



HAL
open science

Generalized Null Space and Restricted Isometry Properties

Tomer Peleg, Rémi Gribonval, Mike E. Davies

► **To cite this version:**

Tomer Peleg, Rémi Gribonval, Mike E. Davies. Generalized Null Space and Restricted Isometry Properties. Signal Processing with Adaptive Sparse Structured Representations 2013 (2013), EPFL, Jul 2013, Lausanne, Switzerland. hal-00811673

HAL Id: hal-00811673

<https://inria.hal.science/hal-00811673v1>

Submitted on 10 Apr 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Generalized Null Space and Restricted Isometry Properties

Tomer Peleg

Department of Electrical Engineering
Technion - Israel Institute of Technology
Haifa 32000, Israel
Email: tomerfa@tx.technion.ac.il

Rémi Gribonval

INRIA Rennes – Bretagne Atlantique
Campus de Beaulieu
F-35042 Rennes Cedex, France
Email: remi.gribonval@inria.fr

Mike E. Davies

School of Engineering and Electronics
The University of Edinburgh, The King's Buildings
Mayfield Road, Edinburgh EH9 3JL, UK
Email: mike.davies@ed.ac.uk

Abstract—We propose a theoretical study of the conditions guaranteeing that a decoder will obtain an *optimal* signal recovery from an underdetermined set of linear measurements. This special type of performance guarantee is termed *instance optimality* and is typically related with certain properties of the dimensionality-reducing matrix \mathbf{M} . Our work extends traditional results in sparse recovery, where instance optimality is expressed with respect to the set of sparse vectors, by replacing this set with an arbitrary finite union of subspaces. We show that the suggested instance optimality is equivalent to a generalized null space property of \mathbf{M} and discuss possible relations with generalized restricted isometry properties.

I. BACKGROUND

Traditional results in sparse recovery relate certain properties of a dimensionality-reducing matrix \mathbf{M} , considered as *encoder* to performance guarantees of certain explicit or implicit decoders Δ . A popular family of performance guarantees is coined *instance optimality*: a decoder is *instance optimal* at order k (with respect to \mathbf{M} and two norms $\|\cdot\|_X$ and $\|\cdot\|_Y$) if, for every vector \mathbf{x}

$$\|\Delta(\mathbf{M}\mathbf{x}) - \mathbf{x}\|_X \leq C\sigma_k(\mathbf{x})_Y \quad (1)$$

where $\sigma_k(\mathbf{x})_Y$ measures the *distance* (in the sense of the Y norm) from \mathbf{x} to the set of k -sparse vectors $\Sigma_k \doteq \{\mathbf{z}, \|\mathbf{z}\|_0 \leq k\}$.

Cohen *et al* [1] have shown that, for a given \mathbf{M} , the existence of an instance optimal decoder is equivalent to a null space property (NSP),

$$\|\mathbf{h}\|_X \leq C'\sigma_{2k}(\mathbf{h})_Y, \quad \forall \mathbf{h} \in \text{Ker}\mathbf{M}. \quad (2)$$

However, the corresponding decoder is somewhat inconvenient to handle. To prove the instance optimality of more convenient decoders such as the minimum ℓ_1 norm decoder, one typically exploits restricted isometry properties (RIP) [2]. In recent years, several extensions to the classical RIP have been suggested, e.g. **D**-RIP [3].

II. OUR WORK CONTRIBUTION

Our main goal is to replace Σ_k , which is a finite union of k -dimensional subspaces, by an *arbitrary finite union of low-dimensional subspaces (UoS)* denoted by Σ . Such an extension to general UoS has been considered in a recent paper by Blumensath [4], which suggested generalized RIP for a specific choice of the Y norm. We address the broader class of generalized NSP¹.

One special case of interest covered by our study is that of the cospase analysis model [5], which has recently drawn considerable attention. In this model the set Σ is the union of low-dimensional spaces associated with the analysis operator Ω . A popular example is the Ω_{DIF} operator, associated to finite differences on the edges of a graph, e.g., a 2D regular grid [5]. This operator generates a UoS Σ covering the family of piecewise constant 2D signals.

We extend the results of Cohen *et al* and derive a more general instance optimality and its equivalent NSP, as stated below,

Theorem 1. *Given a dimensionality-reducing matrix \mathbf{M} , norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and an arbitrary UoS Σ , the following are equivalent:*

I. *Instance optimality – there exists some decoder Δ such that*

$$\|\Delta(\mathbf{M}\mathbf{x}) - \mathbf{x}\|_X \leq Cd(\mathbf{x}, \Sigma)_Y, \quad \forall \mathbf{x} \quad (3)$$

II. *Generalized NSP:*

$$\|\mathbf{h}\|_X \leq C'd(\mathbf{h}, \Sigma + \Sigma)_Y, \quad \forall \mathbf{h} \in \text{Ker}\mathbf{M} \quad (4)$$

Here, $\Sigma + \Sigma = \{\mathbf{z} : \mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2, \mathbf{z}_1, \mathbf{z}_2 \in \Sigma\}$ and $d(\mathbf{x}, \Sigma)_Y$ is the distance (in the sense of the Y norm) from \mathbf{x} to the set Σ .

A typical choice for X is the ℓ_2 norm and we also focus on this setup. It is well known from [1] that for the special case of $\Sigma = \Sigma_k$ and $Y = \ell_2$, instance optimality (with a constant C independent of the dimension) is only possible if \mathbf{M} does not significantly decrease dimension. Under seemingly “mild” assumptions on Σ , we extend this result to more general UoS. One simple example where the required conditions are trivially met is when Σ is described by the Ω_{DIF} analysis operator. In light of the “negative” result we just mentioned, it seems natural to seek for norms $\|\cdot\|_Y$ that can be used in a dimensionality reduction context.

One possible choice for the Y norm is the \mathbf{M} -norm, where $\|\mathbf{u}\|_{\mathbf{M}} \doteq \|\mathbf{u}\|_2 + \frac{\|\mathbf{M}\mathbf{u}\|_2}{\sqrt{\alpha}}$. For this specific choice, it was shown in [4] that the *lower $\Sigma + \Sigma$ -RIP*, expressed as $\alpha\|\mathbf{x}\|_2^2 \leq \|\mathbf{M}\mathbf{x}\|_2^2$, for every $\mathbf{x} \in \Sigma + \Sigma$, implies the existence of a decoder satisfying $\|\Delta(\mathbf{M}\mathbf{x}) - \mathbf{x}\|_2 \leq 2d(\mathbf{x}, \Sigma)_{\mathbf{M}}$. Despite its apparent similarity with the ℓ_2 - ℓ_2 instance optimality, this type of instance optimality can actually hold for highly dimensionality-reducing \mathbf{M} . Moreover, the \mathbf{M} -norm is essentially the best possible norm for instance optimality.

In practice, one would like to exhibit more concrete decoders with the appropriate instance optimality. One approach is through iterative projection algorithms [4] with instance optimality with respect to the \mathbf{M} -norm. We intend to explore a different approach – through the minimization of some convex penalty. Our talk will discuss possible choices for the Y norm and their relations with generalized RIP.

REFERENCES

- [1] A. Cohen, W. Dahmen, and R.A. DeVore, “Compressed sensing and best k -term approximation,” *J. Amer. Math. Soc.*, vol. 22, no. 1, pp. 211–231, 2009.
- [2] E.J. Candès, J. Romberg, and T. Tao, “Stable signal recovery from incomplete and inaccurate measurements,” *Comm. Pure Appl. Math.*, vol. 59, no. 8, pp. 1207–1223, 2006.
- [3] E.J. Candès, Y.C. Eldar, D. Needell, and P. Randall, “Compressed sensing with coherent and redundant dictionaries,” *Applied Computational Harmonic Analysis*, vol. 31, no. 1, pp. 59–73, 2011.
- [4] T. Blumensath, “Sampling and reconstructing signals from a union of linear subspaces,” *IEEE Trans. Inf. Theory*, vol. 57, no. 7, pp. 4660–4671, 2011.
- [5] S. Nam, M.E. Davies, M. Elad, and R. Gribonval, “The cospase analysis model and algorithms,” *Applied Computational Harmonic Analysis*, vol. 34, no. 1, pp. 30–56, 2013.

¹This work has not been reported yet.