

Projection onto the k -Cosparse Set is NP-Hard

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Abstract—We investigate the computational complexity of a problem arising in the context of sparse optimization, namely, the projection onto the set of k -cosparse vectors w.r.t. some given matrix Ω . We show that this projection problem is (strongly) NP-hard, even in the special cases where the matrix Ω contains only ternary or bipolar coefficients. Interestingly, this is in stark contrast to the projection onto the set of k -sparse vectors, which is trivially solved by keeping only the k largest coefficients.

Index Terms—Compressed Sensing, Computational Complexity, Cosparsity, Projection

I. INTRODUCTION

A central problem in compressed sensing (CS) is the task of finding a sparsest solution to an underdetermined linear system, i.e.,

$$\min \|x\|_0 \quad \text{s.t.} \quad Ax = b, \quad (P_0)$$

for a given matrix $A \in \mathbb{R}^{m \times n}$ with $m < n$, where $\|x\|_0$ denotes the number of nonzero entries in x . This problem is known to be strongly NP-hard; the same is true for the variant with $Ax = b$ replaced by $\|Ax - b\|_2 \leq \varepsilon$.

Two related problems arise in signal and image processing, where the unknown signal x to be estimated from a low-dimensional observation $b = Ax$ cannot directly be modeled as being sparse. In the most standard approach, x is assumed to be built from the superposition of few building blocks or *atoms* from an overcomplete dictionary D , i.e., $x = Dz$ where the representation vector z is sparse. Minimizing $\|z\|_0$ s.t. $ADz = b$ is obviously also NP-hard.

The alternative *cosparse analysis model* [1] assumes that Ωx has many zeros, where Ω is an analysis operator. Typical examples include finite difference operators; they are closely connected to total variation minimization and defined as computing the difference between adjacent sample values (for a signal) or pixel values (for an image). The cosparse optimization problem of interest reads

$$\min \|\Omega x\|_0 \quad \text{s.t.} \quad Ax = b \quad (1)$$

and was also shown to be NP-hard [1, Section 4.1].

A popular approach to solve (P_0) is the Iterative Hard Thresholding (IHT) algorithm, which iterates a gradient descent step to decrease the error $\|A\tilde{x} - b\|_2$ and a hard-thresholding step. In recent adaptations of IHT and related algorithms to the cosparse analysis setting (e.g., [2]), a key step is the projection onto the set of k -cosparse vectors, as a replacement for hard-thresholding (which is the projection onto the set of k -sparse vectors).

We show that k -cosparse projection is strongly NP-hard in general, which contrasts with the synthesis case where the corresponding operation (hard-thresholding) is extremely simple and fast.

II. COMPLEXITY OF COSPARSE PROJECTION PROBLEMS

Given an $r \times n$ matrix Ω (with $r > n$), an n -vector ω , and a positive integer k , the (Euclidean) *projection of ω onto the set of*

vectors that are k -cosparse w.r.t. Ω is formally given by

$$\Pi_{\Omega,k}(\omega) := \arg \min_{z \in \mathbb{R}^n} \{ \|\omega - z\|_2 : \|\Omega z\|_0 \leq k \}. \quad (k\text{-CoSP})$$

Our main result is the following.

Theorem 1: Given $\Omega \in \mathbb{R}^{r \times n}$ ($r > n$), $\omega \in \mathbb{R}^n$, and a positive integer $k \in \mathbb{N}$, for any $p \in \mathbb{N} \cup \{\infty\}$, $p > 1$, it is NP-hard in the strong sense to solve the k -cosparse ℓ_p -norm projection problem

$$\min_{z \in \mathbb{R}^n} \{ \|\omega - z\|_p^q : \|\Omega z\|_0 \leq k \}, \quad (k\text{-CoSP}_p)$$

where $q = p$ if $p < \infty$ and $q = 1$ if $p = \infty$. The problem remains strongly NP-hard even if ω has only binary coefficients in $\{0, 1\}$ (with exactly one entry nonzero) and Ω has only ternary or bipolar coefficients in $\{-1, 0, +1\}$ or $\{-1, 1\}$, respectively.

Our proof (see [3] for the details) works with the MIN-ULR $^{\pm}_0(A, K)$ problem: Given a matrix $A \in \mathbb{Q}^{m \times n}$ and a positive integer $K \in \mathbb{N}$, decide whether there exists a vector $z \in \mathbb{R}^n$ such that $z \neq 0$ and at most K of the m equalities in the system $Az = 0$ are violated. This problem was proven to be (strongly) NP-complete even for ternary or bipolar matrices A in [4]. We show that one can answer MINULR $^{\pm}_0(A, K)$ by at most n calls to an algorithm solving $(k\text{-CoSP}_p)$, implying that such a projection algorithm cannot be polynomial in general unless $P=NP$.

Clearly, if a minimizer was known, we would also know the optimal value of $(k\text{-CoSP}_p)$. Hence, computing a minimizer is at least as hard as solving $(k\text{-CoSP}_p)$, and the complexity results of Theorem 1 carry over directly. In particular, we obtain the following result for the usual Euclidean projection:

Corollary 1: It is strongly NP-hard to compute $\Pi_{\Omega,k}(\omega)$.

In theoretical algorithmic applications of $(k\text{-CoSP})$, it had so far been assumed that the Euclidean projection problem $(k\text{-CoSP}_2)$ can be approximated efficiently, see, e.g., [2]. Our results refute this assumption to a certain degree, since NP-hardness in the strong sense implies that no *fully polynomial-time approximation scheme* (FPTAS) can exist unless $P=NP$. Thus, it remains a challenge to find (practically) efficient *approximation* algorithms for the k -cosparse projection problem $(k\text{-CoSP}_p)$, or to establish further (perhaps negative) results concerning its approximability.

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