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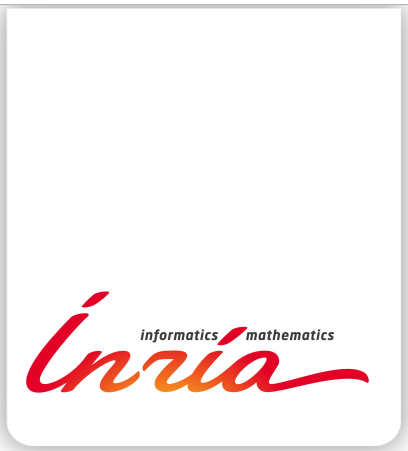

Weighted Coloring in Trees*

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Abstract: A proper coloring of a graph is a partition of its vertex set into stable sets, where each part corresponds to a *color*. For a vertex-weighted graph, the *weight of a color* is the maximum weight of its vertices. The *weight of a coloring* is the sum of the weights of its colors. Guan and Zhu defined the *weighted chromatic number* of a vertex-weighted graph G as the smallest weight of a proper coloring of G (1997). If vertices of a graph have weight 1, its weighted chromatic number coincides with its chromatic number. Therefore, the problem of computing the weighted chromatic number is NP-complete in general graphs. This problem remains NP-complete in some particular graph classes as bipartite graphs.
In their seminal paper, Guan and Zhu asked whether the weighted chromatic number of bounded tree-width graphs (partial k-trees) can be computed in polynomial-time. Escoffier et al. designed a polynomial-time approximation scheme for computing the weighted chromatic number of partial k-trees (2006), and Kavitha and Mestre provided polynomial-time exact algorithms for sub-classes of trees (2009). Surprisingly, the time-complexity of computing this parameter in trees is still open. The Exponential Time Hypothesis (ETH) states that 3-SAT cannot be solved in sub-exponential time. We show that, assuming ETH, the best algorithm to compute the weighted chromatic number of n-node trees has time-complexity $n^{\Theta(\log n)}$. Our result mainly relies on proving that, when computing an optimal proper weighted coloring of a graph G, it is hard to combine colorings of its connected components, even when G is a forest.

Key-words: Weighted Coloring; Exponential Time Hypothesis; 3-SAT; Trees.

* This work was partly funded by the ANR projects AGAPE and GRATEL, and promoted by the INRIA/FUNCAP project ALERTE and the INRIA associate-team AlDyNet. Due to lack of space, most of the proofs have been sketched. The full paper may be found in [1].

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Coloration pondérée dans les arbres

Résumé : Nous prouvons que, en supposant qu'il n'existe aucun algorithme sous-exponentiel pour résoudre 3-SAT (ETH), alors le meilleur algorithme pour résoudre le problème de coloration pondérée dans les arbres a une complexité de $n^{\Theta(\log n)}$, où n est la taille de l'entrée.

Mots-clés : Coloration pondérée; ETH; 3-SAT; arbre

1 Introduction

Given a loop-less graph $G = (V, E)$, a *(proper) k-coloring* of G is a surjective function $c : V \to \{1, \cdots, k\}$ that assigns to each vertex $v \in V$ a *color* $c(v) \in \{1, \cdots, k\}$, such that, for any $\{u, v\} \in E$, $c(u) \neq c(v)$. Equivalently, a k-coloring of G is a partition $c = (S_1, \ldots, S_k)$ of V such that, for any $1 \leq i \leq k$, S_i is a non-empty independent set of vertices that have the same color i. One of the most studied problems in Graph Theory consists in minimizing the number of colors of a proper coloring of a graph. Namely, Graph Coloring aims at computing the *chromatic number* of a graph G, denoted by $\chi(G)$, which is the minimum k for which G has a k-coloring. This is one of the Karp's NP-hard problems [7].

In [5], Guan and Zhu generalized Graph Coloring to vertex-weighted graphs. A *(vertex) weighted graph* (G, w) consists of a loop-less graph $G = (V, E)$ and a weight function $w : V \to \mathbb{R}_+$ over the vertices of G. Given a k-coloring $c = (S_1, \ldots, S_k)$ of a weighted graph (G, w), the *weight of color* i $(1 \leq i \leq k)$ is defined by $w(i) = \max_{v \in S_i} w(v)$. The *weight of coloring* c is $w(c) = \sum_{i=1}^{k} w(i)$. The *weighted chromatic number* of (G, w), denoted by $\chi_w(G)$, is the minimum weight of a proper coloring of (G, w). The Weighted Coloring Problem takes a weighted graph (G, w) as input and aims at computing $\chi_w(G)$ [5].

Observe that if the weight of each of the vertices of a graph (G, w) is equal to one, then the weight of a coloring is the number of its colors and thus, $\chi_w(G) = \chi(G)$. Therefore, Weighted Coloring generalizes Graph Coloring to weighted graphs, and, as a consequence, this problem is NP-hard in general graphs. Moreover, Weighted Coloring has been shown NP-hard in bipartite graphs [3], where Graph Coloring is trivial. In the last years, the Weighted Coloring Problem has been addressed several times, however the complexity of this problem is surprisingly still unknown in the class of trees.

Here, we show that, if 3-SAT cannot be solved in sub-exponential time (Exponential Time Hypothesis), then Weighted Coloring in trees is not in P.

Related work. Guan and Zhu defined the Weighted Coloring Problem to model various problems of shared resources allocation. For instance, many problems consider a set of processes that use some common resources (memory, medium of communication, etc.) and aim at minimizing the total amount of used resources. Indeed, it is possible to mutualize resources: given a subset of processes that do not simultaneously use the resources, they can be assigned the same resources. The amount of resources necessary for this subset of processes is given by the process that uses the maximum amount of resources. Such problems can clearly be modeled as Weighted Coloring in a conflict graph where processes are the nodes, the weight of a node is the amount of resources required by the corresponding process, and two nodes are adjacent if and only if the corresponding processes use resources simultaneously. More precisely, Guan and Zhu studied practical applications such as the Dynamic Storage Allocation Problem and the Distributed Dual Bus Network Media Access Control Protocol, which is a standard IEEE802.6 for metropolitan networks [5].

Weighted Coloring has been shown to be NP-hard in the classes of split graphs, interval graphs, triangle-free planar graphs with bounded degree, and bipartite graphs [3, 10, 4]. On the other hand, the weighted chromatic number of cographs and of some subclasses of bipartite graphs can be found in polynomial-time [3, 10]. In their seminal paper, Guan and Zhu showed that, given a fixed parameter $r \in \mathbb{N}$, the minimum weight of a coloring using at most r colors can be computed in polynomial-time[1] in the class of bounded treewidth graphs (a.k.a. partial k-trees) [5]. They let open the question of the time-complexity of the Weighted Coloring Problem in this class (partial k-trees) and, in particular, in trees. Later on, Escoffier et al. proposed a polynomial-time approximation scheme to compute the weighted chromatic number of bounded treewidth

[1] We emphasis that this algorithm is exponential in r

graphs [4]. Kavitha and Mestre recently presented polynomial-time algorithms for subclasses of trees [8]. More precisely, they show that computing the weighted chromatic number can be done in linear time in the class of trees where nodes with degree at least three induce a stable set [8].

In the last years, many studies have been done on the WEIGHTED COLORING Problem, however the complexity of this natural problem was still unknown on trees. Indeed, WEIGHTED COLORING in trees has some intriguing properties: on the one hand, a reduction to another NP-hard problem was unlikely to exists due to the existence of a sub-exponential algorithm that we present in Section 2; on the other hand, all the classical methods to derive polynomial-time algorithms on trees failed [4, 8]. We provide here some explanation for these facts.

Our results. We show that, under the Exponential Time Hypothesis (ETH) (see Section 2), the best algorithm to compute the weighted chromatic number of trees has time-complexity $n^{\Theta(\log n)}$, where n is the number of vertices of the input tree.

The existence of an algorithm that solves the WEIGHTED COLORING Problem in time $n^{\Theta(\log n)}$ follows easily from previous results. The difficulty is to prove that it is optimal under ETH. For this purpose, we show that computing the weighted chromatic number of an n-node tree is as hard as deciding whether a 3-SAT formula with size $log^2 n$ can be satisfied. So, our reduction is rather complex, but we hope that it contains ideas that may be used in other contexts. Along the line of our reduction, one will discover another surprising aspect: the difficulty of the problem not only comes from the graph structure, but rather relies on the way weights are structured. This implies that choosing the right color for a node is hard. We indeed use non binary constraint satisfaction formulae as main tool. Lastly, our reduction also proves that computing an optimal weighted coloring of a disconnected graph may be hard even if optimal colorings of each of its components can be done in polynomial-time.

Organization of the paper. The reminder of the paper is organized as follows. In Section 2, we formally state the main results of the paper: in Section 2.1, an $n^{\mathcal{O}(\log n)}$-time algorithm is derived from previous works, and in Section 2.2 we prove our main result assuming a technical reduction (Proposition 2). The remaining part of the paper is devoted to the proof of Proposition 2. In Section 3, we give the main ideas of its proof. Finally, in Section 4, we prove the Proposition 3 which allows to prove Proposition 2.

2 Preliminaries

2.1 Sub-exponential algorithm

In this section, we show that there exists a sub-exponential algorithm to solve the WEIGHTED COLORING Problem in trees and more generally in the class of bounded treewidth graphs. Actually, this is an almost trivial consequence of previous works that mainly relies on the number of colors used by weighted colorings in this class of graphs.

It is easy to see that there exist weighted graphs G for which any optimal weighted coloring uses strictly more than $\chi(G)$ colors. For instance, let us consider the 4-node path P_4 with $V(P_4) = \{a, b, c, d\}$, $w(a) = w(d) = 4$ and $w(b) = w(c) = 1$ (see Figure 1). Any coloring of P_4 with $2 = \chi(P_4)$ colors has weight 8, and the optimal coloring $\{\{a, d\}, \{b\}, \{c\}\}$ of P_4 has weight $\chi_w(P_4) = 6$ but uses 3 colors.

Hopefully, the number of colors used by optimal weighted colorings can be bounded by $O(\log n)$ in the class of bounded treewidth graphs with n nodes. Indeed, in their seminal paper, Guan and Zhu studied the number of colors used by an optimal weighted coloring [5]. More precisely, they proved that the maximum number of colors of an optimal weighted coloring of a weighted graph (G, w) is its first-fit chromatic number $\chi_{FF}(G)$ (a.k.a., *Grundy number*) [5].

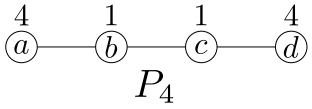

Figure 1: The (unique) optimal weighted coloring of the path P_4 uses strictly more than $\chi(P_4)$ colors.

Moreover, for any graph G, there exists a weight function w such that an optimal weighted coloring of (G, w) uses $\chi_{FF}(G)$ colors. On the other hand, for any n-node graph G with tree-width at most k, $\chi_{FF}(G) = \mathcal{O}(k \log n)$ [9]. In particular, this implies that, for any n-node tree, there is an optimal weighted coloring using $\mathcal{O}(\log n)$ colors. Finally, in the class of bounded treewidth graphs and when the number $r \in \mathbb{N}$ of colors is fixed, there is an algorithm (using dynamic programming on the tree-decomposition) that computes the minimum weight of a coloring using at most r colors in time $O(n^r)$ where n is the number of vertices of the input graph [5].

By combining these results, the following proposition is straightforward:

Proposition 1 *There exists an algorithm that solves the* Weighted Coloring *Problem in time $n^{\mathcal{O}(\log n)}$ in the class of bounded treewidth graphs, where n is the number of vertices of the input graph.*

2.2 Main Result

We now state formally our results that are subject to the Exponential Time Hypothesis.

Recall that an instance of the 3-SAT Problem is any Boolean formula $\Phi(v_1, \cdots, v_\eta)$ over the variables $v_1, \cdots, v_\eta$ in the conjunctive normal form (CNF) where each clause involves three variables. The *size* of Φ is η if it depends on η variables and its number of clauses is polynomial in η. The 3-SAT Problem aims at deciding whether there exists a truth assignment to the variables such that $\Phi(v_1, \cdots, v_\eta)$ is true. It is well known that the 3-SAT Problem is NP-complete [2]. A fundamental question is to know whether it can be solved in sub-exponential time.

Conjecture 1 (Exponential Time Hypothesis (ETH)) *3-SAT cannot be solved in time $2^{o(\eta)}$ where η is the size of the instance [6].*

Observe that proving ETH would imply that $P \neq NP$.

The main part of this paper is devoted to prove the following result.

Proposition 2 *For any Boolean formula Φ of size η, there exist a weighted tree (T, w) with $n = 2^{\mathcal{O}(\sqrt{\eta})}$ vertices and $M \in \mathbb{R}$ such that Φ is satisfiable if and only if $\chi_w(T) < M$. Moreover, (T, w) and M are computable in polynomial time in n, i.e., in time $2^{o(\eta)}$.*

Proposition 2 allow us to prove that there is no polynomial-time algorithm to solve the Weighted Coloring Problem in trees, unless ETH fails.

Theorem 2 *If ETH is true, then the best algorithm to compute the weighted chromatic number of a tree T has time-complexity $n^{\Theta(\log n)}$ where n is the number of nodes of T.*

Proof. The existence of such an algorithm directly follows from Proposition 1.

For purpose of contradiction, let us assume that there exists an algorithm $\mathcal{A}$ that solves the Weighted Coloring Problem in time $n^{o(\log n)}$ in the class of trees, where n is the number of vertices of the input tree.

Let Φ be any Boolean formula of size η. By Proposition 2, there exists a weighted tree (T, w) with $n = 2^{\mathcal{O}(\sqrt{\eta})} = 2^{o(\eta)}$ vertices and $M \in \mathbb{R}$ such that Φ is satisfiable if and only if $\chi_w(T) < M$.

Consider the following algorithm to solve 3-SAT. For any Boolean formula Φ of size η, first compute (T, w) and M in time $2^{o(\eta)}$, then use Algorithm $\mathcal{A}$ to compute $\chi_w(T)$ in time $n^{o(\log n)} = 2^{o((\log n)^2)} = 2^{o(\eta)}$. By definition, Φ is satisfiable if and only if $\chi_w(T) < M$.

Therefore, the above algorithm solves the 3-SAT Problem in time $2^{o(\eta)}$ where η is the size of the instance. This contradicts ETH. ■

The remaining part of the paper is devoted to the proof of Proposition 2.

3 Guideline of the proof of Proposition 2

Proposition 2 establishes a link between the Weighted Coloring Problem and 3-SAT. Our proof is quite technical, so this section is devoted to describe the main difficulties and ingredients of the proof. We also try to give some intuitions.

3.1 From boolean variables to integral variables

3.1.1 Weighted Coloring vs. SAT.

When computing a coloring of a weighted graph, we are facing choices of the following kind: to color a node v, either we use a color c that is already used without increasing its weight, i.e., c has already been assigned to a vertex with weight greater than v, or the color c used for v increases the global weight, i.e., either c has not been used yet or the weight of v is greater than the weights of nodes previously colored with c. For instance, in the example of Figure 1, once the first two colors have been assigned to a and b, we have to decide either to assign a new color to c which increases the global weight, or to use again the color of a. Of course, there are other ways to proceed, but this example illustrates a choice that consists in either increasing the weight of one color or not doing it. In the example, we can choose to increase the weight of a new color from 0 to $w(c)$ (by assigning this new color to c) or not to do it; this latter choice would however lead to a worse coloring. Intuitively, for a boolean formula Φ, we will build a tree such that the choices of the weights of the colors, in any coloring with small (depending on Φ) weight, will reflect the truth assignment of the variables of Φ.

Informally, to evaluate the time-complexity of the Weighted Coloring Problem, the ideal way would be to reduce any 3-SAT formula Φ to a weighted tree (T, w) such that (1) there is a correspondence between truth assignments of the variables of Φ and the optimal colorings of T, and (2) Φ is satisfiable if and only if $\chi_w(T)$ is at most some pre-defined value M (depending on Φ). To do such a reduction, we would like to proceed as follows: given a boolean formula Φ of size η, we build a tree T such that any truth assignment of Φ for which Φ is satisfied, we have a coloring of T or bounded weight, where the weight of a color reflects the truth assignment of a variable. However, proceeding that way, since the number of colors in an optimal weighted coloring of an n-node tree is at most $O(\log n)$, T must have at least $n = 2^\eta$ nodes. Hence, a polynomial-time algorithm to solve the Weighted Coloring Problem in T would only lead to an exponential-time algorithm for deciding whether Φ is satisfiable.

3.1.2 From 3-SAT to INT-SAT.

To bypass this problem, we will use an auxiliary formula with less choices. Intuitively, given a 3-SAT formula with η boolean variables, we will translate it into another logical formula with $\sqrt{\eta}$ *integral variables*. Using this new formula, we build a tree with $2^{\sqrt{\eta}}$ nodes, where the weights

of the colors in coloring of bounded weight will correspond to the integral values of the variables. More formally,

Definition 1 *Given a set of $n \times m$ boolean variables $(y_j^i)_{i<n,j<m}$, an integral assignment of these variables is a truth assignment such that, for any $0 \leq i < n$, at most one variable y_j^i, $\leq j < m$, receives value 1.*

A boolean formula Φ with $n \times m$ boolean variables $(y_j^i)_{i<n,j<m}$ is integrally satisfiable w.r.t. $(y_j^i)_{i<n,j<m}$ if there is an integral assignment of its variables that satisfies Φ.

The INT-SAT Problem takes a formula Φ with variables $(y_j^i)_{i<n,j<m}$ as input and aims at deciding whether Φ is integrally satisfiable w.r.t. $(y_j^i)_{i<n,j<m}$.

It is important to note that there is a one-to-one mapping between any integral assignment of a set of $n \times m$ boolean variables $(y_j^i)_{i<n,j<m}$ and the set of n-tuples $(x_1, \cdots, x_n)$ of integers in $\{0, \cdots, m\}$. Indeed, for any $i < n$, $x_i = j$ if and only if $y_j^i = 1$, and $x_i = 0$ if $y_j^i = 0$ for any $j < m$.

We now show that 3-SAT can be sub-exponentially reduced to INT-SAT. This is an important ingredient of the proof of Proposition 2. We also think this result has its own interest and could be used in other context.

Theorem 3 *For any instance Φ of 3-SAT with size η, there is a Boolean formula Φ_{int} of size $n = 2^{O(\sqrt{\eta})}$, with variables $(y_j^i)_{i<\sqrt{\eta},j<2\sqrt{\eta}}$, such that Φ is satisfiable if and only if Φ_{int} is integrally satisfiable w.r.t. $(y_j^i)_{i,j}$. Moreover, Φ_{int} can be computed in time n and it is a CNF formula where all variables appear positively.*

Proof. Let $\Phi(u_1, \cdots, u_\eta)$ be a instance of 3-SAT of size $\eta = N^2$ (if $\eta \neq N^2$, we can add dummy variables). For any two integers $a < N$ and $b < 2^N$, let $bit(a, b)$ correspond to the a-th bit of the binary representation of b.

Let Φ_{int} be the formula obtained from Φ by replacing each literal u_{iN+j}, $0 \leq i < N$ and $0 \leq j < N$, by $\bigvee_{\{\ell | bit(j,\ell)=1,\ 0 \leq \ell < 2^N\}} v_\ell^i$. Then, each literal $\overline{u}_{iN+j}$, $0 \leq i < N$ and $0 \leq j < N$ is replaced by $\bigvee_{\{\ell | bit(j,\ell)=0,\ 0 \leq \ell < 2^N\}} v_\ell^i$. Hence, Φ_{int} has $N \cdot 2^N$ variables $(v_0^1, \cdots, v_{2^N-1}^1, v_0^2, \cdots, v_{2^N-1}^2, \cdots, v_0^N, \cdots, v_{2^N-1}^N)$ and $poly(N)$ clauses of size $O(2^N)$. Because Φ is in CNF, it is also the case for Φ_{int}. Moreover, all variables appear positively in Φ_{int}.

It remains to show that Φ_{int} is integrally satisfiable if and only if Φ is satisfiable.

First, let us assume that Φ is satisfiable. Let $u_1, \cdots, u_\eta$ be a valid assignment of its variables and, for any $0 \leq i < N$, let x_i be the integer with $(u_{N(i-1)+1}, \cdots, u_{N(i-1)+N})$ as binary representation. Finally, for any $i < N$ and $j < 2^N$, let us define $v_j^i = 1$ if $x_i = j$ and $v_j^i = 0$ otherwise. By definition of Φ_{int}, $(v_j^i)_{0 \leq i < N,\ 0 \leq j < 2^N}$ is a valid assignment and Φ_{int} is therefore integrally satisfiable.

Conversely, let us assume that Φ_{int} is integrally satisfiable and let $(x_1, \cdots, x_N)$ be N integers representing a valid assignment for it. Let $u_1, \cdots, u_\eta$ be defined such that, for any $0 \leq i < N$, $(u_{N(i-1)+1}, \cdots, u_{N(i-1)+N})$ is the binary representation of x_i. Then, $u_1, \cdots, u_\eta$ is a valid assignment for Φ which is satisfiable. ■

3.1.3 Proof of Proposition 2.

Hence, the above Theorem allows us to reduce any 3-SAT instance Φ of size η into an INT-SAT instance Φ_{int} with size $2^{O(\sqrt{\eta})}$. The key point is that this reduction allows us to turn the choice of η boolean variables into the choice of $\sqrt{\eta}$ integers in $\{0, \cdots, 2^{\sqrt{\eta}}\}$. Then, in further

sections, we build a tree T with $2^{O(\sqrt{\eta})}$ vertices from the formula Φ_{int}, such that there is a one to one mapping between any optimal weighted coloring of T and the $\sqrt{\eta}$-tuples of integers in $\{0, \cdots, 2^{\sqrt{\eta}}\}$. Finally, our reduction ensures that the value of $\chi_w(T)$ depends on the integral satisfiability of Φ_{int} and therefore, on the satisfiability of Φ. More formally, the next sections are devoted to prove the following lemma:

Proposition 3 *For any CNF Boolean formula Φ_{int} of size n where all variables $(y^i_j)_{i,j}$ appear positively, there exist a weighted tree $(T(\Phi_{int}), w)$ with a number of vertices polynomial in $\mathcal{O}(n)$ and $M \in \mathbb{R}$ such that Φ_{int} is integrally satisfiable w.r.t. $(y^i_j)_{i,j}$ if and only if $\chi_w(T(\Phi_{int})) < M$.*

Moreover, $(T(\Phi_{int}), w)$ and M are computable in time polynomial in $O(n)$.

The proof of Proposition 2 is straightforward from Theorem 3 and Proposition 3.

3.2 Main ingredients of the proof of Proposition 3

In this section, we give the main ideas of the proof of Proposition 3. In particular, we describe the main property of the tree that we aim at building. The formal proof of it can be found in Section 4.

Let $n, m \in \mathbb{N}$. Let Φ_{int} be any CNF Boolean formula with variables $(y^i_j)_{i<n, j<m}$. Let $(T(\Phi_{int}), w)$ be a weighted tree, with $\mathcal{O}(n \cdot m)$ nodes and weights can be encoded with $\mathcal{O}(n \cdot m)$ bits. Let $M \in \mathbb{R}$, be a positive real value obtained from Φ_{int}. The precise definitions of $(T(\Phi_{int}), w)$ and M are given in next section.

Let $\mathcal{C}$ be the set of all colorings of $(T(\Phi_{int}), w)$ with weight at most M. We show in Section 4 that $\chi_w(T(\Phi_{int})) \leq M$ and, therefore, $\mathcal{C} \neq \emptyset$.

3.2.1 Weights structure and coloring constraints.

We ensure that the weights of the nodes of $T(\Phi_{int})$ belong to some particular set that can be partitioned into $O(n)$ classes. Our construction ensures that, for any coloring in $\mathcal{C}$ and for any class of weights, there is at most one color with weight in this class (see Corollary 1).

This property is very useful because it implies that, in any such coloring, each color is well identified by its weight. More precisely, any color of a coloring in $\mathcal{C}$ is well identified as the unique one having weight in some interval. In particular, when building a coloring, we can affect a color c to a node v if the weight of v is not larger than the maximum weight that can take color c (and, of course, if v is not adjacent to any node with color c).

The second important property ensured by the construction is that there is a correspondence between $\mathcal{C}$, i.e., the set of colorings of $(T(\Phi_{int}), w)$ with weight at most M, and Y, the set of the all integral assignments of the boolean variables $(y^i_j)_{i<n, j<m}$. More precisely, we will show that:

- from any integral assignment $y \in Y$, if y is valid for Φ_{int} (i.e., $\Phi_{int}(y)$ is true), then we can derive a coloring c_y of $T(\Phi_{int})$ with weight strictly less than M (Lemma 6);
- on the other hand, for any coloring with weight at most M, there is an integral assignment $y_c \in Y$ such that $\Phi(y_c)$ is true if and only if $w(c) < M$ (Lemma 7).

This is the cornerstone of the proof of Proposition 3. Indeed, if Φ_{int} is integrally satisfiable, then an integral assignment $y \in Y$ that satisfies it (i.e., such that $\Phi_{int}(y)$ is true) leads to a coloring c_y with weight strictly less than M. On the other hand, if Φ_{int} is not integrally satisfiable, for any coloring c with weight at most M, then y_c does not satisfy Φ_{int} and therefore $w(c) \geq M$. This means that, in that case, no coloring of weight strictly smaller than M exists.

3.3 A bit further into the proof of Proposition 3.

To conclude this section, we give some hints of the proof of the previous two properties.

3.3.1 A tree per boolean variable.

The tree $T(\Phi_{int})$ will be built from a family of subtrees $(T(y_i^j))_{i<n,j<m}$, each of which corresponds to a variable y_i^j ($i < n$, $j < m$) of the formula Φ_{int}. Then, these subtrees will be combined in order to represent Φ_{int}. More precisely, for any $i < n$, $j < m$, and for each occurrence of y_i^j in Φ_{int}, there will be a copy of a rooted tree T_i^j as a subtree of $T(\Phi_{int})$. We say that the root of such a subtree T_i^j is *representing* the variable y_i^j. Moreover, they will satisfy the following properties.

Let W_0 and W_1 be two particular colors that we will prove to be used in any coloring in $\mathcal{C}$ (recall that, we can univocally identify the colors by their weight).

- First, let $y \in Y$ be any integral assignment of the boolean variables $(y_j^i)_{i<n,j<m}$ of Φ_{int}. For any node v representing a variable $y_i^j = 1$, let c_v be arbitrarily chosen in $\{W_0, W_1\}$ (possibly, two roots of two distinct (isomorphic) subtrees representing the same variable receive distinct colors). Then, there is a coloring $c \in \mathcal{C}$ such that, for any v representing a variable $y_i^j = 0$, $c(v) = W_1$ and, for any v representing a variable $y_i^j = 1$, $c(v) = c_v$. This means that we can color $T(\Phi_{int})$, without exceeding a weight of M, and such that, for any $i < n$, $j < m$ such that $y_i^j = 1$, we can arbitrary choose the color of each node representing y_i^j to be either W_0 or W_1. Moreover, for any $i < n$, $j < m$ such that $y_i^j = 0$, the color of each node representing y_i^j is W_1.
- On the other hand, we will show that, for any coloring $c \in \mathcal{C}$, there exists an integral assignment $y \in Y$ of the boolean variables $(y_j^i)_{i<n,j<m}$ of Φ such that, for any node v representing a variable y_i^j, then $c(v) = W_1$ if $y_i^j = 0$, and $c(v) \in \{W_0, W_1\}$ otherwise (Lemma 3). Intuitively, it means that any coloring of $T(\Phi_{int})$ with weight at most M has a particular form that will be used to show that the coloring exceeds some weight when the formula is not satisfiable.

Intuitively, above properties say that a coloring of $T(\Phi_{int})$ with weight at most M can be seen as an integral assignment of the variables. For any node v representing a variable, having a choice to color v means that the corresponding variable is assigned to true, while if the color of v is forced (to be W_1), the corresponding variable is assigned to false.

3.3.2 Combining the subtrees.

We now briefly describe how $T(\Phi_{int})$ is obtained by combining the subtrees corresponding to the occurrences of the variables in Φ_{int}. First, recall that Φ_{int} is in Conjunctive Normal Form with all variables appearing positively. That is, each clause Q with $|Q|$ variables is of the form $\vee_{1\leq k\leq |Q|} u_k$ where $u_k \in \bigcup_{i<n,j<m}\{y_j^i\}$ for any $k \leq |Q|$. For any $1 < k \leq |Q|$, we recursively build a rooted subtree $T(Q^k)$ from the rooted subtree $T(Q^{k-1})$ and a copy of the rooted subtree representing u_k.

The key point is that, we can choose a coloring $c \in \mathcal{C}$ with the color of the root of $T(Q^k)$ being arbitrary W_0 or W_1 if and only if there is a coloring in $\mathcal{C}$ where either the root of $T(Q^{k-1})$ or the root of the subtree representing u_k can arbitrary have color W_0 or W_1. Otherwise, the color of the root of $T(Q^k)$ is forced to be W_1. That is, our construction simulates an OR where a true variable corresponds to a choice of a color, while a false variable corresponds to no choice.

By using this fact, the combination of these subtrees is done in such a way that we extend the properties of the subtrees representing the variables to the subtrees representing the clauses. Roughly, we ensure that, in any coloring $c \in \mathcal{C}$ corresponding to an assignment $y \in Y$, the root of the subtree representing the clause Q must have color W_1 if $Q(y) = 0$ and can be arbitrary chosen in $\{W_0, W_1\}$ otherwise.

To conclude, we have one tree per clause, such that, to any coloring in $\mathcal{C}$ corresponds an integral assignment such that the color of the root of the clause-trees is forced to be W_1 if the corresponding clause is false and can be arbitrary chosen in $\{W_0, W_1\}$ otherwise. Finally, we connect all the roots of the clause-trees to a same node r whose color, in any coloring $c \in \mathcal{C}$, must be either W_1 or a color W_3 not used yet. Moreover, using W_3 will imply that $w(c) \geq M$. Therefore, if at least one clause is false, W_3 is forced in r and $w(c) \geq M$. Finally, if all clauses are true, r can be colored W_1 and $w(c) < M$.

It is interesting to note that to optimally color the trees in the forest is easy and that the difficulty of the coloring of the final tree will arise from their combination.

4 Proof of Proposition 3

This section is devoted to the proof of Proposition 3. Due to lack of space, most of the proofs have been sketched. The complete proof can be found in [1].

Proposition 3 *Let $m = 2^n$. For any CNF Boolean formula Φ_{int} of size $O(nm)$ where all variables $(y_j^i)_{i \leq n, j \leq m}$ appear positively, there exist a weighted tree $(T(\Phi_{int}), w)$ with size polynomial in $O(nm)$ and $M \in \mathbb{R}$ such that Φ_{int} is integrally satisfiable w.r.t. $(y_j^i)_{i,j}$ if and only if $\chi_w(T(\Phi_{int})) < M$.*

Moreover, $(T(\Phi_{int}), w)$ and M are computable in time polynomial in $O(nm)$.

In order to prove Proposition 3, let us introduce some notations. Let $n \in \mathbb{N}$ and let $m = 2^n$.

Let Φ_{int} be a Boolean formula with $n \times m$ variables $\{y_i^j \mid 0 \leq i < n, 0 \leq j < m\}$ and L clauses, where L is polynomial in m. We assume that Φ_{int} is in the Conjunctive Normal Form and that each variable appears positively. Moreover, we may assume that each variable appears at least once. That is, $\Phi_{int} = \wedge_{\ell \leq L} Q_\ell$ and, for any $\ell \leq L$, Q_ℓ is the disjunction of $p_\ell \geq 1$ positive variables.

Let $\epsilon \in \mathbb{R}^*$ such that $nm\epsilon = o(\frac{1}{2^{4n}})$. Let

$$M = \sum_{i=0}^{4n+3} \frac{1}{2^i} + n(m-1)\epsilon < 2.$$

Let $w_i^j = 1/2^i + j\epsilon$, for any $0 \leq i \leq 4n+3$ and $0 \leq j < m$. Let $\mathcal{W} = \{w_i^j \mid 0 \leq i \leq 4n+3, 0 \leq j < m\}$ denote a set of weights. For any $0 \leq k \leq 3$, let $W_k = w_{4i+k}^0 = 1/2^{4n+k}$.

Finally, for any rooted tree T, let $r(T)$ denote its root. A rooted tree S is a *(proper) subtree* of a rooted tree T if there is an edge e of T such that S is the connected component of $T \setminus \{e\}$ that does not contain $r(T)$. We now define various subtrees required to build $(T(\Phi_{int}), w)$.

4.1 Binomial trees.

We now define a particular family of *binomial trees* B_i^j, $0 \leq i \leq 4n+2, 0 \leq j < m$. They will be used in the construction of $T(\Phi_{int})$. Their role is to force the color of most of the nodes in any coloring c of $T(\Phi_{int})$ with $w(c) \leq M$.

Definition 2 *For any $0 \leq i \leq 4n+2, 0 \leq j < m$,*
let B_i^j be the weighted rooted tree defined recursively as follows (see Figure 2):

- *if $i = 0$, then B_0^j has a unique node with weight w_0^j;*
- *otherwise, B_i^j has a root of weight w_i^j whose children are the roots of $B_0^0, B_1^0, \cdots, B_{i-1}^0$.*

Note that B_i^j has 2^i nodes.

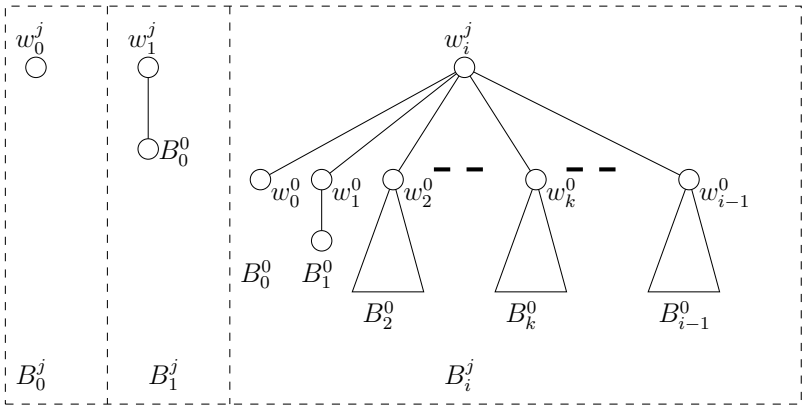

Figure 2: The construction of the binomial tree $B_{i,j}$.

Lemma 1 *For any $0 \leq i \leq 4n+2, 0 \leq j < m$, there is a unique coloring c of B_i^j with $w(c) \leq M$: $c = (S_0, \cdots, S_i)$ and, $\forall 0 \leq k \leq i$, S_k is the set of nodes with weight in $\{w_k^j \mid 0 \leq j < m\}$*

Sketch of proof. The proof is by induction on i. It uses the fact that, for any $0 \leq k < i$, the set of nodes with weight in $\{w_k^j \mid 0 \leq j < m\}$ is an independent set that dominates the nodes with smaller weights (i.e., in $\{w_{k'}^j \mid k < k' \leq i, 0 \leq j < m\}$). Finally, if $\forall 0 \leq k < i$, S_k is a set of nodes with weight in $\{w_k^j \mid 0 \leq j < m\}$, then S_i cannot contain nodes with weight $> w_i^{m-1}$, because otherwise, the weight of c would be at least $\frac{1}{2^i} + \sum_{k=0}^{i} \frac{1}{2^k} = 2 > M$. ■

Corollary 1 *Let (T, w) be any weighted tree having B_{4n+2}^0 as subtree. Let c be any coloring of (T, w) such that $w(c) \leq M$. Then, $c = (S_0, \cdots, S_k)$ with $k \geq 4n+2$ and, for any $0 \leq i \leq 4n+2$, S_i is the unique color with weight in $\{w_i^j \mid 0 \leq j < m\}$.*

The trees we consider below will always satisfy the requirements of Corollary 1. Therefore, we are able to identify a color by its weight. In other words, in what follows, for any coloring $c = (S_0, \cdots, S_k)$ of weight $\leq M$, for any $i \leq 4n+2$, S_i will be the unique color such that $w(S_i) \in \{w_i^j \mid 0 \leq j < m\}$. By extension, for any $0 \leq k \leq 2$, W_k will be the unique color with weight W_k.

4.2 Auxiliary trees and Variable-trees.

The family of *auxiliary trees* A_i^j, $0 \leq i < 4n, 0 \leq j < m$, will be used to introduce some choice when coloring $T(\Phi_{int})$.

Definition 3 *For any $0 \leq i < 4n, 0 \leq j < m$,*
let A_i^j be the weighted rooted tree defined as follows (see Figure 3(a)):

1. *let u be its root with weight $w(u) = W_0$, and connect it to a node v (its subroot) with weight $w(v) = w_i^j$;*
2. *v is made adjacent to the root of a copy of B_ℓ^0, for any $0 \leq \ell < i-1$;*
3. *u is made adjacent to the root of a copy of B_ℓ^0, for any $0 \leq \ell < 4n$, $\ell \neq i-1$.*

Note that A_i^j consists of 2^{4n} nodes.

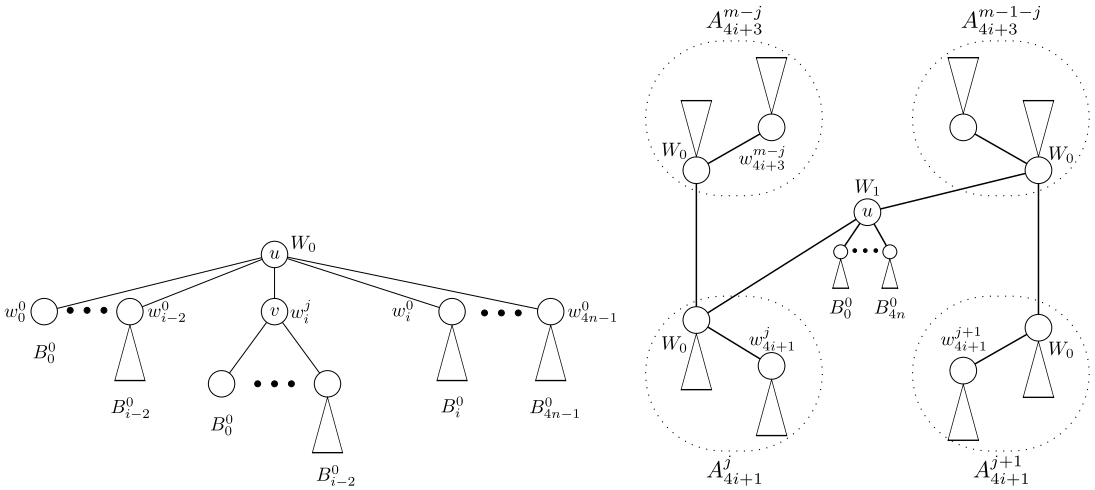

Figure 3: (a) (left) Auxiliary tree A_i^j. It is made from $B_0^0, \cdots, B_{i-2}^0, B_i^0, \cdots, B_{4n-1}^0$ and a subtree of B_i^j (i.e., $B_i^j \setminus B_{i-1}^0$). (b) (right) The variable tree $T(y_i^j)$, Case $0 \leq j < m-1$.

Lemma 2 *Let $0 \leq i < 4n, 0 \leq j < m$. Let (T, w) be any weighted tree having B_{4n+2}^0 and A_i^j as subtrees. Let c be any coloring of T with weight $w(c) \leq M$ (if any). Then, for any $0 \leq k < 4n$, there is one color S_k with weight in $\{w_k^j \mid 0 \leq j < m\}$. Moreover,*

- *either v is colored S_{i-1} and u must be colored with the color W_0*
- *or v is colored S_i (and therefore, $w(S_i) \geq w_i^j$) and u can be colored with S_{i-1}.*

Sketch of proof. The first statement mainly comes from the fact that A_i^j contains B_{4n-1}^0 and then, by Corollary 1, we can identify the colors and their weight. Then, since the root of each subtree B_k^0, $0 \leq k < 4n$, must be colored with S_k, the node v can be colored only with color S_{i-1} or S_i. Note that, if v is colored with color S_p for some $p > i$, then $w(S_p) \geq w_i^j$, contradicting the fact that $w(c) \leq M$. In the first case, u is adjacent to a node with color S_k, for any $k < 4n$. Therefore, v must be colored with color $S_{4n} = W_0$. Otherwise, u may be colored with color S_{i-1}. ■

Intuitively, the previous lemma states that, either we "pay" $j\epsilon$ in the weight of color S_i, or v must be colored with the color W_0. We now define the *variable-trees* $T(y_i^j)$ using the auxiliary trees.

Definition 4 *For any $0 \leq i < n, 0 \leq j < m$,*

let $T(y_i^j)$ be the weighted rooted tree, representing the variable y_i^j*, defined as follows (see Figure 3(b)):*

1. *let u be its root with weight $w(u) = W_1$ and connected to the root of a copy of B_ℓ^0, for any $0 \leq \ell < 4n$.*
2. *if $0 < j < m-1$, let us take one copy of $A_{4i+1}^j, A_{4i+1}^{j+1}, A_{4i+3}^{m-j}$ and A_{4i+3}^{m-1-j}.*
   - *connect $r(A_{4i+1}^j)$ to $r(A_{4i+3}^{m-j})$, and $r(A_{4i+1}^{j+1})$ to $r(A_{4i+3}^{m-1-j})$.*
   - *connect u with $r(A_{4i+1}^j)$ and $r(A_{4i+3}^{m-j-1})$.*
3. *if $j \in \{0; m-1\}$, take two copies of each of A_{4i+1}^j and A_{4i+3}^{m-1-j}.*
   - *connect one copy of $r(A_{4i+1}^j)$ to a copy of $r(A_{4i+3}^{m-j-1})$, and the other copy of $r(A_{4i+1}^j)$ to the other copy of $r(A_{4i+3}^{m-1-j})$.*
   - *connect u with the roots of the two copies of A_{4i+1}^j if $j = m-1$ (resp., of A_{4i+3}^{m-j-1} if $j = 0$).*

Note that $T(y_i^j)$ consists of $O(2^{4n})$ nodes.

Lemma 3 *Let (T, w) be any weighted tree having B_{4n+2}^0 as subtree and, for any $0 \leq i < n$ and any $0 \leq j < m$, T contains (at least) one copy of $T(y_i^j)$ as subtree.*

Let c be any coloring of T with weight $w(c) \leq M$ (if any). Then, there are $(j_0, \cdots, j_{n-1}) \in \{0, \cdots, m-1\}^n$ such that, for any $0 \leq i < n$ and any $0 \leq j < m$, and for any root u of a subtree $T(y_i^j)$, then

- *the color of u must be W_1 if $j \neq j_i$, and*
- *the color of u may be either W_0 or W_1 otherwise.*

Sketch of proof. Let c be any coloring with $w(c) \leq M$. Because T contains B_{4n+2}^0, by Corollary 1, $c = (S_0, \cdots, S_k)$ of weight $\leq M$ and, for any $i \leq 4n+2$, S_i is the unique color such that $w(S_i) \in \{w_k^j \mid 0 \leq j < m\}$. In particular, $w(c) \geq \sum_{i=0}^{4n+2} 1/2^i = M - n(m-1)\epsilon$.

For any $0 \leq i < n$, let j_i be such that $w(S_{4i+1}) = w_{4i+1}^{j_i}$. In particular, this means that any root x of a subtree A_{4i+1}^r ($j_i < r < m$) must be colored W_0 (Lemma 2). Therefore, by construction of the variable-trees, any root x' of a subtree A_{4i+3}^{m-r} ($j_i < r < m$) cannot be colored W_0 because x' is adjacent to a node x. By Lemma 2, this implies that $w(S_{4i+3}) \geq w_{4i+3}^{m-(j_i+1)}$.

Hence, for any $0 \leq i < n$, $w(S_{4i+3}) + w(S_{4i+1}) \geq w_{4i+1}^{j_i} + w_{4i+3}^{m-(j_i+1)} = (m-1)\epsilon + 1/2^{4i+1} + 1/2^{4i+3}$. Since $w(c) \leq M$, it implies that $w(S_{4i+3}) = w_{4i+3}^{m-j_i-2}$ for any $0 \leq i < n$, and $w(S_{2k}) = w_{2k}^0$, for any $0 \leq 2k < 4n$.

Let $0 \leq i < n$ and any $0 < j < m-1$ (the cases $j \in \{0; m-1\}$ are similar). Let us consider a subtree $T(y_i^j)$ of T. If $j \neq j_i$, then both the roots of A_{4i+1}^j and A_{4i+3}^{m-1-j} must be colored W_0 (otherwise, by Lemma 2, it would contradict the previous paragraph). In that case, the color of the root u of $T(y_i^j)$ must be W_1. Indeed, u is adjacent to the root of B_k^0, $0 \leq k \leq 4n$, and therefore cannot use any color S_k, and, moreover, if u use color W_2, then $w(c) > M$ because $w(u) = W_1$.

Similarly, if $j = j_i$, none of the roots of A_{4i+1}^j and A_{4i+3}^{m-1-j} are colored W_0 (for instance, if the root of A_{4i+1}^j is colored W_0, then the copy of A_{4i+3}^{m-1-j} that is adjacent to it would imply

$w(S_{4i+3}) \geq w_{4i+3}^{m-j_i-1}$, a contradiction). Hence, the color of the root u of $T(y_i^j)$ may be either W_0 or W_1. ■

4.3 Clause-trees and definition of $T(\Phi_{int})$.

Definition 5 *Let $Q_\ell = \vee_{1 \leq k \leq p_\ell} u_k$ be any clause of Φ_{int} ($\ell \leq L$).*

For any $1 \leq k \leq p_\ell$, let $T(Q_\ell^k)$ be the rooted weighted tree defined recursively as follows (see Figure 4(a)):

1. *$T(Q_\ell^1) = T(u_1)$ (recall that, for any $1 \leq k \leq p_\ell$, $u_k \in \{y_i^j \mid 0 \leq i < n, 0 \leq j < m\}$);*
2. *for any $k > 1$, start with one copy of $T(Q_\ell^{k-1})$ with root a and one copy of $T(u_k)$ with root b.*

   Let c, d be two nodes with weight W_2 and e, f be two nodes with weight W_3. For each node $v \in \{a, b, c, d\}$, and for any $0 \leq i \leq 4n$, add one copy of B_i^0 and make its root adjacent to v. Add one copy of B_{4n+1}^0 and make its root adjacent to e.

   Finally, we add the edges $\{\{a, f\}, \{b, c\}, \{c, f\}, \{d, e\}, \{e, f\}\}$ and d is chosen as the root.

Let us note $T(Q_\ell) = T(Q_\ell^{p_\ell})$ the clause-tree corresponding to Q_ℓ and that consists of $O(p_\ell 2^{4n})$ nodes (by previous Lemmas).

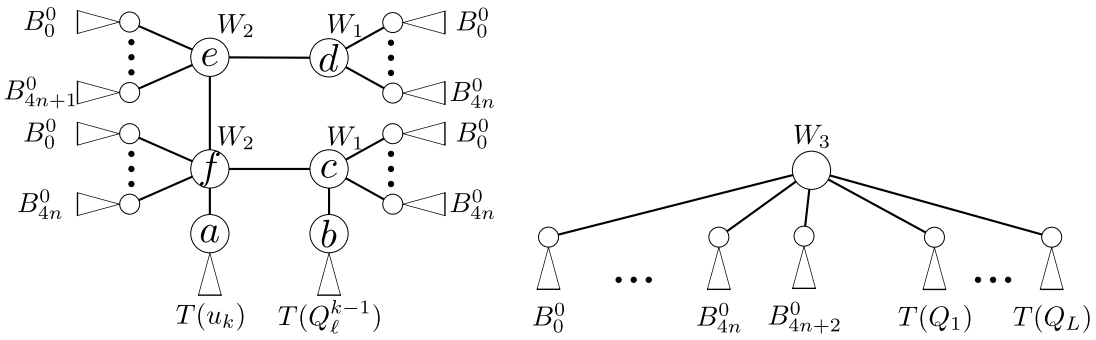

Figure 4: (a) (left) The clause tree $T(Q_\ell^k)$. (b) (right) The final tree $T(\Phi_{int})$.

Lemma 4 *Let (T, w) be any weighted tree having B_{4n+2}^0 as subtree and containing a subtree $T(Q_\ell^k)$ as a subtree ($\ell \leq L$, $k \leq p_\ell$). Let c be any coloring of T with weight $w(c) \leq M$ (if any). Then,*

- *if the color of a and b are identical, then the color of the root d of $T(Q_\ell^k)$ must be W_1;*
- *otherwise, the color of the root d of $T(Q_\ell^k)$ may be either W_0 or W_1.*

Sketch of proof. By Lemma 3, the color of a is either W_1 or W_0. By induction on $k \leq p_\ell$, the color of b is also either W_0 or W_1. Then, the result follows a simple case analysis. ■

Definition 6 *Let $T(\Phi_{int})$ be the weighted rooted tree obtained as follows (see Figure 4(b)). Let r be the root with weight W_3. For any $1 \leq \ell \leq L$, the root of one copy of $T(Q_\ell)$ is made adjacent to r. For any $0 \leq i \leq 4n+2$, $i \neq 4n+1$, r is made adjacent to the root of one copy of B_i^0.*

Lemma 5 *$T(\Phi_{int})$ has size polynomial in $m = 2^n$.*

Sketch of proof. Because each clause-tree $T(Q_\ell)$ has size $O(p_\ell 2^{4n})$ (see Definition 5) where p_ℓ polynomial in m (since $p_\ell \leq nm$, the number of variables). Moreover, the number L of clauses is polynomial in m by hypothesis. ■

Lemma 6 *If Φ_{int} is integrally satisfiable, then $\chi_w(T(\Phi_{int})) < M$.*

Sketch of proof. Let $(y_i^j)_{i<n,j<m}$ be a valid integral assignment for Φ_{int}. For any $0 \leq i < n$, let j_i be the (unique) index such that $y_i^{j_i}$ is true.

By Lemma 1 and because we want a coloring c with $w(c) \leq M$, the colors of all nodes of the binomial subtrees of $T(\Phi_{int})$ are imposed. Consequently, we only need to decide the colors of the following nodes: the roots and sub-roots of any copy of A_i^j (for all i and j), the roots of the trees $T(y_i^j)$, and the nodes added to connect the variables-trees into clause-trees (the nodes a, b, c, d, e, f in Figure 4(a)).

To do so, we first set the weight of color S_i for any $0 \leq i < 4n$. In particular, for any $0 \leq i < n$, the color S_{4i+1} must have weight w_{4i+1}^j. As in the proof of Lemma 3, this fixes almost everything. In particular, it is possible and it implies that all roots of subtrees $T(y_i^j)$ with $y_i^j = 0$ must receive color W_1. On the other hand, we still have the choice between W_0 and W_1 for all roots of subtrees $T(y_i^j)$ with $y_i^j = 1$.

We aim at fixing these colors such that the roots of all the clause-trees can be colored W_0. This is possible since $(y_i^j)_{i<n,j<m}$ is a valid integral assignment and using Lemma 4 (by induction on the length of a clause). To conclude, we can extend the coloring to the root of $T(\Phi_{int})$ with the color W_1.

We can check that the given coloring has weight $w(c) = \sum_{i=0}^{4n+2} \frac{1}{2^i} + n(m-1)\epsilon < M$. ■

Lemma 7 *If Φ_{int} is not integrally satisfiable, then $\chi_w(T(\Phi_{int})) \geq M$.*

Sketch of proof. Let c be any coloring of $T(\Phi_{int})$ with weight at most M. By Lemma 3, there are integers $(j_0, \cdots, j_{n-1})$ such that the color of the root of any subtree $T(v_i^j)$ is forced to be W_1 if $j \neq j_i$. Let $Y = (y_i^j)_{i<n,j<m}$ be the corresponding integral assignment. In other words, for any variable y_i^j $(0 \leq i < n, 0 \leq j < m)$, $y_i^j = 0$ if $j \neq j_i$.

Since Φ_{int} is not integrally satisfiable, there is a clause Q that is not satisfied by this assignment. Let us consider the clause-subtree $T(Q)$. It has been built from variable-trees corresponding to the variables constituting the clause Q. Because all these variables are assigned to false, the roots of these variable-trees are all colored with W_1.

By a simple induction (on the length of Q) using Lemma 4, the color of the root of $T(Q_\ell)$ must be W_1. Thus, the root of $T(\Phi_i nt)$ can just be colored W_3.

We can check that the given coloring has weight $w(c) = \sum_{i=0}^{4n+3} \frac{1}{2^i} + n(m-1)\epsilon = M$. ■

Proposition 3 follows directly from Lemmas 5, 6 and 7.

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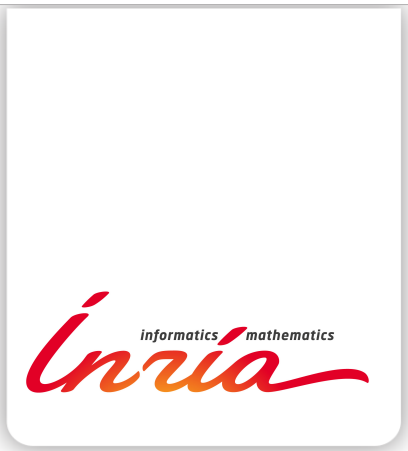

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