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# Stability and Observer Design for a Class of Nonsmooth Dynamical Systems with Nonconvex State Constraints

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**Abstract:** This paper deals with the stability and observer design for Lur'e systems with multivalued nonlinearities, which are not necessarily monotone or time-invariant. Such differential inclusions model the motion of state trajectories which are constrained to evolve inside time-varying non-convex sets. Using Lyapunov-based analysis, sufficient conditions are proposed for local stability in such systems, while specifying the basin of attraction. If the sets governing the motion of state trajectories are moving with bounded variation, then the resulting state trajectories are also of bounded variation, and unlike the convex case, the stability conditions depend on the size of jumps allowed in the sets. Based on the stability analysis, a Luenberger-like observer is proposed which is shown to converge asymptotically to the actual state, provided the initial value of the state estimation error is small enough. In addition, a practically convergent state estimator, based on the high-gain approach, is designed to reduce the state estimation error to the desired accuracy in finite time for larger initial values of the state estimation error. The two approaches are then combined to obtain global asymptotically convergent state estimates.

**Key-words:** Nonsmooth systems, sweeping process, prox-regular sets, locally bounded variation, Lyapunov stability, observer design, differential measure.

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## Stabilité et Observateurs pour les Systèmes Nonsmooth avec les Contraintes Nonconvexes.

**Résumé :** This paper deals with the stability and observer design for Lur'e systems with multivalued nonlinearities, which are not necessarily monotone or time-invariant. Such differential inclusions model the motion of state trajectories which are constrained to evolve inside time-varying non-convex sets. Using Lyapunov-based analysis, sufficient conditions are proposed for local stability in such systems, while specifying the basin of attraction. If the sets governing the motion of state trajectories are moving with bounded variation, then the resulting state trajectories are also of bounded variation, and unlike the convex case, the stability conditions depend on the size of jumps allowed in the sets. Based on the stability analysis, a Luenberger-like observer is proposed which is shown to converge asymptotically to the actual state, provided the initial value of the state estimation error is small enough. In addition, a practically convergent state estimator, based on the high-gain approach, is designed to reduce the state estimation error to the desired accuracy in finite time for larger initial values of the state estimation error. The two approaches are then combined to obtain global asymptotically convergent state estimates.

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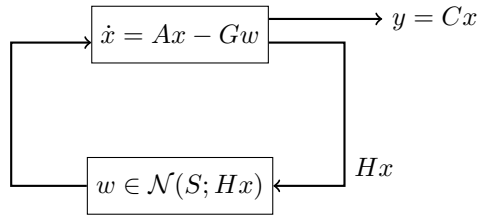


Figure 1: Lur'e systems with multivalued nonlinearities in feedback.

## 1 Introduction

Lur'e systems comprise an asymptotically stable linear system with an additive nonlinearity in the feedback loop as shown in Figure 1. Due to the practical significance, the problem of stability in such systems has received much attention in the literature. The basic question is: for what kind of nonlinearities, the resulting feedback system is asymptotically stable? For the most part, researchers have assumed that the nonlinearities are monotone and satisfy certain sector condition. However, if the monotonicity assumption is relaxed for the nonlinearities, then one can obtain certain sufficient conditions on linear dynamics which render the system stable. Our paper follows this line of thought, by addressing the stability and observer design for a certain type of Lur'e systems where nonlinearities belong to the normal cones associated with time-varying non-convex sets. Such multivalued nonlinearities don't necessarily satisfy monotone condition and may induce jumps in the state trajectory.

The kind of models considered in this paper could also be classified as the *sweeping processes*, introduced in [36] (see [29] for tutorial exposition). Such models have found several applications in non-smooth dynamics such as impact mechanics [34], electrical circuits [1], and could be used to generalize several existing classes of discontinuous dynamical systems such as linear complementarity systems (LCS) [10]. The perturbed version of Moreau's sweeping process is described by the following differential inclusion:

$$-\dot{x}(t) + f(t, x(t)) \in \mathcal{N}(S(t); x(t)), \quad \text{for almost all } t \in [t_0, \infty), \quad (1a)$$

$$x(t_0) = x_0 \in S(t_0), \quad (1b)$$

where  $S : [t_0, \infty) \rightrightarrows \mathbb{R}^n$  is a set-valued map, and  $\mathcal{N}(S(t); x(t))$  denotes the Fréchet normal cone to the set  $S(t) \subset \mathbb{R}^n$  at a point  $x(t) \in S(t)$ . Intuitively speaking, the aforementioned differential inclusion states that the state  $x(t)$  moves in the direction given by  $f(t, x(t))$  when  $x(t)$  is in the interior of the set  $S(t)$  since, in that case,  $\mathcal{N}(S(t); x(t)) = \{0\}$ . However, when  $x(t)$  is on the boundary of  $S(t)$ , the Fréchet normal at  $x(t)$  is no longer just zero, and it acquires the value from the normal cone defined at that boundary point so that  $x(t)$  stays inside the set  $S(t)$ , for all  $t \geq t_0$ .

The important considerations in solving the differential inclusion (1) are the geometry of sets  $S(t)$ , and the variation (measured using Hausdorff metric) of these sets with time. The case where  $S(t)$  is convex renders itself amenable to analysis; see [7, 8, 9] for well-posedness and our preliminary results on observer design for such cases. However, in this paper, we will assume that there exists an  $r > 0$  such that  $S(t)$  is an *r-prox-regular* set and closed for each  $t \geq t_0$ . Some simple examples include: the complement of an open ball of radius greater than  $r$  in  $\mathbb{R}^n$ ; union of disconnected closed intervals on the real line separated by a distance greater than  $2r$ ; and sub-level sets of any twice differentiable function. Every convex set is *r-prox-regular* where one can take  $r$  to be arbitrarily large. Several characterizations of prox-regular sets appear in [40], which

indicate that a set  $S$  is  $r$ -prox-regular if and only if there exists a unique  $\bar{z} := \arg \min_{z \in S} |y - z|$  in  $S$  for every  $y \in \mathbb{R}^n$  satisfying  $\min_{z \in S} |y - z| < r$ . An application of differential inclusions with prox-regular sets appears in [33] for modeling the motion of a crowd where the agents in the crowd are required to keep a certain distance among each other, say  $2r$ . If we lump the positions of  $N$  agents in the crowd into a single vector, say  $x$ , then this vector must belong to a prox-regular set  $\{x \in \mathbb{R}^N : |x_i - x_j| > r, 1 \leq i, j \leq N, i \neq j\}$ . One can see the application of prox-regular sets in optimal control problems [16], as they appear to be the first step in studying problems in non-convex setup. Solution concepts for differential inclusions of type (1), where  $S(t)$  is assumed to be  $r$ -prox-regular, are discussed in [10, 16, 17], and we recall some of them later on.

This paper first addresses the problem of stability for the perturbed Moreau's sweeping process described by (1), where we fix  $f(t, x(t)) := Ax(t)$ . Depending on how the function  $S(\cdot)$  evolves, the state  $x(\cdot)$  evolves accordingly. As a simpler case, we first consider the case where  $S(\cdot)$  varies in an absolutely continuous manner which allows the resulting state trajectory to be absolutely continuous. The major difference in the stability analysis, compared to the case where  $S(\cdot)$  is convex-valued, is that the mapping  $x \mapsto \mathcal{N}(S(t); x)$  is *maximal monotone* when  $S(t)$  is convex, which is not the case for a prox-regular set  $S(t)$ , even though the set  $\mathcal{N}(S(t); x)$  is convex. Unlike the global stability obtained in the convex case, the region of stability in an  $r$ -prox-regular case is determined by  $r$ . Nonetheless, we recover the results for convex sets as a specific case of the results presented.

We then address the problem of designing observers for such systems. Observer design using passivity approach for single valued Lur'e systems was considered in [3, 18] and later generalized to Lur'e systems with multivalued nonlinearity by [8, 9, 39]. Similar approach has been adopted for observer design in linear complementarity systems by [22], and several generalizations of these results have appeared recently in [4, 12, 26, 27]. The stability results developed in this paper are utilized to design a state estimator where we show that the dynamics of state estimation error are asymptotically stable under certain conditions. We remark that, compared to the earlier work on observer design [9], the major contribution of this work is to study the observer problem for non-convex sweeping processes within the general solution framework comprising functions of bounded variations (which allow state jumps with Zeno behavior), and to derive stability criteria for error dynamics that depend on the parameter  $r$  which, roughly speaking, measures the lack of convexity.

The paper is organized as follows: In Section 2, we give a formal definition of the prox-regular set along with some of its properties, and recall the result on existence and uniqueness of solution for system (1) when the system admits absolutely continuous state trajectories. For such systems, the problems of stability and observer design are studied in Section 3. The conditions for stability appear in Section 3.1, followed by a two-stage observer design in Section 3.2. In Section 4, we develop parallel results when the trajectories of the system are of bounded variation, and hence possibly discontinuous. Sufficient conditions for stability and the results on observer design appear in Section 4.2 and Section 4.3, respectively. After some concluding remarks, we have collected the proofs of some technical results in the Appendix.

## 2 Preliminaries

In this section, we introduce the prox-regular sets and derive some properties which will be used in the development of main results later. Before proceeding with the formal treatment, we introduce some standard notation that will be used throughout the paper.

**Notations:** We denote the Euclidean norm of a vector  $x \in \mathbb{R}^n$  by  $|x|$  and the induced norm of a matrix  $A$  with  $\|A\|$ . The maximum and minimum eigenvalues of a matrix  $P$  are denoted by  $\lambda_{\max}(P)$  and  $\lambda_{\min}(P)$ , respectively. For a matrix  $H$ ,  $\sigma_H$  denotes its least positive singular value. The preimage of a set  $S \subseteq \mathbb{R}^l$  under a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^l$  is denoted by  $f^{-1}(S)$ , that is,  $f^{-1}(S) := \{z \in \mathbb{R}^n \mid f(z) \in S\}$ , and the range of  $f(\cdot)$  is defined as  $\text{range } f := \{y \in \mathbb{R}^l \mid f(z) = y \text{ for some } z \in \mathbb{R}^n\}$ . Let the total variation of a function  $f : [t_0, T] \rightarrow \mathbb{R}^n$  be defined for  $t_0 \leq t \leq T$  as  $\text{var}_f(t) = \sup \sum_{i=1}^k |f(t_i) - f(t_{i-1})|$ , where the supremum is taken over all integers  $k$ , and all possible choices of the sequence  $\{t_i\}$  such that  $t_0 < t_1 < \dots < t_k = t$ . For an interval  $I \subseteq \mathbb{R}$ , the function  $f : I \rightarrow \mathbb{R}$  is said to be of bounded variation (BV) on  $I$  if  $\text{var}_f(t) < \infty$ , for all  $t \in I$ . If it is right continuous with bounded variation we denote it as *rcbv*. We say that  $f(\cdot)$  is locally *rcbv* if it is right-continuous and has bounded variation on every bounded interval  $[t_0, T] \subset I$ . If  $f(\cdot)$  is BV on  $I$  then it has at most a countable number of jump discontinuities. Moreover, it has right and left limits everywhere. The right and left limits of a function at  $t$  are denoted by  $f(t^+) := \lim_{s \searrow t} f(s)$  and  $f(t^-) := \lim_{s \nearrow t} f(s)$ , respectively, provided they exist. In this notation, right continuity of  $f(\cdot)$  in  $t$ , means that  $f(t^+) = f(t)$ .

The distance between a point  $x \in \mathbb{R}^n$  and a set  $S \subseteq \mathbb{R}^n$  is denoted by  $d(x, S) := \inf_{z \in S} |x - z|$  and we let  $\text{prox}_S(x) := \arg \min_{z \in S} |x - z|$ . For any two sets  $S, S' \subset \mathbb{R}^n$ , we denote their Hausdorff distance by  $d_H(S, S')$  which is defined as:

$$d_H(S, S') := \max\left\{\sup_{x \in S'} d(x, S), \sup_{x \in S} d(x, S')\right\}. \quad (2)$$

The variation of a set-valued map  $S : [t_0, T] \rightarrow \mathbb{R}^l$  at time  $t$ , denoted by  $v_S(t)$ , is obtained by replacing  $|f(t_i) - f(t_{i-1})|$  with  $d_H(S(t_i), S(t_{i-1}))$  in the definition of  $\text{var}_f(t)$ , that is,

$$v_S(t) := \sup \sum_{i=1}^k d_H(S(t_i), S(t_{i-1}))$$

where the supremum is taken over the set of all partitions of  $[t_0, T]$ . We denote by  $\mathcal{L}^1(I, \mathbb{R}^n; \nu)$  and  $\mathcal{L}_{loc}^1(I, \mathbb{R}^n; \nu)$  the space of integrable and locally integrable functions, respectively, from the interval  $I$  to  $\mathbb{R}^n$  with respect to the measure  $\nu$ . If the measure is not specified then the integration is with respect to the Lebesgue measure. An absolutely continuous (AC) function  $f : [t_0, T] \rightarrow \mathbb{R}^n$  is a function that can be written as  $f(t) - f(t_0) = \int_{t_0}^t \dot{f}(s) ds$  for any  $t_0 \leq t \leq T$ , and some  $\dot{f}(\cdot) \in \mathcal{L}^1([t_0, T], \mathbb{R}^n)$ , which is considered as its derivative.

## 2.1 Non-convex Analysis

To formally define the notion of a normal cone for non-convex sets, the definition of normal cone from convex analysis is extended as follows.

**Definition 1** (Fréchet Normals [35]). For a closed set  $S \subset \mathbb{R}^n$ , and  $x \in S$ , the vector  $w \in \mathbb{R}^n$  is called a Fréchet normal to the set  $S$  at  $x$  if, for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\langle w, x' - x \rangle \leq \epsilon |x' - x| \quad \forall x' \in S, \quad |x - x'| < \delta. \quad (3)$$

The set of all Fréchet normals at a point  $x \in S$  form a cone denoted by  $\mathcal{N}(S; x)$ .

**Definition 2** (Uniformly Prox-Regular Set [13, 40]). A set  $S$  is called uniformly prox-regular with constant  $1/r$ , or simply  $r$ -prox-regular, if for each  $x \in S$ , and each  $w \in \mathcal{N}(S; x)$  with  $|w| < 1$ , it holds that  $\text{prox}_S(x + rw) = \{x\}$ , that is,  $x$  is the unique nearest vector to  $x + rw$  in the set  $S$ .



Thus, it follows from the definition that  $S$  is an  $r$ -prox-regular set, if and only if, for each  $x, x' \in S$ , and each  $w \in \mathcal{N}(S; x)$ , with  $|w| < 1$ , we have

$$|rw|^2 = |x + rw - x|^2 < |x + rw - x'|^2 = |x - x'|^2 + 2 \langle rw, x - x' \rangle + |rw|^2,$$

or equivalently for each  $w \in \mathcal{N}(S; x)$ ,

$$\left\langle \frac{w}{|w|}, x - x' \right\rangle \geq -\frac{1}{2r} |x - x'|^2, \quad \forall x' \in S. \quad (4)$$

In the above inequality, if we let  $r \rightarrow \infty$ , then the expression on the right-hand side becomes zero and we see that  $w$  is the normal vector at  $x \in S$  in the classical sense of convex analysis. For that reason, we say that the case  $r \rightarrow \infty$  corresponds to  $S$  being convex. In our development, the convex sets will be treated as a particular case of the  $r$ -prox-regular sets by taking  $r \rightarrow \infty$ .

**Remark 1.** For closed sets, one may find various notions of normals such as proximal normals, Clarke normals, and limiting (Mordukhovich) normals. It has been proven that, see for example [5, Theorem 3.2], these notions of normals coincide in case the set is uniformly prox-regular. So we choose to denote the normal cone simply by  $\mathcal{N}(S; x)$ , implicitly assuming that it may refer to any of the existing notion of a normal cone.

We now recall a fundamental result from non-convex analysis that gives a characterization of the cone normal to the preimage of a set under a constraint qualification. It is used later in rewriting the system dynamics under a coordinate transformation. For proof, see [41, Theorem 6.14].

**Lemma 1** (Chain rule). *Consider a nonempty, closed,  $r$ -prox-regular set  $S \subseteq \mathbb{R}^l$ ,  $r > 0$ , and a linear map  $H : \mathbb{R}^n \rightarrow \mathbb{R}^l$ , so that  $S \subseteq \text{range } H$ . Let  $S' := H^{-1}(S)$ , and assume that the following constraint qualification holds:*

$$\text{For each } z \in S' \text{ and } w \in \mathcal{N}(S; Hz), H^\top w = 0 \text{ only if } w = 0. \quad (5)$$

Then, for each  $z \in \mathbb{R}^n$ , and  $v = Hz$ , it holds that

$$\mathcal{N}(S'; z) := \{H^\top w \mid w \in \mathcal{N}(S; v)\} = H^\top \mathcal{N}(S; Hz). \quad (6)$$

**Remark 2.** If  $H$  has full row rank, then the constraint qualification (5) holds automatically. However, (5) does not imply that  $H$  has full row rank. An equivalent way of expressing (5) is

$$\ker(H^\top) \cap \mathcal{N}(S; Hz) = \{0\}, \quad \forall z \in H^{-1}(S).$$

Two further properties of the prox-regular sets required for stability analysis appear in Lemma 3 and Lemma 4a. These properties are derived using the metric regularity of a linear surjective map stated in the following lemma:

**Lemma 2.** *Consider a linear map  $H : \mathbb{R}^n \rightarrow \mathbb{R}^l$  and a nonempty closed set  $S \subseteq H$ , then it holds that*

$$d(x, H^{-1}(S)) \leq \frac{1}{\sigma_H} d(Hx, S), \quad \text{for } x \in \mathbb{R}^n, \quad (7)$$

where  $\sigma_H$  denotes the least positive singular value of  $H$ .

To arrive at (7), we recall from [15, Example 1.1] that the regularity modulus of a linear mapping is  $\sup\{d(0, H^{-1}y) \mid y \in \mathbb{R}^l, |y| = 1\}$ . Comparing this characterization to the definition of the singular values, we immediately obtain (7). A more direct proof is worked out in Appendix A.

The next result states that the preimage of a prox-regular set under a metric regular map is also prox-regular. The proof, which is formally developed in Appendix A, is based on relating the normal cone to a set with the subdifferential of the distance function to that set and using the chain rule for subdifferentials.

**Lemma 3.** *Consider a nonempty, closed,  $r$ -prox-regular set  $S \subset \mathbb{R}^l$ ,  $r > 0$ , and a linear map  $H : \mathbb{R}^n \rightarrow \mathbb{R}^l$ , so that  $S$  is in the range space of  $H$ . Then the set  $S' := H^{-1}(S)$  is uniformly  $r'$ -prox-regular with  $r' := r\sigma_H/\|H\|^2$ .*

The last lemma in this section states how the variation of a set changes under the preimage of a metric regular map, and is also proved in Appendix A.

**Lemma 4a.** *For a multivalued function  $S : [t_0, \infty) \rightrightarrows \text{range } H$ , assume that  $v_S(\cdot)$  is locally absolutely continuous. Let  $S'(t) := H^{-1}(S(t))$ , then  $v_{S'}(\cdot)$  is also locally absolutely continuous and furthermore,  $\dot{v}_{S'}(t) \leq \frac{1}{\sigma_H} \dot{v}_S(t)$ , for Lebesgue almost all  $t \in [t_0, \infty)$ .*

## 2.2 Solution Concepts

To study the solution of system (1), we consider  $S : [t_0, \infty) \rightrightarrows \mathbb{R}^n$  and introduce the following hypotheses:

- (H1) There exists a constant  $r > 0$  such that, for each  $t \in [t_0, \infty)$ ,  $S(t)$  is a non-empty, closed and  $r$ -prox-regular set.
- (H2) The function  $v_S(\cdot)$  is locally absolutely continuous and  $|\dot{v}_S(t)|$  is bounded by  $v$  for all  $t$  except for a set of Lebesgue measure zero, i.e.,  $\text{ess sup}_{t \geq t_0} |\dot{v}_S(t)| = v$ .

Let us state a general result on the existence of solutions for perturbed Moreau's sweeping process (1).

**Theorem 1** (AC solutions [16, Theorem 1]). *Consider system (1) over some interval  $I \subseteq [t_0, \infty)$  and assume that  $f(t, x)$  satisfies the following assumptions:*

- $f(\cdot, x)$  is a Lebesgue-measurable function for each  $x \in \mathbb{R}^n$ ,
- $f(t, \cdot)$  is Lipschitz continuous for each  $t \in I$ , and
- There exists  $\alpha(\cdot) \in \mathcal{L}^1(I, \mathbb{R})$  such that  $|f(t, x)| \leq \alpha(t)(1 + |x|)$  for all  $x \in \cup_{s \in I} S(s)$ .

*Then under the hypotheses (H1) and (H2), for each  $x_0 \in S(t_0)$ , there exists a unique locally absolutely continuous solution  $x(\cdot)$  that satisfies (1), and that  $x(t) \in S(t)$  for all  $t \in I$ . Furthermore, the following bound holds:*

$$|\dot{x}(t) + f(t, x(t))| \leq |f(t, x(t))| + |\dot{v}_S(t)| \quad \text{a.e. } t \in I. \quad (8)$$

**Remark 3.** In the formulation of Theorem 1 in [16], the bound on right-hand side in (8) was expressed differently in terms of certain system parameters. However, the more compact expression used in (8) appears in [16, page 358, eq. (3.15)], which we find more suitable for results in this paper.

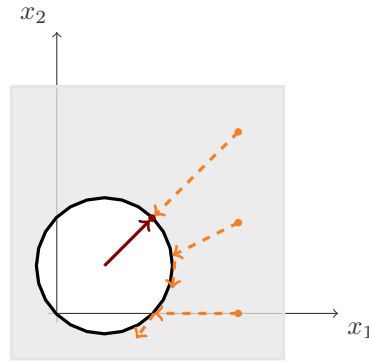


Figure 2: Some state trajectories for the system in Example 1.

### 3 Absolutely Continuous Solutions

For our initial results, we start with the following system class where the state admits absolutely continuous solutions:

$$\dot{x}(t) = Ax(t) - Gw_t \quad \text{for almost all } t \in [t_0, \infty), \quad (9a)$$

$$w_t \in \mathcal{N}(S(t); Hx(t)) \quad (9b)$$

where  $x(t) \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $G \in \mathbb{R}^{n \times l}$ , and  $H \in \mathbb{R}^{l \times n}$ . The initial state  $x(t_0)$  is assumed to satisfy  $Hx(t_0) \in S(t_0)$ . For such systems, the following is assumed so that the lemmas developed in the previous section could be applied.

**Assumption 1.** The matrix  $H \in \mathbb{R}^{l \times n}$  satisfies (5), and  $S(t) \subseteq \mathbb{R}^l$  is contained in the range space of  $H$  for all  $t \geq t_0$ .

The following example motivates how the stability of a constrained dynamical system raises some interesting issues that are not seen in the case of unconstrained dynamical systems.

**Example 1.** In this example, we consider  $S$  to be a complement of an open disk such that the origin is contained in the boundary of  $S$ . For example,  $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 - 1)^2 + (x_2 - 1)^2 - 2 \geq 0\}$ , which is  $r$ -prox-regular for  $r < \sqrt{2}$ . Take  $A := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $G = H = I_{2 \times 2}$ . Then it is seen that no state trajectory starting from the initial condition  $x_1(0) = x_2(0) > 0$  converges to the origin. This is because when such a state trajectory hits the boundary of the set, the only way it can remain in the set is if it acquires static equilibrium at the point of contact with the boundary of the set. But from any other initial condition, the resulting state trajectory continues to slide along the boundary of the set, hence continuing its motion towards the origin.

For constrained dynamical systems described in the above example, where the constraints are not necessarily convex and hence the multivalued part doesn't satisfy the maximal monotone property, we are interested in studying the problem of stability. In the next section, we derive a formula for the basin of attraction of system (9) based on a sufficient condition.

#### 3.1 Sufficient Conditions for Stability

Before stating the result on stability of system (9), we recall some terminology. The system is called asymptotically stable (with respect to the origin) if, for every  $\epsilon > 0$  there exists  $\delta > 0$  such

that  $|x(t)| < \epsilon$  for all  $t \geq t_0$ , and  $\lim_{t \rightarrow \infty} |x(t)| = 0$ , whenever  $|x(t_0)| < \delta$ . An asymptotically stable system has the basin of attraction  $\mathcal{R}$ , if for every  $x(t_0) \in \mathcal{R}$ , the corresponding trajectory converges to the origin.

**Theorem 2.** Consider system (9) under the hypotheses (H1), (H2) and Assumption 1. Suppose there exists a symmetric positive definite matrix  $P$  that satisfies the following for some  $\theta > 0$ :

$$A^\top P + PA \leq -\theta P \quad (10)$$

$$PG = H^\top. \quad (11)$$

For  $0 < \varepsilon < 1$ , define

$$\mathcal{R}_\rho := \{x \in \mathbb{R}^n \mid x^\top P x \leq \rho^2\}, \quad \rho := \frac{\varepsilon \theta r \sigma_{\tilde{H}}}{\|RAR^{-1}\| \|\tilde{H}\|^2}, \quad (12)$$

where  $R$  is the symmetric positive definite matrix such that  $P = R^2$ , and  $\tilde{H} := HR^{-1}$ . Letting  $b := \frac{\|\tilde{H}\|^2}{\sigma_{\tilde{H}}^2}$ , if  $\theta$  is large enough such that

$$(1 - \varepsilon)\theta > \frac{b}{r}v, \quad (13)$$

and  $0 \in S(t)$  for all  $t \geq t_0$ , then system (9) is asymptotically stable and the basin of attraction contains the set  $\mathcal{R}_\rho \cap H^{-1}(S(t_0))$ .

*Proof.* We proceed with the proof in two steps.

*Step 1:* Let  $z := Rx$ , then system (9) in new coordinates becomes

$$\begin{aligned} \dot{z}(t) &= RAR^{-1}z(t) - RGw_t \\ w_t &\in \mathcal{N}(S(t); HR^{-1}z(t)). \end{aligned}$$

From (11), we have  $RG = R^{-1}H^\top$ , so that system (9) is equivalently written as:

$$\dot{z}(t) = RAR^{-1}z(t) - \bar{w}_t \quad (14a)$$

$$\bar{w}_t \in \mathcal{N}(S'(t); z(t)), \quad (14b)$$

where  $S'(t) = \{z \in \mathbb{R}^n \mid HR^{-1}z(t) \in S(t)\}$  is  $r'$ -prox-regular with  $r' := r\sigma_{\tilde{H}}/\|\tilde{H}\|^2$  due to Lemma 3. From Theorem 1, it follows that (14) admits a unique locally absolutely continuous solution over  $[t_0, \infty)$  and from (8), we have

$$|\bar{w}_t| \leq |RAR^{-1}z(t)| + |\dot{v}_{S'}(t)|. \quad (15)$$

*Step 2:* Consider the Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined as  $V(z) = z^\top z$ , then  $V(\cdot)$  is continuously differentiable and its derivative along the trajectories of (14) satisfies the following for almost all  $t \in [t_0, \infty)$ :

$$\begin{aligned} \dot{V}(z(t)) &= z(t)^\top (R^{-1}A^\top R + RAR^{-1})z(t) - 2z(t)^\top \bar{w}_t \\ &\leq -z(t)^\top (R^{-1}A^\top R + RAR^{-1})z(t) + \frac{1}{r'}|\bar{w}_t| \cdot |z(t)|^2, \end{aligned}$$

where we used (4), along with the hypothesis  $0 \in S'(t)$ , to arrive at the last inequality. Since equation (10) is equivalent to  $R^{-1}A^\top R + RAR^{-1} \leq -\theta I$ , using the bound on  $|\bar{w}_t|$  from (15) and  $|\dot{v}_{S'}|$  from Lemma 4a, we get

$$\begin{aligned} \dot{V}(z(t)) &\leq -\theta z(t)^\top z(t) + \frac{1}{r'} (|RAR^{-1}z(t)| + |\dot{v}_{S'}(t)|) \cdot |z(t)|^2 \\ &\leq -\theta |z(t)|^2 + \frac{\|\tilde{H}\|^2}{r\sigma_{\tilde{H}}} (\|RAR^{-1}\| \cdot |z(t)| + \frac{1}{\sigma_{\tilde{H}}} |\dot{v}_S(t)|) |z(t)|^2. \end{aligned}$$

If  $|z(t)| \leq \rho$ , with  $\rho$  defined in (12), then

$$\dot{V}(z(t)) \leq -((1 - \varepsilon)\theta - \frac{b}{r} |\dot{v}_S(t)|) |z(t)|^2,$$

which using (13) gives

$$\dot{V}(z(t)) \leq -\beta |z(t)|^2 \quad \forall z(t) \in S'(t), \quad |z(t)| \leq \rho,$$

for some  $\beta > 0$ . It is further noted that if  $x_0 \in \mathcal{R}_\rho \cap H^{-1}(S(t_0))$ , then  $z_0 = Rx_0 \in S'(t_0)$  with  $|z_0| \leq \rho$ , and since the derivative of  $V(z) = |z|^2$  is negative for all  $t \geq t_0$ , the value of  $|Rx|^2 = x^\top Px$  is also decreasing. The matrix  $P$  being positive definite guarantees that  $\lim_{t \rightarrow \infty} x(t) = 0$ .  $\square$

The conditions for stability given in (10), (11) mean that the triplet  $(A, G, H)$  is dissipative in Willem's sense [46]. Dissipativity has always been an essential property of the linear part in Lur'e systems depicted in Fig. 1. This dissipative relation allows the change of variables  $z = Rx$  in the proof of Theorem 2, which was first introduced in [7] in the context of differential inclusions and used for the sake of analysis in [8, 10, 21].

**Example 2** (Revisiting Example 1). For this example, we may take  $P = I$ , so that  $R = I$ , and  $\theta = 2$ . This gives  $\|RAR^{-1}\| = 1$ , and since the set is not moving, we let  $\dot{v}_S = 0$ . Condition (13) is satisfied for any  $\varepsilon$  arbitrarily close to 1. It then follows that we may take

$$\mathcal{R}_\rho := \{x \in \mathbb{R}^n \mid |x| < 2\sqrt{2}\}.$$

Thus, the region of attraction for this problem is  $S \cap \mathcal{R}_\rho$ .

### 3.1.1 Convex Case

As already pointed out, every convex set is an  $r$ -prox-regular set with  $r$  being arbitrarily large. Thus, we would like to see what shape the result of Theorem 2 acquires when the underlying sets are convex. It is noted that if  $S(\cdot)$  is constant and convex-valued, then the dynamics in (9) are equivalently written as a system of differential variational inequalities, for which the stability has been studied in [21].

**Corollary 1.** *Consider system (9) under the hypotheses (H1), (H2) and Assumption 1. Suppose that  $S(t)$  is convex,  $0 \in S(t)$ , for each  $t \geq t_0$ . If there exist positive definite matrices  $P > 0$ , and a scalar  $\theta > 0$  that satisfy (10) and (11), then the origin of (9) is asymptotically stable with the basin of attraction equal to  $H^{-1}(S(t_0))$ .*

The proof is a direct consequence of Theorem 2 since one can let  $r \rightarrow \infty$  for convex sets. As a result,  $\mathcal{R}_\rho$  is the entire state space, so that every admissible initial condition leads to a trajectory converging to the origin.

### 3.1.2 Linear Complementarity Systems

Linear complementarity systems form an important class of nonsmooth dynamical systems and have been widely used in the modeling of physical systems such as electrical circuits. The general framework proposed in this paper covers a certain subclass of such dynamical systems described as follows:

$$\dot{x}(t) = Ax(t) + H^\top w_t \quad (16a)$$

$$0 \leq w_t \perp v(t) = Hx(t) + Du(t) \geq 0. \quad (16b)$$

Using a basic result from convex analysis, one may write

$$0 \leq w_t \perp Hx(t) + Du(t) \geq 0 \Leftrightarrow -w_t \in \partial\psi_Q(Hx(t) + Du(t)),$$

where  $Q = \mathbb{R}_+^m$ , and  $\psi_Q(\cdot)$  denotes the indicator function of the set  $Q$ , i.e.,  $\psi_Q(x) = 0$  if  $x \in Q$  and  $\psi_Q(x) = +\infty$  otherwise, while  $\partial$  denotes the subdifferential of convex analysis. For each  $t \in [0, \infty)$ , the closed set  $S(t) := \{x \in \mathbb{R}^n \mid Hx + Du(t) \geq 0\}$  and  $\mathbb{R}_+^m$  are convex polyhedral sets and  $\psi_{S(t)}(x) = (\psi_{\mathbb{R}_+^m - Du(t)} \circ H)(x)$ . Thus, using the chain rule,  $H^\top \partial\psi_{\mathbb{R}_+^m}(Hx(t) + Du(t)) = \partial\psi_{S(t)}(x(t)) = \mathcal{N}(S(t); x(t))$ . Using this relation, system (16) can be equivalently described by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - w_t, \\ w_t &\in \mathcal{N}(S(t), x(t)). \end{aligned}$$

which is of the same form as (9). It is noted that  $v_S(\cdot)$  is a locally absolutely continuous function when  $u(\cdot)$  is locally absolutely continuous and that  $S(t)$  is a polyhedral set, and hence convex, for each  $t$ . The stability conditions in Theorem 2 boil down to the existence of a solution to equations (10),(11), which in turn requires the symmetric part of  $A$  to be Hurwitz. This observation is consistent with the stability results obtained in [11, Remark 3.1]. In general, a complementarity system with  $v(t)$  in (16b) of the form  $v(t) = Hx(t) + Ew(t) + Du(t)$  can also be described as a sweeping process, where  $E \geq 0$  has a special structure [8].

### 3.1.3 Scalar Case

Making further comparisons with the convex case, we recall that a normal vector  $w \in \mathcal{N}(S; x)$ , when  $S$  is convex, satisfies the following inequality:

$$\langle w, x' - x \rangle \leq 0 \quad \forall x' \in S,$$

which is similar to the sector condition imposed on the feedback nonlinearity in determining the absolute stability of a Lur'e system.

For an  $r$ -prox-regular set, we merely have the characterization (4) for normal vectors, and it follows from (4) that, in scalar case, for  $x \in S \subseteq \mathbb{R}$ ,

$$\langle -w, x' - x \rangle \geq 0 \quad \text{if } |x - x'| < 2r.$$

Thus, the nonlinearity in the system due to the state constraints actually satisfies the monotonicity condition locally. Also, in the scalar case, a closed set is  $r$ -prox-regular if it is a disjoint union of closed intervals where the distance between one interval and the other is greater than  $2r$ . Thus, in the light of Theorem 2, and the above observation, a scalar system modeled as (9) is asymptotically stable if one of the components contains the origin at all times and the initial condition is within the interval containing the origin.

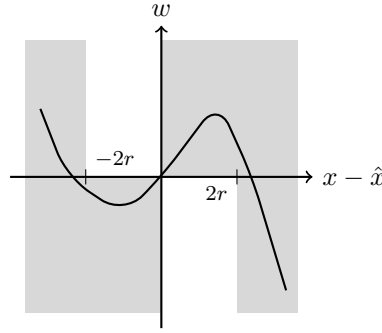


Figure 3: Each element of the Fréchet normal cone satisfies the sector condition locally in scalar case.

### 3.2 Observer Design

We now shift our focus to designing observers for the differential inclusions considered in (9). We suppose that the output equation associated with system (9) is the following one:

$$y(t) = Cx(t) \quad (17)$$

where  $C \in \mathbb{R}^{p \times n}$  and it is assumed that  $p \leq n$ . For our observer design, it will be assumed throughout that the state trajectory stays bounded.

**Assumption 2.** The state trajectory  $x(\cdot)$ , that satisfies (9), is bounded by a constant  $M$  at all times, that is,  $\sup_{t \geq t_0} |x(t)| \leq M$ .

Two different approaches are adopted for observer design: In the first case (Section 3.2.1), we obtain an estimate  $\hat{x}(\cdot)$  such that  $\lim_{t \rightarrow \infty} |x(t) - \hat{x}(t)| = 0$ , under the assumption that the initial value of the state estimation error  $|x(t_0) - \hat{x}(t_0)|$  can be chosen to be small enough (in the sense to be made precise later). In the second approach (Section 3.2.2), there is no a priori bound on the initial value of state estimation error, and the observer is designed to achieve *practical* convergence, that is, for every  $\epsilon > 0$ , there exists  $T > t_0$  such that for all  $t \geq T$ , we have  $|x(t) - \hat{x}(t)| < \epsilon$ . These two approaches are then combined in Section 3.2.3 to obtain an observer that results in asymptotic convergence of the error dynamics to the origin without any bounds on the initial value of the state estimation error.

#### 3.2.1 Locally convergent observer

In the previous section, we saw that even if  $A$  is Hurwitz, the system is asymptotically stable only locally in the presence of non-convex state constraints. For this reason, the Luenberger-like observer proposed in this section generates converging estimates only when the initial error is small enough. The observer we propose, is the following dynamical system:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - C\hat{x}(t)) - G\hat{w}_t \quad (18a)$$

$$\hat{w}_t \in \mathcal{N}(S(t); H\hat{x}(t)). \quad (18b)$$

We next state the criteria to compute the output-injection gain  $L$ , and derive the bound on the initial value of the error between  $x(\cdot)$  and  $\hat{x}(\cdot)$  so that the state estimate of (18) converges to the true state.

**Proposition 1.** Consider system (9) under the hypotheses (H1), (H2) and suppose that Assumptions 1 and 2 hold. If there exist some constants  $\rho, \theta, c_1, c_2 > 0$ ,  $c_3 \geq \frac{c_2}{c_1} \frac{2\|\tilde{H}\|^2}{r\sigma_{\tilde{H}}} (\sigma_H M \|A\| + v)$ , and a matrix  $P$  that satisfy the following inequalities:

$$c_1 I \leq P \leq c_2 I \quad (19a)$$

$$A^\top P + PA - 2\rho C^\top C \leq -(\theta + c_3)P \quad (19b)$$

$$PG = H^\top, \quad (19c)$$

then the state estimate  $\hat{x}(\cdot)$  obtained from (18) with initial condition  $\hat{x}(t_0) \in H^{-1}(S(t_0))$ , and  $L = \rho P^{-1} C^\top$ , satisfies  $\lim_{t \rightarrow \infty} |x(t) - \hat{x}(t)| = 0$ , provided that the initial value of the state estimation error  $e(t_0) := x(t_0) - \hat{x}(t_0)$  belongs to the compact set  $\mathcal{R}_e$ , where

$$\mathcal{R}_e := \{e \in \mathbb{R}^n \mid e^\top P e \leq \rho_e^2\}, \quad \text{and} \quad \rho_e := \frac{\varepsilon \theta r \sigma_{\tilde{H}}}{\|R(A - LC)R^{-1}\| \cdot \|\tilde{H}\|^2} \quad (20)$$

for some  $0 < \varepsilon < 1$ ,  $R = P^{1/2}$ , and  $\tilde{H} = HR^{-1}$ .

*Proof.* Choosing  $R$  such that  $R^2 = P$ , and introducing the coordinate transformation  $z = Rx$  yields (14), along with  $y(t) = CR^{-1}z(t)$ . A similar transformation for the observer with  $\hat{z} = R\hat{x}$  gives:

$$\dot{\hat{z}}(t) = R(A - LC)R^{-1}\hat{z}(t) + RLCR^{-1}z(t) - \hat{w}_t \quad (21a)$$

$$\hat{w}_t \in \mathcal{N}(S'; \hat{z}(t)), \quad (21b)$$

where  $S'(t) := (HR^{-1})^{-1}(S(t)) = RH^{-1}(S(t))$  is  $r'$ -prox-regular, for  $r' = r\sigma_{\tilde{H}}/\|\tilde{H}\|^2$ , due to Lemma 3. Under Theorem 2, the observer is thus well-posed and admits a unique locally absolutely continuous solution since  $H\hat{x}(t_0) \in S(t_0)$ , or equivalently,  $\hat{z}(t_0) \in S'(t_0)$ . Let  $e := x - \hat{x}$ , then  $e(\cdot)$  is a locally absolutely continuous function, and its time derivative satisfies the following equation for almost all  $t \in [t_0, \infty)$ :

$$\dot{e}(t) = (A - LC)e(t) - G(w_t - \hat{w}_t), \quad (22)$$

where  $w_t, \hat{w}_t$  are given by (9b) and (18b), respectively. Choose the candidate Lyapunov function to be  $V(\bar{e}) = \bar{e}^\top \bar{e}$ , where  $\bar{e} = Re$ , then using (19b), the time derivative of  $V(\cdot)$  along the trajectories of (22) for almost all  $t \geq t_0$  is computed as follows:

$$\dot{V}(\bar{e}(t)) = -\bar{e}(t)^\top (\theta + c_3) \bar{e}(t) - 2(z(t) - \hat{z}(t))^\top (\bar{w}_t - \hat{w}_t) \quad (23a)$$

$$\leq -(\theta + c_3)|\bar{e}(t)|^2 + \frac{1}{r'} (|\bar{w}_t| + |\hat{w}_t|) |z(t) - \hat{z}(t)|^2 \quad (23b)$$

$$\leq -(\theta + c_3)|\bar{e}(t)|^2 + \frac{\|\tilde{H}\|^2}{r\sigma_{\tilde{H}}} (|RAR^{-1}z(t)| + |RAR^{-1}\hat{z}(t) + RLCR^{-1}\bar{e}(t)| + \frac{2}{\sigma_{\tilde{H}}} \dot{v}_S(t)) |\bar{e}(t)|^2 \quad (23c)$$

where we arrived at (23b) using (4), and (23c) was obtained by substituting the value of  $r'$  obtained from Lemma 3. Also, in arriving at (23c), the upper bounds on  $|\bar{w}_t|$ , and  $|\hat{w}_t|$ , were obtained by applying (8) to system (14), and (21), respectively. Using the inequalities [6, Corollary 9.6.6],  $\sigma_H \lambda_{\min}(R^{-1}) \leq \sigma_{\tilde{H}} \leq \sigma_H \lambda_{\max}(R^{-1})$ , and the fact that the eigenvalues of  $R$  and  $R^{-1}$



are square roots of eigenvalues of  $P$  and  $P^{-1}$ , respectively, we obtain the following using (19a):

$$\begin{aligned} \dot{V}(\bar{e}(t)) &\leq -(\theta + c_3)|\bar{e}(t)|^2 + \frac{\|R^{-1}\|^2}{\lambda_{\min}(R^{-1})} \frac{\|H\|^2}{r\sigma_H} (2|RAR^{-1}z| + \frac{2}{\lambda_{\min}(R^{-1})\sigma_H}v)|\bar{e}(t)|^2 \\ &\quad + \frac{\|\tilde{H}\|^2}{r\sigma_{\tilde{H}}} \|R(A - LC)R^{-1}\| \cdot |\bar{e}(t)|^3 \end{aligned} \quad (23d)$$

$$\leq -(\theta + c_3)|\bar{e}(t)|^2 + \frac{c_2}{c_1} \frac{\|H\|^2}{r\sigma_H} (2\|A\|M + \frac{2}{\sigma_H}v)|\bar{e}(t)|^2 + \frac{\|\tilde{H}\|^2}{r\sigma_{\tilde{H}}} \|R(A - LC)R^{-1}\| \cdot |\bar{e}(t)|^3 \quad (23e)$$

$$\leq -\theta|\bar{e}(t)|^2 + \frac{\|\tilde{H}\|^2}{r\sigma_{\tilde{H}}} \|R(A - LC)R^{-1}\| \cdot |\bar{e}(t)|^3, \quad (23f)$$

where we used  $|R^{-1}z(t)| = |x(t)| \leq M$  in (23e) and the lower bound on  $c_3$  given in the theorem statement in (23f). The above expression of  $\dot{V}(t)$  is negative definite at each time  $t$ , only if the quadratic term dominates the cubic term, which is the case when  $|\bar{e}(t)|$  is sufficiently small. Assuming that  $|e(t)| \leq \rho_e$ , where  $\rho_e$  is defined in (20), we obtain

$$\dot{V}(\bar{e}(t)) \leq -(1 - \varepsilon)\theta|\bar{e}(t)|^2,$$

and hence  $|\bar{e}(t)|$  converges asymptotically to the origin. It is noted that the condition  $|e(t)| < \rho_e$  holds for all  $t \geq t_0$  if  $e(t_0) \in \mathcal{R}_e$ , since in that case  $|\bar{e}(t_0)| = |Re(t_0)| \leq \rho_e$ , and  $V(t) = |\bar{e}(t)|^2$  decreases at all times.  $\square$

**Remark 4.** In the convex case with  $r \rightarrow \infty$ , the lower bound on the constant  $c_3$  shrinks and (19a), (19b) admit a solution for every observable pair  $(A, C)$ . Moreover the set  $\mathcal{R}_e$  is the entire state space  $\mathbb{R}^n$ , so that one recovers the results given in [9]. Also, for a certain class of linear complementarity systems specified in Section 3.1.2, the proposed state estimator coincides with the observer studied in [22].

**Remark 5** (Positive observers for positive systems). One thing to note in the proposed observer is that the state estimate obtained from the observer respects the same constraints as the actual state. As an implication, if the original system is positive, that is,  $S(t) = \mathbb{R}_+^n$ , for all  $t \geq t_0$ , and hence convex at each time instant with  $G = H = I$ , then the state estimate obtained from (18) has the property that each component of  $\hat{x}(t)$  is nonnegative for  $t \geq t_0$ .

**Example 3** (Local observer for motion on a circle). Consider system (9) with  $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - r^2 = 0\}$ , for some  $r > 0$ , which represents a circle of radius  $r$  centered at the origin and is, by definition,  $r$ -prox-regular. Let the system matrices be  $A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$  and  $C = [1 \ 0]$ . To represent the motion on the circle defined by  $S$ , one may take  $G = H = I_{2 \times 2}$ . We now use the estimator (18) and proceed to compute the output injection gain  $L$  and the basin of attraction for error dynamics  $\mathcal{R}_e$ . In order to satisfy (19c), we must pick  $P = I_{2 \times 2}$ , and hence  $c_1 = c_2 = 1$ . Note that, once the system is initialized in  $S$ , the state  $x(t) \in S, \forall t \geq t_0$ , and thus one may take  $M = r$ , which implies that we must pick  $c_3 \geq \|A\| = 1 + \sqrt{2}$ . Letting  $\varrho = 2$  allows us to choose  $\theta \leq 3 - \sqrt{2}$  in order to satisfy (19b). Thus, with  $L = [2 \ 0]^T$ , and  $\rho_e < \frac{3 - \sqrt{2}}{\sqrt{5}}r$ , Proposition 1 guarantees that the state estimate obtained from (18) converges to actual state of the system.

### 3.2.2 Practically convergent observer

The state estimator proposed in the previous section converges to the actual state only when the initial estimation error is small enough, and it follows the same constraints as the system state.

When the initial error is not small, we are interested in designing an estimator which caters for large (but bounded) values of  $|e(t_0)|$ , and has the property that  $e(t) \in \mathcal{R}_e$  after some finite time, for  $\mathcal{R}_e$  defined in (20). If this is achieved, then one can run the local observer and obtain the asymptotic convergence to the actual state. This motivates us to design a nonlinear state estimator to reduce the state estimation error up to the desired accuracy in finite time, which is primarily adopted from the idea of applying high-gain control to attenuate disturbances [14, 45]. However, in order to implement that idea, the system is required to have relative degree one between  $w_t$  and the output  $y$ , which may be a very strict requirement. A relaxation of this condition was proposed in [19] by letting some higher order derivative of the outputs depend on  $w_t$ . The following assumption, which includes this condition, is now introduced for the development of results in this section.

**Assumption 3.** There exist positive integers  $d_1, \dots, d_p$  such that  $d_i \geq 1$  is the smallest positive integer that satisfies  $c_i A^{d_i-1} G \neq 0$ , where  $c_i$  denotes the  $i$ -th row vector of matrix  $C$ . Also, letting  $C_d := \text{col}(c_1, \dots, c_1 A^{d_1-1}, \dots, c_p, \dots, c_p A^{d_p-1})$ , it is assumed that  $\text{rank}(C_d G) = \text{rank} G$ , where  $\text{col}(x, y) := \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{1 \times 2n}$  for  $x, y \in \mathbb{R}^{1 \times n}$ . Furthermore, the matrix  $H \in \mathbb{R}^{l \times n}$  has rank  $l$  and there exists  $P > 0$  such that  $PG = H^\top$ .

The vector  $y_d = C_d x$  acts as an auxiliary output, whose derivative depends upon  $w_t$ , and if  $y_d$  were directly available, we could have directly implemented the disturbance-attenuating observer. Since this is not the case, we first use a high-gain observer to estimate the derivatives of the output and then use these estimated derivatives in a disturbance-attenuating observer to reduce the state-estimation error within some ball around the origin in finite time. Inspired by the development in [28], we proceed with the following high-gain observer to estimate the vector  $y_d$ :

$$\dot{\bar{y}}_d = \bar{A} \bar{y}_d + \bar{L}(y - \bar{C} \bar{y}_d) \quad (24)$$

$$\bar{A} = \text{block diag}(\bar{A}_1, \dots, \bar{A}_p), A_i \in \mathbb{R}^{d_i \times d_i}, \quad \bar{C} = \text{block diag}(\bar{C}_1, \dots, \bar{C}_p), C_i \in \mathbb{R}^{1 \times d_i},$$

where  $(\bar{A}_i, \bar{C}_i)$  are in Brunovsky form. The gain  $\bar{L}$  is designed as

$$\bar{L} = \text{block diag}(\bar{L}_1, \dots, \bar{L}_2), \quad \bar{L}_i = \text{col}(l_1^i/\epsilon, l_2^i/\epsilon^2, \dots, l_{d_i}^i/\epsilon^{d_i}) \in \mathbb{R}^{d_i \times 1}, \quad (25)$$

where the coefficients  $l_j^i$  are defined such that the roots of  $s^{d_i} + l_1^i s^{d_i-1} + \dots + l_{d_i}^i = 0$  are in the open left-half plane for each  $i = 1, \dots, p$ , and  $\epsilon > 0$  is a design parameter to be specified. Using  $y_{d_{ij}}$  to denote the  $(d_{i-1} + j)$ -th element of  $y_d$ , with  $d_0 = 0$ , the scaled state estimation error is introduced as follows:

$$\eta_{ij} := \frac{y_{d_{ij}} - \bar{y}_{d_{ij}}}{\epsilon^{d_i-j}}, \quad 1 \leq i \leq p, 1 \leq j \leq d_i.$$

Define

$$\bar{D} := \text{block diag}(D_1, \dots, D_p), \quad D_i = \text{diag}(\epsilon^{d_i-1}, \dots, 1) \in \mathbb{R}^{d_i \times d_i},$$

which gives

$$y_d - \bar{y}_d = \bar{D} \eta$$

where  $\eta := \text{col}(\eta_{11}, \dots, \eta_{1d_1}, \dots, \eta_{p1}, \dots, \eta_{pd_p})$ , with  $\|\bar{D}\| = 1$ . Following the idea in [32, Lemma 3], we show that there exists a time  $T_1(\epsilon)$ , such that  $|\eta(t)| \leq c\epsilon$ , for some  $c \geq 0$  and each  $t \geq T_1(\epsilon)$ . Towards this end, we start by writing the dynamics for  $\eta$  as follows [28]:

$$\epsilon \dot{\eta}(t) = (\bar{A} - \bar{L} \bar{C}) \eta(t) + \epsilon \bar{B} w_t,$$

where  $\bar{B} = \text{block diag}(\bar{B}_1, \dots, \bar{B}_p)$ , with  $\bar{B}_i := (0, \dots, 0, c_i A^{d_i-1} G) \in \mathbb{R}^{d_i \times 1}$ . In the sequel, we would need the bound on  $|w_t|$  which is derived in terms of  $|\bar{w}_t|$ . Under Assumption 3, there exists

$P > 0$  that satisfies (11), so that one can introduce the coordinate transformation to arrive at (14) where  $|\bar{w}_t|$  admits the bound (15). Since  $H$  is assumed to have full row rank, and we have the relation  $\bar{w}_t = HR^{-1}w_t =: \tilde{H}w_t$  from Lemma 1, we get  $|w_t| \leq \frac{1}{\sigma_{\tilde{H}}}|\bar{w}_t|$ , and hence

$$|w_t| \leq \frac{1}{\sigma_{\tilde{H}}}(\|RA\| \cdot M + \frac{1}{\sigma_{\tilde{H}}}v) =: \gamma, \quad (26)$$

where we recall that  $M$  is the bound on  $x(\cdot)$  and  $v := \text{ess sup}_{t \geq t_0} |\dot{v}_S(t)|$ . Since  $(\bar{A} - \bar{L}\bar{C})$  is Hurwitz, we take  $\bar{P}$  to be the solution of the following Lyapunov equation:

$$(\bar{A} - \bar{L}\bar{C})^\top \bar{P} + \bar{P}(\bar{A} - \bar{L}\bar{C}) = -I,$$

and let  $\bar{V}(\eta) := \eta^\top \bar{P} \eta$ , whose derivative satisfies the following inequalities:

$$\begin{aligned} \dot{\bar{V}}(\eta(t)) &\leq -\frac{\lambda_{\min}(\bar{P})}{\epsilon} \bar{V}(\eta(t)) + 2\sqrt{\lambda_{\max}(\bar{P})} \|\bar{P}\bar{B}\| |w_t| \sqrt{\bar{V}(\eta(t))} \\ &\leq -\frac{\lambda_{\min}(\bar{P})}{2\epsilon} \bar{V}(\eta(t)), \text{ for } \bar{V}(\eta(t)) \geq \epsilon^2 k_1, \end{aligned}$$

where  $k_1 := 16\|\bar{P}\bar{B}\|^2 \lambda_{\max}(\bar{P}) |w_t|^2 / \lambda_{\min}(\bar{P})^2$ . Using the notation  $d := \max_{1 \leq i \leq p} d_i$ , the following inequality then results from the Gronwall-Bellman lemma:

$$\bar{V}(\eta(t)) \leq \bar{V}(\eta(t_0)) e^{-\lambda_{\min}(\bar{P})t/2\epsilon} \leq \frac{k_2}{\epsilon^{2d-2}} e^{-\lambda_{\min}(\bar{P})t/2\epsilon},$$

where  $k_2 > 0$  is the upper bound on  $\lambda_{\max}(\bar{P})|y_d(t_0) - \bar{y}_d(t_0)|^2$ , which is finite since  $|y_d| = |C_d x| \leq \|C_d\|M$  by Assumption 2. Thus, for  $t > T_1(\epsilon)$ , we have  $\bar{V}(\eta(t)) \leq \epsilon^2 k_1$ , or equivalently,  $|\eta(t)| \leq c\epsilon$ , for all  $t \geq T_1(\epsilon)$ , where  $c := \sqrt{k_1}/\lambda_{\min}(\bar{P})$ , and

$$T_1(\epsilon) = \frac{2\epsilon}{\lambda_{\min}(\bar{P})} \log \left( \frac{k_2}{k_1 \epsilon^{2d}} \right).$$

We now inject  $\bar{y}_d$  obtained from (24) in the following state estimator which is run over an interval  $[t_0, T)$ :

$$\dot{\hat{x}} = A\hat{x} + L_1(y - C\hat{x}) + Gg_\delta(\bar{y}_d, \hat{x}) \quad (27)$$

where

$$g_\delta(\bar{y}_d, \hat{x}) := \frac{F_1(\bar{y}_d - \hat{y}_d)}{|F_1(\bar{y}_d - \hat{y}_d)|\gamma + \delta} \gamma^2, \quad (28)$$

for some  $\delta > 0$ ,  $\hat{y}_d = C_d \hat{x}$ , and  $\gamma \geq |w_t|$  defined in (26). The matrix  $L_1$  in (27) is chosen to satisfy

$$(A - L_1 C)^\top P_1 + P_1(A - L_1 C) \leq -Q_1 \quad (29)$$

for some  $Q_1 > 0$ , and the matrix  $F_1$  in (28) is chosen such that  $P_1 G = (F_1 C_d)^\top$ , which always exists under Assumption 3. The dynamics of the error  $e = x - \hat{x}$  are:

$$\dot{e} = (A - L_1 C)e - Gw_t - Gg_\delta(\bar{y}_d, \hat{x}).$$

Choosing the Lyapunov function  $V_1(e) = e^\top P_1 e$ , we get

$$\begin{aligned} \dot{V}_1 &= e^\top ((A - L_1 C)^\top P_1 + P_1(A - L_1 C))e - e^\top P_1 G w_t - e^\top P_1 G g_\delta(\bar{y}_d, \hat{x}) \\ &= -e^\top Q_1 e - (F_1 C_d e - F_1 \bar{D} \eta)^\top w_t - (F_1 C_d e - F_1 \bar{D} \eta)^\top g_\delta - (F_1 \bar{D} \eta)^\top w_t - (F_1 \bar{D} \eta)^\top g_\delta \\ &\leq -e^\top Q_1 e + \frac{|F_1 C_d e - F_1 \bar{D} \eta| |w_t| (|F_1 C_d e - F_1 \bar{D} \eta| \gamma + \delta) - |F_1 C_d e - F_1 \bar{D} \eta|^2 \gamma^2}{|F_1 C_d e - F_1 \bar{D} \eta| \gamma + \delta} \\ &\quad - (F_1 \bar{D} \eta)^\top (w_t + g_\delta) \\ &\leq -e(t)^\top Q_1 e(t) + \delta - (F_1 \bar{D} \eta)^\top (w_t + g_\delta), \end{aligned}$$

where we used that  $F_1(\bar{y}_d - \hat{y}_d) = F_1(\bar{y}_d - y_d + y_d - \hat{y}_d) = F_1(-\bar{D}\eta + C_d(x - \hat{x})) = F_1 C_d e - F_1 \bar{D}\eta$  in (28), and the fact that  $|w_t| \leq \gamma$ . Since  $|\eta(t)| < c\epsilon$  for  $t \geq T_1(\epsilon)$ , it follows using  $\|\bar{D}\| = 1$  and  $|w_t + g_\delta| \leq 2\gamma$  that

$$\dot{V}_1(e(t)) \leq -\lambda_{\min}(Q_1)|e(t)|^2 + \delta' = -\lambda_{\min}(Q_1)|e(t)| \left( |e(t)| - \frac{\delta'}{\lambda_{\min}(Q_1)} \right) \quad \text{for } t \geq T_1(\epsilon),$$

where  $\delta' := \delta + 2c\|F_1\|\epsilon\gamma$ , which can be made arbitrarily small by picking  $\delta$  and  $\epsilon$  small. Thus,  $\dot{V}_1(e(t)) < 0$  if  $|e(t)| > \frac{\delta'}{\lambda_{\min}(Q_1)}$ . If for some pre-assigned  $\rho^*, \bar{\delta} > 0$ , the constants  $\epsilon$  in (25) and  $\delta$  in (28) are chosen such that

$$0 < \delta' < \bar{\delta} \leq \frac{\rho^{*2} \lambda_{\min}(P_1) \lambda_{\min}(Q_1)}{\lambda_{\max}(P_1)}, \quad (30)$$

then  $|e(t)| \leq \rho^*$ , for all  $t \geq T$ , where

$$T > T_1(\epsilon) + \frac{k_3 \lambda_{\max}(P_1) - \rho^{*2} \lambda_{\min}(P_1)}{\bar{\delta} - \delta'}, \quad (31)$$

and  $k_3$  is the finite upper bound on  $|e(t_0 + T(\epsilon))|^2$ .

**Proposition 2.** *Consider system (9) under Assumptions 1 – 3, and hypotheses (H1), (H2). The state estimate  $\hat{x}(\cdot)$  obtained from (24) – (28), using the design parameters  $\epsilon, \delta$ , and  $\rho^*$  that satisfy (30), has the property that  $|x(t) - \hat{x}(t)| < \rho^*$  for all  $t \geq T$ , where  $T$  is given by (31).*

### 3.2.3 Global Asymptotic convergence

Running the state estimator (24) – (28) makes the state estimation small after some finite time, and in order to achieve convergence to the actual state, we can now activate the locally convergent observer (18) for the interval  $[T, \infty)$ . However, to run (18), we must have  $\hat{x}(T) \in H^{-1}(S(T))$ , which can be done by picking the nearest point to the vector  $\hat{x}(T^-)$  in the set  $\bar{S}(T) := H^{-1}(S(T))$ . As  $S(T)$  is  $r$ -prox-regular,  $\bar{S}(T)$  is  $\bar{r}$ -prox-regular with  $\bar{r} = r\sigma_H/\|H\|^2$  (derived in Lemma 3), and the nearest point in  $\bar{S}(T)$  to the vector  $\hat{x}(T^-)$  is uniquely defined if  $d(\hat{x}(T^-), \bar{S}(T)) < \bar{r}$ . This is done by choosing an appropriate value of  $\rho^*$  in (32). Noting from Lemma 2 that  $d(\hat{x}(T^-), \bar{S}(T)) \leq \frac{1}{\sigma_H} d(H\hat{x}(T^-), S(T)) \leq \frac{\|H\|}{\sigma_H} \cdot |x(T^-) - \hat{x}(T^-)| \leq \frac{\|H\|}{\sigma_H} \rho^*$ , and recalling the definition of  $\rho_e$  in (20), if  $\rho^*$  is chosen such that

$$\rho^* < \min \left\{ \frac{\bar{r}\sigma_H}{\|H\|}, \frac{\rho_e}{2\sqrt{\lambda_{\max}(P)}} \right\}, \quad (32)$$

then  $d(\hat{x}(T^-), \bar{S}(T)) < \bar{r}$ , so that

$$\hat{x}(T) = \text{prox}_{\bar{S}(T)}(\hat{x}(T^-)) \quad (33)$$

is well-defined, and

$$|\hat{x}(T) - x(T)| \leq |x(T^-) - \hat{x}(T^-)| + d(\hat{x}(T^-), \bar{S}(T)) \leq \rho^* + \frac{\|H\|}{\sigma_H} \rho^* \leq 2\rho^* < \frac{\rho_e}{\sqrt{\lambda_{\max}(P)}},$$

which in turn implies that  $e(T) \in \mathcal{R}_e$ , with  $\mathcal{R}_e$  defined in (20). Thus running the estimator (18) for the time interval  $[T, \infty)$ , guarantees that  $\hat{x}(t) \rightarrow x(t)$  under the hypotheses of Proposition 1 without the constraints on initial value of the state estimation error. This way combining the two estimators results in global (with respect to state estimation error) convergence, and this result is summed-up as follows.

**Theorem 3.** Consider system (9) under Assumptions 1 – 3, and hypotheses (H1), (H2). Let us suppose that there exists  $P$  satisfying (19). If the state estimate  $\hat{x}(\cdot)$  is constructed as follows:

- $\hat{x}(\cdot)$  is obtained from (24) – (28) over a time interval  $[t_0, T)$  using the parameters  $\bar{\delta}$  and  $\rho^*$  as in (30) and (32),
- $\hat{x}(T)$  is given by (33), for  $T$  satisfying (31),
- $\hat{x}(\cdot)$  satisfies (18) over the interval  $[T, \infty)$ ,

then  $\lim_{t \rightarrow \infty} |\hat{x}(t) - x(t)| = 0$ .

## 4 Bounded Variations and State Jumps

In this section, we relax (H2) to allow  $v_S(\cdot)$  to be a function of bounded variation, which may introduce discontinuities in the state trajectory. If  $x : I \rightarrow \mathbb{R}^n$  is a function of bounded variation, then one can associate with it a Lebesgue-Stieltjes measure or the so-called *differential measure*  $dx$  on  $I$ . Also, if  $x(\cdot)$  is *rcbv* on  $[s, t]$ , then we have the relation that  $x(t) = x(s) + \int_{]s, t]} dx$ . Since the derivatives of functions of bounded variation do not exist in the classical sense, we use the notion of differential measure to describe the dynamics of the state trajectory<sup>1</sup>:

$$dx \in f(x)d\lambda - \mathcal{N}(S(t); x), \quad \text{with } x(t_0) = x_0 \in S(t_0), \quad (34)$$

where  $\lambda$  denotes the Lebesgue measure on the interval  $[t_0, \infty)$ , and it is assumed that the variation of the set-valued map  $S : [t_0, \infty) \rightrightarrows \mathbb{R}^n$  satisfies the following hypothesis:

(H3) The variation of the set-valued map  $S(\cdot)$ , denoted by  $v_S(\cdot)$ , is *locally rcbv* on  $[t_0, \infty)$ .

The function  $v_S(\cdot)$ , being *locally rcbv*, has countably many discontinuities over the interval  $[t_0, \infty)$ , and we let  $\mathcal{I}$  denote a countable set that indexes the discontinuities of  $v_S(\cdot)$ . The measure  $\mu$  associated with  $v_S(\cdot)$ , i.e.  $\mu = dv_S$ , admits the following decomposition [42, Theorem 6.10]:  $\mu = \mu_{ac} + \mu_{sing} + \mu_d$ , where  $\mu_{ac}$  is absolutely continuous with respect to Lebesgue measure,  $\mu_{sing}$  is associated with a singular function which is continuous everywhere and differentiable almost everywhere with zero derivative (e.g., the Cantor function). We use the notation  $\mu_c := \mu_{ac} + \mu_{sing}$  to denote the continuous part of  $\mu$ . The discontinuous part is given by  $\mu_d := \sum_{i \in \mathcal{I}} \mu(\{t_i\})$ , where  $\mu(\{t_i\})$  denotes the measure of the singleton  $\{t_i\}$  and corresponds to the size of jump in  $v_S(\cdot)$  at  $t_i$ . To clarify some notation later and better understand the definition of the measure  $\mu$ , note that if  $v_S(\cdot)$  is *rcbv* on an interval  $[s, t]$  with a single discontinuity at  $t_i$ ,  $s < t_i < t$ , then  $\mu_d([s, t]) = \mu(\{t_i\}) = v_S(t_i^+) - v_S(t_i^-)$ , and  $\mu_c([s, t]) = v_S(t) - v_S(t_i^+) + v_S(t_i^-) - v_S(s)$ . We remark that  $\mu$  is non-negative since  $v_S(\cdot)$  is nondecreasing.

By introducing (H3), we allow jumps in the variation of the set-valued function  $S(\cdot)$  which in turn induces jumps in the state trajectory  $x(\cdot)$ . The framework of absolutely continuous solutions does not capture this richer class of state trajectories, thus motivating us to consider solutions with bounded variation. It is noteworthy that (H3) also allows for Zeno behavior, i.e., an infinite number of jumps in the state trajectory over finite time intervals. Such functions are used to model behavior of nonsmooth mechanical systems and there are several references [7, 31] which adopt the control theoretic framework to analyze systems with solutions of bounded variation. The major difference between the stability analysis of measure differential inclusions adopted in [7, 31] and the framework of this paper is that the earlier approach uses maximal monotonicity of the multivalued map, and in our case the multivalued operator is not maximal monotone.

<sup>1</sup>The notation  $d\lambda$  corresponds to the usual notation  $dt$  and has been used to avoid confusion later on.

## 4.1 Solution Concepts

The details on existence and precise notion of solutions for system (34) appear in [10, 17], which we now briefly recall. The density of  $dx$  with respect to a positive Radon measure  $\nu$  over an interval  $I$  is defined as:

$$\frac{dx}{d\nu}(t) := \lim_{\varepsilon \rightarrow 0} \frac{dx(I(t, \varepsilon))}{\nu(I(t, \varepsilon))}, \quad (35)$$

where  $I(t, \varepsilon) := I \cap [t - \varepsilon, t + \varepsilon]$ .

**Definition 3.** A mapping  $x : [t_0, \infty) \rightarrow \mathbb{R}^n$  is called a solution of (34) if:

- (S1) the map  $x(\cdot)$  is *locally rcbv*,  $x(t_0) = x_0$ , and  $x(t) \in S(t)$  for all  $t \in [t_0, \infty)$ ,
- (S2) there exists a Radon measure  $\nu$  absolutely continuously equivalent<sup>2</sup> to  $\mu + \lambda$  such that the differential measure  $dx$  is absolutely continuous with respect to  $\nu$ ,  $\frac{dx}{d\nu}(\cdot) \in \mathcal{L}_{loc}^1(I, \mathbb{R}^n; \nu)$ , and

$$-\frac{dx}{d\nu}(t) + f(x(t)) \frac{d\lambda}{d\nu}(t) \in \mathcal{N}(S(t); x(t)), \quad \nu - \text{a.e.}, \quad t \in [t_0, \infty) \quad (36)$$

where

$$\frac{d\lambda}{d\nu}(t) = \lim_{\varepsilon \rightarrow 0} \frac{\lambda(I(t, \varepsilon))}{\nu(I(t, \varepsilon))} \quad (37)$$

denotes the density of Lebesgue measure  $\lambda$  relative to  $\nu$ .

**Observations:** We remark some properties of the foregoing solution framework to give a better understanding of how the state trajectories of (34) evolve:

1. The state  $x(\cdot)$  is discontinuous at a time instant  $t_i$  only if there is a jump in  $v_S(\cdot)$  at that time. If not, then there exists  $t_i$  such that  $\lim_{\varepsilon \rightarrow 0} \nu(I(t_i, \varepsilon)) = 0$ , whereas  $\lim_{\varepsilon \rightarrow 0} dx(I(t_i, \varepsilon))$  denotes the size of jump which does not equal zero, and (S2) is violated as  $\frac{dx}{d\nu}(\cdot) \notin \mathcal{L}_{loc}^1(I, \mathbb{R}^n; \nu)$ .
2. If the set valued mapping  $S(\cdot)$  is not continuous at  $t_i$ , then  $\mu(\{t_i\}) = d_H(S(t_i^-), S(t_i))$ . In that case,  $\frac{d\lambda}{d\nu}(t_i) = 0$ , and (36) becomes

$$-\frac{dx}{d\nu}(t_i) \in \mathcal{N}(S(t_i); x(t_i)).$$

From (35) and recalling  $x(\cdot)$  is *locally rcbv* so that  $x(t_i^+) = x(t_i)$ , it then follows that

$$\frac{x(t_i) - x(t_i^-)}{\mu(\{t_i\})} \in -\mathcal{N}(S(t_i), x(t_i)).$$

Since the right-hand side is a cone and  $\mu(\{t_i\}) > 0$ , we obtain

$$x(t_i^-) - x(t_i) \in \mathcal{N}(S(t_i), x(t_i)).$$

Thus, a possible choice for  $x(t_i)$  is

$$x(t_i) \in \text{prox}_{S(t_i)}(x(t_i^-)),$$

where  $\text{prox}(x, S) := \arg \min_{z \in S} |x - z|$  is uniquely defined for an  $r$ -prox-regular set  $S$  provided that  $d(x, S) < r$ .

<sup>2</sup> A measure  $\mu$  is absolutely continuous with respect to another measure  $\nu$  if  $\nu(I) = 0$  implies  $\mu(I) = 0$ . The measures  $\mu$  and  $\nu$  are absolutely equivalent if they are absolutely continuous with respect to each other.

3. The solution concept adopted in Definition 3 is a natural extension of the absolutely continuous case. If  $v_S(\cdot)$  is absolutely continuous, then we may take  $\mu = \dot{v}_S \lambda$ , so that  $\mu + \lambda = (1 + \dot{v}_S) \lambda$  is absolutely continuously equivalent to  $\lambda$ , and (36) corresponds to the usual differential inclusion (1).

Let us use these observations to formally state the conditions for existence of solution to system (34). The following result basically follows from [10, Theorem 4.4(b)] and [17, Theorem 3.1 and Corollary 3.1] where the existence of solution is proved. The property (38) stated here is not explicitly mentioned in those references, so the proof has been included in Appendix A.

**Theorem 4.** *Consider system (34) where  $f(t, x)$  satisfies the basic assumptions listed in Theorem 1. If (H1), (H3) hold, and  $\sup_{s \geq t_0} \mu(\{s\}) < r/4$ , then, for each  $x_0 \in S(t_0)$ , there exists a unique locally rcbv solution to system (34) that satisfies*

$$\sup_{t \geq t_0} |x(t^+) - x(t^-)| \leq \mu(\{t\}). \quad (38)$$

Furthermore, corresponding to that solution, it holds that

$$\left| \frac{dx}{d\nu}(t) - f(t, x(t)) \frac{d\lambda}{d\nu}(t) \right| \leq |f(t, x(t))| \frac{d\lambda}{d\nu}(t) + \frac{d\mu}{d\nu}(t). \quad (39)$$

**Remark 6.** In the formulation of Theorem 4, we assume a bound on  $\mu(\{t_i\})$  that constrains the size of jumps in the variation of the set-valued mapping  $S(\cdot)$ . This is done so that  $x(t_i)$  could be obtained by projecting  $x(t_i^-)$  onto  $S(t_i)$  because the projections are uniquely defined *only* locally for prox-regular sets. If  $\mu(\{t_i\})$  is too large, then there may be more than one possible choice for  $x(t_i) \in S(t_i)$  that solves (34) and minimizes  $|x(t_i^+) - x(t_i^-)|$ . Even with the constraint  $\mu(\{t_i\}) < r/4$ , it is possible that there is more than one solution to (34), but this constraint ensures that there is only one solution that satisfies (38). For such a solution,  $x(t_i)$  is obtained by projecting  $x(t_i^-)$  on the set  $S(t_i)$  at jump instants, and hence this choice of  $x(t_i) \in S(t_i)$  minimizes  $|x(t_i) - x(t_i^-)|$ . To illustrate these arguments, we recall an example from [17] in which the set-valued mapping  $S(\cdot)$  is given by:

$$S(t) = \begin{cases} [0, 1] \cup \{10\}, & t \in [0, 1) \\ \{1, 10\}, & t \geq 1 \end{cases}$$

so that  $S(t)$ , for each  $t \geq 0$ , is  $r$ -prox regular for  $r < r^* := 4.5$ . Moreover,  $\sup_{t \geq 0} \mu(\{t\}) = \mu(\{1\}) = 1 < \frac{r^*}{4}$ , so that the condition imposed on the measure associated with the variation of  $S(\cdot)$  in Theorem 4 holds. Let us consider the two functions  $x^1, x^2 : [t_0, \infty) \rightarrow \mathbb{R}$  defined as:

$$x^1(t) = \begin{cases} 0, & t \in [0, 1) \\ 1, & t \geq 1 \end{cases} \quad \text{and} \quad x^2(t) = \begin{cases} 0, & t \in [0, 1) \\ 10, & t \geq 1 \end{cases}$$

then  $x^1(\cdot), x^2(\cdot)$  are two different solutions to the inclusion

$$dx^i \in -\mathcal{N}(S(t); x^i) \quad \text{with} \quad x^i(0) = 0 \in S(0), \quad i = 1, 2.$$

But, it is noted that,

$$\sup_{t \geq 0} |x^1(t^+) - x^1(t^-)| = 1 < \frac{r^*}{4} \quad \text{and} \quad \sup_{t \geq 0} |x^2(t^+) - x^2(t^-)| = 10 \geq \frac{r^*}{4}.$$

Thus, *only one* solution satisfies the inequality (38) appearing in the statement of Theorem 4.

Another way to state this result is to say that *the solution with the least variation is unique*. In the rest of the paper, whenever we talk about the solution of (34), we only consider the unique solution with least variation. This issue was not stated in the result given in [10, Theorem 4.4(b)], due to which the proof in the Appendix follows different arguments.

## 4.2 Stability Considerations

From stability viewpoint, we need to analyze whether the jumps introduce any destabilizing effect. The fundamental difference between the convex and prox-regular sets is that the projection onto a convex set is a non-expansive map, that is, a Lipschitz map with Lipschitz constant at most 1. However, the projection on a prox-regular set doesn't satisfy such nice property: It is single-valued and Lipschitz only within a neighborhood of the set, with the Lipschitz constant depending on the size of neighborhood under consideration, which is greater than one (proved in Lemma 5 below). Thus, the jumps in systems with prox-regular constraints introduce discontinuities that may increase the norm of the state. The basic idea in studying the stability of such systems is to limit the number of jumps over an interval such that the system has enough time in-between the discontinuities to compensate for the destabilizing jumps. We will also use the notion of average dwell-time to characterize the frequency of jumps that preserve stability of the system.

The system class we consider is defined by the following equations:

$$dx(t) = Ax(t)d\lambda - Gw_t \quad (40a)$$

$$w_t \in \mathcal{N}(S(t); Hx(t)), \quad (40b)$$

with initial condition  $x(t_0) = x_0$  satisfying  $Hx_0 \in S(t_0)$ . Before proceeding with the stability result, we first state a result parallel to Lemma 4a to describe a bound on the variation of the preimage of a set in the case of bounded variation. Its proof is given in Appendix A.

**Lemma 4b.** *Let  $S(\cdot), S'(\cdot)$  and  $H$  be as in Lemma 4a. If  $v_S(\cdot)$  is locally rcbv, then  $v_{S'}(\cdot)$  is also locally rcbv; moreover,  $\frac{d\mu'_c}{d\nu} \leq \frac{1}{\sigma_H} \frac{d\mu_c}{d\nu}$  and  $\frac{d\mu'_d}{d\nu} \leq \frac{1}{\sigma_H} \frac{d\mu_d}{d\nu}$ , where  $\mu := dv_S$ ,  $\mu' := dv_{S'}$ , and  $\nu := \mu' + \lambda$ .*

The next two lemmas characterize the size and effect of jumps in state trajectories due to discontinuities in the variation of the set valued map  $S(\cdot)$ , and would be used in analyzing the stability of system (40) at the jump instants.

**Lemma 5.** *Let  $S^a, S^b$  be  $r$ -prox-regular sets, and  $\alpha := d_H(S^a, S^b) < \frac{r}{4}$ . Suppose that  $z_i^a \in S^a$ , and  $z_i^b := \text{prox}_{S^b}(z_i^a) \in S^b$ , for  $i = 1, 2$ . Then, using the notation  $\gamma := \left(1 - \frac{\alpha}{r}\right)^{-2}$ , we have*

$$|z_1^b - z_2^b|^2 \leq \gamma |z_1^a - z_2^a|^2. \quad (41)$$

*Proof.* Since the distance between  $S^a$  and  $S^b$  is less than  $r$  by hypothesis, the nearest point to  $z_i^a \in S^a$  in the set  $S^b$  is uniquely defined and given by  $z_i^b$ , for  $i = 1, 2$ . We can write  $z_i^b = z_i^a + \xi_i$ , where  $\xi_i \in \mathcal{N}(S^b, z_i^b)$ , and  $|\xi_i| \leq \alpha$ . Then,

$$\begin{aligned} |z_1^b - z_2^b| \cdot |z_1^a - z_2^a| &\geq \langle z_1^b - z_2^b, z_1^a - z_2^a \rangle = \langle z_1^b - z_2^b, \xi_1 - \xi_2 \rangle + |z_1^b - z_2^b|^2 \\ &\geq -\frac{1}{2r} (|\xi_1| + |\xi_2|) |z_1^b - z_2^b|^2 + |z_1^b - z_2^b|^2 \\ &\geq \left(1 - \frac{\alpha}{r}\right) |z_1^b - z_2^b|^2, \end{aligned}$$

where (4) has been used and whence (41) follows.  $\square$



**Lemma 6.** Let  $\alpha_i := \mu(\{t_i\}) = d_H(S(t_i^+), S(t_i^-)) < \frac{r}{4}$ , and  $\gamma_i := (1 - \frac{\alpha_i}{r})^{-2}$  for each  $i$  belonging to a countable set  $\mathcal{I}$ . Using the notation  $\mu_d := \sum_{i \in \mathcal{I}} \alpha_i$ , there exists a constant  $b_d > 0$  such that

$$\prod_{i \in \mathcal{I}} \gamma_i \leq e^{\frac{b_d}{r} \mu_d}. \quad (42)$$

*Proof.* For the desired claim, we first find an upper bound on the series  $\sum_{i \in \mathcal{I}} \frac{2r\alpha_i - \alpha_i^2}{(r - \alpha_i)^2}$ . We use the fact that  $0 < \alpha_i < \frac{r}{4}$  for each  $i \in \mathcal{I}$ , so that  $(r - \alpha_i) > \frac{3r}{4}$  and  $(2r - \alpha_i) < 2r$ . This in turn gives:

$$\sum_{i \in \mathcal{I}} \frac{2r\alpha_i - \alpha_i^2}{(r - \alpha_i)^2} < \frac{32}{9r} \sum_{i \in \mathcal{I}} \alpha_i = \frac{b_d}{r} \mu_d,$$

where  $b_d = \frac{32}{9}$ , which in general depends on the upper bound on  $\mu(\{t_i\})$ . This inequality is now used to derive the desired result as follows:

$$\prod_{i \in \mathcal{I}} \gamma_i = \prod_{i \in \mathcal{I}} \left( 1 + \frac{2r\alpha_i - \alpha_i^2}{(r - \alpha_i)^2} \right) \leq e^{\sum_{i \in \mathcal{I}} \frac{2r\alpha_i - \alpha_i^2}{(r - \alpha_i)^2}} \leq e^{\frac{b_d}{r} \mu_d}.$$

where the exponential bound was obtained using the fact that  $\ln(1 + x) \leq x$  for  $x \geq 0$ .  $\square$

The bound obtained in (42) in terms of a constant  $b_d$  depends upon the upper bound of  $\mu(\{t_i\})$ ; the smaller the value of  $\mu(\{t_i\})$ , the smaller the constant  $b_d$ .

**Theorem 5.** Consider system (40) under the hypotheses (H1), (H3) and Assumption 1. Suppose there exists a symmetric positive definite matrix  $P$  that satisfies (10)–(11) for some constant  $\theta > 0$ , and take  $R := P^{1/2}$ ,  $\tilde{H} := HR^{-1}$ . Let

$$\kappa(t) := -a(t - t_0) + \frac{b_c}{r} \mu_c([t_0, t]) + \frac{b_d b_c}{r} \mu_d([t_0, t]), \quad (43)$$

where  $a := (1 - \varepsilon)\theta$  for  $0 < \varepsilon < 1$ ,  $b_c := \frac{\|\tilde{H}\|^2}{\sigma^2}$ , and  $b_d := \frac{32}{9}$ . If  $\kappa(t) \leq 2\bar{\kappa}$  for some finite  $\bar{\kappa} \geq 0$ ,  $0 \in S(t)$  for all  $t \geq t_0$ ,  $\sup_{s \geq t_0} \mu(\{s\}) < \frac{r}{4b_c}$ , and  $x(t_0) \in H^{-1}(S(t_0)) \cap \mathcal{R}_\rho$ , then  $x(t) \in \mathcal{R}_{e^{\bar{\kappa}}\rho}$  for all  $t \geq t_0$ , where

$$\mathcal{R}_\rho := \{x \in \mathbb{R}^n \mid x^\top P x \leq \rho^2\}, \quad \text{and} \quad \rho := \frac{\varepsilon \theta r \sigma \tilde{H}}{e^{\bar{\kappa}} \|RAR^{-1}\| \|\tilde{H}\|^2}. \quad (44)$$

If, in addition,  $\kappa(t) \rightarrow -\infty$  as  $t \rightarrow \infty$  (which is the case when  $\mu_c([t_0, \infty))$  and  $\mu_d([t_0, \infty))$  are finite<sup>3</sup>), then system (40) is locally asymptotically stable with  $H^{-1}(S(t_0)) \cap \mathcal{R}_\rho$  contained in the basin of attraction.

*Proof.* We proceed with the proof in two steps.

*Step 1:* Using (11), we have  $RG = R^{-1}H^\top =: \tilde{H}^\top$ , where  $R$  is such that  $R^2 = P$ . Letting  $z := Rx$ , system (40) in new coordinates is written as:

$$dz(t) = RAR^{-1}z(t) - \bar{w}_t \quad (45a)$$

$$\bar{w}_t \in \mathcal{N}(S'(t); z(t)), \quad (45b)$$

<sup>3</sup> If  $v_s(\cdot)$  is *rcbv*, that is,  $\mu([t_0, \infty))$  is finite, then  $\mu_c([t_0, t])$  and  $\mu_d([t_0, t])$  are bounded by some constant for all  $t \geq t_0$ , which may not be the case when  $v_S(\cdot)$  is only *locally rcbv*.

where  $S'(t) = \tilde{H}^{-1}(S(t))$  is  $r'$ -prox-regular with  $r' := r \frac{\sigma_{\tilde{H}}}{\|\tilde{H}\|^2}$  as a result of Lemma 3. The solution of this system is taken in terms of densities with respect to the measure  $\nu = \mu' + \lambda$ , where  $\mu'$  is the measure associated with the variation of  $S'(\cdot)$ . Under the hypothesis that  $\sup_{s \geq t_0} \mu(\{s\}) < \frac{r}{4b_c}$ , Lemma 4b guarantees that  $\sup_{s \geq t_0} \mu'(\{s\}) \leq \frac{r}{4\sigma_{\tilde{H}}b_c} = \frac{r'}{4}$ , and from Theorem 4, we obtain

$$|\bar{w}_t| \leq |RAR^{-1}z(t)| \frac{d\lambda}{d\nu}(t) + \frac{d\mu'}{d\nu}(t). \quad (46)$$

*Step 2:* Consider the Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  defined as  $V(z) = z^\top z$ , then  $V \circ z$  is locally *rcbv* and it has at most countably many jump discontinuities at time instants  $t_i$ , where  $i$  belongs to the countable set  $\mathcal{I}$  that indexes the discontinuities in  $v_S(\cdot)$ . The density of the differential measure of  $V \circ z$  relative to  $\nu$  for each  $t \in ]t_i, t_{i+1}[$  is computed as [37, Section 11]:

$$\begin{aligned} \frac{dV}{d\nu}(z(t)) &= (z(t^+) + z(t^-))^\top \frac{dz}{d\nu}(t) \\ &= z(t)(R^{-1}A^\top R + RAR^{-1})z(t) \frac{d\lambda}{d\nu}(t) - 2z(t)^\top \bar{w}_t \\ &\leq -\theta z(t)^\top z(t) \frac{d\lambda}{d\nu}(t) + \frac{1}{r'} |\bar{w}_t| \cdot |z(t)|^2 \end{aligned} \quad (47a)$$

$$\leq -\theta V(z(t)) \frac{d\lambda}{d\nu}(t) + \frac{\|\tilde{H}\|^2}{r\sigma_{\tilde{H}}} \left( \|RAR^{-1}\| \cdot |z(t)| \frac{d\lambda}{d\nu}(t) + \frac{1}{\sigma_{\tilde{H}}} \frac{d\mu_c}{d\nu}(t) \right) V(z(t)), \quad (47b)$$

where we arrived at (47a) using (4) and (10) under the hypothesis that  $0 \in S(t)$  for all  $t \geq t_0$ . Equation (47b) was obtained using (46), the value of  $r'$ , and the bound on  $\frac{d\mu'_c}{d\nu}$  given in Lemma 4b. If  $|z(t)| < e^{\bar{\kappa}} \rho$ , for each  $t \geq t_0$ , then  $|z(t)|^3 \leq e^{\bar{\kappa}} \rho |z(t)|^2$ , and the above inequality becomes:

$$\frac{dV}{d\nu}(z(t)) \leq -aV(z(t)) \frac{d\lambda}{d\nu}(t) + \frac{b_c}{r} V(z(t)) \frac{d\mu_c}{d\nu}(t), \quad (48)$$

where  $a := (1 - \varepsilon)\theta$ , and  $b_c := \frac{\|\tilde{H}\|^2}{\sigma_{\tilde{H}}^2}$ . Now define for  $t \in ]t_i, t_{i+1}[$ ,

$$W(t) := e^{a(t-t_i) - \frac{b_c}{r} \mu_c([t_i, t])} V(z(t)).$$

The use of product rule and chain rule for differential of functions of bounded variation (recalled from [34, 38] as Lemma 7 and Theorem 6 in Appendix B) gives:

$$\frac{dW}{d\nu}(t) = e^{a(t-t_i) - \frac{b_c}{r} \mu_c([t_i, t])} \frac{dV}{d\nu}(z(t)) + V(z(t)) e^{a(t-t_i) - \frac{b_c}{r} \mu_c([t_i, t])} \left( a \frac{d\lambda}{d\nu}(t) - \frac{b_c}{r} \frac{d\mu_c}{d\nu}(t) \right) \leq 0.$$

Thus,  $W(\cdot)$  is non-increasing on the interval  $]t_i, t_{i+1}[$  for each  $i \in \mathcal{I}$ , and since  $W(t)$  is right-continuous,  $W(t) \leq W(t_i)$ , which implies that

$$V(z(t)) \leq e^{-a(t-t_i) + \frac{b_c}{r} \mu_c([t_i, t])} V(z(t_i)). \quad (49)$$

Also, for each  $t_i$ ,  $i \in \mathcal{I}$ , it follows from Lemma 4b that  $\alpha'_i := \mu'(\{t_i\}) = d_H(S'(t_i^+), S'(t_i^-)) < \frac{\mu(\{t_i\})}{\sigma_{\tilde{H}}} < \frac{r'}{4}$ , which combined with Lemma 5 gives

$$V(z(t_i)) \leq \gamma_i V(z(t_i^-)), \quad (50)$$

where  $\gamma_i = \left(1 - \frac{\alpha'_i}{r'}\right)^{-2}$ . Combining (49) and (50), we get:

$$V(z(t)) \leq \prod_{i \in \mathcal{I}} \gamma_i \cdot e^{-a(t-t_0) + \frac{b_c}{r'} \mu_c([t_0, t])} V(z(t_0)).$$

From Lemma 6,  $\prod_{i \in \mathcal{I}} \gamma_i \leq e^{\frac{b_d}{r'} \mu'_d}$ . Using  $\mu'_d([t_0, t]) \leq \frac{1}{\sigma_H} \mu_d([t_0, t])$  (due to Lemma 4b), and the value of  $r'$ , it follows that

$$V(z(t)) \leq e^{-a(t-t_0) + \frac{b_c}{r'} \mu_c([t_0, t]) + \frac{b_d b_c}{\sigma_H} \mu_d([t_0, t])} V(z(t_0)) =: e^{\kappa(t)} V(z(t_0)). \quad (51)$$

If  $x(t_0) \in \mathcal{R}_\rho \cap H^{-1}(S(t_0))$ , then  $V(z(t_0)) = |z(t_0)|^2 \leq \rho^2$  and combining this with the bound on  $\kappa(t)$  in (51), it is guaranteed that  $|z(t)| \leq e^{\bar{\kappa}} \rho$ , or equivalently  $x(t) \in \mathcal{R}_{e^{\bar{\kappa}} \rho}$ . Indeed,

$$|z(t)|^2 = V(z(t)) \leq e^{2\bar{\kappa}} V(z(t_0)) \leq e^{2\bar{\kappa}} \rho^2.$$

Now, if  $\kappa(t) \rightarrow -\infty$ , then it follows from (51) that  $V(t)$ , and hence  $x(t)$ , converges to zero. This is true, in particular, when  $v_S(\cdot)$  is *rcbv* because its associated measure  $\mu([t_0, \infty))$  is upper bounded by a constant; that is,  $\mu_c([t_0, t])$  and  $\mu_d([t_0, t])$  are bounded by some constant for  $t$  large enough, whereas  $a(t - t_0)$  gets large, and thus  $\kappa(t) \rightarrow -\infty$ .  $\square$

**Remark 7.** In the proof of Theorem 5, we used the differential measure of the Lyapunov function  $V(\cdot)$  to analyze the stability in-between the discontinuities even though  $V(\cdot)$  is continuous and its derivative exists almost everywhere on such intervals. That was done to handle the singular part of the measure  $\mu_c$ . To understand this, consider the function  $f_i : [0, 1] \rightarrow [0, 1]$ ,  $i = 1, 2$ , such that  $f_1(x) = -\alpha x$ , where  $\alpha \in (0, 1)$  and  $f_2(\cdot)$  is the Cantor function on the interval  $[0, 1]$ . Let  $f = f_1 + f_2$ , then  $f(\cdot)$  is a continuous function of bounded variation, and  $\dot{f} = -\alpha < 0$  almost everywhere with respect to Lebesgue measure, but  $f(\cdot)$  is monotonically increasing. However,  $df([0, 1]) = (1 - \alpha) > 0$ , that is, the differential measure of the function shows that the function is increasing on the interval  $[0, 1]$ .

**Remark 8** (Absolutely Continuous  $v_S$ ). We can recover the results of Theorem 2 if it is assumed that the variation in sets, given by  $v_S(\cdot)$ , is absolutely continuous with essentially bounded derivative. In that case,  $\mu_d([t_0, \infty)) = 0$  over each interval and  $\mu_c([t_0, t]) = v_S(t) - v_S(t_0) \leq v(t - t_0)$ .

As stated earlier, for the convex sets, we let  $r \rightarrow \infty$  and the stability results given in Theorem 5 are in coherence with the existing results in the literature. By letting  $r$  go to infinity, we see that  $\rho$  can be chosen to be arbitrarily large. The positive terms in the expression for  $\kappa(t)$  in (43) vanish to zero and hence the system is asymptotically stable starting from every initial condition in  $H^{-1}(S(t_0))$  as long as (10) and (11) are satisfied.

#### 4.2.1 Special Cases

Requiring the value of  $\kappa$  in (43) to be negative or decrease in time represents a tradeoff between the *stable* and *unstable* elements of the system. The instability in the system is due to non-dissipative jumps that are represented by  $\mu_d$ , and the variation of the set  $S(\cdot)$  denoted by  $\mu_c$ , as both these terms increase the value of  $\kappa$ . It is also noted that increase in the value of parameter  $r$  not only enlarges the basin of attraction but also shrinks down the effect of  $\mu_d$  and  $\mu_c$ . Several special cases of Theorem 5 can be derived depending on how much the hypotheses are strengthened.

We consider two special cases where the variation in  $S(\cdot)$  is only due to jump discontinuities so that  $\mu_c = 0$ . In essence, system (40) then behaves like an impulsive system with state constraints and in this case we arrive at a result similar to stability of impulsive systems reported in [23, Theorem 1].

**Corollary 2** (Jumps in Stationary Sets). *Consider system (40) under the hypotheses (H1), (H3) and Assumption 1 with  $\mu_c([t_0, \infty)) = 0$ , and  $t_i, i \in \mathcal{I}$ , denoting the time of discontinuity in  $v_S(\cdot)$ . Let the matrices  $P, R, \tilde{H}$  and the constants  $a, b_c, b_d$  be defined as in the statement of Theorem 5. It follows that for  $|x(t_0)|$  sufficiently small, and  $V(x) = x^\top P x$ , we have*

$$\frac{dV}{d\lambda}(x(t)) \leq -aV(x(t)), \quad t \in ]t_i, t_{i+1}[, \quad (52)$$

and at jump instants  $t_i$ ,

$$V(x(t_i)) \leq \gamma_i V(x(t_i^-)), \quad (53)$$

where  $\gamma_i$  satisfy  $\prod_{i \in \mathcal{I}} \gamma_i \leq e^{\frac{b_d b_c}{r} \mu_d([t_0, \infty))}$ . If, for all  $t \geq t_0$ ,  $0 \in S(t)$ ,  $\mu(\{t\}) < \frac{r}{4b_c}$ , and the following holds:

$$\kappa_d(t) := \frac{b_d b_c}{r} \mu_d([t_0, t]) - a(t - t_0) \leq 2\bar{\kappa}_d \quad (54)$$

for some  $\bar{\kappa}_d > 0$ , then  $V(x(t)) \leq e^{2\bar{\kappa}_d} V(x(t_0))$ . If, in addition,  $\mu_d([t_0, \infty))$  is finite, or  $\kappa_d(t) \rightarrow -\infty$ , then  $V(x(t)) \rightarrow 0$ .

In the above result, the assumptions on system data and the initial condition allow us to arrive at (52) and (53). In general, if a system with *locally rcbv* state trajectories admits a function  $V(\cdot)$  which is continuously differentiable with respect to the state variable, and satisfies (52), (53) then the boundedness of  $V(\cdot)$  still holds under (54), and the convergence of  $V(\cdot)$  to the origin follows if  $\kappa_d(t) \rightarrow -\infty$ . Note that the above result includes the case where the state trajectories may exhibit Zeno phenomenon and thus it can be used independently for systems without state constraints provided the flow and jumps in Lyapunov function satisfy the required assumptions stated in (52) and (53) with some finite bound on the product of  $\gamma_i$ . The result is also in contrast to the existing conditions for stability of an accumulation point (or, Zeno equilibrium) reported in [20, 30], since our work deals with the stability of the origin where the system trajectories may exhibit infinite jumps in finite time away from the origin.

The next special case of Theorem 5 is obtained by excluding the accumulation of discontinuities and assuming that there is an upper bound on average number of discontinuities over each interval. The notion was originally introduced in [24] and is widely used in stability of switched systems where the discrete part tends to bring instability to the system.

**Corollary 3** (Average dwell-time condition). *Consider system (40) under the hypotheses (H1), (H3) and Assumption 1. Let the matrices  $P, R, \tilde{H}$  and the constants  $a, b_c, b_d$  be defined as in the statement of Theorem 5. Assume that the discontinuities of  $v_S(\cdot)$  satisfy the following inequality:*

$$N_\mu(t, t_0) \leq N_0 + \frac{t - t_0}{\tau_a}, \quad (55)$$

where  $N_\mu(t, t_0)$  denotes the number of jumps in  $v_S(\cdot)$  over the open interval  $]t_0, t[$  and  $N_0 > 0$  is a constant. If  $0 \in S(t)$  and  $\mu_c([t_0, t]) = 0$  for all  $t \geq t_0$ ;  $\alpha := \sup_{s \geq t_0} \mu(\{s\}) < \frac{r}{4b_c}$ ; and  $\tau_a$  is such that

$$\tau_a > \frac{\log \gamma}{a}, \quad (56)$$

where  $\gamma := \left(1 - \frac{\alpha}{b_c r}\right)^{-2}$ , then system (40) is locally asymptotically stable with  $\mathcal{R}_\rho \cap H^{-1}(S(t_0))$  contained in the basin of attraction, and  $\mathcal{R}_\rho$  defined in (44) using  $\bar{\kappa} = \frac{N_0 \log \gamma}{2}$ .

*Proof.* Consider  $\gamma_i$  as introduced in (50). Since  $\mu(\{t_i\}) < \alpha$ , for each  $i$ , we get  $1 < \gamma_i = \left(1 - \frac{\mu'(\{t_i\})}{r'}\right)^{-2} \leq \left(1 - \frac{\mu(\{t_i\})}{b_c r}\right)^{-2} < \gamma$ , and hence

$$\prod_{i \in \mathcal{I}} \gamma_i < \gamma^{N_\mu(t, t_0)} = e^{N_\mu(t, t_0) \log \gamma}.$$

Thus, equation (51) gets modified as follows:

$$V(z(t)) \leq e^{N_\mu(t, t_0) \log \gamma - a(t-t_0) + \frac{b_c}{r} \mu_c([t_0, t])} V(z(t_0)).$$

Since the variation is constant between two discontinuities, i.e.,  $\mu_c([t_0, t]) = 0$ , the right-hand side simplifies as:

$$V(z(t)) \leq e^{N_\mu(t, t_0) \log \gamma - a(t-t_0)} V(z(t_0)).$$

The bound on  $N_\mu(t, t_0)$  given in (55) then results in

$$V(z(t)) \leq e^{N_0 \log \gamma} e^{\left(\frac{\log \gamma}{\tau_a} - a\right)(t-t_0)} V(z(t_0)).$$

It follows from (56) that  $\frac{\log \gamma}{\tau_a} - a < 0$ , hence  $V(z(t)) \leq e^{N_0 \log \gamma} V(z(t_0))$  and  $V(z(t)) \rightarrow 0$  as  $t \rightarrow \infty$ .  $\square$

### 4.3 Observer Design

We now consider the problem of state estimation for system (40) using the output equation (17). The observer construction intrinsically follows the same design procedure as discussed in the absolutely continuous case. The care must be taken however due to the destabilizing effect of state jumps. Recent works dealing with the observers for discontinuous systems [43, 44] assume that the jump maps are globally Lipschitz, so that the state estimation error is scaled by the corresponding Lipschitz constant and its effect can be minimized by choosing fast enough convergence rate between the jumps. In our setup, the jumps are introduced by taking the nearest point to  $x(t^-)$  on the set  $H^{-1}(S(t))$ . Since these sets are prox-regular, and not necessarily convex, this map is only locally Lipschitz and not even uniquely defined for large jumps in the sets. This intrinsic difficulty in the problem setup allows us to look at the local results for the state estimators when the sets evolve with bounded variation. For that, we propose the following estimator:

$$d\hat{x} = (A - LC)\hat{x}d\lambda + Ly(t)d\lambda - G\hat{w}_t \quad (57a)$$

$$\hat{w}_t \in \mathcal{N}(S(t); Hx(t)). \quad (57b)$$

The result concerning the convergence of this state estimator is stated next.

**Proposition 3.** *Consider system (40) under hypotheses (H1), (H3) and suppose that Assumptions 1 and 2 hold. Also, assume that there exist some matrix  $P$ , and constants  $\varrho, \theta_e > 0$  that satisfy the following:*

$$c_1 I \leq P \leq c_2 I \quad (58)$$

$$A^\top P + PA - 2\varrho C^\top C \leq -(\theta_e + c_3)P \quad (59)$$

$$PG = H^\top, \quad (60)$$

for some  $c_1, c_2 > 0$  and  $c_3 \geq \frac{c_2}{c_1} \frac{\|H\|^2 \|A\| M}{r \sigma_{\tilde{H}}}$ . With  $R := P^{1/2}$ , and  $\tilde{H} = HR^{-1}$ , let

$$\mathcal{R}_e := \{e \in \mathbb{R}^n \mid e^\top P e \leq \rho_e^2\}, \quad \rho_e := \frac{\varepsilon \theta_e r \sigma_{\tilde{H}}}{\exp(\bar{\kappa}_e) \cdot \|R(A - LC)R^{-1}\| \cdot \|\tilde{H}\|^2} \quad (61)$$

for some  $0 < \varepsilon < 1$ , and  $\kappa_e(t) \leq 2\bar{\kappa}_e$ , for all  $t \geq t_0$  and some  $\bar{\kappa}_e > 0$ , where

$$\kappa_e(t) := -a(t - t_0) + \frac{b_c}{r} \mu_c([t_0, t]) + \frac{b_d b_c}{r} \mu_d([t_0, t]),$$

with  $a := (1 - \varepsilon)\theta_e$ ,  $b_c := \frac{\|\tilde{H}\|^2}{\sigma_{\tilde{H}}^2}$ ,  $b_d := \frac{32}{9}$ . If  $\sup_{s \geq t_0} \mu(\{s\}) \leq \frac{r}{4b_c}$ ;  $e(t_0) := x(t_0) - \hat{x}(t_0) \in \mathcal{R}_e$ ; and  $\kappa_e(t) \rightarrow -\infty$  as  $t \rightarrow \infty$ , then the state estimate  $\hat{x}(t)$  obtained from (57) with initial condition  $H\hat{x}(t_0) \in S(t_0)$ , and  $L = \varrho P^{-1} C^\top$ , satisfies  $\lim_{t \rightarrow \infty} |\hat{x}(t) - x(t)| = 0$ .

*Proof.* Picking  $R = P^{1/2}$  and introducing the coordinate transformation  $z = Rx$ , and  $\hat{z} = R\hat{x}$  shows that the system and observer dynamics are well-posed and yield a unique solution in the sense of Definition 3 using Theorem 4. Consider the state estimation error  $e := x - \hat{x}$ , which is locally *rcbv*. The differential measure of  $e(\cdot)$  satisfies the following inclusion:

$$de \in (A - LC)e(t)d\lambda - G(w_t - \hat{w}_t), \quad (62)$$

where  $w_t, \hat{w}_t$  are given by (40b) and (57b), respectively. Choose the candidate Lyapunov function to be  $V = \bar{e}^\top \bar{e}$ , where  $\bar{e} = Re$ . Then using the computations similar to (47) and (23), the differential measure of  $V \circ \bar{e}$  between two discontinuities over the interval  $]t_i, t_{i+1}[$  is given by:

$$\begin{aligned} \frac{dV}{d\nu}(\bar{e}(t)) &= -(\theta + c_3)\bar{e}(t)^\top \bar{e}(t) \frac{d\lambda}{d\nu} - 2(z(t) - \hat{z}(t))^\top (\bar{w}_t - \hat{w}_t) \\ &\leq -(\theta + c_3)|\bar{e}(t)|^2 \frac{d\lambda}{d\nu}(t) + 2 \frac{c_2}{c_1} \frac{\|H\|^2}{r \sigma_{\tilde{H}}} \|A\| M |\bar{e}(t)|^2 \frac{d\lambda}{d\nu}(t) \\ &\quad + \frac{\|\tilde{H}\|^2}{r \sigma_{\tilde{H}}} \left( \|R(A - LC)R^{-1}\| \cdot |\bar{e}(t)| \frac{d\lambda}{d\nu}(t) + \frac{2}{\sigma_{\tilde{H}}} \frac{d\mu_c}{d\nu}(t) \right) |\bar{e}(t)|^2 \\ &\leq -\theta |\bar{e}(t)|^2 \frac{d\lambda}{d\nu}(t) + \frac{\|\tilde{H}\|^2}{r \sigma_{\tilde{H}}} \left( \|R(A - LC)R^{-1}\| \cdot |\bar{e}(t)| \frac{d\lambda}{d\nu}(t) + \frac{2}{\sigma_{\tilde{H}}} \frac{d\mu_c}{d\nu}(t) \right) |\bar{e}(t)|^2. \end{aligned}$$

If  $|\bar{e}(t)| \leq \exp(\bar{\kappa}_e)\rho_e$ , then using the same arguments used in the proof of Theorem 5, we get

$$\frac{dV}{d\nu}(\bar{e}(t)) \leq -aV(\bar{e}(t)) \frac{d\lambda}{d\nu}(t) + \frac{b}{r} V(\bar{e}(t)) \frac{d\mu_c}{d\nu}(t),$$

which gives

$$V(\bar{e}(t)) \leq \exp(-a(t - t_0) + \frac{b_c}{r} \mu_c([t_0, t]) - \frac{b_d b_c}{r} \mu_d([t_0, t])) V(\bar{e}(t_0)) = \exp(\kappa_e(t)) V(\bar{e}(t_0)). \quad (63)$$

In arriving at (63), it was assumed that  $|e(t)| \leq \exp(\bar{\kappa}_e)\rho_e$ , which holds due to the assumption on  $e(t_0)$  in the theorem statement. Indeed, if  $e(t_0) \in \mathcal{R}_e$ , then  $V(\bar{e}(t_0)) = |\bar{e}(t_0)|^2 \leq \rho_e^2$  and combining this with the bound on  $\kappa_e(t)$ , we get

$$|\bar{e}(t)|^2 = V(\bar{e}(t)) \leq e^{2\bar{\kappa}} V(\bar{e}(t_0)) \leq \exp(2\bar{\kappa}_e)\rho_e^2.$$

It then follows from (63) that if  $\kappa_e(t) \rightarrow -\infty$  with time, then  $V(\bar{e}(t))$  converges to zero.  $\square$

**Example 4** (A linear complementary set with inputs of locally bounded variation). We apply our observer design to linear complementarity systems defined in (16) in which we take,

$$A = \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad D = 1$$

along with the output equation  $y = Cx = [1 \ 1]x$ . The input  $u(\cdot)$  is chosen to be a locally *rcbv* function, defined as,

$$u(t) = \begin{cases} 0 & t \in [0, 1), \\ c_u 2^{k+1} t - c_u(1 + 2^{k+1}(3 - 2^{1-k})) & t \in [3 - 2^{1-k}, 3 - 2^{-k}), \quad 0 \leq k \leq N, \\ 0 & t \in [3 - 2^{-N}, 3) \\ c_u & t \in [3 + 2k, 4 + 2k), \quad k \geq 0 \\ -c_u & t \in [4 + 2k, 5 + 2k), \quad k \geq 0 \end{cases} \quad (64)$$

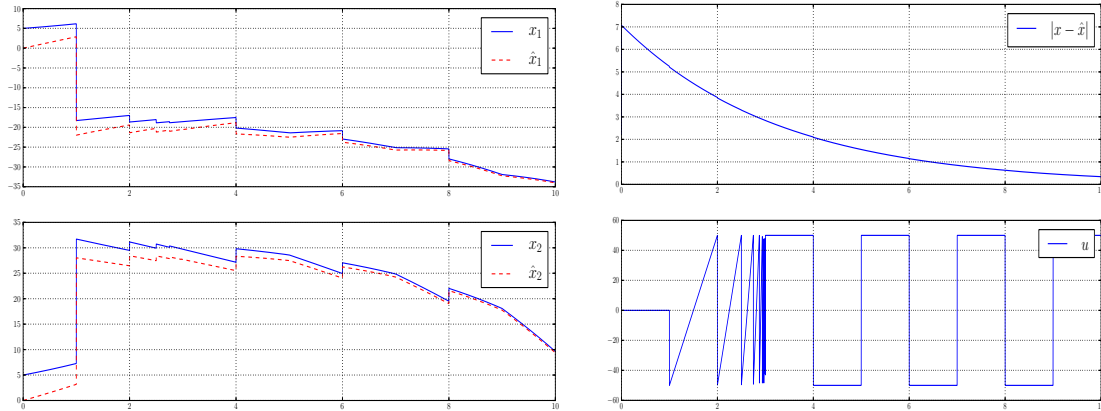
where  $c_u > 0$  is some constant and  $N$  can be an arbitrarily large positive integer. Using the same arguments as in Section 3.1.2 that such systems can be rewritten in the form (40) with  $S(t) := \{x \in \mathbb{R}^n \mid Hx + Du(t) \geq 0\}$  and  $G = H^\top$ . It is verified that  $v_S(\cdot)$  is indeed locally *rcbv* because  $u(\cdot)$  in (16) is locally *rcbv* [10]. We can thus realize the observer (57), in which we pick  $L = [0.3 \ 0.3]^\top$ , so that  $A-LC = \begin{bmatrix} -0.2 & -0.2 \\ 0 & -0.2 \end{bmatrix}$  and  $(A-LC) + (A-LC)^\top = \begin{bmatrix} -0.4 & -0.2 \\ -0.2 & -0.4 \end{bmatrix} \leq -0.2 I_{2 \times 2}$ . Since  $S(t)$  is a polyhedral, and hence convex, for each  $t \geq 0$ , we can let  $r \rightarrow \infty$  and the solution of the estimator (57) initialized with  $\hat{x}_0$  satisfying  $H\hat{x}_0 \in S(0)$  converges to the actual state of the system.

The simulations for this example were carried out on SICONOS platform [2] and the results are shown in Figure 4. The numerical integration schemes implemented in SICONOS are based on Moreau's time-stepping algorithm which were also used to prove the existence of solutions for system (40) in [17]. Because these algorithms are numerically convergent, we can numerically approximate the accumulation of jumps in our simulations. It is seen that the jumps in the input  $u(\cdot)$  introduce the jumps in the state trajectories and despite the accumulation of jumps, the proposed estimator continues to converge to the actual state.

**Remark 9.** If there is no knowledge about the initial state of the system, so that the initial error does not satisfy the bound specified in Proposition 3, then the transposition of high-gain observer (proposed in Section 3.2.3 for the absolutely continuous case) for the systems with bounded variation is not straightforward. The primary reason is that we cannot replicate the jump map with the high-gain approach because, in that case, the state estimate does not follow the set constraints of the actual system and the projection map (which causes jumps in the state of the system) is well-defined only within the neighborhood of the sets  $S(\cdot)$ . Furthermore, if we estimate the state of system (40) using an absolutely continuous trajectory following the same approach, as in Section 3.2.3, we observe that the state estimation error would decrease over an interval without jumps, but due to the jump in the state  $x(\cdot)$  at some time instant  $\{t_i\}$ , it can be shown that

$$\bar{V}(t_i^+) \leq \bar{V}(t_i^-) + 2k_2 \sqrt{\bar{V}(t_i^-)} \mu(\{t_i\}) + k_1 \mu(\{t_i\})^2. \quad (65)$$

The last term in the above inequality shows that the error increases by an amount corresponding to the size of jump in the sets, even though the error before  $t_i$  was small. To avoid this, one may be able to construct a high-gain observer to approximate the state up to the desired accuracy (given by  $\mathcal{R}_e$ ) in finite time under the added assumption that initially  $v_S(\cdot)$  is absolutely continuous, by repeating the procedure outlined in Section 3.2.3. Once the state estimation is made small enough using such techniques, one may project the estimate on the set  $S$  at that time and then



(a) The two plots show the evolution of the state  $x(\cdot)$  and its estimate  $\hat{x}(\cdot)$  for each of their components, where the estimate converges to the actual value.

(b) The top plot shows the norm of the state estimation error converging to zero. The bottom plot shows the input  $u(\cdot)$  in (64) with parameters  $c_u = 50$  and  $N = 100$ .

Figure 4: Simulation results for state estimation in linear complementarity systems with locally *rcbv* inputs.

run the observer (57) to obtain asymptotic convergence to the actual state. This approach is just an extension of the results already developed in this paper, and hence we avoid the details.

## 5 Conclusions

In this paper, we addressed the problem of stability and observer design for a class of measure differential inclusions that model the evolution of state trajectories constrained by non-convex and time-varying sets. The framework of this paper generalizes some previous approach for systems constrained by convex sets. The stability results are local where the basin of attraction is parameterized by a scalar  $r$  which, in some sense, characterizes the lack of convexity such that when  $r \rightarrow \infty$ , we recover the global results of the convex case. We apply the stability analysis to design a locally convergent observer. In addition, a practically convergent observer is designed based on the high-gain approach, which in combination with the locally convergent observer yields asymptotic convergence of the state estimate to the actual state.

The work could be extended in several directions. In system described by (1), the multivalued part is due to the subdifferential of the *indicator* function associated with a set-valued mapping and one could investigate how far these ideas extend when treating the subdifferentials of general time-varying nonconvex set-valued mappings. Another possible extension could be to address the performance of the state estimators in closed-loop with state feedback. Also, as a possible future work, one could consider set-valued maps which are not functions of time but rather the state. This would allow for discontinuities in the state trajectories which are not known to the estimator, and hence a different approach needs to be adopted. A classical example of such a system would be a bouncing ball, which is modeled as a second order sweeping process. Designing estimators for such kind of mechanical systems is an ongoing work.

**Acknowledgements:** The authors would like to thank Olivier Huber for his help with simulations in Example 4.



## A Proofs of Preliminary Results

We collect the proofs of Lemmas 2 – 4b, and Theorem 4 in this Appendix.

### A.1 Proof of Lemma 2

*Proof.* If  $x \in H^{-1}(S)$ , then (7) holds trivially. So it is assumed that  $x \notin H^{-1}(S)$ . By definition,  $\sigma_H$  is the square root of the smallest nonzero eigenvalue of  $H^\top H$ , and using [25, Theorem 4.2.11, page 179],

$$\sigma_H = \max_{v_1, v_2, \dots, v_{n-k} \in \mathbb{R}^n} \min_{\substack{x \neq 0, \\ x \perp v_1, v_2, \dots, v_{n-k}}} \frac{|Hx|}{|x|},$$

where  $k$  is the rank of matrix  $H$ . It is seen that the outer maximization is achieved if  $v_1, v_2, \dots, v_{n-k}$  span the null space of  $H$ . Hence,

$$\sigma_H = \min_{\substack{x \neq 0, \\ x \perp \ker H}} \frac{|Hx|}{|x|}. \quad (66)$$

Also, since  $S$  is in the range space of  $H$ , we have  $H(H^{-1}(S)) = S$ , and thus,

$$\begin{aligned} \frac{d(Hx, S)}{d(x, H^{-1}(S))} &= \min_{y \in S} \max_{z \in H^{-1}(S)} \frac{|Hx - y|}{|x - z|} = \min_{y \in H(H^{-1}(S))} \max_{z \in H^{-1}(S)} \frac{|Hx - y|}{|x - z|} \\ &= \min_{z \in H^{-1}(S)} \max_{\bar{z} \in H^{-1}(S)} \frac{|Hx - H\bar{z}|}{|x - \bar{z}|}. \end{aligned}$$

Since  $H^{-1}(S)$  is closed because  $H$  defines a continuous mapping, therefore, for each  $\bar{z} \in H^{-1}(S)$  there exists a point  $\bar{z}^*$  (possibly non-unique) which achieves the maximum and satisfies  $x - \bar{z}^* \perp \ker H$ , so that

$$\frac{|Hx - H\bar{z}|}{|x - \bar{z}^*|} \geq \frac{|Hx - H\bar{z}|}{|x - \bar{z}|}, \quad \forall \bar{z} \in H^{-1}(S), x - \bar{z} \perp \ker H.$$

Using (66), it then follows that

$$\begin{aligned} \frac{d(Hx, S)}{d(x, H^{-1}(S))} &= \min_{z \in H^{-1}(S)} \max_{\bar{z} \in H^{-1}(S)} \frac{|Hx - H\bar{z}|}{|x - \bar{z}|} \geq \min_{\substack{z \in H^{-1}(S) \\ x - z \perp \ker H}} \frac{|Hx - H\bar{z}|}{|x - \bar{z}|} \\ &\geq \min_{\substack{x \neq z \\ x - z \perp \ker H}} \frac{|H(x - \bar{z})|}{|x - \bar{z}|} = \sigma_H, \end{aligned}$$

which is the desired result.  $\square$

### A.2 Proof of Lemma 3

*Proof.* For the second item, we introduce the following definition: A vector  $w$  belongs to the Fréchet subdifferential of a lower semicontinuous function  $f(\cdot)$ , denoted  $w \in \partial_F f(x)$ , if

$$\liminf_{x' \rightarrow x} \frac{f(x') - f(x) - \langle w, x' - x \rangle}{|x' - x|} \geq 0.$$

Let  $\mathbb{B}$  denote the closed unit ball in  $\mathbb{R}^n$  with respect to the Euclidean norm and consider  $\bar{w} \in \mathcal{N}(S'; \bar{z})$ , then  $\partial_F d_{S'}(\bar{z}) = \mathcal{N}(S'; \bar{z}) \cap \mathbb{B}$ , so that  $\frac{\bar{w}}{|\bar{w}|} =: \hat{w} \in \partial_F d_{S'}(\bar{z})$ , where  $d_{S'}$  denotes the distance function from the set  $S'$ ,  $d_{S'}(z) = d(S', z)$ ,  $\forall z \in \mathbb{R}^n$ . Using the definition of Fréchet subdifferential, it follows that  $\sigma_H \hat{w} \in \partial_F (d_S \circ H)(\bar{z})$ . Indeed, since  $\hat{w}$  satisfies

$$\liminf_{z \rightarrow \bar{z}} \frac{d_{S'}(z) - d_{S'}(\bar{z}) - \langle \hat{w}, z - \bar{z} \rangle}{|z - \bar{z}|} \geq 0,$$

where  $d_{S'}(\bar{z}) = 0 = (d_S \circ H)(\bar{z})$  for all  $\bar{z} \in S'$  and  $(d_S \circ H)(z) \geq \sigma_H d_{S'}(z)$  due to (7), it follows that

$$\liminf_{z \rightarrow \bar{z}} \frac{(d_S \circ H)(z) - (d_S \circ H)(\bar{z}) - \langle \sigma_H \hat{w}, z - \bar{z} \rangle}{|z - \bar{z}|} \geq \sigma_H \liminf_{z \rightarrow \bar{z}} \frac{d_{S'}(z) - d_{S'}(\bar{z}) - \langle \hat{w}, z - \bar{z} \rangle}{|z - \bar{z}|} \geq 0.$$

From the basic subdifferential chain rule [41, Theorem 10.6], it follows under (5) that  $\partial_F (d_S \circ H)(\bar{z}) = H^\top \partial_F d_S(H\bar{z})$ . Hence there exists  $w \in \mathcal{N}(S; H\bar{z})$ , with  $|w| \leq 1$ , such that  $\sigma_H \hat{w} = H^\top w$ . Using the prox-regularity of  $S$ , we combine the aforementioned arguments to arrive at the desired result using the following steps: for any  $z' \in S'$ ,

$$\begin{aligned} \langle \bar{w}, \bar{z} - z' \rangle &= \frac{|\bar{w}|}{\sigma_H} \langle H^\top w, \bar{z} - z' \rangle = \frac{|\bar{w}|}{\sigma_H} \langle w, H(\bar{z} - z') \rangle \\ &\geq -\frac{|\bar{w}|}{2r\sigma_H} |H(\bar{z} - z')|^2. \end{aligned}$$

The above inequality holds for all  $\bar{w} \in \mathcal{N}(S'; \bar{z})$ , and each  $\bar{z} \in S'$ . From the definition of prox-regular sets,  $S'$  is prox-regular with  $r' > 0$ .  $\square$

### A.3 Proof of Lemmas 4a and 4b

*Proof.* Consider the value of  $S(\cdot)$  at two time instants  $t_{i-1} \neq t_i$ , and without loss of generality, assume that  $S(t_{i-1}) \neq S(t_i)$ . Under the given hypotheses, we obtain

$$\sup_{w \in S(t_{i-1})} d(w, S(t_i)) = \sup_{Hy \in S(t_{i-1})} d(Hy, S(t_i)) = \sup_{y \in S'(t_{i-1})} d(Hy, S(t_i)) \geq \sigma_H \sup_{y \in S'(t_{i-1})} d(y, S'(t_i)),$$

where the first and second equalities followed from the fact that  $S(t)$  is contained in the range space of  $H$ , for all  $t$ , and the last inequality is a direct implication of Lemma 2. From the definition of Hausdorff distance (2), it then follows that  $d_H(S'(t_{i-1}), S'(t_i)) \leq \frac{1}{\sigma_H} d_H(S(t_{i-1}), S(t_i))$ . Next, for a fixed  $t \in (t_0, \infty)$ , consider  $h > 0$  to be small enough such that  $t_0 \leq t - h$ , and sequences  $\{t_i\}_{i=1}^k \in [t - h, t + h]$ , for  $k \in \mathbb{N}$ , then

$$\begin{aligned} v_{S'}(t + h) - v_{S'}(t - h) &= \sup_{\{t_i\}_{i=1}^k \in [t-h, t+h]} d_H(S'(t_i), S'(t_{i-1})) \\ &\leq \frac{1}{\sigma_H} \sup_{\{t_i\}_{i=1}^k \in [t-h, t+h]} d_H(S(t_i), S(t_{i-1})) \\ &= \frac{1}{\sigma_H} (v_S(t + h) - v_S(t - h)). \end{aligned}$$

Dividing both sides of above expression by  $h$ , and letting  $h$  go to zero, we get  $\dot{v}_{S'}(t) \leq \frac{1}{\sigma_H} \dot{v}_S(t)$  which proves Lemma 4a. Using the definition of density of measures (37), and taking  $\nu = \mu' + \lambda$ ,

we prove Lemma 4b as follows:

$$\begin{aligned} \frac{d\mu'}{d\nu}(t) &= \lim_{\varepsilon \rightarrow 0} \frac{\mu'([t-\varepsilon, t+\varepsilon])}{\nu([t-\varepsilon, t+\varepsilon])} = \lim_{\varepsilon \rightarrow 0} \frac{v_{S'}(t+\varepsilon) - v_{S'}(t-\varepsilon)}{\nu([t-\varepsilon, t+\varepsilon])} \\ &\leq \frac{1}{\sigma_H} \lim_{\varepsilon \rightarrow 0} \frac{v_S(t+\varepsilon) - v_S(t-\varepsilon)}{\nu([t-\varepsilon, t+\varepsilon])} = \frac{1}{\sigma_H} \lim_{\varepsilon \rightarrow 0} \frac{\mu([t-\varepsilon, t+\varepsilon])}{\nu([t-\varepsilon, t+\varepsilon])} = \frac{1}{\sigma_H} \frac{d\mu}{d\nu}(t), \end{aligned}$$

where the limit in the last equality is well-defined since  $\mu$  is absolutely continuous with respect to  $\nu$ . In the above inequality, if  $t$  is a time instant between two consecutive discontinuities of  $v_S$ , then we have  $d\mu'_c \leq \frac{1}{\sigma_H} d\mu_c$ , and if there is a jump discontinuity in  $v_S(\cdot)$  at time  $t$ , then  $d\mu'_d \leq \frac{1}{\sigma_H} d\mu_d$ .  $\square$

#### A.4 Proof of Theorem 4

*Proof.* Using the solution concept proposed in [17], for  $s < t$  sufficiently close to  $t$ , we have

$$x(t) = \text{prox}_{S(t)}(x(s) + (t-s)f(s, x(s)))$$

which gives

$$\begin{aligned} |x(t) - x(s) - (t-s)f(s, x(s))| &\leq d(x(s) + (t-s)f(s, x(s)), S(t)) \\ &\leq d(x(s), S(t)) + (t-s)|f(s, x(s))| \\ &\leq \mu([s, t]) + (t-s)|f(s, x(s))|. \end{aligned}$$

Using the Cauchy-Schwarz inequality, for  $\xi \neq 0$  we get

$$\begin{aligned} \left\langle \frac{\xi(t)}{|\xi(t)|}, x(t) - x(s) - (t-s)f(s, x(s)) \right\rangle &\leq |x(t) - x(s) - (t-s)f(s, x(s))| \\ &\leq \mu([s, t]) + (t-s)|f(s, x(s))|. \end{aligned}$$

Dividing both sides by  $\nu([s, t])$ , we get

$$\left\langle \frac{\xi(t)}{|\xi(t)|}, \frac{x(t) - x(s)}{\nu([s, t])} - \frac{(t-s)}{\nu([s, t])} f(s, x(s)) \right\rangle \leq \frac{\mu([s, t])}{\nu([s, t])} + \frac{(t-s)}{\nu([s, t])} |f(s, x(s))|$$

Since  $\nu([s, t]) \geq \lambda([s, t]) = t-s$ , taking the limit  $s \rightarrow t$  gives

$$\left\langle \frac{\xi(t)}{|\xi(t)|}, \frac{dx}{d\nu} - f(t, x(t)) \frac{d\lambda}{d\nu}(t) \right\rangle \leq \frac{d\mu}{d\nu}(t) + |f(t, x(t))| \frac{d\lambda}{d\nu}(t)$$

Letting

$$\xi(t) := \frac{dx}{d\nu}(t) - f(t, x(t)) \frac{d\lambda}{d\nu}(t),$$

we get

$$|\xi(t)| = \left\langle \frac{\xi(t)}{|\xi(t)|}, \xi(t) \right\rangle \leq \frac{d\mu}{d\nu}(t) + |f(t, x(t))| \frac{d\lambda}{d\nu}(t).$$

If  $t_i$  is a point of discontinuity in the variation of  $S(\cdot)$ , then  $d\lambda(t_i) = 0$ , and

$$|\xi(t_i)| = \left| \frac{dx}{d\nu}(t_i) \right| \leq \frac{d\mu}{d\nu}(t_i)$$

and hence

$$|x(t_i^+) - x(t_i^-)| = \left| \int_{\{t_i\}} dx \right| \leq \int_{\{t_i\}} \left| \frac{dx}{d\nu} \right| d\nu \leq \int_{\{t_i\}} d\mu = \mu(\{t_i\}).$$

The arguments for unicity of the solutions can be found in the proof of [10, Theorem 4.4].  $\square$

## B Calculus of BV functions

The following results relate to computation of differential of the product and composition of functions.

**Theorem 6** (Chain Rule [38, Theorem 3]). *Let  $f : I \rightarrow \mathbb{R}^n$  be locally BV with differential measure  $df = f'_\mu d\mu$ ; here  $d\mu$  is a nonnegative real measure on  $I$  and  $f \in \mathcal{L}_1^{loc}(I, \mathbb{R}^n; d\mu)$ . Let  $\Omega$  be an open convex subset of  $\mathbb{R}^n$  and suppose that  $\gamma : \Omega \rightarrow \mathbb{R}$  is continuously Fréchet-differentiable, with gradient at point  $x$  denoted by  $\nabla\gamma(x)$ . Then  $\gamma \circ f$  is locally BV and the differential measure  $d(\gamma \circ f)$  admits as density relative to  $d\mu$  the function  $t \rightarrow \langle \theta(t), f'_\mu(t) \rangle$ , where  $\theta : I \rightarrow \mathbb{R}^n$  is defined as:*

$$\theta(t) = \int_0^1 \nabla\gamma[(1-s)f(t^-) + sf(t^+)] ds.$$

*In particular, the function  $f^\top f$  is locally BV, and its differential measure admits the following function as density relative to  $d\mu$ :*

$$\frac{d(f^\top f)}{d\mu}(t) = \langle f'_\mu(t), f(t^+) + f(t^-) \rangle.$$

The following result is taken from [34, Page 8].

**Lemma 7** (Product Rule). *If  $u, v$  are  $\mathbb{R}^n$  valued functions of bounded variation over an interval  $I$ , then their inner product  $\langle u, v \rangle : I \rightarrow \mathbb{R}$  is also a function of bounded variation, and*

$$d(\langle u, v \rangle) = \left\langle \frac{v^+ + v^-}{2}, du \right\rangle + \left\langle \frac{u^+ + u^-}{2}, dv \right\rangle.$$

*In particular when  $u, v$  are continuous, then*

$$d(\langle u, v \rangle) = \langle v, du \rangle + \langle u, dv \rangle.$$

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