Projection Onto The k-Cosparse Set is NP-Hard

Andreas M. Tillmann, Rémi Gribonval, Marc E. Pfetsch

Projection onto the k-Cosparse Set is NP-Hard

Andreas M. Tillmann
Research Group Optimization
Technical University Darmstadt
64293 Darmstadt, Germany
Email: tillmann@mathematik.tu-darmstadt.de

Rémi Gribonval
PANAMA Project Team
Inria Rennes-Bretagne Atlantique
35042 Rennes, France
Email: remi.gribonval@inria.fr

Marc E. Pfetsch
Research Group Optimization
Technical University Darmstadt
64293 Darmstadt, Germany
Email: pfetsch@mathematik.tu-darmstadt.de

***Abstract*—We investigate the computational complexity of a problem arising in the context of sparse optimization, namely, the projection onto the set of k-cosparse vectors w.r.t. some given matrix $\boldsymbol{\Omega}$. We show that this projection problem is (strongly) NP-hard, even in the special cases where the matrix $\boldsymbol{\Omega}$ contains only ternary or bipolar coefficients. Interestingly, this is in stark contrast to the projection onto the set of k-sparse vectors, which is trivially solved by keeping only the k largest coefficients.**

***Index Terms*—Compressed Sensing, Computational Complexity, Cosparsity, Projection**

I. Introduction

A central problem in compressed sensing (CS) is the task of finding a sparsest solution to an underdetermined linear system, i.e.,

$$\min \|\boldsymbol{x}\|_0 \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{P$_0$}$$

for a given matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $m < n$, where $\|\boldsymbol{x}\|_0$ denotes the number of nonzero entries in $\boldsymbol{x}$. This problem is known to be strongly NP-hard; the same is true for the variant with $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ replaced by $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2 \leq \varepsilon$.

Two related problems arise in signal and image processing, where the unknown signal $\boldsymbol{x}$ to be estimated from a low-dimensional observation $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$ cannot directly be modeled as being sparse. In the most standard approach, $\boldsymbol{x}$ is assumed to be built from the superposition of few building blocks or *atoms* from an overcomplete dictionary $\boldsymbol{D}$, i.e., $\boldsymbol{x} = \boldsymbol{D}\boldsymbol{z}$ where the representation vector $\boldsymbol{z}$ is sparse. Minimizing $\|\boldsymbol{z}\|_0$ s.t. $\boldsymbol{A}\boldsymbol{D}\boldsymbol{z} = \boldsymbol{b}$ is obviously also NP-hard.

The alternative *cosparse analysis model* [1] assumes that $\boldsymbol{\Omega}\boldsymbol{x}$ has many zeros, where $\boldsymbol{\Omega}$ is an analysis operator. Typical examples include finite difference operators; they are closely connected to total variation minimization and defined as computing the difference between adjacent sample values (for a signal) or pixel values (for an image). The cosparse optimization problem of interest reads

$$\min \|\boldsymbol{\Omega}\boldsymbol{x}\|_0 \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

and was also shown to be NP-hard [1, Section 4.1].

A popular approach to solve (P$_0$) is the Iterative Hard Thresholding (IHT) algorithm, which iterates a gradient descent step to decrease the error $\|\boldsymbol{A}\tilde{\boldsymbol{x}} - \boldsymbol{b}\|_2$ and a hard-thresholding step. In recent adaptations of IHT and related algorithms to the cosparse analysis setting (e.g., [2]), a key step is the projection onto the set of k-cosparse vectors, as a replacement for hard-thresholding (which is the projection onto the set of k-sparse vectors).

We show that k-cosparse projection is strongly NP-hard in general, which contrasts with the synthesis case where the corresponding operation (hard-thresholding) is extremely simple and fast.

II. Complexity of Cosparse Projection Problems

Given an $r \times n$ matrix $\boldsymbol{\Omega}$ (with $r > n$), an n-vector $\boldsymbol{\omega}$, and a positive integer k, the (Euclidean) *projection of $\boldsymbol{\omega}$ onto the set of vectors that are k-cosparse w.r.t. $\boldsymbol{\Omega}$* is formally given by

$$\Pi_{\boldsymbol{\Omega},k}(\boldsymbol{\omega}) := \arg\min_{\boldsymbol{z}\in\mathbb{R}^n} \{\, \|\boldsymbol{\omega} - \boldsymbol{z}\|_2 \,:\, \|\boldsymbol{\Omega}\boldsymbol{z}\|_0 \leq k \,\}. \tag{k-CoSP}$$

Our main result is the following.

Theorem 1: Given $\boldsymbol{\Omega} \in \mathbb{R}^{r\times n}$ ($r > n$), $\boldsymbol{\omega} \in \mathbb{R}^n$, and a positive integer $k \in \mathbb{N}$, for any $p \in \mathbb{N} \cup \{\infty\}$, $p > 1$, it is NP-hard in the strong sense to solve the k-cosparse ℓ_p-norm projection problem

$$\min_{\boldsymbol{z}\in\mathbb{R}^n} \{\, \|\boldsymbol{\omega} - \boldsymbol{z}\|_p^q \,:\, \|\boldsymbol{\Omega}\boldsymbol{z}\|_0 \leq k \,\}, \tag{k-CoSP$_p$}$$

where $q = p$ if $p < \infty$ and $q = 1$ if $p = \infty$. The problem remains strongly NP-hard even if $\boldsymbol{\omega}$ has only binary coefficients in $\{0, 1\}$ (with exactly one entry nonzero) and $\boldsymbol{\Omega}$ has only ternary or bipolar coefficients in $\{-1, 0, +1\}$ or $\{-1, 1\}$, respectively.

Our proof (see [3] for the details) works with the MinULR$_0^=(\boldsymbol{A},K)$ problem: Given a matrix $\boldsymbol{A} \in \mathbb{Q}^{m\times n}$ and a positive integer $K \in \mathbb{N}$, decide whether there exists a vector $\boldsymbol{z} \in \mathbb{R}^n$ such that $\boldsymbol{z} \neq 0$ and at most K of the m equalities in the system $\boldsymbol{A}\boldsymbol{z} = 0$ are violated. This problem was proven to be (strongly) NP-complete even for ternary or bipolar matrices $\boldsymbol{A}$ in [4]. We show that one can answer MinULR$_0^=(\boldsymbol{A},K)$ by at most n calls to an algorithm solving (k-CoSP$_p$), implying that such a projection algorithm cannot be polynomial in general unless P=NP.

Clearly, if a minimizer was known, we would also know the optimal value of (k-CoSP$_p$). Hence, computing a minimizer is at least as hard as solving (k-CoSP$_p$), and the complexity results of Theorem 1 carry over directly. In particular, we obtain the following result for the usual Euclidean projection:

Corollary 1: It is strongly NP-hard to compute $\Pi_{\boldsymbol{\Omega},k}(\boldsymbol{\omega})$.

In theoretical algorithmic applications of (k-CoSP), it had so far been assumed that the Euclidean projection problem (k-CoSP$_2$) can be approximated efficiently, see, e.g., [2]. Our results refute this assumption to a certain degree, since NP-hardness in the strong sense implies that no *fully polynomial-time approximation scheme* (FPTAS) can exists unless P=NP. Thus, it remains a challenge to find (practically) efficient *approximation* algorithms for the k-cosparse projection problem (k-CoSP$_p$), or to establish further (perhaps negative) results concerning its approximability.

References

[1] S. Nam and M. E. Davies and M. Elad and R. Gribonval, "The cosparse analysis model and algorithms", *Appl. Comput. Harmon. Anal.*, vol. 34, pp. 30–56, 2013.

[2] R. Giryes and S. Nam and R. Gribonval and M. E. Davies, "Iterative Cosparse Projection Algorithms for the Recovery of Cosparse Vectors," in *Proc. EUSIPCO 2011*, Barcelona, Spain, 2011.

[3] A. M. Tillmann and R. Gribonval and M. E. Pfetsch, "Projection Onto the k-Cosparse Set is NP-hard", preprint, arXiv:1303.5305[cs.CC], 2013.

[4] E. Amaldi and V. Kann, "The complexity and approximability of finding maximum feasible subsystems of linear relations," *Theor. Comput. Sci.*, vol. 147, pp. 181–210, 1995.