



**HAL**  
open science

## Projection Onto The $k$ -Cosparse Set is NP-Hard

Andreas M. Tillmann, Rémi Gribonval, Marc E. Pfetsch

► **To cite this version:**

Andreas M. Tillmann, Rémi Gribonval, Marc E. Pfetsch. Projection Onto The  $k$ -Cosparse Set is NP-Hard. Signal Processing with Adaptive Sparse Structured Representations 2013 (2013), Jul 2013, Lausanne, Switzerland. hal-00811671

**HAL Id: hal-00811671**

**<https://inria.hal.science/hal-00811671v1>**

Submitted on 22 Apr 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Projection onto the $k$ -Cosparse Set is NP-Hard

ANDREAS M. TILLMANN  
Research Group Optimization  
Technical University Darmstadt  
64293 Darmstadt, Germany

Email: tillmann@mathematik.tu-darmstadt.de

RÉMI GRIBONVAL  
PANAMA Project Team  
Inria Rennes-Bretagne Atlantique  
35042 Rennes, France

Email: remi.gribonval@inria.fr

MARC E. PFETSCH  
Research Group Optimization  
Technical University Darmstadt  
64293 Darmstadt, Germany

Email: pfetsch@mathematik.tu-darmstadt.de

**Abstract**—We investigate the computational complexity of a problem arising in the context of sparse optimization, namely, the projection onto the set of  $k$ -cosparse vectors w.r.t. some given matrix  $\Omega$ . We show that this projection problem is (strongly) NP-hard, even in the special cases where the matrix  $\Omega$  contains only ternary or bipolar coefficients. Interestingly, this is in stark contrast to the projection onto the set of  $k$ -sparse vectors, which is trivially solved by keeping only the  $k$  largest coefficients.

**Index Terms**—Compressed Sensing, Computational Complexity, Cosparsity, Projection

## I. INTRODUCTION

A central problem in compressed sensing (CS) is the task of finding a sparsest solution to an underdetermined linear system, i.e.,

$$\min \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{Ax} = \mathbf{b}, \quad (\text{P}_0)$$

for a given matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  with  $m < n$ , where  $\|\mathbf{x}\|_0$  denotes the number of nonzero entries in  $\mathbf{x}$ . This problem is known to be strongly NP-hard; the same is true for the variant with  $\mathbf{Ax} = \mathbf{b}$  replaced by  $\|\mathbf{Ax} - \mathbf{b}\|_2 \leq \varepsilon$ .

Two related problems arise in signal and image processing, where the unknown signal  $\mathbf{x}$  to be estimated from a low-dimensional observation  $\mathbf{b} = \mathbf{Ax}$  cannot directly be modeled as being sparse. In the most standard approach,  $\mathbf{x}$  is assumed to be built from the superposition of few building blocks or *atoms* from an overcomplete dictionary  $\mathbf{D}$ , i.e.,  $\mathbf{x} = \mathbf{Dz}$  where the representation vector  $\mathbf{z}$  is sparse. Minimizing  $\|\mathbf{z}\|_0$  s.t.  $\mathbf{ADz} = \mathbf{b}$  is obviously also NP-hard.

The alternative *cosparse analysis model* [1] assumes that  $\Omega\mathbf{x}$  has many zeros, where  $\Omega$  is an analysis operator. Typical examples include finite difference operators; they are closely connected to total variation minimization and defined as computing the difference between adjacent sample values (for a signal) or pixel values (for an image). The cosparse optimization problem of interest reads

$$\min \|\Omega\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{Ax} = \mathbf{b} \quad (1)$$

and was also shown to be NP-hard [1, Section 4.1].

A popular approach to solve (P<sub>0</sub>) is the Iterative Hard Thresholding (IHT) algorithm, which iterates a gradient descent step to decrease the error  $\|\mathbf{Ax} - \mathbf{b}\|_2$  and a hard-thresholding step. In recent adaptations of IHT and related algorithms to the cosparse analysis setting (e.g., [2]), a key step is the projection onto the set of  $k$ -cosparse vectors, as a replacement for hard-thresholding (which is the projection onto the set of  $k$ -sparse vectors).

We show that  $k$ -cosparse projection is strongly NP-hard in general, which contrasts with the synthesis case where the corresponding operation (hard-thresholding) is extremely simple and fast.

## II. COMPLEXITY OF COSPARSE PROJECTION PROBLEMS

Given an  $r \times n$  matrix  $\Omega$  (with  $r > n$ ), an  $n$ -vector  $\omega$ , and a positive integer  $k$ , the (Euclidean) *projection of  $\omega$  onto the set of*

*vectors that are  $k$ -cosparse w.r.t.  $\Omega$*  is formally given by

$$\Pi_{\Omega,k}(\omega) := \arg \min_{\mathbf{z} \in \mathbb{R}^n} \{ \|\omega - \mathbf{z}\|_2 : \|\Omega\mathbf{z}\|_0 \leq k \}. \quad (k\text{-CoSP})$$

Our main result is the following.

**Theorem 1:** Given  $\Omega \in \mathbb{R}^{r \times n}$  ( $r > n$ ),  $\omega \in \mathbb{R}^n$ , and a positive integer  $k \in \mathbb{N}$ , for any  $p \in \mathbb{N} \cup \{\infty\}$ ,  $p > 1$ , it is NP-hard in the strong sense to solve the  $k$ -cosparse  $\ell_p$ -norm projection problem

$$\min_{\mathbf{z} \in \mathbb{R}^n} \{ \|\omega - \mathbf{z}\|_p^q : \|\Omega\mathbf{z}\|_0 \leq k \}, \quad (k\text{-CoSP}_p)$$

where  $q = p$  if  $p < \infty$  and  $q = 1$  if  $p = \infty$ . The problem remains strongly NP-hard even if  $\omega$  has only binary coefficients in  $\{0, 1\}$  (with exactly one entry nonzero) and  $\Omega$  has only ternary or bipolar coefficients in  $\{-1, 0, +1\}$  or  $\{-1, 1\}$ , respectively.

Our proof (see [3] for the details) works with the MIN-ULR<sub>0</sub><sup>−</sup>( $\mathbf{A}, K$ ) problem: Given a matrix  $\mathbf{A} \in \mathbb{Q}^{m \times n}$  and a positive integer  $K \in \mathbb{N}$ , decide whether there exists a vector  $\mathbf{z} \in \mathbb{R}^n$  such that  $\mathbf{z} \neq 0$  and at most  $K$  of the  $m$  equalities in the system  $\mathbf{Az} = 0$  are violated. This problem was proven to be (strongly) NP-complete even for ternary or bipolar matrices  $\mathbf{A}$  in [4]. We show that one can answer MINULR<sub>0</sub><sup>−</sup>( $\mathbf{A}, K$ ) by at most  $n$  calls to an algorithm solving ( $k$ -CoSP <sub>$p$</sub> ), implying that such a projection algorithm cannot be polynomial in general unless P=NP.

Clearly, if a minimizer was known, we would also know the optimal value of ( $k$ -CoSP <sub>$p$</sub> ). Hence, computing a minimizer is at least as hard as solving ( $k$ -CoSP <sub>$p$</sub> ), and the complexity results of Theorem 1 carry over directly. In particular, we obtain the following result for the usual Euclidean projection:

**Corollary 1:** It is strongly NP-hard to compute  $\Pi_{\Omega,k}(\omega)$ .

In theoretical algorithmic applications of ( $k$ -CoSP), it had so far been assumed that the Euclidean projection problem ( $k$ -CoSP<sub>2</sub>) can be approximated efficiently, see, e.g., [2]. Our results refute this assumption to a certain degree, since NP-hardness in the strong sense implies that no *fully polynomial-time approximation scheme* (FPTAS) can exist unless P=NP. Thus, it remains a challenge to find (practically) efficient *approximation* algorithms for the  $k$ -cosparse projection problem ( $k$ -CoSP <sub>$p$</sub> ), or to establish further (perhaps negative) results concerning its approximability.

## REFERENCES

- [1] S. Nam and M. E. Davies and M. Elad and R. Gribonval, “The cosparse analysis model and algorithms”, *Appl. Comput. Harmon. Anal.*, vol. 34, pp. 30–56, 2013.
- [2] R. Giryes and S. Nam and R. Gribonval and M. E. Davies, “Iterative Cosparse Projection Algorithms for the Recovery of Cosparse Vectors,” in *Proc. EUSIPCO 2011*, Barcelona, Spain, 2011.
- [3] A. M. Tillmann and R. Gribonval and M. E. Pfetsch, “Projection Onto the  $k$ -Cosparse Set is NP-hard”, preprint, arXiv:1303.5305[cs.CC], 2013.
- [4] E. Amaldi and V. Kann, “The complexity and approximability of finding maximum feasible subsystems of linear relations,” *Theor. Comput. Sci.*, vol. 147, pp. 181–210, 1995.