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► **To cite this version:**

Joachim Lebovits. From Stochastic Integration wrt Fractional Brownian Motion to Stochastic Integration wrt Multifractional Brownian Motion. 2013. hal-00795494v1

**HAL Id: hal-00795494**

**<https://inria.hal.science/hal-00795494v1>**

Preprint submitted on 28 Feb 2013 (v1), last revised 26 Apr 2013 (v2)

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# From Stochastic Integral *w.r.t.* Fractional Brownian Motion to Stochastic Integral *w.r.t.* Multifractional Brownian Motion

JOACHIM LÉBOVITS

**Abstract** - Stochastic integration w.r.t. fractional Brownian motion (fBm) has raised strong interest in recent years, motivated in particular by applications in finance and Internet traffic modelling. Since fBm is not a semimartingale, stochastic integration requires specific developments. Multifractional Brownian motion (mBm) is a Gaussian process that generalizes fBm by letting the local Hölder exponent vary in time. This is useful in various areas, including financial modelling and biomedicine.

In this work we start from the fact, established in [9, Thm 2.1.(i)], that an mBm may be approximated, in law, by a sequence of "tangent" fBms. We used this result to show how one can define a stochastic integral w.r.t. mBm from the stochastic integral w.r.t. fBm, defined in [4], in the White Noise Theory sense.

**Key words and phrases** : Fractional and multifractional Brownian motions, Gaussian processes, convergence in law, white noise theory, Wick-Itô integral.

**Mathematics Subject Classification** (2010) : 60G22, 60G15, 60H40, 60H05.

## 1 Background and motivation

Fractional Brownian motion (fBm) is a centred Gaussian process with features that make it a useful model in various applications such as financial and Internet traffic modeling, image analysis and synthesis, physics, geophysics and more.

These features include self-similarity, long range dependence and the ability to match any prescribed constant local regularity. Its covariance function  $R_H$  reads:

$$R_H(t, s) := \frac{\gamma_H}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where  $\gamma_H$  is a positive constant and  $H$ , which is usually called the Hurst exponent, belongs to  $(0, 1)$ . When  $H = \frac{1}{2}$ , fBm reduces to standard Brownian motion. Various integral representations of fBm are known, including the harmonizable and moving average ones [15], as well as representations by integrals over a finite domain [1, 7].

The fact that most of the properties of fBm are governed by the single real  $H$  restricts its application in some situations. In particular, its Hölder exponent remains the same all along its trajectory. This does not seem to be adapted to describe adequately a phenomenon for which the Hölder regularity varies along the time. Moreover it can not be used to model any phenomenon that presents in the same time a long range dependance and irregular trajectories (*i.e.* trajectories less irregular than the ones of Brownian motion). indeed, long range dependance requires  $H > 1/2$  and irregular trajectories requires  $H < 1/2$ . Multifractional Brownian motion was introduced to overcome these limitations. The basic idea is to replace the real  $H$  by a function  $t \mapsto h(t)$  ranging in  $(0, 1)$ .

Several definitions of multifractional Brownian motion exist. The first ones were proposed in [14] and in [3]. They then have been extended in [16]. Finally, a more general definition of mBm, that includes all the definitions given in [14, 3, 16], has been introduced in [9].

In the present work, and for sake of simplicity, we will only deal with the harmonizable version of mBm, which we will call “the” mBm.

More precisely, a deterministic function  $h : \mathbf{R} \rightarrow (0, 1)$  being fixed, the mBm of functional parameter  $h$ , noted  $B^h := (B_t^h)_{t \in \mathbf{R}}$ , is defined by:

$$B_t^h := \frac{1}{c_{h(t)}} \int_{\mathbf{R}} \frac{e^{itu} - 1}{|u|^{h(t)+1/2}} \widetilde{W}(du), \quad (1.1)$$

where  $c_x := \left( \frac{2 \cos(\pi x) \Gamma(2-2x)}{x(1-2x)} \right)^{\frac{1}{2}}$ , for every  $x$  in  $(0, 1)$ , and where  $W$  denotes an independently scattered standard Gaussian measure on  $\mathbf{R}$ , and  $\widetilde{W}$  denotes the complex-valued Gaussian measure which can be associated in a unique way to  $W$  (see [16, p.203-204] and [15, p.325-326] for more details). Denote  $R_h$  the covariance function of  $B^h$ . Thanks to [2], we know that:

$$R_h(t, s) = \frac{c_{h_{t,s}}^2}{c_{h(t)}c_{h(s)}} \left[ \frac{1}{2} (|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}) \right],$$

where  $h_{t,s} := \frac{h(t)+h(s)}{2}$ .

A word on notation:  $B^H$  or  $B^{h(t)}$  will always denote an fBm with Hurst index  $H$  or  $h(t)$ , while  $B^h$  will stand for an mBm. Note that  $B_t^h := B_t^{h(t)}$ , for every real  $t$ . Besides, and in order to simplify some notations in the sequel, denote  $\mathbf{B}(t, H) := \frac{1}{c_H} \int_{\mathbf{R}} \frac{e^{itu} - 1}{|u|^{H+1/2}} \widetilde{W}(du)$ , for every  $H$  in  $(0, 1)$ .

The result of the next proposition will be useful in all this paper.

**Proposition 1.1** *For every  $[a, b] \times [c, d]$  in  $\mathbf{R} \times (0, 1)$ , there exists  $\Lambda$  in  $\mathbf{R}_+^*$  such that for every  $(t, s, H, H')$  in  $[a, b]^2 \times [c, d]^2$ ,*

$$\mathbf{E}[(\mathbf{B}(t, H) - \mathbf{B}(s, H'))^2] \leq \Lambda (|t-s|^{2c} + |H-H'|^2),$$

where  $\mathbf{E}[Y]$  denotes the expectation of a real random variable  $Y$ , which belongs to  $L^1(\Omega, \mathcal{F}, P)$ .

**Proof.** Let  $(s, t, H, H')$  be fixed in  $[a, b]^2 \times [c, d]^2$ . Since, for every  $H$  in  $[c, d]$ , the process  $\{\mathbf{B}(t, H), t \in \mathbf{R}\}$  is a fractional Brownian motion of Hurst index  $H$ , we know, using the triangular inequality, that it is sufficient to show that  $I_t^{H, H'} := \mathbf{E}[(\mathbf{B}(t, H) - \mathbf{B}(t, H'))^2] \leq \Lambda |H - H'|^2$ . One has:

$$I_t^{H, H'} = \int_{\mathbf{R}} \left| \frac{e^{it\xi} - 1}{\xi} \right|^2 \left| \frac{1}{c_H} |\xi|^{1/2-H} - \frac{1}{c_{H'}} |\xi|^{1/2-H'} \right|^2 d\xi.$$

For  $\xi$  in  $\mathbf{R}^*$ , the map  $f_\xi : [c, d] \rightarrow \mathbf{R}_+$ , defined by  $f_\xi(H) := \frac{1}{c_H} |\xi|^{1/2-H}$  is  $C^1$ , since  $H \mapsto c_H$  is  $C^\infty$  on  $(0, 1)$ . Thus there exists a positive real  $D$  such that, for all  $(\xi, H)$  in  $\mathbf{R}^* \times [c, d]$ ,

$$|f'_\xi(H)| \leq D |\xi|^{1/2-H} (1 + |\ln(|\xi|)|) \leq D (|\xi|^{1/2-c} + |\xi|^{1/2-d}) (1 + |\ln(|\xi|)|).$$

Thanks to the mean-value theorem, one can write

$$\begin{aligned} I_t^{H, H'} &\leq C |H - H'|^2 \int_{\mathbf{R}} \frac{|e^{it\xi} - 1|^2}{|\xi|^2} (|\xi|^{1/2-c} + |\xi|^{1/2-d})^2 (1 + |\ln(|\xi|)|)^2 d\xi \\ &\leq C |H - H'|^2 \left( \int_{|\xi|>1} \frac{(1 + \ln(|\xi|))^2}{|\xi|^{1+2c}} d\xi + \int_{|\xi|\leq 1} |\xi|^{1-2d} (1 + |\ln(|\xi|)|)^2 d\xi \right) \\ &\leq C |H - H'|^2, \end{aligned}$$

where  $C$  stands for a constant, independent of  $t$  and  $H$ , whose precise value is unimportant and which may change from line to line.  $\square$

**Remark 1.1** Since  $\mathbf{B} := (\mathbf{B}(t, H))_{(t, H) \in \mathbf{R} \times [0, 1]}$  is a Gaussian field, the previous result, as well as Kolmogorov's criterion, entail that the field  $\mathbf{B}$  has a  $d$ -Hölder continuous version for any  $d$  in  $(0, \frac{1}{2} \wedge c)$ . In the sequel we will always work with such a version.

## Outline of the paper

The remaining of this paper is organized as follows. We explain in the first part of Section 2 how an mBm can be approximated in law by a sequence of "tangent" fBms. The second part of Section 2 is devoted to some heuristics about the way to define a stochastic integral w.r.t. mBm as a limit of integrals w.r.t. approximating fBms. Section 3 provides the background on White noise Theory that will allow us to define, in a rigorous manner, the limiting fractional Wick-Itô integral in Section 4. The main result of Section 4, being Theorem 4.1. Finally, we compare in Section 5 the limiting fractional Wick-Itô integral to the stochastic integral with respect to mBm that has been defined in [13].

## 2 Approximation of mBm and some heuristics

### 2.1 Approximation of multifractional Brownian motion

Since a mBm is generalization of fBm, a natural question is to enquire whether it may be approximated by patching adequately chosen fBMs, and in which sense. For notational simplicity we take  $[a, b] := [0, 1]$  in the sequel. Heuristically, we divide  $[0, 1]$  into “small” intervals  $[t_i, t_{i+1})$ , and replace on each of these  $B^h$  by the fBm  $B^{H_i}$  where  $H_i = h(t_i)$ . It seems reasonable to expect that the resulting process  $\sum_i B_t^{H_i} \mathbb{1}_{[t_i, t_{i+1})}(t)$  will converge, in law, to  $B^h$  when the sizes of the intervals  $[t_i, t_{i+1})$  go to 0. The notations remains the same as previously. Let  $h : [0, 1] \rightarrow (0, 1)$  be a continuous deterministic function and denote  $B^h$  the fixed mBm of functional parameter  $h$ . We aim to explain how this mBm can be approximated on  $[0, 1]$  by patching together fractional Brownian motions defined on a sequence of partitions of  $[0, 1]$ .

In that view, we choose an increasing sequence  $(q_n)_{n \in \mathbf{N}}$  of integers such that  $q_0 := 1$  and  $2^n \leq q_n \leq 2^{2^n}$  for all  $n$  in  $\mathbf{N}^*$ . For any  $n$  in  $\mathbf{N}$ , define  $x^{(n)} := \{x_k^{(n)}; k \in \llbracket 0, q_n \rrbracket\}$  where  $x_k^{(n)} := \frac{k}{q_n}$  for  $k$  in  $\llbracket 0, q_n \rrbracket$  (for integers  $p$  and  $q$  with  $p < q$ ,  $\llbracket p, q \rrbracket$  denotes the set  $\{p; p+1; \dots; q\}$ ). Define, for  $n$  in  $\mathbf{N}$ , the partition  $\mathcal{A}_n := \{[x_k^{(n)}, x_{k+1}^{(n)}]; k \in \llbracket 0, q_n - 1 \rrbracket\} \cup \{x_{q_n}^{(n)}\}$ . It is clear that  $\mathcal{A} := (\mathcal{A}_n)_{n \in \mathbf{N}}$  is a decreasing nested sequence of subdivisions of  $[a, b]$  (*i.e.*  $\mathcal{A}_{n+1} \subset \mathcal{A}_n$ , for every  $n$  in  $\mathbf{N}$ ).

For  $t$  in  $[0, 1]$  and  $n$  in  $\mathbf{N}$  there exists a unique integer  $p$  in  $\llbracket 0, q_n - 1 \rrbracket$  such that  $x_p^{(n)} \leq t < x_{p+1}^{(n)}$ . We will note  $x_t^{(n)}$  the real  $x_p^{(n)}$  in the sequel. The sequence  $(x_t^{(n)})_{n \in \mathbf{N}}$  is increasing and converges to  $t$  as  $n$  tends to  $+\infty$ . Besides, define for  $n$  in  $\mathbf{N}$ , the function  $h_n : [0, 1] \rightarrow (0, 1)$  by setting  $h_n(1) = h(1)$  and, for any  $t$  in  $[0, 1]$ ,  $h_n(t) := h(x_t^{(n)})$ . The sequence of step functions  $(h_n)_{n \in \mathbf{N}}$  converges pointwise to  $h$  on  $[0, 1]$ .

Define, for  $t$  in  $[0, 1]$  and  $n$  in  $\mathbf{N}$ , the process

$$B_t^{h_n} := \mathbf{B}(t, h_n(t)) = \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \mathbf{B}(t, h(x_k^{(n)})) + \mathbb{1}_{\{1\}}(t) \mathbf{B}(1, h(1)). \quad (2.1)$$

Note that, despite the notation, the process  $B^{h_n}$  is not an mBm, as  $h_n$  is not continuous.

The following theorem, the proof of which can be found in [9, Thm 2.1.(i)], shows that mBm appears naturally as a limit object of sums of fBMs.

**Theorem 2.1 (Approximation theorem)** *Let  $\mathcal{A}$  be a sequence of partitions of  $[0, 1]$  as defined above, and consider the sequence of processes defined in (2.1).*

*If  $h$  is  $\beta$ -Hölder continuous for some positive real  $\beta$ , then the sequence of processes  $(B^{h_n})_{n \in \mathbf{N}}$  converges, in law, to the process  $B^h$ . In other words,*

$$\{B_t^{h_n}; t \in [a, b]\} \xrightarrow[n \rightarrow +\infty]{law} \{B_t^h; t \in [a, b]\}.$$

## 2.2 Some Heuristics about Stochastic integrals w.r.t. mBm as limits of integrals w.r.t. fBm

In the remaining of this paper, we consider a  $C^1$  deterministic function  $h : \mathbf{R} \rightarrow (0, 1)$ . Moreover  $\mathbf{B}$  still denotes the Gaussian field defined just above Proposition 1.1 and  $B^h$  the mBm defined by (1.1). The mBm  $B^h$  then verifies  $B_t^h \stackrel{\text{a.s.}}{=} \mathbf{B}(t, h(t))$ , for every  $t$ .

The result of Theorem 2.1 suggests that one may define stochastic integrals with respect to mBm as limits of integrals with respect to approximating fBms. Let first notice the following fact, the proof of which can be found in [9, Appendix B], that will be useful to define the integral with respect to mBm.

**Proposition 2.1** *For every real  $t$ , the map  $H \mapsto \mathbf{B}(t, H)$ , from  $(0, 1)$  to  $L^2(\Omega)$ , is  $C^1$ . Moreover, it fulfills the following Hölder condition. For all  $[a, b] \times [c, d] \subset \mathbf{R} \times (0, 1)$  there exists  $\Delta$  in  $\mathbf{R}_+^*$  such that:*

$$\mathbf{E} \left[ \left( \frac{\partial \mathbf{B}}{\partial H}(t, H) - \frac{\partial \mathbf{B}}{\partial H}(s, H') \right)^2 \right] \leq \Delta (|t - s|^2 + |H - H'|^2),$$

where  $\frac{\partial \mathbf{B}}{\partial H}(t, H)$  denotes, for every real  $t$ , the derivative with respect to  $H$  of the map  $H \mapsto \mathbf{B}(t, H)$ .

Let us explain now, in a heuristic way, how to define an integral with respect to mBm using approximating fBms. Write the “differential” of  $\mathbf{B}(t, H)$ :

$$d\mathbf{B}(t, H) = \frac{\partial \mathbf{B}}{\partial t}(t, H) dt + \frac{\partial \mathbf{B}}{\partial H}(t, H) dH.$$

Of course, this is only formal as  $t \mapsto \mathbf{B}(t, H)$  is not differentiable in the  $L^2$ -sense nor almost surely with respect to  $t$ . It is, however, in the sense of Hida distributions (this will be made precise in Section 3, see in particular (3.2)). With a differentiable function  $h$  in place of  $H$ , this (again formally) yields

$$d\mathbf{B}(t, h(t)) = \frac{\partial \mathbf{B}}{\partial t}(t, h(t)) dt + h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \quad (2.2)$$

Of course the first term of the right hand side of (2.2) has no meaning *a priori* since mBm is not differentiable with respect to  $t$ . However, continuing with our heuristic reasoning, we then approximate  $\frac{\partial \mathbf{B}}{\partial t}(t, h(t))$  by  $\lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \frac{\partial \mathbf{B}}{\partial t}(t, h_n(t))$ . This formally yields:

$$d\mathbf{B}(t, h(t)) \approx \lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \frac{\partial \mathbf{B}}{\partial t}(t, h_n(t)) dt + h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \quad (2.3)$$

For the sake of notational simplicity, we will consider integrals over the interval  $[0, 1]$ . Let us note  $\int_{[0,1]} Y_t d^\circ B_t^H$  the integral of a process  $Y$  with respect to a fBm of Hurst index  $H$ , in the White Noise theory sense (that will be fully detailed in the next section), assuming it exists.

From (2.3), and assuming we may exchange integrals and limits, we would thus like to define, for suitable processes  $Y$ , the integral with respect to the mBm  $B^h$ , noted  $\int_0^1 Y_t dB_t^h$ , by setting,

$$\int_0^1 Y_t dB_t^h = \lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_t d^\circ B_t^{h(x_k^{(n)})} + \int_0^1 Y_t h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (2.4)$$

where the first term of the right-hand side of (2.4) is a limit, in a sense to be made precise of a sum of integrals with respect to fBms and the second term is a weak integral (see Section 3). The following notation will be useful:

**Notation (integral with respect to lumped fBms)** Let  $Y := (Y_t)_{t \in [0,1]}$  be a real-valued process on  $[0, 1]$  which is integrable with respect to all fBms of index  $H$  in  $h([0, 1])$  in the White Noise Theory sense. We denote the integral with respect to lumped fBms by:

$$\int_0^1 Y_t d^\circ B_t^{h_n} := \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) Y_t d^\circ B_t^{h(x_k^{(n)})}, \quad n \in \mathbf{N} \quad (2.5)$$

where  $(q_n)_{n \in \mathbf{N}}$  and  $x^{(n)} := \{x_k^{(n)}; k \in \llbracket 0, q_n \rrbracket\}$  have been defined in Section 2.1. With this notation, our tentative definition of an integral w.r.t. to mBm (2.4) reads:

$$\int_0^1 Y_t dB_t^h = \lim_{n \rightarrow +\infty} \int_0^1 Y_t d^\circ B_t^{h_n} + \int_0^1 Y_t h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (2.6)$$

The drawback with the previous definition is that, when  $\int_0^1 Y_t h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$  will belong to  $L^2(\Omega)$ , it will not be centred, *a priori*. For this reason and because one can see the Wick product as a centered product (see Remark 3.1 below), we would rather choose, as a definition of  $\int_0^1 Y_t dB_t^h$ , the following one:

$$\int_0^1 Y_t dB_t^h := \lim_{n \rightarrow +\infty} \int_0^1 Y_t d^\circ B_t^{h_n} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (2.7)$$

where  $\diamond$  denotes the Wick product (that will be rigorously defined in the next section).

In order to make the previous statement rigorous, we need to give a precise meaning of the right hand side of (2.7). In particular the meaning of  $\int_0^1 Y_t d^\circ B_t^H$  and thus of  $\int_0^1 Y_t d^\circ B_t^{h_n}$  is crucial. This is the aim of the next section.

### 3 Backgrounds on White Noise Theory and Fractional Wick-Itô Integral in the Bochner sense

This section is divided into two parts. In the first one we briefly recall some basic facts about white noise theory and the Bochner integral.

In the second part we particularize the definition of the Fractional Wick-Itô Integral, defined in [4, 5, 6, 8], into the framewok of Bochner integral.

#### 3.1 Recalls on white noise theory and the Bochner integral

##### 3.1.1 White noise Theory

Define the measurable space  $(\Omega, \mathcal{F})$  by setting  $\Omega := \mathcal{S}'(\mathbf{R})$  and  $\mathcal{F} := \mathcal{B}(\mathcal{S}'(\mathbf{R}))$ , where  $\mathcal{B}$  denotes the  $\sigma$ -algebra of Borel sets. There exists a unique probability measure  $\mu$  on  $(\Omega, \mathcal{F})$  such that, for every  $f$  in  $L^2(\mathbf{R})$ , the map  $\langle \cdot, f \rangle : \Omega \rightarrow \mathbf{R}$  defined by  $\langle \cdot, f \rangle(\omega) = \langle \omega, f \rangle$  (where  $\langle \omega, f \rangle$  is by definition  $\omega(f)$ , *i.e.* the action of the distribution  $\omega$  on the function  $f$ ) is a centred Gaussian random variable with variance equal to  $\|f\|_{L^2(\mathbf{R})}^2$  under  $\mu$ . For every  $n$  in  $\mathbf{N}$ , define  $e_n(x) := (-1)^n \pi^{-1/4} (2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2})$  the  $n$ -th Hermite function. Let  $(|\cdot|_p)_{p \in \mathbf{Z}}$  be the family norms defined by  $|f|_p^2 := \sum_{k=0}^{+\infty} (2k+2)^{2p} \langle f, e_k \rangle_{L^2(\mathbf{R})}^2$ , for all  $(p, f)$  in  $\mathbf{Z} \times L^2(\mathbf{R})$ . The operator  $A$  defined on  $\mathcal{S}'(\mathbf{R})$  by  $A := -\frac{d^2}{dx^2} + x^2 + 1$  admits the sequence  $(e_n)_{n \in \mathbf{N}}$  as eigenfunctions and the sequence  $(2n+2)_{n \in \mathbf{N}}$  as eigenvalues.

We denote  $(L^2)$  the space  $L^2(\Omega, \mathcal{G}, \mu)$  where  $\mathcal{G}$  is the  $\sigma$ -field generated by  $(\langle \cdot, f \rangle)_{f \in L^2(\mathbf{R})}$ . For every random variable  $\Phi$  of  $(L^2)$  there exists, according to the Wiener-Itô theorem, a unique sequence  $(f_n)_{n \in \mathbf{N}}$  of functions  $f_n$  in  $\widehat{L}^2(\mathbf{R}^n)$  such that  $\Phi$  can be decomposed as  $\Phi = \sum_{n=0}^{+\infty} I_n(f_n)$ , where  $\widehat{L}^2(\mathbf{R}^n)$  denotes the set of all symmetric functions  $f$  in  $L^2(\mathbf{R}^n)$  and  $I_n(f)$  denotes the  $n$ -th multiple Wiener-Itô integral of  $f$  with the convention that  $I_0(f_0) = f_0$  for constants  $f_0$ . Moreover the equality  $\mathbf{E}[\Phi^2] = \sum_{n=0}^{+\infty} n! \|f_n\|_{L^2(\mathbf{R}^n)}^2$  holds, where  $\mathbf{E}$  denotes the expectation with respect to  $\mu$ . For any  $\Phi :=$



$\sum_{n=0}^{+\infty} I_n(f_n)$  satisfying the condition  $\sum_{n=0}^{+\infty} n! |A^{\otimes n} f_n|_0^2 < +\infty$ , define the element  $\Gamma(A)(\Phi)$  of  $(L^2)$  by  $\Gamma(A)(\Phi) := \sum_{n=0}^{+\infty} I_n(A^{\otimes n} f_n)$ , where  $A^{\otimes n}$  denotes the  $n$ -th tensor power of the operator  $A$  (see [11, Appendix E] for more details about tensor products of operators). The operator  $\Gamma(A)$  is densely defined on  $(L^2)$ . It is invertible and its inverse  $\Gamma(A)^{-1}$  is bounded. Let, for  $\varphi$  in  $(L^2)$ ,  $\|\varphi\|_0^2 := \|\varphi\|_{(L^2)}^2$  and, for  $n$  in  $\mathbf{N}$ , let  $\mathbb{D}\text{om}(\Gamma(A)^n)$  be the domain of the  $n$ -th iteration of  $\Gamma(A)$ . Define the family of norms  $(\|\cdot\|_p)_{p \in \mathbf{Z}}$  by:

$$\|\Phi\|_p := \|\Gamma(A)^p \Phi\|_0 = \|\Gamma(A)^p \Phi\|_{(L^2)}, \quad \forall p \in \mathbf{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}\text{om}(\Gamma(A)^p).$$

For  $p$  in  $\mathbf{N}$ , define  $(\mathcal{S}_p) := \{\Phi \in (L^2) : \Gamma(A)^p \Phi \text{ exists and belongs to } (L^2)\}$  and define  $(\mathcal{S}_{-p})$  as the completion of the space  $(L^2)$  with respect to the norm  $\|\cdot\|_{-p}$ . As in [12], we let  $(\mathcal{S})$  denote the projective limit of the sequence  $((\mathcal{S}_p))_{p \in \mathbf{N}}$  and  $(\mathcal{S})^*$  the inductive limit of the sequence  $((\mathcal{S}_{-p}))_{p \in \mathbf{N}}$ . This means that we have the equalities  $(\mathcal{S}) = \bigcap_{p \in \mathbf{N}} (\mathcal{S}_p)$  (resp.  $(\mathcal{S})^* = \bigcup_{p \in \mathbf{N}} (\mathcal{S}_{-p})$ ) and that convergence in  $(\mathcal{S})$  (resp. in  $(\mathcal{S})^*$ ) means convergence in  $(\mathcal{S}_p)$  for every  $p$  in  $\mathbf{N}$  (resp. convergence in  $(\mathcal{S}_{-p})$  for some  $p$  in  $\mathbf{N}$ ).

The space  $(\mathcal{S})$  is called the space of stochastic test functions and  $(\mathcal{S})^*$  the space of Hida distributions. One can show that, for any  $p$  in  $\mathbf{N}$ , the dual space  $(\mathcal{S}_p)^*$  of  $\mathcal{S}_p$  is  $(\mathcal{S}_{-p})$ . Thus we will write  $(\mathcal{S}_{-p})$ , in the sequel, to denote the space  $(\mathcal{S}_p)^*$ . Note also that  $(\mathcal{S})^*$  is the dual space of  $(\mathcal{S})$ . We will note  $\ll, \gg$  the duality bracket between  $(\mathcal{S})^*$  and  $(\mathcal{S})$ . If  $\phi, \Phi$  belong to  $(L^2)$  then we have the equality  $\ll \Phi, \phi \gg = \langle \Phi, \phi \rangle_{(L^2)} = \mathbf{E}[\Phi \phi]$ . A function  $\Phi : \mathbf{R} \rightarrow (\mathcal{S})^*$  is called a stochastic distribution process, or an  $(\mathcal{S})^*$ -process, or a Hida process. A Hida process  $\Phi$  is said to be differentiable at  $t_0 \in \mathbf{R}$  if  $\lim_{r \rightarrow 0} r^{-1}(\Phi(t_0 + r) - \Phi(t_0))$  exists in  $(\mathcal{S})^*$ .

For  $f$  in  $L^2(\mathbf{R})$ , we define the *Wick exponential* of  $\langle \cdot, f \rangle$ , noted  $:e^{\langle \cdot, f \rangle}:$ , as the  $(L^2)$  random variable equal to  $e^{\langle \cdot, f \rangle - \frac{1}{2}|f|_0^2}$ . The  $S$ -transform of an element  $\Phi$  of  $(\mathcal{S})^*$ , noted  $S(\Phi)$ , is defined as the function from  $\mathcal{S}(\mathbf{R})$  to  $\mathbf{R}$  given by  $S(\Phi)(\eta) := \ll \Phi, :e^{\langle \cdot, \eta \rangle} : \gg$  for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ . Finally for every  $(\Phi, \Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$ , there exists a unique element of  $(\mathcal{S})^*$ , called the Wick product of  $\Phi$  and  $\Psi$  and noted  $\Phi \diamond \Psi$ , such that  $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta) S(\Psi)(\eta)$  for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ .

**Remark 3.1** It is easy to see that if  $\Phi$  is deterministic then, for all  $\Psi$  in  $(\mathcal{S})^*$ ,  $\Phi \diamond \Psi = \Phi \Psi$ . Moreover, let  $(X_t)_{t \in \mathbf{R}}$  be a Gaussian process and let  $\mathcal{H}$  be the subspace of  $(L^2)$  defined by  $\mathcal{H} := \overline{\text{vect}_{\mathbf{R}}\{X_t; t \in \mathbf{R}\}}^{(L^2)}$ . If  $U$  and  $V$  are two elements of  $\mathcal{H}$  then  $U \diamond V = UV - \mathbf{E}[UV]$ .

### 3.1.2 Fractional and multifractional White noise

We introduce two operators, denoted  $M_H$  and  $\frac{\partial M_H}{\partial H}$ , that will prove useful for the definition of the integral with respect to fBm and mBm.

### Operators $M_H$ and $\frac{\partial M_H}{\partial H}$

Let  $H$  be a fixed real in  $(0, 1)$ . Following [8] and references therein, define the operator  $M_H$ , specified in the Fourier domain, by  $\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{c_H} |y|^{1/2-H} \widehat{u}(y)$  for every  $y$  in  $\mathbf{R}^*$ . This operator is well defined on the homogeneous Sobolev space of order  $1/2 - H$ , denoted  $L_H^2(\mathbf{R})$  and defined by  $L_H^2(\mathbf{R}) := \{u \in \mathcal{S}'(\mathbf{R}) : \widehat{u} \in L_{loc}^1(\mathbf{R}) \text{ and } \|u\|_H < +\infty\}$ , where the norm  $\|\cdot\|_H$  derives from the inner product  $\langle, \rangle_H$  defined on  $L_H^2(\mathbf{R})$  by  $\langle u, v \rangle_H := \frac{1}{c_H^2} \int_{\mathbf{R}} |\xi|^{1-2H} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi$  and where  $c_H$  was given in (1.1).

The definition of the operator  $\frac{\partial M_H}{\partial H}$  is quite similar. More precisely, define for every  $H$  in  $(0, 1)$ , the space  $\Gamma_H(\mathbf{R}) := \{u \in \mathcal{S}'(\mathbf{R}) : \widehat{u} \in L_{loc}^1(\mathbf{R}) \text{ and } \|u\|_{\delta_H(\mathbf{R})} < +\infty\}$ , where the norm  $\|\cdot\|_{\delta_H(\mathbf{R})}$  derives from the inner product on  $\Gamma_H(\mathbf{R})$  defined by  $\langle u, v \rangle_{\delta_H} := \frac{1}{c_H^2} \int_{\mathbf{R}} (\beta_H + \ln |\xi|)^2 |\xi|^{1-2H} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi$ . Following [13], define the operator  $\frac{\partial M_H}{\partial H}$  from  $(\Gamma_H(\mathbf{R}), \langle, \rangle_{\delta_H(\mathbf{R})})$  to  $(L^2(\mathbf{R}), \langle, \rangle_{L^2(\mathbf{R})})$ , in the Fourier domain, by:  $\widehat{\frac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \ln |y|) \frac{\sqrt{2\pi}}{c_H} |y|^{1/2-H} \widehat{u}(y)$ , for every  $y$  in  $\mathbf{R}^*$ . The reader interested in the properties of  $M_H$  and  $\frac{\partial M_H}{\partial H}$  may refer to [13, Sections 2.2 and 4.2].

### Fractional and multifractional White noise

Recall the following result ([13, (5.10)]): Almost surely, for every  $t$ ,

$$B_t^h = \sum_{k=0}^{+\infty} \left( \int_0^t M_{h(t)}(e_k)(s) ds \right) \langle \cdot, e_k \rangle. \quad (3.1)$$

We now define the derivative in the sense of  $(\mathcal{S})^*$  of mBm. Define the  $(\mathcal{S}^*)$ -valued process  $W^h := (W_t^h)_{t \in [0,1]}$  by

$$W_t^h := \sum_{k=0}^{+\infty} \left[ \frac{d}{dt} \left( \int_0^t M_{h(t)}(e_k)(s) ds \right) \right] \langle \cdot, e_k \rangle. \quad (3.2)$$

**Theorem-Definition 3.1** [13, Theorem-definition 5.1] *The process  $W^h$  defined by (3.2) is an  $(\mathcal{S})^*$ -process which verifies, in  $(\mathcal{S})^*$ , the following equality:*

$$W_t^h = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) \langle \cdot, e_k \rangle + h'(t) \sum_{k=0}^{+\infty} \left( \int_0^t \frac{\partial M_H}{\partial H}(e_k)(s) \Big|_{H=h(t)} ds \right) \langle \cdot, e_k \rangle. \quad (3.3)$$

Moreover the process  $B^h$  is  $(\mathcal{S})^*$ -differentiable on  $[0, 1]$  and verifies  $\frac{dB^h}{dt}(t) = W_t^h$  in  $(\mathcal{S})^*$ .

When the function  $h$  is constant and identically equal to  $H$ , we will write  $W^H := (W_t^H)_{t \in [0,1]}$  and call the  $(\mathcal{S})^*$ -process  $W^H$  a fractional white noise. Note that (3.3) may be written as

$$W_t^h = W_t^{h(t)} + h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)), \quad (3.4)$$

where  $W_t^{h(t)}$  is nothing but  $W_t^H|_{H=h(t)}$  and where the equality holds in  $(\mathcal{S})^*$ .

### 3.1.3 Bochner integral

Since the objects we are dealing with are no longer random variables in general, the Riemann or Lebesgue integrals are not relevant here. However, taking advantage of the fact that we are working with vector linear spaces, we may use Pettis or Bochner integrals.

In the frame of the Wick-Itô integral, and in view of the considerations about the norm  $\|\cdot\|_{\Lambda_{p_0}}$ , that will be given right after Proposition 4.1, the use of Bochner integral seems to be relevant. Thus we give here a brief statement on Bochner integral. However, and in order not to weigh down this statement we will only give the necessary tools to proceed (see [12, p.247] for more details about Bochner integral).

**Definition 3.1 (Bochner integral [12], p.247)** *Let  $I$  be a subset of  $[0, 1]$  endowed with the Lebesgue measure. One says that  $\Phi : I \rightarrow (\mathcal{S})^*$  is Bochner integrable of index  $p$  on  $I$  if it satisfies the two following conditions:*

1.  $\Phi$  is weakly measurable on  $I$ , i.e.  $t \mapsto \langle \Phi_t, \varphi \rangle$  is measurable on  $I$  for every  $\varphi$  in  $(\mathcal{S})$ .
2. There exists  $p$  in  $\mathbf{N}$  such that  $\Phi_t \in (\mathcal{S}_{-p})$  for almost every  $t$  in  $I$  and such that  $t \mapsto \|\Phi_t\|_{-p}$  belongs to  $L^1(I, dt)$ .

The Bochner-integral of  $\Phi$  on  $I$  is denoted  $\int_I \Phi_t dt$ .

**Proposition 3.1** *If  $\Phi : I \rightarrow (\mathcal{S})^*$  is Bochner-integrable on  $I$  with index  $p$  then  $\|\int_I \Phi_t dt\|_{-p} \leq \int_I \|\Phi_t\|_{-p} dt$ .*

**Theorem 3.1 ([12], Theorem 13.5)** *Let  $\Phi := (\Phi_t)_{t \in [0, 1]}$  be an  $(\mathcal{S})^*$ -valued process such that:*

- (i)  $t \mapsto S(\Phi_t)(\eta)$  is measurable for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ .
- (ii) There exist  $p$  in  $\mathbf{N}$ ,  $b$  in  $\mathbf{R}^+$  and a function  $L$  in  $L^1([0, 1], dt)$  such that, for a.e.  $t$  in  $[0, 1]$ ,  $|S(\Phi_t)(\eta)| \leq L(t) e^{b|\eta|^2}$ , for every  $\eta$  in  $\mathcal{S}(\mathbf{R})$ .

*Then  $\Phi$  is Bochner integrable on  $[0, 1]$  and  $\int_0^1 \Phi(s) ds \in (\mathcal{S}_{-q})$  for every  $q > p$  such that  $2be^2 D(q-p) < 1$  where  $e$  denotes the base of the natural logarithm and where  $D(r) := \frac{1}{2^{2r}} \sum_{n=1}^{+\infty} \frac{1}{n^{2r}}$  for  $r$  in  $(1/2, +\infty)$ .*

### 3.2 Wick-Itô integral with respect to fBm in the Bochner sense

The fractional Wick-Itô integral with respect to fBm (or integral w.r.t. fBm in the white noise sense) was introduced in [8] and extended in [4] using the Pettis integral. However, in order to use Theorem 4.1, we need that  $Y_s$  belongs to  $(\mathcal{S}_{-p})$  for almost every real  $s$ . It then seems reasonable to assume that  $(Y_s)_{s \in [0,1]}$  is Bochner integrable on  $[0,1]$ . For this reason, we now particularize the fractional Wick-Itô integral with respect to fBm (or Wick-Itô integral w.r.t. fBm) of [8] and [4] in the framework of the Bochner integral.

#### Definition 3.2 (Wick-Itô integral w.r.t fBm in the Bochner sense)

Let  $H \in (0,1)$ ,  $I$  be a Borel subset of  $[0,1]$ ,  $B^H := (B_t^H)_{t \in I}$  be a fractional Brownian motion of Hurst index  $H$ , and  $Y := (Y_t)_{t \in I}$  be an  $(\mathcal{S})^*$ -valued process verifying:

- (i) there exists  $p \in \mathbf{N}$  such that  $Y_t \in (\mathcal{S}_{-p})$  for almost every  $t \in I$ ,
- (ii) the process  $t \mapsto Y_t \diamond W_t^H$  is Bochner integrable on  $I$ .

Then,  $Y$  is said to be Bochner-integrable with respect to fBm on  $I$  and its integral is defined by:

$$\int_I Y_s d^\diamond B_s^H := \int_I Y_s \diamond W_s^H ds. \quad (3.5)$$

**Remark 3.2** In order to keep the name given in [8], we also call this integral fractional Wick-Itô integral.

The following lemma ensures us that every Bochner process is integrable on  $[0,1]$  w.r.t. fBm of any Hurst index  $H$  in  $(0,1)$ .

**Lemma 3.1** Let  $Y := (Y_t)_{t \in [0,1]}$  be an  $(\mathcal{S})^*$ -valued process, Bochner integrable of index  $p_0 \in \mathbf{N}$ . Then  $Y$  is integrable on  $[0,1]$ , with respect to fBm of any Hurst index  $H$ , in the Bochner sense. Moreover, for any  $H$  in  $(0,1)$ ,  $\int_{[0,1]} Y_s d^\diamond B_s^H$  belongs to  $(\mathcal{S}_{-r_0})$  for every  $r_0 \geq p_0 + 1$  if  $p_0 \geq 2$  and for every  $r_0 \geq p_0 + 2$  if  $p_0 \in \{0,1\}$ .

**Proof.** Fix  $H \in (0,1)$ ,  $p_0 \geq 2$  and  $r_0 \geq p_0 + 1$ . The map  $t \mapsto Y_t \diamond W_t^H$  is weakly measurable since  $t \mapsto S(Y_t \diamond W_t^H)(\eta)$  is measurable for all  $\eta \in \mathcal{S}(\mathbf{R})$ . Using [12, Remark 2 p.92], one obtains that, for almost all  $t$  in  $[0,1]$ ,  $\|Y_t \diamond W_t^H\|_{-r_0} \leq \|Y_t\|_{-p_0} \|W_t^H\|_{-p_0} < +\infty$ . Hence  $Y_t \diamond W_t^H$  belongs to  $(\mathcal{S}_{-r_0})$ . Since the map  $t \mapsto \|W_t^H\|_{-r}$  is continuous for all integer  $r \geq 2$  (see [13, Proposition 5.9]), one also gets:

$$\int_0^1 \|Y_t \diamond W_t^H\|_{-r_0} dt \leq \left( \sup_{t \in [0,1]} \|W_t^H\|_{-p_0} \right) \int_0^1 \|Y_t\|_{-p_0} dt < +\infty.$$

This shows that  $t \mapsto Y_t \diamond W_t^H$  is Bochner-integrable of index  $r_0$ .

Let us now assume that  $p_0 \in \{0, 1\}$ . It is sufficient to check that Theorem 3.1 applies. Condition (i) is obviously fulfilled. Moreover, using [12, p.79], we obtain that, for every  $(t, \eta)$  in  $[0, 1] \times \mathcal{S}(\mathbf{R})$ ,

$$|S(Y_t \diamond W_t^H)(\eta)| \leq \|Y_t\|_{-p_0} e^{\frac{1}{2}|\eta|_2^2} \sup_{t \in [0, 1]} \|W_t^H\|_{-2} =: L(t) e^{\frac{1}{2}|\eta|_2^2}.$$

Since  $Y$  is Bochner integrable of index  $p_0$ , it is clear that  $L$  belongs to  $L^1([0, 1], dt)$ . Moreover,  $e^2 D(r_0 - p_0) < 1$ , for every  $r_0 \geq p_0 + 2$ . Theorem 3.1 then allows to conclude that  $t \mapsto Y_t \diamond W_t^H$  is Bochner integrable of index  $r_0$ .  $\square$

We end this section with the following lemma, the proof of which is obvious in view of Proposition 2.1, will be useful in the proof of Proposition 4.1 below.

**Lemma 3.2** *For every  $p$  in  $\mathbf{N}$ , the map  $(t, H) \mapsto \frac{\partial \mathbf{B}}{\partial H}(t, H)$  is continuous from  $[0, 1]$  into  $((\mathcal{S}_{-p}, \|\cdot\|_p)$ . In particular, for every subset  $[a, b]$  of  $(0, 1)$ , there exists a positive real  $\kappa$  such that:*

$$\forall p \in \mathbf{N}, \quad \sup_{(s, H) \in [0, t] \times [a, b]} \left\| \frac{\partial \mathbf{B}}{\partial H}(s, H) \right\|_{-p} \leq \kappa. \quad (3.6)$$

## 4 Integral with respect to mBm through approximating fBms

Our aim in this section, is to construct an integral w.r.t. mBm using approximating integrals w.r.t. fBms. This new integral, that will be named limiting fractional Wick-Itô integral, is defined at the end of this section. The main result of this section is Theorem 4.1. It requires the two results given in Proposition 4.1, as well as Lemma 4.1 above.

Let  $p_0$  be a fixed integer and let  $Y := (Y_t)_{t \in [0, 1]}$  be an  $(\mathcal{S}_{-p_0})$ -valued process (i.e.  $Y_t$  belongs to for every real  $t$  in  $[0, 1]$  and  $t \mapsto Y_t$  is measurable from  $(0, 1)$  to  $(\mathcal{S}_{-p_0})$ , endowed with its Borelian measure. As explained in Section 2.2, we wish to define the integral w.r.t. an mBm  $B^h$ , noted  $\int_0^1 Y_t d^\circ B_t^h$ , by a formula of the kind:

$$\int_0^1 Y_t d^\circ B_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t d^\circ B_t^{h_n} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (4.1)$$

where the limit holds in  $(\mathcal{S})^*$ . For this formula to make sense, it is certainly necessary that  $Y$  be Bochner-integrable with respect to fBm, on  $[0, 1]$ , of all exponents  $\alpha$  in  $h([0, 1])$ . The following proposition ensures us that Bochner-integrability of  $Y$  on  $[0, 1]$  is sufficient to guarantee that both the sequence  $(\int_0^1 Y_t d^\circ B_t^{h_n})_{n \in \mathbf{N}}$  and the quantity  $\int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$  exist and belong to  $(\mathcal{S})^*$ .

**Proposition 4.1** For any  $(\mathcal{S}_{-p_0})$ -valued process  $Y := (Y_t)_{t \in [0,1]}$  that is Bochner integrable on  $[0, 1]$ , the sequence  $(\int_0^1 Y_t \diamond d^\circ B_t^{h_n})_{n \in \mathbf{N}}$  and the quantity  $\int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$  are well defined in  $(\mathcal{S})^*$ . Moreover they both belong to  $(\mathcal{S}_{-s_0})$  for every  $s_0 \geq \max\{p_0 + 1, 3\}$ .

**Proof.** The existence of the sequence  $(\int_0^1 Y_t \diamond d^\circ B_t^{h_n})_{n \in \mathbf{N}}$  in  $(\mathcal{S}_{-s_0})$  is a straightforward consequence of Lemma 3.1 since it has been proven there that  $\int_0^1 Y_t \diamond d^\circ B_t^H$  is well defined and belongs to  $(\mathcal{S}_{-s_0})$  for any  $H$  in  $(0, 1)$ ,  $s_0$  being independent from  $H$ .

One then just needs to show that the map  $t \mapsto h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$  is Bochner integrable on  $[0, 1]$ . Using the same arguments as in the proof of Lemma 3.1 one easily prove that  $t \mapsto h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$  is weakly measurable on  $[0, 1]$ . Lemma 3.2 entails that,  $\sup_{(s, H) \in [0,1] \times h([0,1])} \|\frac{\partial \mathbf{B}}{\partial H}(s, H)\|_{-p_0} \leq \kappa$ , for every  $p_0$ . We hence get, for any  $s_0 \geq \max\{p_0 + 1, 3\}$ ,

$$\|h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\|_{-s_0} \leq \|Y_s\|_{-p_0} \left( \sup_{s \in [0,1]} |h'(s)| \right) \sup_{s \in [0,1]} \|\frac{\partial \mathbf{B}}{\partial H}(s, h(s))\|_{-p_0} < +\infty.$$

Thus there exists  $\delta \in \mathbf{R}_+^*$ , such that  $\int_0^1 \|h'(s) Y_s \diamond \frac{\partial \mathbf{B}}{\partial H}(s, h(s))\|_{-s_0} ds \leq \delta \int_0^1 \|Y_s\|_{-p_0} ds < +\infty$ . As a consequence,  $\int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$  is well defined in the sense of Bochner.  $\square$

From now on,  $(p_0, s_0)$  is fixed in  $\mathbf{N}^2$  such that  $s_0 \geq \max\{p_0 + 1, 3\}$ . Keeping in mind Definition 3.1, one defines the set  $\Lambda_{p_0}$  by setting:

$$\Lambda_{p_0} := \{Y := (Y_t)_{t \in [0,1]} \in (\mathcal{S}_{-p_0})^{\mathbf{R}} : Y \text{ is Bochner integrable of index } p_0 \text{ on } [0, 1]\}.$$

Thanks to Proposition 4.1 we know that, for every  $\alpha$  in  $h([0, 1])$ ,  $\int_{[0,1]} Y_t \diamond d^\circ B_t^\alpha$  exists and belongs to  $(\mathcal{S}_{-s_0})$ , where  $s_0$  is the integer defined in Proposition 4.1. Besides one can define a norm on  $\Lambda_{p_0}$ , noted  $\|\cdot\|_{\Lambda_{p_0}}$ , by setting  $\|\Phi\|_{\Lambda_{p_0}} := \int_0^1 \|\Phi_t\|_{-p_0} dt$ . The following lemma will be useful to establish Theorem 4.1.

**Lemma 4.1** The space  $(\Lambda_{p_0}, \|\cdot\|_{\Lambda_{p_0}})$  is complete. Moreover it has the following property: there exists  $M > 0$  such that for all partitions of  $[0, 1]$  in intervals  $A_1, \dots, A_n$  of equal size  $\frac{1}{n}$ ,

$$\|Y \cdot \mathbb{1}_{A_1}\|_{\Lambda_{p_0}} + \dots + \|Y \cdot \mathbb{1}_{A_n}\|_{\Lambda_{p_0}} \leq M \|Y\|_{\Lambda_{p_0}}. \quad (4.2)$$

**Proof.** The fact that  $(\Lambda_{p_0}, \|\cdot\|_{\Lambda_{p_0}})$  is complete is a straightforward consequence of [10, Theorem 3.7.7 p.82].

Besides (4.2) is obvious by very definition of Bochner integral.  $\square$

When  $Y$  belongs to  $\Lambda_{p_0}$ , the definition of  $\int_0^1 Y_t \diamond d^\circ B_t^h$ , given in (4.1) will be a valid one as soon as the limit on the right hand side exists. It turns out

that a simple sufficient condition guarantees the existence of the limit of the integral w.r.t. lumped fBms.

$$\begin{aligned} L_n &: \Lambda_{p_0} \rightarrow (\mathcal{S}_{-s_0}) \\ Y &\mapsto \int_{[0,1]} Y_t d^\circ B_t^{h_n}. \end{aligned} \quad (4.3)$$

For any sequence  $(a_n)_{n \in \mathbf{N}}$  of positive integers and every positive integer  $l$ , denote  $\widehat{a}_l := \prod_{0 \leq n \leq l-1} a_n$ . The following theorem, which constitutes the main result of this section, provides a sufficient condition under which  $(L_n(Y))_{n \in \mathbf{N}}$  converges in  $(\mathcal{S}_{-s_0})$ .

**Theorem 4.1** *Let  $(a_n)_{n \in \mathbf{N}}$  be an increasing sequence of positive integers such that  $2^n \leq \widehat{a}_n \leq 2^{2^n}$ , for every  $n$  in  $\mathbf{N}$ , and such that  $\lim_{n \rightarrow +\infty} (n(a_n - 1)) (\widehat{a}_n)^{-1} = 0$ . Choose the sequence  $(q_n)_{n \in \mathbf{N}}$  used in (2.5) such that  $q_0 = 1$  and  $q_{n+1} = a_n q_n$  for all  $n$  in  $\mathbf{N}$ . Then the sequence of functions  $(L_n)_{n \in \mathbf{N}}$  defined in (4.3) converges pointwise to a function  $L : \Lambda_{p_0} \rightarrow (\mathcal{S}_{-s_0})$ .*

The proof of this theorem requires the following lemma.

**Lemma 4.2** *The map  $\mathcal{I} : \Lambda_{p_0} \times (0, 1) \rightarrow (\mathcal{S}_{-s_0})$  defined by*

$$\forall Y \in \Lambda_{p_0}, \forall \alpha \in (0, 1), \quad \mathcal{I}(Y, \alpha) := \int_{[0,1]} Y_t d^\circ B_t^\alpha,$$

where  $\int_{[0,1]} Y_t d^\circ B_t^\alpha$  is the Wick-Itô integral w.r.t. fBm, is Hölder continuous with respect to  $\alpha$  uniformly in  $Y$ , i.e. there exists  $K > 0$  such that

$$\forall Y \in \Lambda_{p_0}, \forall (\alpha, \alpha') \in (0, 1)^2, \quad \|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_{-s_0} \leq K |\alpha - \alpha'| \|Y\|_{\Lambda_{p_0}}. \quad (4.4)$$

**Proof of Theorem 4.1.** Let us admit the proof of the previous lemma for the moment. For the sake of simplicity, we will establish the result in the case where  $a_n \equiv 2$  for every  $n$ , which obviously fulfils the required conditions. The general case is similar. For  $n \in \mathbf{N}$  and  $Y \in \Lambda_{p_0}$ ,  $L_n(Y)$  may be decomposed as

$$L_n(Y) = \sum_{k=0}^{2^n-1} \left( \int_{[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}]} Y_t d^\circ B_t^{h(\frac{k}{2^n})} + \int_{[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}]} Y_t d^\circ B_t^{h(\frac{k}{2^n})} \right). \text{ Now}$$

$$L_{n+1}(Y) = \sum_{k=0}^{2^{n+1}-1} \left( \int_{[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}]} Y_t d^\circ B_t^{h(\frac{2k}{2^{n+1}})} + \int_{[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}]} Y_t d^\circ B_t^{h(\frac{2k+1}{2^{n+1}})} \right).$$

Setting  $D_n := \|L_n(Y) - L_{n+1}(Y)\|_{-p_0}$  and using assumptions (4.2) and (4.4), one gets,

$$\begin{aligned}
D_n &= \left\| \sum_{k=0}^{2^n-1} (\mathcal{I}(Y \cdot \mathbb{1}_{[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}]}, h(\frac{k}{2^n})) - \mathcal{I}(Y \cdot \mathbb{1}_{[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}]}, h(\frac{2k+1}{2^{n+1}}))) \right\|_{-s_0} \\
&\leq \sum_{k=0}^{2^n-1} \left\| (\mathcal{I}(Y \cdot \mathbb{1}_{[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}]}, h(\frac{k}{2^n})) - \mathcal{I}(Y \cdot \mathbb{1}_{[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}]}, h(\frac{2k+1}{2^{n+1}}))) \right\|_{-s_0} \\
&\leq K \sum_{k=0}^{2^n-1} \|Y \cdot \mathbb{1}_{[(2k+1) \cdot 2^{-(n+1)}, (k+1) \cdot 2^{-n}]} \|_{\Lambda_{p_0}} |h((2k+1) \cdot 2^{-(n+1)}) - h(k \cdot 2^{-n})| \\
&\leq K 2^{-(n+1)} \sup_{t \in [0,1]} |h'(t)| \sum_{k=0}^{2^n-1} \left\| Y \cdot \mathbb{1}_{[(2k+1) \cdot 2^{-(n+1)}, (k+1) \cdot 2^{-n}]} \right\|_{\Lambda_{p_0}} \\
&\leq K M 2^{-\theta(n+1)} \sup_{t \in [0,1]} |h'(t)| \|Y\|_{\Lambda_{p_0}}. \tag{4.5}
\end{aligned}$$

It follows that the series  $\sum_{n \in \mathbf{N}} (L_{n+1}(Y) - L_n(Y))$  converges absolutely for any fixed  $Y \in \Lambda_{p_0}$ , and consequently  $(L_n(Y))_{n \in \mathbf{N}}$  converges to a limit  $L(Y)$  as  $n$  goes to infinity.  $\square$

It then remains to prove Lemma 4.2.

**Proof of Lemma 4.2.** Since  $(\mathcal{S}_{-p_0}) \subset (\mathcal{S}_{-2})$  if  $p_0$  belongs to  $\{0; 1\}$ , we assume from now on that  $p_0 \geq 2$  and that  $s_0 \geq p_0 + 1$ .

Let  $Y \in \Lambda_{p_0}$ . Lemma 3.1 entails that  $Y$  is integrable with respect to fBm in the Bochner sense, for all  $\alpha$  in  $(0, 1)$  and that  $\mathcal{I}(Y, \alpha) = \int_{[0,1]} Y_t d^\alpha B_t^\alpha$  belongs to  $(\mathcal{S}_{-s_0})$ . Now for all  $(\alpha, \alpha')$  in  $(0, 1)^2$ , using the same arguments we used in the proof of Lemma 3.1, one gets:

$$\begin{aligned}
\|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_{-s_0} &= \left\| \int_{[0,1]} Y_t \diamond (W_t^\alpha - W_t^{\alpha'}) dt \right\|_{-s_0} \\
&\leq \int_0^1 \|Y_t\|_{-p_0} \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0} dt \leq \left( \sup_{t \in [0,1]} \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0} \right) \|Y\|_{\Lambda_{p_0}},
\end{aligned}$$

and thus

$$\sup_{\|Y\|_{\Lambda_{p_0}} \leq 1} \|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_{-s_0} \leq \sup_{t \in [0,1]} \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0}. \tag{4.6}$$

By definition,  $\|W_t^\alpha - W_t^{\alpha'}\|_{-p_0}^2 = \sum_{k=0}^{+\infty} \frac{(M_\alpha(e_k)(t) - M_{\alpha'}(e_k)(t))^2}{(2k+2)^{2p_0}}$ . Besides, for all  $(t, k)$  in  $[0, 1] \times \mathbf{N}$ , the function  $\alpha \mapsto M_\alpha(e_k)(t)$  is differentiable on  $(0, 1)$  (this is in [13, Lemma 5.5]). Using point 1 of [13, Lemma 5.6] and the mean value theorem, one obtains the following fact: for all  $[a, b] \subset (0, 1)$ , there exists a positive real  $\rho$  such that for all  $(t, \alpha, \alpha', k) \in [0, 1] \times [a, b]^2 \times \mathbf{N}$ :

$$|M_\alpha(e_k)(t) - M_{\alpha'}(e_k)(t)| \leq \rho (k+1)^{2/3} \ln(k+1) |\alpha - \alpha'|.$$

As a consequence, we get  $\|W_t^\alpha - W_t^{\alpha'}\|_{-p_0}^2 \leq \rho^2 |\alpha - \alpha'|^2 \sum_{k=0}^{+\infty} \frac{(k+1)^{4/3} \ln^2(k+1)}{2^{2p_0} (k+1)^{2p_0}}$ .

With (4.6), one obtains

$$\sup_{\|Y\|_{\Lambda_{p_0}} \leq 1} \|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_{-s_0} \leq |\alpha - \alpha'| \gamma_{p_0}, \tag{4.7}$$



where  $\gamma_{p_0} := \rho \left( \sum_{k=1}^{+\infty} \frac{\ln^2 k}{k^{2(p_0-2/3)}} \right)^{1/2}$  is finite since  $p_0 \geq 2$ .  $\square$

**Corollary 4.1** *Let  $Y \in \Lambda_{p_0}$ . Then*

$$\int_0^1 Y_t d^\diamond B_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t d^\diamond B_t^{h_n} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (4.8)$$

where the limit and the equality hold in  $(\mathcal{S}_{-s_0})$ , is well-defined and belongs to  $(\mathcal{S}_{-s_0})$ .

As a consequence of Theorem 4.1 and Corollary 4.1, the integral w.r.t. mBm exists as a limit of integrals w.r.t. fBms plus a second term. Thus, we are finally able to define our integral:

**Definition 4.1 (Limiting fractional Wick-Itô integral)** *For any fixed integers  $p_0$  and  $s_0$  such that  $s_0 \geq \max\{p_0 + 1, 3\}$ , and any element  $Y := (Y_t)_{t \in [0,1]}$  of  $\Lambda_{p_0}$ , the integral of  $Y$  with respect to  $B^h$  can be obtained as limits of fractional Wick-Itô integral. We note  $\int_0^1 Y_t dB_t^h$  this integral and call it limiting fractional Wick-Itô integral plus a second term. It is defined by:*

$$\int_0^1 Y_t dB_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t d^\diamond B_t^{h_n} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (4.9)$$

where the limit and equality both hold in  $(\mathcal{S}_{-s_0})$ .

In view of (4.1) and of Remark 3.2, and even if  $\int_0^1 Y_t dB_t^h$  is not only a limit of sums of fractional Wick-Itô integrals, the name limiting fractional Wick-Itô integral to call  $\int_0^1 Y_t dB_t^h$  seems to be indicate since it give the essence of it.

**Remark 4.1** The linearity of limiting fractional Wick-Itô integral as well as the equality  $\int_a^b dB_u^h \stackrel{\text{a.s.}}{=} B_b^h - B_a^h$ , for any  $(a, b)$  in  $\mathbf{R}^2$  such that  $a < b$ , are consequences of Definition 4.1.

Moreover, for any  $(\mathcal{S}_{-p_0})$ -valued process  $Y := (Y_t)_{t \in [0,1]}$  that admits a limiting fractional Wick-Itô integral over a Borel subset  $I$  of  $\mathbf{R}$ , if  $\int_I X_s dB_s^h$  belongs to  $(L^2)$ . Then  $\mathbf{E}[\int_I X_s dB_s^h] = 0$ .

We shall compare the integral w.r.t. mBm, obtained in Definition 4.1, to the one defined with the direct approach of [13]. This the goal of the next section.

## 5 A comparison between multifractional Wick-Itô integral and limiting fractional Wick-Itô integral

A multifractional Wick-Itô integral with respect to mBm was defined in [13]. In addition Itô formulas (in both weak and strong senses) and a Tanaka formula were provided. It is interesting to check whether it coincides with the one provided by Definition 4.1.

In that view, we need to adapt the definition of [13], which used Pettis integrals, to deal with Bochner integrals.

### 5.1 Multifractional Wick-Itô integral in Bochner sense

#### Definition 5.1 (Multifractional Wick-Itô integral in Bochner sense)

Let  $I$  be a Borelian connected subset of  $[0, 1]$ ,  $B^h := (B_t^h)_{t \in I}$  be a multifractional Brownian motion and  $Y := (Y_t)_{t \in I}$  be a  $(\mathcal{S})^*$ -valued process such that:

- (i) There exists  $p \in \mathbf{N}$  such that  $Y_t \in (\mathcal{S}_{-p})$  for almost every  $t \in I$ ,
- (ii) the process  $t \mapsto Y_t \diamond W_t^h$  is Bochner integrable on  $I$ .

$Y$  is then said to be integrable on  $I$  with respect to mBm in the Bochner sense or to admit a multifractional Wick-Itô integral. This integral, noted  $\int_I Y_s d^\diamond B_s^h ds$ , is defined by  $\int_I Y_s d^\diamond B_s^h ds = \int_I Y_s \diamond W_s^h ds$ .

**Remark 5.1** From the definition of  $(W_t^h)_{t \in [0,1]}$  [13, Proposition 5.9], and the proof of Lemma 3.1, it is clear that every  $(\mathcal{S})^*$ -valued process  $Y := (Y_t)_{t \in I}$  which is Bochner integrable on  $I$  of index  $p_0$ , is integrable on  $I$  with respect to mBm, in the Bochner sense. Moreover  $\int_{[0,1]} Y_t d^\diamond B_t^h$  belongs to  $(\mathcal{S}_{-r_0})$ , where  $r_0$  was defined in Lemma 3.1.

### 5.2 A comparison between multifractional Wick-Itô integral and limiting fractional Wick-Itô integral

In order to compare our two integrals with respect to mBm when they both exist, it seems natural to assume that  $Y = (Y_t)_{t \in [0,1]}$  is a Bochner integrable process of index  $p_0 \in \mathbf{N}$ . We keep notations of the previous sections (in particular for  $p_0$  and  $s_0$ ).

**Theorem 5.1** Let  $Y = (Y_t)_{t \in [0,1]}$  be a Bochner integrable process of index  $p_0 \in \mathbf{N}$ . Then  $Y$  is integrable with respect to mBm in both senses of Definition 4.1 and Definition 5.1. Moreover  $\int_I Y_t dB_t^h dt$  and  $\int_{[0,1]} Y_t d^\diamond B_t^h$  are equal in  $(\mathcal{S}^*)$ .

**Proof:** Since  $Y$  is a Bochner integrable process of index  $p_0 \in \mathbf{N}$ , Proposition 4.1 and Remark 5.1 entail that  $\int_{[0,1]} Y_s d^\diamond B_s^h$  exists in  $(\mathcal{S}_{-s_0})$  and that

$\int_I Y_s \diamond W_s^h ds$  exist in  $(\mathcal{S}_{-r_0})$ , where  $s_0$  has been defined as being a positive integer such that  $s_0 \geq \max\{p_0 + 1, 3\}$  and  $r_0$  has been defined in Lemma 3.1. Moreover, thanks to (2.5) and (3.5), we may write, in  $(\mathcal{S}_{-s_0})$ ,

$$\begin{aligned} \int_{[0,1]} Y_t d^\diamond B_t^{h_n} &= \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t d^\diamond B_t^{h(x_k^{(n)})} \\ &= \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t \diamond W_t^{h(x_k^{(n)})} dt. \end{aligned}$$

In view of (4.8), this allows us to write that,

$$\begin{aligned} \int_{[0,1]} Y_s dB_s^h &= \lim_{n \rightarrow \infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t \diamond W_t^{h(x_k^{(n)})} dt \\ &\quad + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \end{aligned} \quad (5.1)$$

Besides, Definition 5.1 and (3.4) entail that, in  $(\mathcal{S}_{-r_0})$ ,

$$\int_0^1 Y_t \diamond W_t^h dt = \int_0^1 Y_t \diamond W_t^{h(t)} dt + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \quad (5.2)$$

Since  $s_0 \geq r_0$  we have  $(\mathcal{S}_{-r_0}) \subset (\mathcal{S}_{-s_0})$ . Thus it remains to show that, in  $(\mathcal{S}_{-s_0})$ ,  $L(Y) = M(Y)$ , where we noted:

$$L(Y) := \lim_{n \rightarrow \infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t \diamond W_t^{h(x_k^{(n)})} dt$$

and

$$M(Y) := \int_0^1 Y_t \diamond W_t^{h(t)} dt.$$

Since  $L(Y)$  and  $M(Y)$  both belong to  $(\mathcal{S}_{-s_0})$ , it is sufficient to show that they have the same  $S$ -transform. Using [12, Theorem 8.6], one has, for  $\eta$  in  $\mathcal{S}(\mathbf{R})$ ,  $S(L(Y))(\eta) = \lim_{n \rightarrow \infty} S\left(\sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t \diamond W_t^{h(x_k^{(n)})} dt\right)(\eta)$ . Using now (ii) of [13, Theorem 5.12], one gets

$$\begin{aligned} S(L(Y))(\eta) &= \lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} S(Y_t)(\eta) S(W_t^{h(x_k^{(n)})})(\eta) dt \\ &= \lim_{n \rightarrow +\infty} \int_{[0,1]} S(Y_t)(\eta) \left(\sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) M_{h(x_k^{(n)})}(\eta)(t)\right) dt. \end{aligned} \quad (5.3)$$

The map  $(t, H) \mapsto M_H(\eta)(t)$  being continuous (see [13, Lemma 5.5]), one defines  $K_\eta := \sup_{(t,H) \in [0,1] \times h([0,1])} |M_H(\eta)(t)|$ . For all  $n$  in  $\mathbf{N}$  and  $t$  in  $[0, 1]$ , one has

$$|S(Y_t)(\eta) \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) M_{h(x_k^{(n)})}(\eta)(t)| \leq K_\eta e^{\frac{1}{2}|\eta|_{p_0}^2} \|Y_t\|_{-p_0}. \quad (5.4)$$

The map  $t \mapsto K_\eta e^{|\eta|_{p_0}^2} \|Y_t\|_{-p_0}$  belongs to  $L^1(\mathbf{R}, dt)$ . In addition, for almost every  $t$  in  $[0, 1]$ ,

$$\lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \mathbf{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) M_{h(x_k^{(n)})}(\eta)(t) = M_{h(t)}(\eta)(t),$$

Thus, by dominated convergence and using (5.4), one gets,

$$\begin{aligned} S(L(Y))(\eta) &= \int_{[0,1]} S(Y_t)(\eta) M_{h(t)}(\eta)(t) dt = \int_{[0,1]} S(Y_t)(\eta) S(W_t^{h(t)})(\eta) dt \\ &= S\left(\int_{[0,1]} Y_t \diamond W_t^{h(t)} dt\right)(\eta), \end{aligned}$$

Since the map  $S : \Phi \mapsto S(\Phi)$  from  $(\mathcal{S})^*$  into itself is injective, one deduces that  $L(Y) = \int_{[0,1]} Y_t \diamond W_t^{h(t)} dt$ , *i.e.* that  $L(Y) = M(Y)$ . The proof is then complete.  $\square$

## Acknowledgments

The author thanks Jacques Lévy Véhel for many helpful remarks and comments.

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