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# Projection Onto The $k$ -Cosparse Set is NP-Hard

Andreas M. Tillmann, Rémi Gribonval, Marc E. Pfetsch

## Abstract

The computational complexity of a problem arising in the context of sparse optimization is considered, namely, the projection onto the set of  $k$ -cosparse vectors w.r.t. some given matrix  $\Omega$ . It is shown that this projection problem is (strongly) NP-hard, even in the special cases where the matrix  $\Omega$  contains only ternary or bipolar coefficients. Interestingly, this is in contrast to the projection onto the set of  $k$ -sparse vectors, which is trivially solved by keeping only the  $k$  largest coefficients.

## Index Terms

Compressed Sensing, Computational Complexity, Cosparsity, Projection

## I. INTRODUCTION

A central problem in compressed sensing (CS), see, e.g., [1], [2], [3], is the task of finding a sparsest solution to an underdetermined linear system, i.e.,

$$\min \|\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad (\text{P}_0)$$

for a given matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  with  $m < n$  and right hand side vector  $\mathbf{b} \in \mathbb{R}^m$ , where  $\|\mathbf{x}\|_0$  denotes the  $\ell_0$ -quasi-norm, i.e., the number of nonzero entries in  $\mathbf{x}$ . This problem is known to be strongly NP-hard, cf. [MP5] in [4]; the same is true for the variant with  $\mathbf{A}\mathbf{x} = \mathbf{b}$  replaced by  $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \leq \varepsilon$  [5], [6].

Two related problems arise in signal and image processing, where the unknown signal  $\mathbf{x}$  to be estimated from a low-dimensional observation  $\mathbf{b} = \mathbf{A}\mathbf{x}$  cannot directly be modeled as being sparse. In the most standard approach,  $\mathbf{x}$  is assumed to be built from the superposition of few building blocks or *atoms* from an overcomplete dictionary  $\mathbf{D}$ , i.e.,  $\mathbf{x} = \mathbf{D}\mathbf{z}$ , where the representation vector  $\mathbf{z}$  is sparse. The problem of minimizing  $\|\mathbf{z}\|_0$  such that  $\mathbf{A}\mathbf{D}\mathbf{z} = \mathbf{b}$  is obviously also NP-hard.

The alternative *cosparse analysis model* [7] assumes that  $\Omega\mathbf{x}$  has many zeros, where  $\Omega$  is an analysis operator. Typical examples include finite difference operators: they are closely connected to total variation minimization and defined as computing the difference between adjacent sample values (for a signal) or pixel values (for an image). The cosparse optimization problem of interest reads

$$\min \|\Omega\mathbf{x}\|_0 \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad (\text{C}_0)$$

and was also shown to be NP-hard [7, Section 4.1].

Due to the hardness of problem (P<sub>0</sub>), various heuristics have been developed. A popular approach is well-illustrated by the Iterative Hard Thresholding (IHT) algorithm, which iterates a gradient descent step to decrease the error  $\|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_2$  and a hard-thresholding step, i.e., the (Euclidean) projection onto the set of  $k$ -sparse vectors. Under restricted isometry properties (RIP) on  $\mathbf{A}$ , the IHT algorithm can be proven to converge to the solution of (P<sub>0</sub>) [10], [11]. A desirable RIP often holds with high probability when  $\mathbf{A}$  has i.i.d. sub-Gaussian entries, see, e.g., [9], but is strongly NP-hard to verify in general [8].

Adaptations of IHT and related algorithms to the cosparse analysis setting have been proposed [12], [13], based on a general scheme for unions of subspaces [14]. A key step is the projection onto the set of  $k$ -cosparse vectors, as an analogous replacement for hard-thresholding.

The main contribution of this note is to show that this projection is in fact strongly NP-hard in general, which contrasts with the extremely simple and fast hard-thresholding operation.

## II. COMPLEXITY OF COSPARSE PROJECTION PROBLEMS

This section is mainly devoted to proving the following central result.

*Theorem 1:* Consider any  $p \in \mathbb{N} \cup \{\infty\}$ ,  $p > 1$ , and let  $q = p$  if  $p < \infty$  and  $q = 1$  if  $p = \infty$ . Given  $\Omega \in \mathbb{Q}^{r \times n}$  ( $r > n$ ),  $\omega \in \mathbb{Q}^n$ , and a positive integer  $k \in \mathbb{N}$ , it is NP-hard in the strong sense to solve the  $k$ -cosparse  $\ell_p$ -norm projection problem

$$\min_{\mathbf{z} \in \mathbb{R}^n} \{ \|\omega - \mathbf{z}\|_p^q : \|\Omega\mathbf{z}\|_0 \leq k \}, \quad (k\text{-CoSP}_p)$$

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even when  $\omega \in \{0, 1\}^n$  (with exactly one nonzero entry) and  $\Omega$  contains only entries in  $\{-1, 0, +1\}$  or  $\{-1, 1\}$ , respectively.

Let us first provide some comments on the statement of the theorem.

*Remark 1:* Theorem 1 shows the NP-hardness of  $(k\text{-CoSP}_p)$  in the strong sense (i.e.,  $(k\text{-CoSP}_p)$  is strongly NP-hard). Thus, since the objective is always polynomially bounded (never exceeding  $\|\omega\|_p^q$  or  $\|\omega\|_\infty$ ), not even a *fully polynomial-time approximation scheme* (FPTAS) can exist, unless  $\text{P}=\text{NP}$ . An FPTAS is an algorithm that, given an approximation factor  $\epsilon > 0$ , computes a solution that has an objective function value at most  $(1+\epsilon)$  times the minimum value, with running time polynomial in the encoding length of the instance and  $1/\epsilon$ . Moreover, there also cannot exist a *pseudo-polynomial (exact) algorithm*, i.e., one that runs in time polynomial in the *numeric* value of the input, but not its *encoding length*, unless  $\text{P}=\text{NP}$ ; for details, see [4], and also [18].

*Remark 2:* It is perhaps not immediately clear why the “min” in  $(k\text{-CoSP}_p)$  is attained, since the constraint set is not bounded, in general. Nevertheless, the infimum is finite since the objective is lower-bounded by zero. Moreover, since  $z = 0$  is always feasible, the optimal value is upper-bounded by  $\|\omega\|_p^q$ . Since the level set  $\{z : \|\Omega z\|_0 \leq k, 0 \leq \|\omega - z\|_p^q \leq \|\omega\|_p^q\}$  is compact, the infimum is attained, justifying the use of “min” instead of “inf” in  $(k\text{-CoSP}_p)$ . (See also Remark 6 below.)

Before stating the proof of Theorem 1, we need some preliminaries. Let  $\text{MINULR}_0^-(\mathbf{A}, K)$  be the problem to decide, for a given matrix  $\mathbf{A} \in \mathbb{Q}^{m \times n}$  and a positive integer  $K \in \mathbb{N}$ , whether there exists a vector  $z \in \mathbb{R}^n$  such that  $z \neq 0$  and at most  $K$  of the  $m$  equalities in the system  $\mathbf{A}z = 0$  are violated (MINULR stands for “minimum number of unsatisfied linear relations”). This homogeneous equality version, in which the trivial all-zero solution is excluded, was proven to be NP-complete [15], and even NP-hard to approximate within any constant factor [16]; more results about the problem’s approximation complexity can be found in [17]. For the proof of Theorem 1, we utilize the following result:

*Theorem 2 (Theorem 1 and Corollary 2 in [15]):*  $\text{MINULR}_0^-(\mathbf{A}, K)$  is strongly NP-hard, even when  $\mathbf{A}$  contains only ternary entries, i.e.,  $\mathbf{A} \in \{-1, 0, +1\}^{m \times n}$ , or for bipolar  $\mathbf{A} \in \{-1, +1\}^{m \times n}$ .

The cited results from [15] are in fact more general than the statements we use here in that they also hold for non-homogeneous systems, where the right hand side vector is part of the input. Moreover, [15] deals with the complementary problem of  $\text{MINULR}_0^-$ , namely the *Maximum Feasible Subsystem* (MAXFS) problem (w.r.t. homogeneous linear equality systems) in which one asks for the largest possible number of simultaneously satisfied equalities from a given linear system. It is easy to see that MINULR and MAXFS are equivalent when solved to optimality (see also [17]).

Note also that the above two results state NP-hardness in the strong sense, although this is not made explicit in the original version [15]. The NP-hardness proofs, however, are by reduction from the *Exact Cover by 3-Sets* (X3C) problem, which is well-known to be strongly NP-hard, see, e.g., [4].

*Proof of Theorem 1:* Let  $p \in \mathbb{N} \cup \{\infty\}$ ,  $p > 1$ , and  $q = p$  if  $p < \infty$  and  $q = 1$  if  $p = \infty$ . Given an instance  $(\mathbf{A}, K)$  of  $\text{MINULR}_0^-$  (w.l.o.g.  $\mathbf{A} \in \mathbb{Q}^{r \times n}$  with  $r > n$ ), we will reduce it to  $n$  instances of  $(k\text{-CoSP}_p)$ .

For all  $i = 1, \dots, n$ , we define a  $k$ -cosparse projection instance given by  $\Omega = \mathbf{A}$ ,  $\omega = e_i$  and  $k = K$  (where  $e_i$  denotes the  $i$ -th unit vector in  $\mathbb{Q}^n$ ). Note that each such instance obviously has encoding length polynomially bounded by that of  $\mathbf{A}$  and  $K$ . Writing

$$f_{i,k}^{(p)} := \min_{z \in \mathbb{R}^n} \{ \|e_i - z\|_p^q : \|\Omega z\|_0 \leq k \} \quad \text{and} \quad f_k^{(p)} := \min_{1 \leq i \leq n} f_{i,k}^{(p)},$$

we observe that since  $z = 0$  is always a feasible point,  $f_{i,k}^{(p)} \leq 1$  for all  $i$ , hence  $f_k^{(p)} \leq 1$ . We claim that  $f_k^{(p)} < 1$  if and only if there exists a nonzero vector  $z$  that violates at most  $k = K$  equations in  $\mathbf{A}z = 0$ .

- 1) If  $f_k^{(p)} < 1$  (i.e., there exists some  $i \in \{1, \dots, n\}$  such that  $f_{i,k}^{(p)} < 1$ ) then there exists a vector  $z$  such that  $\|\Omega z\|_0 \leq k$  and  $\|e_i - z\|_p^q < 1$  (in particular,  $z_i \neq 0$ ). But this of course means that at most  $K$  equalities in the system  $\mathbf{A}z = 0$  are violated, i.e.,  $\text{MINULR}_0^-(\mathbf{A}, K)$  has a positive answer.
- 2) Conversely, assume that there exists a nonzero vector  $z$  that violates at most  $K$  equations in  $\mathbf{A}z = 0$ , i.e.,  $\|\Omega z\|_0 \leq k$  and  $\text{MINULR}_0^-(\mathbf{A}, K)$  has a positive answer. We will prove that

$$f_k^{(p)} \leq \min_{i, \lambda} \|e_i - \lambda z\|_p^q < 1. \tag{1}$$

In fact, for an arbitrary scalar  $\lambda \in \mathbb{R}$ ,  $z' := \lambda z$  obeys  $\|\Omega z'\|_0 \leq k$  as well. If  $z$  contains only one nonzero component  $z_i$ ,  $\lambda$  can be chosen such that  $z' = e_i$  and consequently  $f_k^{(p)} = f_{i,k}^{(p)} = 0$ , which shows the claim. Thus, there remains to deal with the case where  $z$  contains at least two nonzero components. Consider some  $i$  such that  $z_i \neq 0$ . Without loss of generality, we can also assume that  $z_i > 0$  (otherwise we replace  $z$  with  $-z$ ). We consider the case  $p < \infty$  first. It holds that

$$\|e_i - \lambda z\|_p^p = \|\lambda z\|_p^p - |\lambda z_i|^p + |1 - \lambda z_i|^p = |\lambda|^p (\|z\|_p^p - |z_i|^p) + |1 - \lambda z_i|^p$$

for any  $\lambda$ . The function  $|1 - \lambda z_i|^p$  is convex (in  $\lambda$ ), and it is easy to see that we have  $|1 - \lambda z_i|^p \leq 1 - \lambda z_i$  for  $0 \leq \lambda \leq \frac{1}{z_i}$ . Consequently,

$$|\lambda|^p (\|z\|_p^p - |z_i|^p) + |1 - \lambda z_i|^p \leq 1 + |\lambda|^p (\|z\|_p^p - |z_i|^p) - \lambda z_i, \quad \text{for } 0 \leq \lambda \leq \frac{1}{z_i}. \tag{2}$$

Since  $z$  contains at least two nonzero components,  $\alpha := \|z\|_p^p - |z_i|^p > 0$ . Since  $p > 1$ , it is easy to see that  $|\lambda|^p \alpha - \lambda z_i < 0$  for sufficiently small positive  $\lambda$ . In conclusion, there exists a solution  $\lambda z$  with  $\|e_i - \lambda z\|_p^p < 1$  (and thus  $f_k^{(p)} < 1$ ), which proves the reverse direction for  $p < \infty$ .

Finally, for the case  $p = \infty$ , note that

$$\min_{i,\lambda} \|e_i - \lambda z\|_\infty \leq \min_{i,\lambda} \|e_i - \lambda z\|_{\tilde{p}} \quad \text{for all } 1 < \tilde{p} < \infty,$$

whence we can employ the above reasoning to reach the same conclusion.

To summarize, we could reduce the NP-hard problem  $\text{MINULR}_0^-(\mathbf{A}, K)$  to  $n$  instances of the  $k$ -cosparse projection problem ( $k$ -CoSP $_p$ ), which therefore is NP-hard as well. Moreover, by Theorem 2, NP-hardness of ( $k$ -CoSP $_p$ ) holds in the *strong* sense, even for  $\Omega (=A)$  with ternary or bipolar coefficients and, by the construction above, for  $\omega$  binary (and indeed, with only one nonzero entry). ■

*Remark 3:* The reduction in the proof of Theorem 1 does not work for  $0 < p \leq 1$ . Indeed, consider the vector with all entries equal to 1,  $z = \mathbf{1}$ . It is not hard to verify that  $\|e_i - \lambda z\|_p^p = |1 - \lambda|^p + (n-1)|\lambda|^p \geq 1$  for all  $i$  and  $\lambda$ . Thus,  $\min_{i,\lambda} \|e_i - \lambda z\|_p^p \geq 1$  does not imply  $z = 0$  when  $p \in (0, 1]$ ; vice versa,  $z \neq 0$  does not imply that  $\min_{i,\lambda} \|e_i - \lambda z\|_p^p < 1$ .

*Remark 4:* It is noteworthy that the claim in the proof of Theorem 1 is in fact true for every real-valued  $p > 1$ . However, one may encounter irrational (i.e., not finitely representable) numbers when working with  $p$ -th powers for arbitrary real, or even rational,  $p$ . Since retaining finite rational representations is crucial for NP-hardness proofs, we restricted ourselves to integer  $p$  (or  $p = \infty$ ).

The following result is an immediate consequence of Theorem 1.

*Corollary 1:* It is strongly NP-hard to compute a minimizer for the problem ( $k$ -CoSP $_p$ ), even under the input data restrictions specified in Theorem 1. In particular, it is strongly NP-hard to compute the Euclidean projection

$$\Pi_{\Omega,k}(\omega) := \arg \min_{z \in \mathbb{R}^n} \{ \|\omega - z\|_2 : \|\Omega z\|_0 \leq k \}. \quad (k\text{-CoSP})$$

*Proof:* Clearly, if a minimizer was known, we would also know the optimal value of ( $k$ -CoSP $_p$ ). Hence, computing a minimizer is at least as hard as solving ( $k$ -CoSP $_p$ ), and the complexity results of Theorem 1 carry over directly. In particular, computing  $\Pi_{\Omega,k}(\omega)$  is also strongly NP-hard, since Theorem 1 applies to the  $\ell_2$ -norm ( $p = 2$ ) and the minimizers of  $\|\omega - z\|_2$  (the objective in ( $k$ -CoSP)) are of course the same as those of  $\|\omega - z\|_2^2$ . ■

Let us comment on a few more subtle aspects concerning ( $k$ -CoSP $_p$ ) and the proof of Theorem 1.

*Remark 5:* The above reduction from  $\text{MINULR}_0^-$  is an example of what is called a *Turing reduction*; more commonly used are *Karp reductions*, cf. [4], [18]. In the latter, the known NP-hard problem is reduced to a *single instance* of the problem under consideration. For ( $k$ -CoSP $_p$ ) with  $p \in \mathbb{N}, p > 1$  (excluding the case  $p = \infty$ ), we could also obtain a Karp reduction by constructing the instance

$$\tilde{\Omega}^\top := (\mathbf{A}^\top, \dots, \mathbf{A}^\top) \in \mathbb{Q}^{n \times rn}, \quad \tilde{\omega}^\top := (e_1^\top, \dots, e_m^\top) \in \{0, 1\}^{n^2}, \quad \tilde{k} := nK,$$

which is obviously still polynomially related to the input  $(\mathbf{A}, K)$  of the given  $\text{MINULR}_0^-$  instance. Then, defining

$$\tilde{f}_k^{(p)} := \min_{z \in \mathbb{R}^{rn}} \{ \|\tilde{\omega} - z\|_p^p : \|\tilde{\Omega} z\|_0 \leq \tilde{k} \},$$

it is easy to see that  $\tilde{f}_k^{(p)} < n (= \|\tilde{\omega}\|_p^p)$  if and only if  $\text{MINULR}_0^-(\mathbf{A}, K)$  has a positive answer. In particular, a solution can be seen to be comprised of the solutions to the  $n$  separate ( $k$ -CoSP $_p$ )-problems considered in the Turing reduction, stacked on top of each other. We chose the Turing reduction for our proof of Theorem 1 because it allows us to conclude NP-hardness even if  $\omega$  is a unit vector, i.e., binary with exactly one nonzero entry, and also for  $p = \infty$ .

*Remark 6:* The optimization problem ( $k$ -CoSP $_p$ ) could be replaced by its corresponding decision version:

$$\text{Does there exist some } z \in \mathbb{R}^n \text{ such that } \|\omega - z\|_p^q < \gamma \text{ and } \|\Omega z\|_0 \leq k? \quad (k\text{-CoSP}_p\text{-Dec})$$

Indeed, the proof of Theorem 1 corresponds to showing hardness of this decision problem for  $\gamma = 1$ ; the Karp reduction sketched in the previous remark would use  $\gamma = n$ .

Note that ( $k$ -CoSP $_p$ -Dec) is in fact contained in NP (at least for  $p = 1, 2$ , or  $\infty$ ): the constraint set  $\{z : \|\Omega z\|_0 \leq k\}$  yields (exponentially many) affine subspaces of  $\mathbb{R}^n$  defined by  $n - k$  or more homogeneous equalities, and the projection of some  $\omega$  onto it (w.r.t.  $\|\cdot\|_p^q$ ) is clearly equivalent to that onto one (unknown) of these affine subspaces. For  $p = 2$ , the (Euclidean) projection onto such spaces has a known explicit formula which keeps all entries in the solution rational if the input  $(\Omega)$  is rational. Similarly, for  $p = 1$  or  $p = \infty$ , ( $k$ -CoSP $_p$ ) can be seen as a linear program over the unknown correct affine subspace; hence, here, the solution is also rational.

Thus, for  $p = 1, 2$ , or  $\infty$ , a certificate of a positive answer exists that has encoding length polynomially bounded by that of the input. Hence, by Theorem 1 and the above discussion, we have that ( $k$ -CoSP $_p$ -Dec) is in fact strongly NP-complete for  $p = 2$  or  $p = \infty$ . (The case  $p = 1$  is not covered by Theorem 1, and for the remaining values of  $p$  it is not immediately clear how one could guarantee the existence of a certificate with polynomially bounded encoding length.)

### III. CONCLUSIONS

Theorem 1 and Corollary 1 show that no polynomial algorithm to compute the projection onto the set of  $k$ -cosparse vectors can exist unless  $\mathbf{P}=\mathbf{NP}$ .

In theoretical algorithmic applications of the Euclidean  $k$ -cosparse projection operation ( $k$ -CoSP) in [12], [13], it had so far been assumed that the projection problem ( $k$ -CoSP<sub>2</sub>) can be approximated efficiently. While Remark 1 refutes this assumption to a certain degree, it is not clear whether other (general polynomial-time) approximation algorithms exist that may still be useful in practice despite exhibiting theoretical running time bounds that depend exponentially on (at least) the approximation quality  $1/\epsilon$ .

Moreover, as for most NP-hard problems, there are specific instances which are known to be much easier to handle than the general case. For instance, when  $\Omega$  is the identity (or, more generally, a unitary) matrix, hard-thresholding—i.e., zeroing all but the  $k$  entries with largest absolute values—achieves the projection onto the  $k$ -cosparse set w.r.t. any  $\ell_p$ -norm. Other examples include the Euclidean case ( $p = 2$ ) when  $\Omega$  is the 1D finite difference operator or the 1D fused Lasso operator, respectively: the projection is then achieved using dynamic programming [13].

For the problem  $\text{MINULR}_0^-$  (or its minimization variant, respectively), strong non-approximability results were derived in [16], [15], [17]; for instance, it cannot be approximated within any constant unless  $\mathbf{P}=\mathbf{NP}$ . However, these results do not carry over to the  $k$ -cosparse projection problem, since the objectives differ: in the optimization version of  $\text{MINULR}_0^-$ , we wish to minimize the number  $K$  of violated equalities, while in ( $k$ -CoSP <sub>$p$</sub> ), the goal is minimizing the distance of  $z$  to a given point (under the *constraint* that the number of nonzeros in  $\Omega z$  does not exceed  $K$ ). Thus, despite the link between the two problems exploited in the proof of Theorem 1, (hypothetical) approximation guarantees for this distance unfortunately do not yield any (non-)approximability statements for ( $k$ -CoSP <sub>$p$</sub> ) by means of those for  $\text{MINULR}_0^-$ .

Thus, it remains a challenge to find (practically) efficient approximation schemes for the  $k$ -cosparse projection problem ( $k$ -CoSP <sub>$p$</sub> ), or to establish further (perhaps negative) results concerning its approximability. Other open questions are the complexity of ( $k$ -CoSP <sub>$p$</sub> ) for  $0 < p \leq 1$ , or containment in NP of ( $k$ -CoSP <sub>$p$</sub> -Dec) for  $p \in \mathbb{N} \setminus \{1, 2\}$ , cf. Remarks 3 and 6.

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