\ 0 & 0 \end{pmatrix}, \quad \bar{C}^i_i = \begin{pmatrix} 0 & 0 \\ E_{ii} & 0 \end{pmatrix}, \quad i = 1,2, \tag{1.2}$$

and

$$\bar{B}^1_2 = \begin{pmatrix} 0 & E_{12} + E_{21} \\ 0 & 0 \end{pmatrix}, \quad \bar{C}^1_2 = \begin{pmatrix} 0 & 0 \\ E_{12} + E_{21} & 0 \end{pmatrix},$$

where E_{ij} is the matrix having 1 at the position (i,j) as the only nonzero entry. Let A^j_i, B^j_i and C^j_i be the corresponding 5×5 matrices in $\mathfrak{s}(4)$. Together with the infinitesimal translations

$$T_1 = \left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \ \ldots, \ T_4 = \left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right), \tag{1.3}$$

they form a basis for $\mathfrak{s}(4,\mathbb{R})$. The corresponding infinitesimal generators of the action of $\mathrm{Sp}(4,\mathbb{R})$ on $\mathbb{R}^4$ are the vector fields:

$$\widetilde{A}^j_i = x_j \frac{\partial}{\partial x_i} - x_{2+i} \frac{\partial}{\partial x_{2+j}}, \quad i,j = 1,2,$$
$$\widetilde{B}^i_i = x_{2+i} \frac{\partial}{\partial x_i}, \quad \widetilde{C}^i_i = x_i \frac{\partial}{\partial x_{2+i}}, \quad i = 1,2$$
$$\widetilde{B}^2_1 = x_3 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2}, \quad \widetilde{C}^2_1 = x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4}$$

and

$$\widetilde{T}_a = \frac{\partial}{\partial x_a}, \quad a = 1, ..., 4.$$

Structure in terms of frames: A basis (E_1, E_2, E_3, E_4) of $\mathbb{R}^4$ is said to be symplectic if $\Lambda(E_a, E_b) = J_{ab}$, for every $a, b = 1, ..., 4$. This is equivalent to the matrix $\mathbf{E}$ with column vectors E_1, E_2, E_3, E_4 to belong to $\mathrm{Sp}(4,\mathbb{R})$.

An *affine symplectic frame* (p, E) consists of a point $p \in \mathbb{R}^4$, the origin, and a symplectic basis E. The manifold of all these frames can be identified with

$\mathcal{S}(4,\mathbb{R})$, the matrix representation of the affine symplectic group defined as (1.1). Differentiating the maps

$$p : (p,E) \in \mathcal{S}(4,\mathbb{R}) \to p \in \mathbb{R}^4, \quad E_a : (p,E) \in \mathcal{S}(4,\mathbb{R}) \to E_a \in \mathbb{R}^4,$$

and taking into account the identities $\Lambda(E_a, E_b) = J_{ab}$ we find

$$\begin{aligned} dp &= \tau^1 E_1 + \tau^2 E_2 + \tau^3 E_3 + \tau^4 E_4, \\ dE_j &= \alpha^1_j E_1 + \alpha^2_j E_2 + \beta^1_j E_3 + \beta^2_j E_4, \quad j = 1,2, \\ dE_{2+j} &= \eta^1_j E_1 + \eta^2_j E_2 - \alpha^1_i E_3 - \alpha^2_i E_4, \quad j = 1,2, \end{aligned} \tag{1.4}$$

where $\beta^i_j = \beta^j_i$ and $\eta^i_j = \eta^j_i$ for every $i,j = 1,2$. Note that

$$(\tau^1, \tau^2, \tau^3, \tau^4, \alpha^1_1, \alpha^2_2, \alpha^1_2, \alpha^2_1, \beta^1_1, \beta^2_2, \beta^1_2, \eta^1_1, \eta^2_2, \eta^1_2) \tag{1.5}$$

is a basis for the vector space of the left-invariant 1-forms of $\mathcal{S}(4,\mathbb{R})$ that is dual to the infinitesimal generators associated to

$$\big(T_1, T_2, T_3, T_4, A^1_1, A^2_2, A^1_2, A^2_1, B^1_1, B^2_2, B^1_2, C^1_1, C^2_2, C^1_2\big)$$

when the action of the one dimensional group determined by $\mathfrak{a} \in \mathfrak{s}$ is given by $B \in \mathcal{S} \to Be^{t\mathfrak{a}}$.

Structure in terms of the Maurer-Cartan form: Another way to organize the structural information is to consider the left Maurer-Cartan form

$$\hat{\Omega} = \begin{pmatrix} 1 & 0 \\ -A^{-1}a & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathrm{d}a & \mathrm{d}A \end{pmatrix}$$

that is the left invariant form on on $\mathcal{S}(4,\mathbb{R})$ with values in $\mathfrak{s}(4,\mathbb{R})$. Hence

$$\hat{\Omega} = \sum_{1\le i,j\le 2} \alpha^j_i\, A^j_i + \sum_{1\le i\le j\le 2} \beta^j_i\, B^j_i + \sum_{1\le i\le j\le 2} \eta^j_i\, C^j_i + \sum_{a=1}^{4} \tau_a\, T_a$$

that is

$$\hat{\Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \tau_1 & \alpha^1_1 & \alpha^2_1 & \beta^1_1 & \beta^2_1 \\ \tau_2 & \alpha^1_2 & \alpha^2_2 & \beta^2_1 & \beta^2_2 \\ \tau_3 & \eta^1_1 & \eta^2_1 & -\alpha^1_1 & -\alpha^1_2 \\ \tau_3 & \eta^2_1 & \eta^2_2 & -\alpha^2_1 & -\alpha^2_2 \end{pmatrix}.$$

The left Maurer-Cartan form satisfies the structure equation

$$\mathrm{d}\hat{\Omega} + \frac{1}{2}\left[\hat{\Omega}, \hat{\Omega}\right] = 0, \quad \text{that is} \quad \mathrm{d}\hat{\Omega} + \hat{\Omega} \wedge \hat{\Omega} = 0$$

in the case of matrix groups. It is an essential tool for the classification of manifolds.

Theorem 1.1 *[43, Theorem 5.2 and 6.1], [22, Theorem 1.6.10] Let $\mathcal{M}$ be a manifold endowed with a $\mathfrak{s}$-valued one-form $\tilde{\Omega}$ satisfying $\mathrm{d}\tilde{\Omega} = -\tilde{\Omega} \wedge \tilde{\Omega}$. Then for any point $z \in \mathcal{M}$ there exists a neighborhood U of z and a map $\tilde{\rho} : U \to \mathcal{S}$ s.t. $\tilde{\rho}^*\Omega = \tilde{\Omega}$. Any two such maps $\tilde{\rho}_1$, $\tilde{\rho}_2$ satisfies $\tilde{\rho}_1 = g \cdot \tilde{\rho}_2$ for some fixed $g \in \mathcal{S}$.*

The proof proceeds by showing the existence of a solution $\tilde{\rho} : \mathcal{M} \to \mathcal{S}$ to the differential system $\mathrm{d}\tilde{\rho} = \tilde{\rho}\,\tilde{\Omega}$.

We will alternatively use the right Maurer-Cartan form

$$\Omega = \begin{pmatrix} 0 & 0 \\ \mathrm{d}a & \mathrm{d}A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1}a & A^{-1} \end{pmatrix}$$

that satisfies the structure equation $\mathrm{d}\Omega = \Omega \wedge \Omega$. This is motivated by the fact that the infinitesimal generators are implicitly defined with a right invariant structure on the group [40].

1.2 Symplectic curvatures

Let $\mathrm{J}^k(\mathbb{R}, \mathbb{R}^4)$ be the space of kth-order jets of smooth parameterized curves $\gamma : \mathbb{R} \to \mathbb{R}^4$, with coordinates $X^{(0)}$, $X^{(1)}, \ldots, X^{(k)}$. We can represent a point in J^k by an ordered pair (t, X), consisting of the independent variable $t \in \mathbb{R}$ and a $5 \times (k+1)$ matrix

$$\mathrm{X} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ X^{(0)} & X^{(1)} & \ldots & X^{(k)} \end{pmatrix}$$

so that the action of $\mathcal{S}(4, \mathbb{R})$ is given by matrix multiplication:

$$\begin{pmatrix} 1 & 0 \\ a & A \end{pmatrix} \begin{pmatrix} 1 & 0 & \ldots & 0 \\ X^{(0)} & X^{(1)} & \ldots & X^{(k)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ AX^{(0)} + a & AX^{(1)} & \ldots & AX^{(k)} \end{pmatrix}$$

The maps

$$\Lambda(X^{(i)}, X^{(j)}) = -x_3^{(i)} x_1^{(j)} - x_4^{(i)} x_2^{(j)} + x_1^{(i)} x_3^{(j)} + x_2^{(i)} x_4^{(j)}$$

are differential invariants for the action of the affine symplectic group on the jet space J^k. We single out the *symplectic curvatures* $\kappa_i = \Lambda\left(X^{(i)}, X^{(i+1)}\right)$, for $i \geq 1$. Any other differential invariant can be expressed in terms of those. For instance:

$$\Lambda(X^{(1)}, X^{(2)}) = \kappa_1, \quad \Lambda(X^{(1)}, X^{(3)}) = \kappa_1', \quad \Lambda(X^{(1)}, X^{(4)}) = \kappa_1'' - \kappa_2,$$

$$\Lambda(X^{(1)}, X^{(5)}) = \kappa_1''' - 2\kappa_2', \quad \Lambda(X^{(2)}, X^{(3)}) = \kappa_2, \quad \Lambda(X^{(2)}, X^{(4)}) = \kappa_2',$$

$$\Lambda(X^{(1)}, X^{(5)}) = \kappa_2' - \kappa_3, \quad \Lambda(X^{(3)}, X^{(4)}) = \kappa_3, \quad \Lambda(X^{(3)}, X^{(5)}) = \kappa_3'.$$

We shall come across another differential invariant

$$\phi = \det(X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}) = \kappa_2^2 - \kappa_1\kappa_3 + \kappa_1'\kappa_2' - \kappa_2\kappa_1''. \tag{1.6}$$

[29, 24, 44] considered curves $\gamma : I \to \mathbb{R}^4$ s.t. $\kappa_1 \circ j^h(\gamma) = \Lambda(\gamma', \gamma'')$ does not vanish anywhere on I, where $j^h(\gamma)$ denotes the canonical lifting of γ to any of the jet spaces J^h, $h \geq 2$. The present article focuses on the curves where κ_1 is identically zero.

2 Lagrangian curves

In this section we define Lagrangian curves and show some of their properties. We make explicit their relevance to conformal and affine geometry. We first exhibit a natural parametrization based on which we can define a symplectic arc length. We briefly discuss the standard conformal structure of the Grassmannian of the oriented Lagrangian planes of $\mathbb{R}^4$ (we refer to [2],[12] for more details). Subsequently we define the osculating curve and the phase portraits of a Lagrangian curve. We prove that the osculating curve is a null-curve in the Grassmannian of Lagrangian planes and we observe that any such curve satisfying some mild conditions arises in this fashion. This fact highlights the links between Lagrangian curves in $\mathbb{R}^4$ with null curves in a 3-dimensional Lorentzian space form. Secondly, we characterize Lagrangian curves in terms of the behavior of the phase portraits. We prove that the phase portraits of a Lagrangian curve are second order deformations of each other with respect to the action of the special affine group. The converse is also true, provided that the two phase portraits do not have inflection points.

2.1 Definition and parametrisation

Definition 2.1 *A smooth parameterized curve $\gamma : I \to \mathbb{R}^4$ is said to be Lagrangian if $\gamma'(t)$ and $\gamma''(t)$ are linearly independent and $\Lambda(\gamma'(t), \gamma''(t)) = 0$.*

The definition does not depend on the parametrization, and is invariant by symplectic transformations and dilations. Indeed, if $h : J \to I$ is a change of parameters, $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ is a symplectic transformation, r is a non-zero real number and $\gamma : I \to \mathbb{R}^4$ is a Lagrangian curve, then $\widetilde{\gamma} : t \in J \to r\Phi[\gamma(h(t))] \in \mathbb{R}^4$ is another Lagrangian curve. More generally, if $\gamma : I \to \mathbb{R}^4$ is a Lagrangian curve and if $r : I \to \mathbb{R}$ is a nowhere vanishing smooth function, then integrating $r\gamma'$ we obtain another Lagrangian curve $\tilde{\gamma} : I \to \mathbb{R}^4$ such that the tangent lines of γ and $\tilde{\gamma}$ at $\gamma(t)$ and $\tilde{\gamma}(t)$ are parallel to each other, for every $t \in I$.

Definition 2.2 *A Lagrangian curve is non-degenerate if $\Lambda(\gamma''(t), \gamma'''(t)) \neq 0$, for every $t \in I$ and linearly full if $\gamma'|_t \wedge ... \wedge \gamma^{(iv)}|_t \neq 0$, for every $t \in I$.*

With (1.6) we see that a linearly full Lagrangian curve is automatically non-degenerate.

If γ is Lagrangian and non-degenerate, then the differential 1-form

$$\sigma_\gamma = \Lambda(\gamma'', \gamma''')^{1/5} dt = ({}^t\gamma''\, J\, \gamma''')^{1/5} dt,$$

is nowhere vanishing. Furthermore, it is independent of the parametrization and invariant by the action of the affine symplectic group. Therefore, we can define an intrinsic orientation on a non-degenerate Lagrangian curve requiring that $\Lambda(\gamma''|_t, \gamma'''|_t) > 0$, for every $t \in I$. From now on a Lagrangian curve is equipped with the intrinsic orientation.

Definition 2.3 *Let γ be a non-degenerate Lagrangian curve, the differential form σ_γ is said to be the symplectic arc-element of γ. If $\Lambda(\gamma'', \gamma''')^{1/5} = 1$ (i.e. if $\sigma_\gamma = dt$), γ is said to be parameterized by the symplectic arc-length.*

The symplectic arc-length parameter will be denoted by s. Obviously, the symplectic arc-length parametrization is unique up to a shift $s \to s + s_0$ of the parameter. If $[a, b] \subset I$ is a closed interval, the quantity

$$\int_a^b \sigma_\gamma$$

is the *symplectic length* of the Lagrangian arc $\gamma([a, b])$. Using standard arguments one can easily prove the following Proposition :

Proposition 2.4 *If $\gamma : I \to \mathbb{R}^4$ is a non-degenerate Lagrangian curve equipped with its intrinsic orientation, then there exists a strictly increasing surjective map $h : J \subset \mathbb{R} \to I$ such that $\gamma \circ h$ is a parametrization by symplectic arc-length.*

2.2 Osculating curves

Here we discuss the links between Lagrangian curves in $\mathbb{R}^4$ and null curves in a 3-dimensional Lorentzian space form. Since the notion of null curve is invariant by conformal transformations, the natural environment is the conformal compactification of the Minkowski 3-space, which can be thought of as the manifold of all oriented Lagrangian vector subspaces of $\mathbb{R}^4$. Such a link between symplectic and conformal geometry is specific to the four-dimensional case.The reason lies in the fact that $\mathrm{Sp}(4, \mathbb{R})$ is a covering group of the connected component of the identity of $\mathrm{O}(3, 2)$. We begin with a brief description of the conformal structure of the Grassmannian of oriented Lagrangian planes in $\mathbb{R}^4$. Then we introduce the concept of osculating curve by which we establish the correlations among Lagrangian and null curves.

Definition 2.5 *An oriented Lagrangian plane is a two dimensional linear subspace $L \subset \mathbb{R}^4$ such that $\Lambda_{|L} = 0$. The Grassmannian Λ^2_+ of all such planes is a smooth manifold diffeomorphic to $S^2 \times S^1$ (see [2, 33]).*

The action of $\mathrm{Sp}(4,\mathbb{R})$ on $\mathbb{R}^4$ induces an action on Λ^2_+ which is transitive. The projection map

$$\pi_\Lambda : \mathbf{E} \in \mathrm{Sp}(4,\mathbb{R}) \to [E_1 \wedge E_2] \in \Lambda^2_+ \tag{2.1}$$

makes $\mathrm{Sp}(4,\mathbb{R})$ into a principal fiber bundle with structure group

$$\mathrm{Sp}(4,\mathbb{R})_1 = \left\{ \mathbf{X}(A,b) = \begin{pmatrix} A & A\cdot b \\ 0 & {}^tA^{-1} \end{pmatrix} \,:\, \det A > 0,\, b \in \mathrm{S}(2,\mathbb{R}) \right\}.$$

From this, it follows that the 1-forms $(\eta^1_1, \eta^2_2, \eta^2_1)$ span the semi-basic forms[1] of the projection π_Λ. Moreover, the symmetric quadratic form $g = -\eta^1_1\eta^2_2 + (\eta^2_1)^2$ and the exterior 3-form $\eta^1_1 \wedge \eta^2_1 \wedge \eta^2_2$ are well defined on Λ^2_+, up to a positive multiple. They determine a conformal structure of signature $(2,1)$ and an orientation, respectively.

Remark 2.6 *The 3-dimensional Minkowski space $\mathbb{R}^{2,1}$ can be identified with the vector space $\mathrm{Q}(2,\mathbb{R})$ of 2×2 symmetric matrices equipped with the non-degenerate inner product of signature $(2,1)$ induced by the quadratic form*

$$B \in \mathrm{Q}(2,\mathbb{R}) \to -\det(B) \in \mathbb{R}.$$

For each $B \in \mathrm{Q}(2,\mathbb{R})$ we associate the oriented Lagrangian plane $L(B)$ spanned by the vectors $(1,0,B^1_1,B^2_1)$ and $(0,1,B^2_1,B^2_2)$. It is easy to check that the map

$$B \in \mathrm{Q}(2,\mathbb{R}) \to L(B) \in \Lambda^2_+$$

is a conformal embedding. This shows that Λ^2_+ can be viewed as the conformal compactification of the 3-dimensional Minkowski space (cf. [2],[12]).

Definition 2.7 *The osculating spaces to a non-degenerate Lagrangian curve γ define a smooth map*

$$\delta_\gamma : t \in I \to [\gamma'(t) \wedge \gamma''(t)] \in \Lambda^2_+$$

to the manifold of the oriented Lagrangian planes. We say that δ is the osculating curve of γ.

Proposition 2.8 *Two non-degenerate Lagrangian curves γ and $\tilde\gamma$ have the same osculating curve if and only if they have parallel tangent lines.*

PROOF: If $\gamma, \tilde\gamma : I \to \mathbb{R}^4$ have the same osculating curves, then $\gamma' = r_1\tilde\gamma' + r_2\tilde\gamma''$, where $r_1, r_2 : I \to \mathbb{R}$ are suitable smooth functions. Then, the acceleration of γ is given by

$$\gamma'' = r_1'\tilde\gamma' + (r_1 + r_2')\tilde\gamma'' + r_2\tilde\gamma'''.$$

[1] An exterior differential form is semi-basic if it annihilates the vertical vectors of the fibration

Differentiating $\Lambda(\tilde{\gamma}', \tilde{\gamma}'') = 0$ we get $\Lambda(\tilde{\gamma}', \tilde{\gamma}''') = 0$. This implies

$$0 = \Lambda(\gamma', \gamma'') = r_2^2 \Lambda(\tilde{\gamma}'', \tilde{\gamma}''').$$

Bearing in mind that $\tilde{\gamma}$ is non-degenerate, it follows that $r_2 = 0$. Therefore $\gamma' = r_1 \tilde{\gamma}'$, i.e. γ and $\tilde{\gamma}$ have parallel tangent lines. Conversely, if γ and $\tilde{\gamma}$ have parallel tangent lines, then $\gamma' = r\tilde{\gamma}'$, where $r : I \to \mathbb{R}$ is a smooth function everywhere different from zero. Therefore we have $\gamma' \wedge \gamma'' = r^2 \tilde{\gamma}' \wedge \tilde{\gamma}''$, from which it follows that $\delta_\gamma = \delta_{\tilde{\gamma}}$.□

Proposition 2.9 *Let $\gamma : I \to \mathbb{R}^4$ be a non-degenerate Lagrangian curve. Then, δ_γ is a null-curve*[2] *of Λ^2_+.*

PROOF: To check if $\delta : I \to \Lambda^2_+$ is a null-curve we choose a lift of δ to $\mathrm{Sp}(4, \mathbb{R})$, i.e. any map $\mathbf{E} = (E_1, E_2, E_3, E_4) : I \to \mathrm{Sp}(4, \mathbb{R})$ such that $\delta = [E_1 \wedge E_2]$. Subsequently we compute the pull-back of the Mauer-Cartan forms $\eta^1_1, \eta^2_2, \eta^2_1$ and we write $\mathbf{E}^*(\eta^i_j) = c^i_j dt$, where $c^i_j : I \to \mathbb{R}$ are smooth functions. Then, δ is a null-curve if and only if $c^1_1 c^2_2 - (c^2_1)^2 = 0$ and $(c^1_1)^2 + (c^2_2) + (c^2_1)^2$ is nowhere vanishing. If γ is a non-degenerate Lagrangian curve, we consider a *second order moving frame* along γ, that is a smooth map

$$\mathbf{E} = (E_1, E_2, E_3, E_4) : I \to \mathrm{Sp}(4, \mathbb{R})$$

such that $\gamma'(t) = E_1(t)$ and $\gamma''(t) = E_2(t)$ for every $t \in I$. We then have

$$\begin{pmatrix} \mathbf{E}^*(\eta^1_1) & \mathbf{E}^*(\eta^2_1) \\ \mathbf{E}^*(\eta^2_1) & \mathbf{E} * (\eta^2_2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & c^2_2 \end{pmatrix} dt,$$

where c^2_2 is a nowhere vanishing smooth function. On the other hand $\mathbf{E}$ is a lift of the osculating curve δ_γ satisfying

$$c^1_1 c^2_2 - (c^2_1)^2 = 0, \quad (c^1_1)^2 + (c^2_2) + (c^2_1)^2 = c^2_2 > 0.$$

This yields the required result.□

It is furthermore not difficult to prove that, under a mild generic condition, any null curve of Λ^2_+ arises as the osculating curve of a non-degenerate Lagrangian curve of $\mathbb{R}^4$.

2.3 Phase portraits

We wish to show here the relation between Lagrangian curves and second order deformation of plane curves under the special affine group. We first recall the classical notion of deformation of a plane curve, with fixed parametrisation, with respect to a group of transformations.

[2] A smooth immersed curve $\delta : I \to \Lambda^2_+$ is *null* if its tangent vectors are isotropic (null) with respect to the conformal structure of Λ^2_+

Definition 2.10 *Let $\mathcal{G}$ be a Lie group and $(g, \mathbf{x}) \in \mathcal{G} \times \mathbb{R}^2 \to g \star \mathbf{x} \in \mathbb{R}^2$ be a (left) action of $\mathcal{G}$ on $\mathbb{R}^2$. Two plane curves $\mathbf{a}, \mathbf{b} : I \to \mathbb{R}^2$ are said to be k-th order deformations of each other with respect to $\mathcal{G}$ if there exists a smooth map $g : I \to \mathcal{G}$ such that $\mathbf{a}$ and $g(t) \star \mathbf{b}$ have the same k-th order jets at t, for every $t \in I$.*

If $\mathcal{G}$ is the Euclidean group of rigid motions, two curves are first order deformations each other if and only if they have the same speed, while second order deformation implies the congruence of the two curves We shall consider the larger *special affine group*, i.e. the semi-direct product of special linear group $\mathrm{SL}(2, \mathbb{R})$ with the group of the translations.

Definition 2.11 *The* phase portraits *of a curve $\gamma : I \to \mathbb{R}^4$ are the plane curves $\mathbf{a}_\gamma$, $\mathbf{b}_\gamma : I \to \mathbb{R}^2$ defined by $\mathbf{a}_\gamma = (\gamma_1, \gamma_3)$, $\mathbf{b}_\gamma = (-\gamma_2, \gamma_4)$.*

Conversely, given two plane curve $\mathbf{a}, \mathbf{b} : I \to \mathbb{R}^2$ we denote by $\gamma_{(\mathbf{a},\mathbf{b})} : I \to \mathbb{R}^4$ the curve defined by $\gamma_{(\mathbf{a},\mathbf{b})} = (a_1, -b_1, a_2, b_2)$.

A curve $\gamma_{(\mathbf{a},\mathbf{b})}$ is Lagrangian if and only if its phase portraits satisfy $\|\mathbf{a}'\| + \|\mathbf{b}'\| > 0$ and $\mathbf{a}' \wedge \mathbf{a}'' = \mathbf{b}' \wedge \mathbf{b}''$.

Proposition 2.12 *Let $\mathbf{a}, \mathbf{b} : I \to \mathbb{R}^2$ be two regular plane curves without inflection points. Then, they are second order deformations each other with respect to the special affine group if and only if $\gamma_{(\mathbf{a},\mathbf{b})}$ is a Lagrangian curve of $\mathbb{R}^4$.*

PROOF: It suffices to prove that $\mathbf{a}$ and $\mathbf{b}$ are second order deformations each other with respect to the special affine group if and only if $\mathbf{a}' \wedge \mathbf{a}'' = \mathbf{b}' \wedge \mathbf{b}''$. If $\mathbf{a}$ and $\mathbf{b}$ are second-order deformations each other, then there exist smooth maps $A : I \to \mathrm{SL}(2, \mathbb{R})$ and $T : I \to \mathbb{R}^2$ such that $A(t_0)\mathbf{a} + T(t_0)$ and $\mathbf{b}$ have the same second order jet at t_0, for every $t_0 \in I$. This implies

$$\mathbf{b}'|_{t_0} = A(t_0)\mathbf{a}'|_{t_0}, \quad \mathbf{b}''|_{t_0} = A(t_0)\mathbf{a}''|_{t_0}.$$

We then have

$$\mathbf{b}'|_{t_0} \wedge \mathbf{b}''|_{t_0} = \det(A(t_0))\mathbf{a}'|_{t_0} \wedge \mathbf{a}''|_{t_0} = \mathbf{a}'|_{t_0} \wedge \mathbf{a}''|_{t_0}, \quad \forall t_0 \in I.$$

This proves that $\gamma_{(\mathbf{a},\mathbf{b})}$ is a Lagrangian curve. Conversely, suppose that $\mathbf{a}$ and $\mathbf{b}$ are two curves without inflection points and that $\gamma_{(\mathbf{a},\mathbf{b})}$ is a Lagrangian curve. Since the two curves do not have inflection points the functions $\det(\mathbf{a}', \mathbf{a}'')$ and $\det(\mathbf{b}', \mathbf{b}'')$ are nowhere vanishing. Moreover, the assumption that $\gamma_{(\mathbf{a},\mathbf{b})}$ is Lagrangian implies $\det(\mathbf{a}', \mathbf{a}'') = \det(\mathbf{b}', \mathbf{b}'')$. Therefore, if we set $A = (\mathbf{b}', \mathbf{b}'') \cdot (\mathbf{a}', \mathbf{a}'')^{-1}$ we have a map with values in the special linear group. If we put $T = \mathbf{b} - A \cdot \mathbf{a}$, we obtain

$$\mathbf{b}(t_0) = A(t_0)\mathbf{a}(t_0) + T(t_0), \quad \mathbf{b}'|_{t_0} = A(t_0)\mathbf{a}'|_{t_0}, \quad \mathbf{b}''|_{t_0} = A(t_0)\mathbf{a}''|_{t_0}.$$

This means that $\mathbf{b}$ and $A(t_0)\mathbf{a}+T(t_0)$ have the same second order jet at t_0, for every $t_0 \in I$. So, $\mathbf{a}$ and $\mathbf{b}$ are second order deformations each other with respect to the special affine group. $\square$

3 Symplectic moving frames

The goal is to determine a $\mathfrak{s}(4,\mathbb{R})$-valued one form $\tilde{\Omega}$ on jets of Lagrangian curves that satisfies $\mathrm{d}\tilde{\Omega} = \tilde{\Omega} \wedge \tilde{\Omega}$. As an intermediate tool we define, following [8], a *moving frame* as an appropriate (right) equivariant map $\rho : \mathrm{J}(\mathbb{R},\mathbb{R}^4) \to \mathcal{S}(4,\mathbb{R})$. Then $\tilde{\Omega} = \rho^*\Omega$, where Ω is the (right) Maurer-Cartan form on $\mathcal{S}(4,\mathbb{R})$

A moving frame is implicitly defined by a choice of *cross-section*. The technology introduced in [8], and further developed in [16, 18, 17, 27], lends itself to an algorithmic treatment. Neither the moving frame, nor the differential invariants, need to be known explicitly to characterize $\tilde{\Omega}$. In the present case of curves in affine symplectic geometry, we can express everything in terms of the symplectic curvatures that we introduced.

In the first subsection we shortly review the results that enable us to perform the computations formally. In each of the three next subsections we examine the results obtained for different choices of cross-section. The computations are lead with the set of Maple routines AIDA [15] that works on top of the libraries *DifferentialGeometry* and *diffalg* [3, 13]. The first cross-section is the one used in [29, 27, 24, 44]. It assumes that the first symplectic curvature does not vanish. Contrary to expectations, we provide a minimal order cross-section that removes this restriction. As a more general construction for $\mathcal{S}(2n,\mathbb{R})$ we can construct a moving frame thanks to a symplectic Gram-Schmidt process. This is illustrated in the last subsection.

3.1 Moving frames from sections

For this review, we place ourself in a slightly more general context. We consider a r dimensional (matrix) Lie group $\mathcal{G}$ acting on (an open set of) $\mathbb{R}^n$. To each element $\mathfrak{a}$ in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ we can associate a vector field $V_{\mathfrak{a}}$ the flow of which is an orbit of the action of a one-dimensional subgroup of $\mathcal{G}$ classically denoted $e^{\mathfrak{a}t}$. To a basis of the Lie algebra thus correspond r *infinitesimal generators* of the action of $\mathcal{G}$ on $\mathbb{R}^n$.

We consider the jets $\mathrm{J}^k(\mathbb{R}^m,\mathbb{R}^n)$, or simply J^k, of parameterized m-dimensional submanifolds. $\mathrm{D}_1,\ldots,\mathrm{D}_m$ are the total derivations with respect to the parameters. The action of $\mathcal{G}$ is prolonged to those jet spaces so as to be compatible with those total derivations. Explicit prolongation formulae for the action and the infinitesimal generators can be found in [40] for instance. Like many such operations, their implementation is available through the Maple library *DifferentialGeometry*. In the following, $V_1,\ldots,V_r$ denote the appropriate prolongation

of the infinitesimal generators.

The prolonged action of $\mathcal{G}$ on J^k is denoted by $\star$. It is given by a smooth map

$$\begin{array}{ccc} \mathcal{G} \times \mathrm{J}^k & \to & \mathrm{J}^k \\ (\lambda, z) & \mapsto & \lambda \star z \end{array}$$

If the group $\mathcal{G}$ acts locally effectively, a mild hypothesis, there exists $s \in \mathbb{N}$ such that the generic orbits of the prolonged action of $\mathcal{G}$ on J^s have the same dimension r as the group [41, Theorem 5.11]. We place ourselves in a neighborhood of a point $z_0 \in \mathrm{J}^s$ where the distribution defined by the prolonged infinitesimal generators $V_1, \ldots, V_r$ of the action has full rank r. The orbits of the points in this neighborhood are of dimension r and the action is locally free there. Through z_0 we can find a *cross-section*, i.e. a manifold $\mathcal{P}$ of codimension r that is transverse to the orbits.

Assume the cross-section $\mathcal{P}$ is determined as the level set $C = (c_1, \ldots, c_r) \in \mathbb{R}^r$ of a map $P = (p_1, \ldots, p_r) : \mathrm{J}^s \to \mathbb{R}^r$. In other words $\mathcal{P}$ is defined by the equations $p_1(z) = c_1, \ldots, p_r(z) = c_r$. Then $\mathcal{P}$ is transverse to the orbits in the neighborhood of one of its point z if the $r \times r$ matrix

$$V(P) = (V_i(p_j))_{i,j}$$

is invertible when evaluated at $z \in \mathcal{P}$. Note that the matrix $V(P)$ is the Jacobian of the map $\mathcal{G} \to \mathbb{R}^r$ defined by $\lambda \mapsto P(\lambda \star z)$ at identity. By virtue of the implicit function theorem, there exists a neighborhood $\mathcal{U}$ of z and a unique smooth map $\rho : \mathcal{U} \to \mathcal{G}$ such that

$$P(\rho(z) \star z) = C \text{ and } \rho|_{\mathcal{P}} = e.$$

This map has the sought equivariant property: $\rho(\lambda \star z) = \rho(z) \cdot \lambda^{-1}$.

Beside the moving frame, a local cross-section allows us to define an *invariantization process* and the *normalized invariants*[3]. Given a smooth function $f : \mathcal{U} \subset \mathrm{J}^{s+k} \to \mathbb{R}$ its invariantization $\bar{\iota} f : \mathcal{U} \subset \mathrm{J}^{s+k} \to \mathbb{R}$ is defined by $\bar{\iota} f(z) = f(\bar{z})$ where $\bar{z}$ is the intersection of the connected part of the orbit of z with the cross-section $\mathcal{P}$. Analytically this is given as $\bar{\iota} f(z) = f(\rho(z) \star z)$. The *normalized invariants* are the invariantization of the coordinate functions [8, 19]. We can compute them algebraically [19] but it is often preferable to work with those formally. This is made possible by the fact that $\bar{\iota} f(z) = f(\bar{\iota} z)$ and $P(\bar{\iota} z) = C$. In particular, if f is an invariant then $f(z) = f(\bar{\iota} z)$. We can therefore work formally with the normalized invariants $\bar{\iota} z$ subjected to the relationships defined by the chosen cross-section. This idea is reinforced by the explicit relation between derivation and invariantization [8, Section 13], [16, Theorem 3.6]:

$$\mathrm{D}(\bar{\iota} f) = \bar{\iota}(\mathrm{D} f) - K \cdot \bar{\iota}\,(V(f)) \tag{3.1}$$

[3] Contrary to the moving frame construction, the invariantization does not restrict to locally free actions. See [19].

where $\mathrm{D} = {}^t\begin{pmatrix}\mathrm{D}_1 & \dots & \mathrm{D}_m\end{pmatrix}$ and K is the $m \times r$ matrix

$$K = \bar{\iota}\,(\mathrm{D}(P)) \cdot \bar{\iota}\,(V(P))^{-1} \tag{3.2}$$

defined with the $m \times r$ and $r \times r$ matrices

$$\mathrm{D}(P) = (\mathrm{D}_i(p_j))_{1 \le i \le m, 1 \le j \le r}\,, \quad V(P) = (V_i(p_j))_{1 \le i,j \le r}\,.$$

Thanks to this formula we can characterize finite sets of differential invariants as generating. The first such set is the set of *normalized invariants* of order $s+1$ and less. Their complete syzygies are described in [16]; They are built on (3.1). Of relevance to the geometric applications, the Maurer-Cartan invariants are the entries of the matrix K. From the formula above, we see that those entries consist of differential invariants of order $s+1$ at most. They form a generating set of differential invariants [18].

Formula (3.2) provides the expression of the Maurer-Cartan invariants in terms of the normalized invariants of order $s+1$. Conversely, normalized invariants, and thus any differential invariants, can be written effectively in terms of Maurer-Cartan invariants thanks to (3.1). The syzygies on a set of generating differential invariants allows to determine smaller sets of generators algorithmically, with differential elimination [14, 16, 20].

The geometric importance of the Maurer-Cartan invariants comes from the fact that they describe the pullback by ρ of the (right) Maurer-Cartan form Ω:

$$\rho^*\Omega \equiv -\sum_{i=1}^{m}\left(\sum_{j=1}^{r} K_{ij}\,\mathfrak{a}_j\right) dt_i \tag{3.3}$$

where $\mathfrak{a}_1, \dots, \mathfrak{a}_r$ form a basis of the Lie algebra of $\mathcal{G}$ and $(dt_1, \dots, dt_m)$ form the horizontal one-form basis dual to $(\mathrm{D}_1, \dots, \mathrm{D}_m)$, i.e. $t_1, \dots, t_m$ are the parameters. In (3.3) the symbol $\equiv$ stands for equality modulo the contact ideal. The Maurer-Cartan invariants K_{ij} are subjected to the syzygies determined by the pullback by ρ of the structure equation $\mathrm{d}\Omega = \Omega \wedge \Omega$.

A difficulty in the presented formalism is to determine the neighborhood $\mathcal{U}$ in which the normalized invariants and the moving frame are well defined. Even for linear actions one quickly encounters algebraic functions. As we shall see in the next subsections, the case of the affine symplectic action is particularly well behave. One already saw that the symplectic curvatures are polynomial functions. One shall discover that, for a relevant choice of cross-section, the moving frame and the normalized invariants are given by rational functions at worst. The open set $\mathcal{U}$ is a subset of the linearly full curves determined by a single equation, either $\kappa_1 \neq 0$ for the first choice of cross-section, or $\kappa_2 \neq 0$ for the second and third choice.

3.2 Section previously considered

The orbits of the action of $\mathcal{S}(4,\mathbb{R})$ prolonged to $\mathrm{J}^4(\mathbb{R},\mathbb{R}^4)$ are 14 dimensional, as can be checked by computing the rank of the prolonged infinitesimal generators. The cross-section chosen in [29, Section 3.2], [27, Example 5.5.2], [24, 44] is given by the equations:

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0,\ x_4^{(0)} = 0,$$

$$x_1^{(1)} = 1,\ x_2^{(1)} = 0,\ x_3^{(1)} = 0, x_4^{(1)} = 0,$$

$$x_1^{(2)} = 0,\ x_2^{(2)} = 0,\ x_4^{(2)} = 0,$$

$$x_2^{(3)} + x_3^{(2)} = 0,\ x_4^{(3)} = 0,$$

$$x_2^{(4)} + 2\,x_3^{(3)} = 0.$$

The manifold in $\mathrm{J}^4(\mathbb{R},\mathbb{R}^4)$ that those equations define is transverse to the orbits in a neighborhood of any of its point, except where $\det(V(P))|_{\mathcal{P}} = \left(x_3^{(2)}\right)^5 x_4^{(4)}$ vanishes.

Applying the replacement properties of normalized invariants defined by this cross-section we immediately see that:

$$\kappa_1 = \Lambda(X^{(1)}, X^{(2)}) = \bar{\imath} x_3^{(2)}, \quad \kappa_2 = \Lambda(X^{(2)}, X^{(3)}) = -\bar{\imath} x_3^{(2)}\, \imath x_1^{(3)}, \ldots$$

and

$$\phi = \det\left(X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}\right) = \left(\bar{\imath} x_3^{(2)}\right)^2 \bar{\imath} x_4^{(4)}.$$

From (3.2) and (3.3) and the basis (1.2,1.3) of the Lie agebra, a couple of steps of differential elimination handled by *diffalg* allow us to write the Maurer-Cartan form in terms of the symplectic curvatures:

$$\rho^*\Omega \equiv -\left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \frac{\kappa_2}{\kappa_1^2} & 1 \\ 0 & 0 & 0 & 1 & \tau \\ 0 & -\kappa_1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\phi}{\kappa_1^3} & 0 & 0 \end{array}\right) \mathrm{d}t \tag{3.4}$$

where

$$\begin{aligned}\tau &= \frac{-\bar{\iota}x_2^{(5)} + 2\bar{\iota}x_3^{(2)}\bar{\iota}x_1^{(3)} - 3\bar{\iota}x_3^{(4)}}{\bar{\iota}x_4^{(4)}} \\ &= \frac{\kappa_1^4}{\phi^2}\,\kappa_4 + \frac{\kappa_1^3\kappa_2'}{\phi^2}\,\kappa_1''' - \frac{(\kappa_1''-\kappa_2)\kappa_1^3}{\phi^2}\,\kappa_2'' + \frac{2\kappa_1'\kappa_1^2}{\phi^2}\,\phi' - 2\frac{\kappa_1^2}{\phi}(2\kappa_1''-\kappa_2) \\ &\quad -\frac{\kappa_1^2}{\phi^2}(\kappa_2\kappa_1'\kappa_2' + \kappa_2\kappa_1''^2 - 2\kappa_1''\kappa_2^2 + 2\kappa_2'^2\kappa_1 - \kappa_1''\kappa_1'\kappa_2' + \kappa_2^3).\end{aligned}$$

We can similarly determine algorithmically all the normalized invariants in terms of the symplectic curvatures. We organise the information in matrix form:

$$\begin{pmatrix}\bar{\iota}X^{(0)} & \bar{\iota}X^{(1)} & \bar{\iota}X^{(2)} & \bar{\iota}X^{(3)} & \bar{\iota}X^{(4)}\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -\frac{\kappa_2}{\kappa_1} & -\frac{\kappa_2'}{\kappa_1} \\ 0 & 0 & 0 & -\kappa_1 & -2\kappa_1' \\ 0 & 0 & \kappa_1 & \kappa_1' & \kappa_1''-\kappa_2 \\ 0 & 0 & 0 & 0 & \frac{\phi}{\kappa_1^2} \end{pmatrix}. \quad (3.5)$$

Here the transversality condition is $\bar{\iota}(\det V(P)) = -\kappa_1^3\,\phi \neq 0$ and both the moving frame and the normalized invariants are well defined on the open set this inequation defines. Consider a linearly full curve $\gamma : I \to \mathbb{R}^4$, its jets $j^{(k)}(\gamma) : I \to \mathrm{J}^k$ and the functions $\bar{\kappa}_i = \kappa_i \circ j^{(i+1)}(\gamma)$. Assume $\bar{\kappa}_1$ does not vanish. The curve $\rho(j^{(4)}(\gamma)) \star j^{(4)}(\gamma)$ belongs to the cross-section $\mathcal{P}$ and is uniquely determined by $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$ as can be seen from (3.5).

Conversely, consider smooth functions $\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\kappa}_4 : I \to \mathbb{R}$ such that $\bar{\kappa}_1$ and $\bar{\phi} = \bar{\kappa}_2^2 - \bar{\kappa}_1\bar{\kappa}_3 + \bar{\kappa}_1'\bar{\kappa}_2' - \bar{\kappa}_2\bar{\kappa}_1''$ do not vanish anywhere. On one hand they define a curve $\bar{\gamma} : I \to \mathcal{P} \subset \mathrm{J}^4$ given by (3.5). On the other hand they define a one-form $\bar{\Omega}$ on I with values in $\mathcal{S}(4,\mathbb{R})$ of the shape (3.4). By Theorem 1.1, there is a curve $\bar{\rho} : I \to \mathcal{S}(4,\mathbb{R})$ that is unique up to an element of $\mathcal{S}(4,\mathbb{R})$ such that $\bar{\rho}^*\Omega = \bar{\Omega}$. Then $\bar{\rho}^{-1} \star \bar{\gamma}$ is the jet of a curve $\gamma : I \to \mathbb{R}^4$ such that the functions $\bar{\kappa}_i$ are its symplectic curvatures.

3.3 Minimal order section with specialization property

The cross-section in J^4 of the previous paragraph is not transverse to the jets of Lagrangian curves. It was claimed in [24, 44] that Lagrangian curves require higher order frames. Such is not the case. We exhibit two cross-sections in J^4 that are transverse to full Lagrangian curves. The first one is even of *minimal order* [42, 16].

The equations

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0,\ x_4^{(0)} = 0$$

$$x_1^{(1)} = 1,\ x_2^{(1)} = 0,\ x_3^{(1)} = 0,\ x_4^{(1)} = 0,$$

$$x_1^{(2)} = 0,\ x_2^{(2)} = 1,\ x_4^{(2)} = 0$$
$$x_1^{(3)} = 0,\ x_2^{(3)} = 0,$$
$$x_1^{(4)} = 0.$$

define a cross-section in the neighborhood of any of its point, except when $(x_4^{(3)}\, x_2^{(4)}\, x_3^{(2)} + x_4^{(4)}\, x_3^{(3)} - x_4^{(3)}\, x_3^{(4)})\, x_4^{(3)} = 0$.

From the replacement properties of normalized invariants defined by this cross-section we see that:

$$\kappa_1 = \Lambda(X^{(1)}, X^{(2)}) = \bar{\imath} x_3^{(2)}, \quad \kappa_2 = \Lambda(X^{(2)}, X^{(3)}) = \bar{\imath} x_4^{(3)}, \quad \kappa_3 = \Lambda(X^{(4)}, X^{(3)}) = -\bar{\imath} x_4^{(3)} \bar{\imath} x_2^{(4)}.$$

Applying (3.3) and (3.2) with the basis (1.2,1.3) of the Lie algebra, and some differential elimination, we algorithmically obtain

$$\rho^*\Omega \equiv -\left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & \kappa_1\,\tau & -\tau & \dfrac{\kappa_1'\tau}{\kappa_2} \\ 0 & 1 & -\dfrac{\kappa_1\kappa_1'\tau}{\kappa_2} & \dfrac{\kappa_1'\tau}{\kappa_2} & -\dfrac{{\kappa_1'}^2\tau + \kappa_3}{\kappa_2^2} \\ 0 & \kappa_1 & 0 & 0 & -1 \\ 0 & 0 & \kappa_2 + \kappa_1^2\tau & -\kappa_1\tau & \dfrac{\kappa_1\kappa_1'\tau}{\kappa_2} \end{array}\right) \mathrm{d}t, \tag{3.6}$$

where

$$\tau = \frac{\bar{\imath} x_4^{(3)}\, \bar{\imath} x_1^{(5)}}{\phi} = \frac{\kappa_2^2}{\phi^2}\,\kappa_4 - \frac{\kappa_2\kappa_2'}{\phi^2}\,\kappa_3' + \frac{\kappa_2\kappa_3}{\phi^2}\,\kappa_2'' - \frac{\kappa_2\kappa_3^2}{\phi^2}.$$

Furthermore, thanks to (3.1), we have:

$$\begin{pmatrix} \bar{\imath}X^{(0)} & \bar{\imath}X^{(1)} & \bar{\imath}X^{(2)} & \bar{\imath}X^{(3)} & \bar{\imath}X^{(4)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{\kappa_3}{\kappa_2} \\ 0 & 0 & \kappa_1 & \kappa_1' & \kappa_1'' - \kappa_2 \\ 0 & 0 & 0 & \kappa_2 & \kappa_2' \end{pmatrix}. \tag{3.7}$$

This latter equality defines the moving frame explicitly since

$$\rho \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ X^{(0)} & X^{(1)} & X^{(2)} & X^{(3)} & X^{(4)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \bar{\imath}X^{(0)} & \bar{\imath}X^{(1)} & \bar{\imath}X^{(2)} & \bar{\imath}X^{(3)} & \bar{\imath}X^{(4)} \end{pmatrix}$$

Here the transversality condition is $\bar{\imath}\,(\det V(P)) = -\kappa_2\,\phi \neq 0$. Consider a linearly full curve $\gamma : I \to \mathbb{R}^4$, its jets $j^{(k)}(\gamma) : I \to \mathrm{J}^k$ and the functions $\bar{\kappa}_i = \kappa_i \circ j^{(i+1)}(\gamma)$. Assume $\bar{\kappa}_2$ does not vanish. The curve $\rho(j^{(4)}(\gamma)) \star j^{(4)}(\gamma)$ belongs to the cross-section $\mathcal{P}$ and is uniquely determined by $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$ as can be seen from (3.7).

Conversely, consider smooth functions $\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\kappa}_4 : I \to \mathbb{R}$ such that $\bar{\kappa}_2$ and $\bar{\phi} = \bar{\kappa}_2^2 - \bar{\kappa}_1\bar{\kappa}_3 + \bar{\kappa}_1'\bar{\kappa}_2' - \bar{\kappa}_2\bar{\kappa}_1''$ do not vanish anywhere. On one hand they define

a curve $\bar{\gamma} : I \to \mathcal{P} \subset \mathrm{J}^4$ given by (3.7). On the other hand they define a one-form $\bar{\Omega}$ on I with values in $\mathcal{S}(4,\mathbb{R})$ of the shape (3.6). By Theorem 1.1, there is a curve $\bar{\rho} : I \to \mathcal{S}(4,\mathbb{R})$ that is unique up to an element of $\mathcal{S}(4,\mathbb{R})$ such that $\bar{\rho}^*\Omega = \bar{\Omega}$. Then $\bar{\rho}^{-1} \star \bar{\gamma}$ is the jet of a curve $\gamma : I \to \mathbb{R}^4$ such that $\bar{\kappa}_i$ are its symplectic curvatures. We shall apply this construction to determine Lagrangian curves of constant symplectic curvatures in Section 4.

3.4 Section corresponding to a Gram-Schmidt process

There is a general method to obtain a moving frame $\rho : \mathrm{J} \to \mathcal{S}(2n,\mathbb{R})$ that can be restricted to jets of Lagrangian curves. It is obtained through a symplectic Gram-Schmidt process for which we provide the case of $n = 2$. The underlying cross-section is not of minimal order, but the entailed Serret-Frenet matrix is sparser than the one proposed in previous paragraph.

The equations

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0,\ x_4^{(0)} = 0$$

$$x_2^{(1)} = 1,\ x_4^{(1)} = 0,$$

$$x_1^{(2)} = 1,\ x_2^{(2)} = 0,\ x_3^{(2)} = 0,\ x_4^{(2)} = 0,$$

$$x_1^{(3)} = 0,\ x_2^{(3)} = 0,\ x_4^{(3)} = 0,$$

$$x_2^{(4)} = 0,$$

define a cross-section in the neighborhood of any of its point, except when $\det V(P)|_{\mathcal{P}} = -\left(x_3^{(3)}\right)^3 x_4^{(4)}$ vanishes.

From the replacement properties of normalized invariants defined by this cross-section we see that:

$$\kappa_1 = \Lambda(X^{(1)}, X^{(2)}) = -\iota x_3^{(1)}, \quad \kappa_2 = \Lambda(X^{(2)}, X^{(3)}) = \iota x_3^{(3)}, \quad \kappa_3 = \Lambda(X^{(4)}, X^{(3)}) = -\iota x_3^{(3)} \iota x_1^{(4)}.$$

The transversality condition is $\iota\,(\det V(P)) = \kappa_2^2\,\phi \neq 0$. Applying (3.3) and (3.2), and some differential elimination, we algorithmically obtain

$$\rho^*\Omega \equiv \left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline -\dfrac{\kappa_1'}{\kappa_2} & 0 & -\dfrac{\phi}{\kappa_2^2} & \dfrac{\kappa_3}{\kappa_2^2} & 0 \\ -1 & 0 & 0 & 0 & \tau \\ \kappa_1 & -\kappa_2 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\kappa_1\phi}{\kappa_2^2} & \dfrac{\phi}{\kappa_2^2} & 0 \end{array}\right) \mathrm{d}t,$$

where τ is a fifth order invariant such that

$$\kappa_4 = \Lambda(X^{(4)}, X^{(5)}) = \frac{\phi^2}{\kappa_2^2}\,\tau + \frac{\kappa_2'\,\kappa_3' - \kappa_3\,\kappa_2'' + \kappa_3^2}{\kappa_2}.$$

The proposed cross-section was actually obtained through a symplectic Gram-Schmidt process on $X^{[+]} = \big(X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}\big)$ to define a moving frame. Starting with the assumption that $\kappa_2 = \Lambda(X^{(2)}, X^{(3)}) \neq 0$ we introduce the matrix[4]

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -\dfrac{\Lambda(X^{(1)}, X^{(3)})}{\Lambda(X^{(2)}, X^{(3)})} & 0 & -\dfrac{\Lambda(X^{(3)}, X^{(4)})}{\phi} \\ 0 & \dfrac{\Lambda(X^{(1)}, X^{(2)})}{\Lambda(X^{(2)}, X^{(3)})} & \dfrac{1}{\Lambda(X^{(2)}, X^{(3)})} & \dfrac{\Lambda(X^{(2)}, X^{(4)})}{\phi} \\ 0 & 0 & 0 & -\dfrac{\Lambda(X^{(2)}, X^{(3)})}{\phi} \end{pmatrix}$$

so that $Q = X^{[+]}\,R \in \mathrm{Sp}(4, \mathbb{R})$.

A moving frame $\rho : \mathrm{J}^4 \to \mathcal{S}(4, \mathbb{R})$ is then given by

$$\rho = \begin{pmatrix} 1 & 0 \\ -Q^{-1}X^{(0)} & Q^{-1} \end{pmatrix} \in \mathcal{S}(4, \mathbb{R}).$$

We can read the equations of the underlying cross-section from R^{-1} since:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \bar{\imath}X^{(0)} & \bar{\imath}X^{(1)} & \bar{\imath}X^{(2)} & \bar{\imath}X^{(3)} & \bar{\imath}X^{(4)} \end{pmatrix} = \rho \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ X^{(0)} & X^{(1)} & X^{(2)} & X^{(3)} & X^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & R^{-1} \end{pmatrix} = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & \dfrac{\kappa_1'}{\kappa_2} & 1 & 0 & -\dfrac{\kappa_3}{\kappa_2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\kappa_1 & 0 & \kappa_2 & \kappa_2' \\ 0 & 0 & 0 & 0 & -\dfrac{\phi}{\kappa_2} \end{array}\right).$$

The process can thus be generalised to determine appropriate cross-sections for all dimensions.

[4] We first write R above in terms of the $\Lambda(X^{(i)}, X^{(j)})$ so has to visualize the Gram-Schmidt process. The expression of R^{-1} in terms of the symplectic curvatures κ_i is given afterwards.

4 Lagrangian curves with constant curvatures

4.1 Serret-Frenet equations

In Section 3.3 (and 3.4) we constructed a moving frame that can be specialized to Lagrangian curves. As discussed there, given functions $\kappa_2, \kappa_3, \kappa_4 : I \to \mathbb{R}$ such that κ_2 does not vanish we obtain the unique curve, up to the action of $\mathcal{S}(4,\mathbb{R})$, with symplectic curvatures $\kappa_1 = 0$ and $\kappa_2, \kappa_3, \kappa_4$ as the first column of ρ^{-1} where $\rho : I \to \mathcal{S}(4,\mathbb{R})$ is a matrix solution of

$$\rho^*\Omega = -A(\kappa)\,\mathrm{d}t, \quad \rho(0) \in \mathcal{S}(4,\mathbb{R})$$

where, in accordance with (3.6)

$$A(\kappa) = \left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -\tau & 0 \\ 0 & 1 & 0 & 0 & -\dfrac{\kappa_3}{\kappa_2^2} \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \kappa_2 & 0 & 0 \end{array}\right) \in \mathfrak{s}(4,\mathbb{R})$$

and

$$\tau = \frac{1}{\kappa_2^2}\,\kappa_4 - \frac{\kappa_2'}{\kappa_2^3}\,\kappa_3' + \frac{\kappa_3}{\kappa_2^3}\,\kappa_2'' - \frac{\kappa_3^2}{\kappa_2^3}.$$

Equivalently we can directly consider the equation for $\tilde{\rho} = \rho^{-1}$ which is the analogue of the Serret-Frenet equation

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = \tilde{\rho}\,A(\kappa).$$

In terms of a frame (p, E_1, E_2, E_1, E_2) those are the linear differential equations

$$p' = E_1,\ E_1' = E_2,\ E_2' = \kappa_2 E_3,\ E_2' = -\tau E_1,\ E_3' = \frac{\kappa_3}{\kappa_2^2}E_2 - E_3.$$

4.2 Classification

In this section we classify the Lagrangian curves parameterized by arc-length that have constant symplectic curvatures. In other words we investigate the curves with symplectic curvatures $\kappa_1 = 0, \kappa_2 = 1$ and κ_3, κ_4 constant. For each pair of constants (κ_3, κ_4) any such curve is given by the first column of $B\,e^{As}$ where $B \in \mathcal{S}(4,\mathbb{R})$ and

$$A = \left(\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \kappa_3^2 - \kappa_4 & 0 \\ 0 & 1 & 0 & 0 & -\kappa_3 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

They can thus be classified according to the roots of the characteristic polynomial $\lambda\,\pi(\lambda) = \lambda(\lambda^4 + \kappa_3\lambda^2 + \kappa_3^2 - \kappa_4)$ of A. In each case we provide a curve in the congruence class. It is described in terms of some $\mu, \nu \in \mathbb{R}$ that are taken such that $\mu > \nu > 0$.

I: $(\kappa_3^2 - \kappa_4)(4\kappa_4 - 3\kappa_3^2) \neq 0$. π has four distinct roots.

I.1: $4\kappa_4 < 3\kappa_3^2$. The roots of π are $\{\pm\mu \pm i\nu\}$.

$$\gamma(s) = \begin{pmatrix} \dfrac{(\mu^2 - 3\nu^2)\mu\cos(\nu s)\sinh(\mu s)}{\mu^2+\nu^2} \\ -\dfrac{1}{2}\dfrac{\sin(\nu s)\sinh(\mu s)}{(\mu^2+\nu^2)^2\mu\nu} + \dfrac{1}{2}\dfrac{(\mu^2-3\nu^2)\cos(\nu s)\cosh(\mu s)}{(\nu^2-3\mu^2)(\mu^2+\nu^2)^2\nu^2} \\ -\dfrac{1}{2}\dfrac{\sin(\nu s)\sinh(\mu s)}{(\mu^2+\nu^2)^2\mu\nu} + \dfrac{1}{2}\dfrac{(3\mu^2-\nu^2)\cos(\nu s)\cosh(\mu s)}{(\mu^2-3\nu^2)(\mu^2+\nu^2)^2\mu^2} \\ \dfrac{(\nu^2 - 3\mu^2)\nu\sin(\nu s)\cosh(\mu s)}{\mu^2+\nu^2} \end{pmatrix}$$

I.2: $4\kappa_4 > 3\kappa_3^2$. The squares of the roots of π are real.

I.2.a: $\pm\sqrt{4\kappa_4 - 3\kappa_3^2} > \kappa_3$. The roots of π are $\{\pm\mu, \pm\nu\}$.

$${}^t\gamma(s) = \left(\frac{\sinh(\nu s)}{\sqrt{\mu^2-\nu^2}\nu^{\frac{3}{2}}},\ \frac{\cosh(\mu s)}{\sqrt{\mu^2-\nu^2}\mu^{\frac{3}{2}}},\ \frac{\cosh(\nu s)}{\sqrt{\mu^2-\nu^2}\nu^{\frac{3}{2}}},\ \frac{\sinh(\mu s)}{\sqrt{\mu^2-\nu^2}\mu^{\frac{3}{2}}}\right)$$

I.2.b: $\pm\sqrt{4\kappa_4 - 3\kappa_3^2} < \kappa_3$. The roots of π are $\{\pm i\mu, \pm i\nu\}$.

$${}^t\gamma(s) = \left(\frac{\sin(\nu s)}{\sqrt{\mu^2-\nu^2}\nu^{\frac{3}{2}}},\ \frac{\cos(\mu s)}{\sqrt{\mu^2-\nu^2}\mu^{\frac{3}{2}}},\ \frac{\cos(\nu s)}{\sqrt{\mu^2-\nu^2}\nu^{\frac{3}{2}}},\ \frac{\sin(\mu s)}{\sqrt{\mu^2-\nu^2}\mu^{\frac{3}{2}}}\right)$$

I.2.c: $-\sqrt{4\kappa_4 - 3\kappa_3^2} < \kappa_3 < \sqrt{4\kappa_4 - 3\kappa_3^2}$. The roots of π are $\{\pm i\mu, \pm\nu\}$.

$${}^t\gamma(s) = \left(-\frac{\sinh(\nu s)}{\sqrt{\mu^2+\nu^2}\nu^{\frac{3}{2}}},\ \frac{\cos(\mu s)}{\sqrt{\mu^2+\nu^2}\mu^{\frac{3}{2}}},\ \frac{\cosh(\nu s)}{\sqrt{\mu^2+\nu^2}\nu^{\frac{3}{2}}},\ \frac{\sin(\mu s)}{\sqrt{\mu^2+\nu^2}\mu^{\frac{3}{2}}}\right)$$

II: $\kappa_4 = \kappa_3^2$, $\kappa_3 \neq 0$. Then $\pi(\lambda) = \lambda^2(\lambda^2 + \kappa_3)$.

II.1: $\kappa_3 < 0$. The non-zero roots of π are $\{\mu, -\mu\}$, where $\mu = \sqrt{-\kappa_3} > 0$.

$${}^t\gamma(s) = \left(s,\ \frac{\cosh(\mu s)}{\mu^{\frac{5}{2}}},\ \frac{s^2}{2\mu^2},\ \frac{\sin(\mu s)}{\mu^{\frac{5}{2}}}\right)$$

II.2: $\kappa_3 > 0$. The non-zero roots of π are $\{i\mu, -i\mu\}$, where $\mu = \sqrt{\kappa_3} > 0$.

$${}^t\gamma(s) = \left(s,\ \frac{\cos(\mu s)}{\mu^{\frac{5}{2}}},\ -\frac{s^2}{2\mu^2},\ \frac{\sin(\mu s)}{\mu^{\frac{5}{2}}}\right)$$

III: $4\kappa_4 = 3\kappa_3^2$ and $\kappa_3 \neq 0$. Then $\pi(\lambda) = (\lambda^2 + \frac{1}{2}\kappa_3)^2$.

III.1: $\kappa_3 > 0$. The roots of π are $\{i\mu, -i\mu\}$ where $\mu = \sqrt{\frac{\kappa_3}{2}} > 0$.

$$^t\gamma(s) = \left(\frac{s\,\cos(\mu s)}{\mu^2}, \quad \frac{s\,\sin(\mu s)}{\mu} + \frac{3\cos(\mu s)}{2\mu^2}, \quad -\frac{1}{2}\frac{\cos(\mu s)}{\mu^2}, \quad -\frac{1}{2}\frac{\sin(\mu s)}{\mu^3} \right)$$

III.2: $\kappa_3 < 0$. The roots of π are $\{\mu, -\mu\}$ where $\mu = \sqrt{-\frac{\kappa_3}{2}} > 0$.

$$^t\gamma(s) = \left(\frac{s\,\cosh(\mu s)}{\mu^2}, \quad \frac{s\,\sinh(\mu s)}{\mu} - \frac{3\cos(\mu s)}{2\mu^2}, \quad -\frac{1}{2}\frac{\cosh(\mu s)}{\mu^2}, \quad \frac{1}{2}\frac{\sinh(\mu s)}{\mu^3} \right)$$

IV: $\kappa_3 = \kappa_4 = 0$. The roots of π are zero.

$$^t\gamma(s) = \left(\frac{t}{\sqrt{24}}, \quad \frac{t^2}{\sqrt{12}}, \quad -\frac{t^4}{\sqrt{24}}, \quad \frac{t^3}{\sqrt{12}} \right)$$

All the curves cataloged in the previous classification are orbits of one-parameter subgroups of the affine symplectic group. For instance, the curve I.2.b is the orbit passing through the point

$$p_{\mu,\nu} = {}^t\left(0, (\mu^3(\mu^2-\nu^2))^{-1/2}, (\nu^3(\mu^2-\nu^2))^{-1/2}, 0\right)$$

of the 1-parameter subgroup of $\mathcal{S}(4,\mathbb{R})$ generated by

$$\mathrm{X}_{\mu,\nu} = \begin{pmatrix} 0 & 0 \\ 0 & J_{\mu,\nu} \end{pmatrix} \in \mathfrak{s}(4,\mathbb{R}),$$

where

$$J_{\mu,\nu} = \mu(C_2^2 - B_2^2) + \nu(B_1^1 - C_1^1) \in \mathfrak{sp}(4,\mathbb{R}).$$

In the previous formula, the 4×4 matrices B_j^i and C_j^i denote the generators of $\mathfrak{sp}(4,\mathbb{R})$ defined in Subsection 1.1. The symplectic curvatures κ_3, κ_4 and the constants μ, ν are related by

$$\kappa_3 = \mu^2 + \nu^2, \quad \kappa_4 = \mu^4 + \mu^2\nu^2 + \nu^4.$$

Also in the other cases it is possible to write explicitly the 1-parameter subgroups that generate the curves.

We would like to highlight here a fact about the curves of type I.2.b : their trajectories are the integral curves of a linear flow on the homogenous Lagrangian torus $\mathrm{T}^2_{\mu,\nu} \subset \mathbb{R}^4$. Such a torus is the orbit through the point $p_{\mu,\nu} \in \mathbb{R}^4$ of the maximal Abelian subgroup $\mathrm{T}^2 \cong \mathrm{SO}(2)\times\mathrm{SO}(2)$ of $\mathcal{S}(4,\mathbb{R})$ consisting of all $\mathrm{X}(\theta_1,\theta_2) \in \mathcal{S}(4,\mathbb{R})$ of the following form

$$\mathrm{X}(\theta_1,\theta_2) = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & \cos(\theta_1) & 0 & -\sin(\theta_1) & 0 \\ 0 & 0 & \cos(\theta_2) & 0 & -\sin(\theta_2) \\ 0 & \sin(\theta_1) & 0 & \cos(\theta_1) & 0 \\ 0 & 0 & \sin(\theta_2) & 0 & \cos(\theta_2) \end{array}\right), \quad \theta_1\theta_2 \in \mathbb{R}.$$

This fact has an immediate geometric consequence : the trajectories of the curves of type I.2.b are either simple closed curves or else are dense in the 2-dimensional torus $\mathrm{T}^2_{\mu,\nu}$. Another noteworthy observation is that the 2-dimensional tori $\mathrm{T}^2_{\mu,\nu}$ are the fibers of the moment map $\mathrm{m} : \mathbb{R}^4 \to \mathfrak{t}^*$ induced by the Hamiltonian action of the maximal compact Abelian subgroup T^2. The symbol $\mathfrak{t}^*$ stands for the dual of the Lie algebra of T^2.

4.3 Closed curves

Closed trajectories occur only for curves of Type I.2.b when $\frac{\mu}{\nu} \in \mathbb{Q}$. The conditions on κ_3, κ_4 can be simplified to

$$\kappa_3^2 > \kappa_4 > \frac{3}{4}\kappa_3^2 \quad \text{and} \quad \kappa_3 > 0.$$

Then

$$\mu = \sqrt{\frac{\kappa_3 + \sqrt{4\kappa_4 - 3\kappa^2}}{2}} \quad \text{and} \quad \nu = \sqrt{\frac{\kappa_3 - \sqrt{4\kappa_4 - 3\kappa^2}}{2}}.$$

The fact that $\frac{\mu}{\nu} = \frac{m}{n}$, for $m, n \in \mathbb{N}$ can be rewritten as follows.

Proposition 4.1 *A Lagrangian curve γ with constant symplectic curvatures κ_3, κ_4 is closed if and only if $\kappa_3 > 0$, $\kappa_3^2 > \kappa_4 > \frac{3}{4}\kappa_3^2$ and there exist $m, n \in \mathbb{N}$*

$$\frac{\kappa_3^2}{(m^2+n^2)^2} = \frac{\kappa_4}{m^4 + m^2 m^2 + n^4}.$$

Taking m and n relatively prime, γ is a torus knot of type (m, n). Its symplectic length is $\frac{2\pi}{\nu\, n} lcm(m, n)$.

The concept of second-order deformations of curves introduced in Section 2.3 can be illustrated on those curves. If $(\mathbf{a}, \mathbf{b})$ is the pair of phase portraits of a Lagrangian curve $\gamma_{(\mathbf{a},\mathbf{b})}$, the curves $\mathbf{a}$ and $\mathbf{b}$ are second-order deformations of each other with respect to the special affine group (d-pairs for short). Given an element $g \in \mathcal{S}(4, \mathbb{R})$, $g \star \gamma_{(\mathbf{a},\mathbf{b})}$ is another Lagrangian curve whose phase portraits $(\tilde{\mathbf{a}}, \tilde{\mathbf{b}})$ make up a new d-pair. In this way $\mathcal{S}(4, \mathbb{R})$ acts on the space of all d-pairs. Such an action is global in nature and is not originated by a pseudo-group of transformations of the plane. Its effects can be rather unpredictable, as we wish to show in Figure 1 and Figure 2.

The classification shows that any closed Lagrangian curve with constant symplectic curvatures is congruent to a curve whose phase portrait consists of a pair of circles. That such a pair of circles are second-order deformations of each other with respect to the special affine group is rather easy to visualise. Other elements in the congruence class provide more surprising d-pairs. Figure 1 reproduces the phase portraits of a closed Lagrangian curve with constant symplectic curvatures $\kappa_3 = \frac{5}{2}$, $\kappa_4 = \frac{91}{16}$, which correspond to the data $\nu = \frac{1}{2}$,

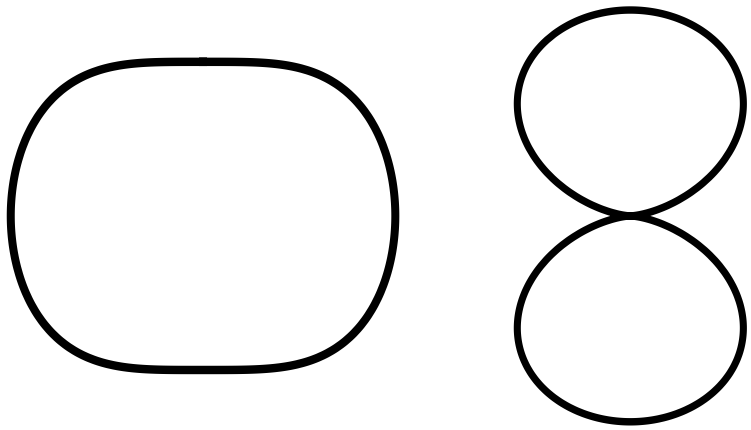

Figure 1: Phase portraits of a Lagrangian curve with $\kappa_3 = 2.5$ and $\kappa_4 = 5.6875$.

$m = 3$ and $n = 1$. The symplectic length of such a curve is $\ell \approx 12.5664$. Figure 2 provides the phase portraits of a closed Lagrangian curve with symplectic curvatures $\kappa_3 = 1.64$ and $\kappa_4 = 2.0496$, which correspond to the values $\nu = 0.8$, $m = 5$, $n = 4$. The symplectic length is $\ell \approx 31.4259$.

4.4 Lagrangian tori

We recall that an immersed surface $f : S \to \mathbb{R}^4$ is Lagrangian if its tangent planes are Lagrangian. This notion plays a central role in the Hamilton-Jacobi theory. Among all Lagrangian surfaces, the Lagrangian tori have a particular significance because they arise in a natural way as the fibers of the momentum map of a Liouville-integrable Hamiltonian system. We now exhibit a procedure to construct families of Lagrangian tori starting from closed Lagrangian curves with constant curvatures.

Let $\gamma : \mathbb{R} \to \mathbb{R}^4$ be a closed Lagrangian curve with constant curvatures and symplectic length ℓ_γ. We assume that γ is parameterized by the symplectic arc-length. Therefore $\kappa_3 > 0$ and $\kappa_3^2 - \kappa_4 > 0$. We fix a real constant h and take any closed regular curve, parameterized by the Euclidean arc-length, $\alpha : \vartheta \in \mathbb{R} \to \mathbb{R}^3$ whose trajectory lies in the quadric $\mathcal{Q}_h \subset \mathbb{R}^3$ defined by the equation $x^2 + \kappa_3 y^2 - z^2 = h$. By construction, $\mathcal{Q}_0$ is a quadratic cone, $\mathcal{Q}_h$, $h < 0$, is a two-sheet hyperboloid and $\mathcal{Q}_h$, $h > 0$, is a one-sheet hyperboloid. Let assume that the trajectory belongs either to $\mathcal{Q}_h^+ = \{(x, y, z) \in \mathcal{Q}_h | z > 0\}$ or else to $\mathcal{Q}_h^- = \{(x, y, z) \in \mathcal{Q}_h | z < 0\}$. If $h < 0$ the condition is automatically fulfilled. When, $h = 0$, we are actually imposing that the vertex of the cone does not belongs to the trajectory of the curve. Finally, when $h > 0$, this imposes that the trajectory lies either in the upper or in the lower parts of the one sheet

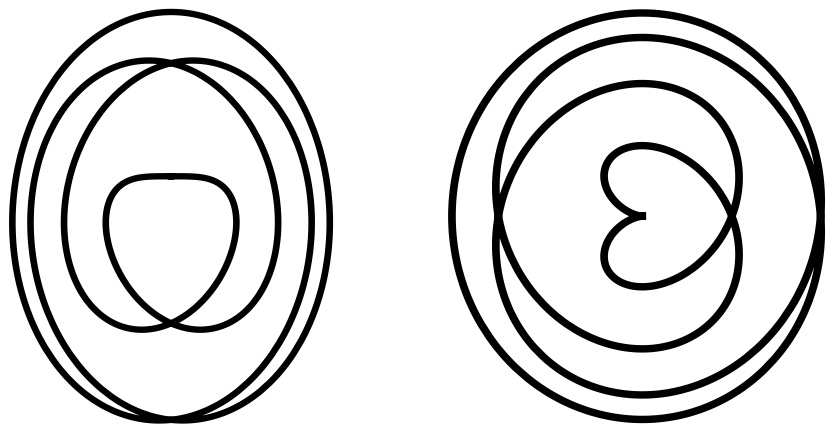

Figure 2: Phase portraits of a Lagrangian curve with $\kappa_3 = 1.64$ and $\kappa_4 = 2.0496$.

hyperboloid.

Denoting by $(E_1, ..., E_4) : \mathbb{R} \to \mathrm{Sp}(4, \mathbb{R})$ the symplectic Frenet frame along γ we define

$$f(s, \vartheta) = \gamma(s) + x(\vartheta)E_2(s) + \frac{1}{\kappa_3^2 - \kappa_4}(\sqrt{\kappa_3^2 - \kappa_4}z(\vartheta) - 1)E_3(s) + y(\vartheta)E_4(s) \in \mathbb{R}^4,$$

where $x(\vartheta), y(\vartheta)$ and $z(\vartheta)$ are the components of α. Geometrically, our surface is a sort of molding surface with directrix curve γ and profile α.

By construction, f is a doubly periodic map and its lattice of periods $L_{(\gamma,\alpha)}$ is generated by $(\ell_\gamma, 0)$ and $(0, \ell_\alpha)$, where ℓ_α is the Euclidean length of α. Thus, f induces a smooth map $\tilde{f} : \mathbb{R}^2/L_{(\gamma,\alpha)} \to \mathbb{R}^4$. Using the Serret-Frenet equations satisfied by the symplectic frame we obtain

$$(\partial_s f)|_{(s,\vartheta)} = \sqrt{\kappa_3^2 - \kappa_4}z(\vartheta)E_1(s) - \kappa_3 y(\vartheta)E_2(s) - y(\vartheta)E_3(s) + x(\vartheta)E_4(s).$$

and

$$(\partial_\vartheta f)|_{(s,\vartheta)} = \dot{x}(\vartheta)E_2(s) + \frac{1}{\sqrt{\kappa_3^2 - \kappa_4}}\dot{z}(\vartheta)E_3(s) + \dot{y}(\vartheta)E_4(s),$$

where $\dot{x}, \dot{y}$ and $\dot{z}$ are the derivatives with respect to the parameter ϑ. Since $z(\vartheta) \neq 0$ and $\dot{x}(\vartheta)^2 + \dot{y}(\vartheta)^2 + \dot{z}(\vartheta)^2 > 0$, for every ϑ, the maps f and $\tilde{f}$ are smooth immersions. Moreover, the two previous equations also imply

$$\begin{aligned} \Lambda(\partial_s f, \partial_\vartheta f)|_{(s,\vartheta)} &= -x(\vartheta)\dot{x}(\vartheta) - \kappa_3 y(\vartheta)\dot{y}(\vartheta) + z(\vartheta)\dot{z}(\vartheta) \\ &= -\frac{1}{2}\frac{d}{d\vartheta}\left(x^2 + \kappa_3 y^2 - z^2\right)|_\vartheta = 0. \end{aligned}$$

This shows that f and $\tilde{f}$ are Lagrangian immersions.

5 Lagrangian geodesics

In this section we introduce a concept of geodesics for Lagrangian curves. Interestingly, they form a subset of the Lagrangian curves with constant curvatures.

Definition 5.1 *Let $\mathcal{L}$ be the space of linearly full Lagrangian curves in $\mathbb{R}^4$. By a Lagrangian variation of $\gamma \in \mathcal{L}$ we mean a mapping $\Gamma : I \times (-\epsilon, \epsilon) \to \mathbb{R}^4$ such that $\gamma_u := \Gamma(-, u) : I \to \mathbb{R}^4$ is a linearly full Lagrangian curve, $\forall u \in (-\epsilon, \epsilon)$ and $\Gamma(t, 0) = \gamma(t)$, for every $t \in I$. The infinitesimal variation of Γ is the vector field along γ defined by*

$$\mathfrak{v} : t \in I \to \partial_u \Gamma|_{(t,0)} \in \mathbb{R}^4.$$

If $\mathfrak{v}$ vanishes outside a closed interval, then Γ is said to be compactly supported.

Definition 5.2 *A curve $\gamma \in \mathcal{L}$ is said to be a Lagrangian geodesic if it is a critical point of the symplectic arclength functional*

$$\ell : \gamma \in \mathcal{L} \mapsto \int \sigma_\gamma \in \mathbb{R}, \tag{5.1}$$

when one considers compactly supported variations.

Accordingly, a curve $\gamma \in \mathcal{L}$ is a Lagrangian geodesic if, for every compactly supported variation Γ, we have that

$$\frac{d}{du}\left(\int_K \sigma_u\right)\Big|_{u=0} = 0,$$

where K is the smallest closed interval which contains the support of the infinitesimal variation $\mathfrak{v}$.

Definition 5.3 *Let $\gamma : I \to \mathbb{R}^4$ be a Lagrangian curve. A vector field $\mathfrak{v} : I \to \mathbb{R}^4$ along γ is said to be an* infinitesimal Lagrangian variation *if $\mathfrak{v}|_t = \partial_u \Gamma|_{(t,0)}$ for some Lagrangian variation $\Gamma : I \times (-\epsilon, \epsilon) \to \mathbb{R}^4$ of γ.*

The set of all infinitesimal Lagrangian variations of γ should be thought of as the tangent space $T_\gamma(\mathcal{L})$ of $\mathcal{L}$ at γ.

Theorem 5.4 *A linearly full Lagrangian curve γ parameterized by the symplectic arc-length is a Lagrangian geodesic if and only if κ_3 is a constant and $\kappa_4 = \kappa_3^2$.*

Consequently geodesics are of Type II or IV in the classification of Section 4.2.

For simplification we introduce $k_1 = -\kappa_3$ and $k_2 = \kappa_3^2 - \kappa_4$ as the symplectic curvatures of a Lagrangian curve when parameterized by symplectic arc-length. As preliminaries to the proof we first characterize the infinitesimal variations of a linearly full Lagrangian curve. We then derive the Euler-Lagrange equations of the symplectic arc-length.

Lemma 5.5 *Let $\gamma : I \to \mathbb{R}^4$ be a linearly full Lagrangian curve parameterized by symplectic arc-length. A vector field*

$$\mathfrak{v} : s \in I \to (v_1(s), ..., v_4(s)) \in \mathbb{R}^4$$

along γ is an infinitesimal Lagrangian variation if and only if $v_2 = \frac{1}{2}(v_3'' - 3v_4')$.

PROOF: Suppose that $\mathfrak{v}$ is the infinitesimal variation induced by the Lagrangian variation $\Gamma(s,u)$ of γ. Possibly restricting the interval $(-\epsilon, \epsilon)$, we can assume that all the curves $\gamma_u : s \in I \to \Gamma(s,u)$ are linearly-full. Let

$$\mathcal{F} : I \times (-\epsilon, \epsilon) \to \mathcal{S}(4, \mathbb{R})$$

be the Frenet frame along the variation, i.e. the map which associate to each (s,u) the symplectic Frenet frame of γ_u at the point $\gamma_u(s)$. If we set $\Omega = \mathcal{F}^{-1} d\mathcal{F}$, then

$$\Omega = \mathcal{K}(s,u)ds + \mathcal{P}(s,u)du,$$

for $\mathfrak{s}(4,\mathbb{R})$-valued functions

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \mathcal{K}_2 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{K}_1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \mathcal{U} & 0 & 0 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \mathcal{V}_1 & \mathcal{A}_1 & \mathcal{A}_2 & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{V}_2 & \mathcal{A}_3 & \mathcal{A}_4 & \mathcal{B}_2 & \mathcal{B}_3 \\ \mathcal{V}_3 & \mathcal{C}_1 & \mathcal{C}_2 & -\mathcal{A}_1 & -\mathcal{A}_2 \\ \mathcal{V}_4 & \mathcal{C}_2 & \mathcal{C}_3 & -\mathcal{A}_3 & -\mathcal{A}_4 \end{pmatrix},$$

such that

$$\mathcal{U}(s,0) = 1, \quad \mathcal{V}_j(s,0) = v_j(s), \quad \mathcal{K}_1(s,0) = k_1(s), \quad \mathcal{K}_2(s,0) = k_2(s), \tag{5.2}$$

where $k_1(s)$ and $k_2(s)$ are the symplectic curvatures at $\gamma(s)$. By construction, Ω satisfies

$$d\Omega = -\Omega \wedge \Omega. \tag{5.3}$$

Equation (5.3) can be rewritten in the form

$$\frac{\partial \mathcal{K}}{\partial u} - \frac{\partial \mathcal{P}}{\partial s} = [\mathcal{K}, \mathcal{P}]. \tag{5.4}$$

In turn, (5.4) is equivalent to the following system of equations

$$\begin{cases} \mathcal{V}_2 = \mathcal{U}^{-1}(\mathcal{V}_3^{(2,0)} - 3\mathcal{V}_4^{(1,0)})/2, \\ \mathcal{A}_J = \sum_{m=1}^4 \sum_{h=1}^4 \mathcal{U}^{-r_1(J,m,h)} A_{J,h}^m(j_s(\mathcal{U}), j_s(\mathcal{K}_1), j_s(\mathcal{K}_2))\mathcal{V}_m^{(h,0)}, \\ \mathcal{C}_p = \sum_{m=1}^4 \sum_{h=1}^3 \mathcal{U}^{-r_2(p,m,h)} C_{p,h}^m(j_s(\mathcal{U}), j_s(\mathcal{K}_1), j_s(\mathcal{K}_2))\mathcal{V}_m^{(h,0)}, \\ \mathcal{B}_p = \sum_{m=1}^4 \sum_{h=1}^5 \mathcal{U}^{-r_3(p,m,h)} B_{p,h}^m(j_s(\mathcal{U}), j_s(\mathcal{K}_1), j_s(\mathcal{K}_2))\mathcal{V}_m^{(h,0)}, \\ \mathcal{U}^{(0,1)} = \sum_{m=1}^4 \sum_{h=1}^4 \mathcal{U}^{-\tilde{r}(m,h)} U_h^m(j_s(\mathcal{U}), j_s(\mathcal{K}_1), j_s(\mathcal{K}_2))\mathcal{V}_m^{(h,0)}, \\ \mathcal{K}_1^{(0,1)} = \sum_{m=1}^4 \sum_{h=1}^6 \mathcal{U}^{-\tilde{r}_1(m,h)} K_{1,h}^m(j_s(\mathcal{U}), j_s(\mathcal{K}_1), j_s(\mathcal{K}_2))\mathcal{V}_m^{(h,0)}, \\ \mathcal{K}_2^{(0,1)} = \sum_{m=1}^4 \sum_{h=1}^5 \mathcal{U}^{-\tilde{r}_2(m,h)} K_{2,h}^m(j_s(\mathcal{U}), j_s(\mathcal{K}_1), j_s(\mathcal{K}_2))\mathcal{V}_m^{(h,0)}, \end{cases} \tag{5.5}$$

where $A^m_{J,h}, C^m_{p,h}, B^m_{p,h}, U^m_{p,h}, K^m_{1,h}, K^m_{2,h}$ are suitable polynomial functions, $j_s(f)$ and is the jet with respect to the variable s of a function $f(s,u)$, $f^{(h,k)} = \partial_s^h \partial_u^k f$ and the exponents $r_1(J,m,h)$, $r_2(p,m,h)$, $r_3(p,m,h)$, $\tilde{r}(m,h)$, $\tilde{r}_1(m,h)$, $\tilde{r}_2(m,h)$ are non-negative integers. Note that all these quantities can be calculated explicitly. However, the only explicit formula that will be useful in the following is the derivative of the function $\mathcal{U}$ with respect to the parameter u, which can be written in the form

$$\begin{aligned}\mathcal{U}^{(0,1)} = \; & \mathcal{U}^{(1,0)}\mathcal{V}_1 + 5\mathcal{U}\mathcal{V}_1^{(1,0)} + 2\mathcal{U}\mathcal{K}_2\mathcal{V}_3 + \\ & \left(\mathcal{U}\mathcal{K}_1 + \tfrac{5}{2}\left(\tfrac{\mathcal{U}^{(1,0)}}{\mathcal{U}}\right)^2 - \tfrac{3}{2}\tfrac{\mathcal{U}^{(2,0)}}{\mathcal{U}}\right)\mathcal{V}_3^{(2,0)} - \tfrac{5}{2}\tfrac{\mathcal{U}^{(1,0)}}{\mathcal{U}}\mathcal{V}_3^{(3,0)} + \\ & 2\mathcal{V}_3^{(4,0)} + \left(\mathcal{K}_1\mathcal{U}^{(1,0)} + 3\mathcal{U}\mathcal{K}_1^{(1,0)}\right)\mathcal{V}_4 + \\ & \left(2\mathcal{U}\mathcal{K}_1 - \tfrac{15}{2}\left(\tfrac{\mathcal{U}^{(1,0)}}{\mathcal{U}}\right)^2 + \tfrac{9}{2}\tfrac{\mathcal{U}^{(2,0)}}{\mathcal{U}}\right)\mathcal{V}_4^{(1,0)} + \\ & \tfrac{15}{2}\tfrac{\mathcal{U}^{(1,0)}}{\mathcal{U}}\mathcal{V}_4^{(2,0)} - 5\mathcal{V}_4^{(3,0)}\end{aligned} . \tag{5.6}$$

Keeping in mind that $\mathcal{V}_m(s,0) = v_m(0)$, $m = 1,...,4$, and $\mathcal{U}(s,0) = 1$, the first formula in (5.5) implies $v_2 = \frac{1}{2}(v_3'' - 3v_4')$. Conversely, let us consider four real-valued smooth functions $v_1,..,v_4$ defined on the open interval I such that $v_2 = \frac{1}{2}(v_3'' - 3v_4')$. We let A_J, $J = 1,..,4$, B_p and C_p, $p = 1,2,3$, be the functions form I to $\mathbb{R}$ obtained by placing $\mathcal{U} = 1$, $\mathcal{K}_1 = k_1$ and $\mathcal{K}_2 = k_2$ in the right hand side of the second, third and fourth equations of (5.5). Similarly, we let $\dot{u}$,$\dot{k}_1$ and $\dot{k}_2$ be the function defined by putting $\mathcal{U} = 1$, $\mathcal{K}_1 = k_1$ and $\mathcal{K}_2 = k_2$ in the right hand side of the last three equations of (5.5). Next we consider the functions $K, P, Q : I \to \mathfrak{s}(4,\mathbb{R})$ defined by

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 & k_1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ v_1 & A_1 & A_2 & B_1 & B_2 \\ v_2 & A_3 & A_4 & B_2 & B_3 \\ v_3 & C_1 & C_2 & -A_1 & -A_2 \\ v_4 & C_2 & C_3 & -A_3 & -A_4 \end{pmatrix}, \tag{5.7}$$

and by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dot{k}_2 & 0 \\ 0 & 0 & 0 & 0 & \dot{k}_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{u} & 0 & 0 \end{pmatrix}.$$

By construction, P is a solution of the o.d.e

$$\frac{dP}{ds} = Q - [K,P].$$

Subsequently, we let $\mathcal{K}$ be the map

$$\mathcal{K} : (s,t) \in I \times \mathbb{R} \to K(s) + tQ(s), \tag{5.8}$$

and we let $\mathcal{P} : I \times \mathbb{R} \to \mathfrak{s}(4, \mathbb{R})$ be the solution of the equation

$$\frac{\partial \mathcal{P}}{\partial s} = Q - [K, \mathcal{P}] - t[Q, \mathcal{P}], \quad \mathcal{P}(s, 0) = P(s_0), \tag{5.9}$$

where s_0 is an element of I. The maps P and $\mathcal{P}(-, 0)$ are solutions of the same o.d.e with the same Cauchy data $P(s_0) = \mathcal{P}(s_0, 0)$. From this we infer that $P(s) = \mathcal{P}(s, 0)$, for every $s \in I$. Then, we consider the $\mathfrak{s}(4, \mathbb{R})$-valued 1-form

$$\Omega = \mathcal{K} ds + \mathcal{P} du \in \Omega^1(I \times \mathbb{R}) \otimes \mathfrak{s}(4, \mathbb{R}). \tag{5.10}$$

From (5.8) and (5.9) it follows that

$$d\Omega = -\Omega \wedge \Omega. \tag{5.11}$$

This implies the existence of a smooth map

$$\mathcal{F} = (\Gamma, \mathbf{E}) : I \times \mathbb{R} \to \mathcal{S}(4, \mathbb{R}) \cong \mathbb{R}^4 \times \mathrm{Sp}(4, \mathbb{R}) \tag{5.12}$$

such that

$$\mathcal{F}^{-1} d\mathcal{F} = \Omega, \quad \mathcal{F}(s_0, 0) = (\gamma(s_0), \mathbf{E}_0), \tag{5.13}$$

where $\mathbf{E}_0 \in \mathrm{Sp}(4, \mathbb{R})$ is the symplectic Frenet frame $\mathbf{E}_\gamma$ of γ evaluated at s_0. From (5.7), (5.8), (5.10) and (5.13) we see that the map

$$(\tilde{\gamma}, \tilde{\mathbf{E}}) : s \in I \to \mathcal{F}(s, 0) = (\Gamma(s, 0), \mathbf{E}(s, 0)) \in \mathcal{S}(4, \mathbb{R})$$

satisfies

$$\tilde{\gamma}' = \mathbf{E}_1, \quad \tilde{\mathbf{E}}' = \tilde{\mathbf{E}} \cdot K, \tag{5.14}$$

with the Cauchy data $\tilde{\gamma}(s_0) = \gamma(s_0)$ and $\tilde{\mathbf{E}}(s_0) = \mathbf{F}_\gamma(s_0)$. On the other hand, (5.14) is just the Frenet system satisfied by the canonical moving frame along γ. From this we deduce that

$$\Gamma(s, 0) = \gamma(s), \quad \mathbf{E}(s, 0) = \mathbf{F}_\gamma(s), \quad \forall s \in I.$$

This means that Γ is a Lagrangian variation of γ. Furthermore, using once again (5.7), (5.8), (5.10) and (5.13) we see that $\frac{\partial \Gamma}{\partial u}|_{(s,0)} = \mathfrak{v}$. We have thus constructed a Lagrangian variation of γ that has $\mathfrak{v}$ as its infinitesimal variation. This concludes the proof of the Lemma. □

Lemma 5.6 *Let $\Gamma : I \times (-\epsilon, \epsilon) \to \mathbb{R}^4$ be a compactly supported variation of a linearly full Lagrangian curve $\gamma : I \to \mathbb{R}^4$ parameterized by the symplectic arc-length and let $\mathfrak{v} : I \to \mathbb{R}^4$ be its infinitesimal variation. Then,*

$$\frac{d}{du}\left(\int_K \sigma_u\right)|_0 = \int_K ((2k_2(s) + k_1''(s))v_3(s) + k_1'(s)v_4(s))\, ds, \tag{5.15}$$

where $v_1, ..., v_4$ are the components of $\mathfrak{v}$, k_1, k_2 are the symplectic curvatures of γ, σ_u is the symplectic arc-element of the Lagrangian curves γ_u swept out by the variation and $K \subset I$ is the smallest closed interval containing the support of $\mathfrak{v}$.

PROOF: We maintain the same notations that have been used in the proof of the previous lemma. Then we have

$$\frac{d}{du}\left(\int_K \sigma_u\right)|_0 = \frac{d}{du}\left(\int_K \mathcal{U}(s,u)ds\right)|_{u=0} = \int_K \mathcal{U}^{(0,1)}(s,0)ds.$$

From (5.6) and keeping in mind that

$$\mathcal{U}(s,0) = 1, \quad \mathcal{K}_1(s,0) = k_1(s), \quad \mathcal{K}_2(s,0) = k_2(s)$$

we have

$$\int_K \mathcal{U}^{(0,1)}(s,0)ds = \int_K \left(2k_2v_3 + k_1v_3'' + 2v_3^{(4)} + 3k_1'v_4 + 2k_1v_4' - 5v_4^{(3)}\right)ds.$$

Taking into account that the supports of the functions $v_1, .., v_4$ are subsets of K and integrating by parts, we obtain

$$\begin{array}{l} \int_K \left(2k_2v_3 + k_1v_3'' + 2v_3^{(4)} + 3k_1'v_4 + 2k_1v_4' - 5v_4^{(3)}\right)ds = \\ = \int_K \left(2k_2v_3 + k_1v_3'' + 3k_1'v_4 + 2k_1v_4'\right)ds = \\ = \int_K \left(2k_2v_3 + (k_1v_3')' - k_1'v_3' + 2(k_1v_4)' + k_1'v_4\right)ds = \\ = \int_K \left((2k_2 + k_1'')v_3 + k_1v_4 - (k_1'v_3)'\right)ds = \int_K \left((2k_2 + k_1'')v_3 + k_1v_4\right)ds \end{array}.$$

This yields the required result. □

PROOF: We are now in a position to prove the proposition. From the Lemma above we see that if k_1 is constant and $k_2 = 0$, then γ is automatically a critical point of the symplectic arc-length functional with respect to compactly supported variations. Conversely, suppose that γ is a Lagrangian geodesic of $\mathbb{R}^4$. Take any compactly supported function $v_4 : I \to \mathbb{R}$, set $v_1 = v_3 = 0$ and $v_2 = -3v_4'/2$. Then, from the first Lemma we know that there is a Lagrangian variation of γ whose infinitesimal variation is given by $\mathfrak{v} = (0, v_2, 0, v_4)$. Using the second Lemma we obtain

$$0 = \frac{d}{du}\left(\int_K \sigma_u\right)|_0 = \int_K k_1v_4ds.$$

Since the function v_4 is arbitrary (provided with compact support), this implies that $k_1' = 0$, i.e. k_1 is a constant. Next, take any compactly supported smooth function $v_3 : I \to \mathbb{R}$ and set $\mathfrak{v} = (0, v_3''/2, v_3, 0)$. Thus, using again the first Lemma, we deduce the existence of a compactly supported Lagrangian variation of γ having $\mathfrak{v}$ as its infinitesimal variation. Therefore, using again the second Lemma we obtain

$$0 = \frac{d}{du}\left(\int_K \sigma_u\right)|_0 = 2\int_K k_2v_3ds.$$

On the other hand, v_3 can be any compactly supported smooth function. Therefore k_2 vanishes identically. We have thus proved the theorem. □

The result could have been inferred using a more conceptual framework, based on the Griffiths' approach to the calculus of variations in one independent variable [11, 31, 32, 35]. However, this point of view require a considerable amount of preliminary work, such as the construction of an appropriate exterior differential system on the configuration space $\mathcal{S}(4,\mathbb{R}) \times \mathbb{R}^2$ and the computation of the so called Euler-Lagrange system, whose integral curves give back the critical points of the functional. Similarly, the Euler-Lagrange operator may be obtained in the framework of [25].

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